Some Inequalities for the Euclidean Operator Radius of Two Operators in Hilbert Spaces

SOME INEQUALITIES FOR THE EUCLIDEAN OPERATOR RADIUS OF TWO OPERATORS IN HILBERT SPACES

SEVER S. DRAGOMIR

Abstract. Some sharp bounds for the Euclidean operator radius of two bounded linear operators in Hilbert spaces are given. Their connection with Kittaneh's recent results which provide sharp upper and lower bounds for the numerical radius of linear operators are also established.

1. Introduction

Let $B(H)$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle\cdot,\cdot\rangle$. For $A \in B(H)$, let $w(A)$ and $\|A\|$ denote the numerical radius and the usual operator norm of A, respectively. It is well known that $w(\cdot)$ defines a norm on $B(H)$, and for every $A \in B(H)$,

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|. \tag{1.1}$$

For other results concerning the numerical range and radius of bounded linear operators on a Hilbert space, see [2] and [3].

In [4], F. Kittaneh has improved (1.1) in the following manner:

$$\frac{1}{4}\|A^*A + AA^*\| \leq w^2(A) \leq \frac{1}{2}\|A^*A + AA^*\|, \tag{1.2}$$

with the constants $\frac{1}{4}$ and $\frac{1}{2}$ as best possible.

Following Popescu's work [5], we consider the *Euclidean operator radius* of a pair (C, D) of bounded linear operators defined on a Hilbert space $(H; \langle\cdot,\cdot\rangle)$. Note that in [5], the author has introduced the concept for an $n-$tuple of operators and pointed out its main properties.

Let (C, D) be a pair of bounded linear operators on H. The *Euclidean operator radius* is defined by:

$$w_e(C, D) := \sup_{\|x\|=1}\left(|\langle Cx, x\rangle|^2 + |\langle Dx, x\rangle|^2\right)^{1/2}. \tag{1.3}$$

As pointed out in [5], $w_e : B^2(H) \to [0, \infty)$ is a norm and the following inequality holds:

$$\frac{\sqrt{2}}{4}\|C^*C + D^*D\|^{1/2} \leq w_e(C, D) \leq \|C^*C + D^*D\|^{1/2}, \tag{1.4}$$

where the constants $\frac{\sqrt{2}}{4}$ and 1 are best possible in (1.4).

Date: 14 February, 2006.

2000 *Mathematics Subject Classification.* Primary 47A12, 47A30, 47A63.

Key words and phrases. Numerical radius, Euclidean radius, Bounded linear operators, Operator norm, Self-adjoint operators.

We observe that, if C and D are self-adjoint operators, then (1.4) becomes

$$\frac{\sqrt{2}}{4}\left\|C^2+D^2\right\|^{1/2} \le w_e(C,D) \le \left\|C^2+D^2\right\|^{1/2}. \tag{1.5}$$

We observe also that if $A \in B(H)$ and $A = B + iC$ is the Cartesian decomposition of A, then

$$\begin{aligned} w_e^2(B,C) &= \sup_{\|x\|=1}\left[|\langle Bx,x\rangle|^2 + |\langle Cx,x\rangle|^2\right] \\ &= \sup_{\|x\|=1}|\langle Ax,x\rangle|^2 = w^2(A). \end{aligned}$$

By the inequality (1.5) and since (see [4])

$$A^*A + AA^* = 2\left(B^2+C^2\right), \tag{1.6}$$

then we have

$$\frac{1}{16}\|A^*A + AA^*\| \le w^2(A) \le \frac{1}{2}\|A^*A + AA^*\|. \tag{1.7}$$

We remark that the lower bound for $w^2(A)$ in (1.7) provided by Popescu's inequality (1.4) is not as good as the first inequality of Kittaneh from (1.2). However, the upper bounds for $w^2(A)$ are the same and have been proved using different arguments.

The main aim of this paper is to extend Kittaneh's result to Euclidean radius of two operators and investigate other particular instances of interest. Related results connecting the Euclidean operator radius, the usual numerical radius of a composite operator and the operator norm are also provided.

2. Some Inequalities for the Euclidean Operator Radius

The following result concerning a sharp lower bound for the Euclidean operator radius may be stated:

Theorem 1. *Let $B, C : H \to H$ be two bounded linear operators on the Hilbert space $(H; \langle\cdot,\cdot\rangle)$. Then*

$$\frac{\sqrt{2}}{2}\left[w\left(B^2+C^2\right)\right]^{1/2} \le w_e(B,C)\left(\le \|B^*B + C^*C\|^{1/2}\right). \tag{2.1}$$

The constant $\frac{\sqrt{2}}{2}$ is best possible in the sense that it cannot be replaced by a larger constant.

Proof. We follow a similar argument to the one from [4].

For any $x \in H$, $\|x\| = 1$, we have

$$\begin{aligned} |\langle Bx,x\rangle|^2 + |\langle Cx,x\rangle|^2 &\ge \frac{1}{2}\left(|\langle Bx,x\rangle| + |\langle Cx,x\rangle|\right)^2 \\ &\ge \frac{1}{2}|\langle (B\pm C)x,x\rangle|^2. \end{aligned} \tag{2.2}$$

Taking the supremum in (2.2), we deduce

$$w_e^2(B,C) \ge \frac{1}{2}w^2(B\pm C). \tag{2.3}$$

Utilising the inequality (2.3) and the properties of the numerical radius, we have successively:

$$
\begin{aligned}
2w_e^2(B,C) &\geq \frac{1}{2}\left[w^2(B+C)+w^2(B-C)\right] \\
&\geq \frac{1}{2}\left\{w\left[(B+C)^2\right]+w\left[(B-C)^2\right]\right\} \\
&\geq \frac{1}{2}\left\{w\left[(B+C)^2+(B-C)^2\right]\right\} \\
&= w\left(B^2+C^2\right),
\end{aligned}
$$

which gives the desired inequality (2.1).

The sharpness of the constant will be shown in a particular case, later on. ∎

Corollary 1. *For any two self-adjoint bounded linear operators B, C on H, we have*

$$\frac{\sqrt{2}}{2}\left\|B^2+C^2\right\|^{1/2} \leq w_e(B,C)\left(\leq \left\|B^2+C^2\right\|^{1/2}\right). \tag{2.4}$$

The constant $\frac{\sqrt{2}}{2}$ is sharp in (2.4).

Remark 1. *The inequality (2.4) is better than the first inequality in (1.5) which follows from Popescu's first inequality in (1.4). It also provides, for the case that B, C are the self-adjoint operators in the Cartesian decomposition of A, exactly the lower bound obtained by Kittaneh in (1.2) for the numerical radius $w(A)$. Moreover, since $\frac{1}{4}$ is a sharp constant in Kittaneh's inequality (1.2), it follows that $\frac{\sqrt{2}}{2}$ is also the best possible constant in (2.4) and (2.1), respectively.*

The following particular case may be of interest:

Corollary 2. *For any bounded linear operator $A : H \to H$ and $\alpha, \beta \in \mathbb{C}$ we have:*

$$
\begin{aligned}
\frac{1}{2}w\left[\alpha^2A^2+\beta^2(A^*)^2\right] &\leq \left(|\alpha|^2+|\beta|^2\right)w^2(A) \\
&\quad \left(\leq \left\||\alpha|^2A^*A+|\beta|^2AA^*\right\|\right).
\end{aligned} \tag{2.5}
$$

Proof. If we choose in Theorem 1, $B = \alpha A$ and $C = \beta A^*$, we get

$$w_e^2(B,C) = \left(|\alpha|^2+|\beta|^2\right)w^2(A)$$

and

$$w\left(B^2+C^2\right) = w\left[\alpha^2A^2+\beta^2(A^*)^2\right],$$

which, by (2.1) implies the desired result (2.5). ∎

Remark 2. *If we choose in (2.5) $\alpha = \beta \neq 0$, then we get the inequality*

$$\frac{1}{4}\left\|A^2+(A^*)^2\right\| \leq w^2(A)\left(\leq \frac{1}{2}\left\|A^*A+AA^*\right\|\right), \tag{2.6}$$

for any bounded linear operator $A \in B(H)$.

If we choose in (2.5), $\alpha = 1$, $\beta = i$, then we get

$$\frac{1}{4}w\left[A^2-(A^*)^2\right] \leq w^2(A), \tag{2.7}$$

for every bounded linear operator $A : H \to H$.

The following result may be stated as well.

Theorem 2. *For any two bounded linear operators B, C on H we have:*

$$\frac{\sqrt{2}}{2}\max\left\{w\left(B+C\right), w\left(B-C\right)\right\} \leq w_e\left(B,C\right) \leq \frac{\sqrt{2}}{2}\left[w^2\left(B+C\right)+w^2\left(B-C\right)\right]^{1/2}. \tag{2.8}$$

The constant $\frac{\sqrt{2}}{2}$ is sharp in both inequalities.

Proof. The first inequality follows from (2.3).

For the second inequality, we observe that

$$\left|\langle Cx,x\rangle \pm \langle Bx,x\rangle\right|^2 \leq w^2\left(C \pm B\right) \tag{2.9}$$

for any $x \in H$, $\|x\| = 1$.

The inequality (2.9) and the parallelogram identity for complex numbers give:

$$\begin{aligned} 2\left[\left|\langle Bx,x\rangle\right|^2 + \left|\langle Cx,x\rangle\right|^2\right] &= \left|\langle Bx,x\rangle - \langle Cx,x\rangle\right|^2 + \left|\langle Bx,x\rangle + \langle Cx,x\rangle\right|^2 \\ &\leq w^2\left(B+C\right) + w^2\left(B-C\right), \end{aligned} \tag{2.10}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum in (2.9) we deduce the desired result (2.8).

The fact that $\frac{\sqrt{2}}{2}$ is the best possible constant follows from the fact that for $B = C \neq 0$ one would obtain the same quantity $\sqrt{2}w\left(B\right)$ in all terms of (2.8). ∎

Corollary 3. *For any two self-adjoint operators B, C on H we have:*

$$\frac{\sqrt{2}}{2}\max\left\{\|B+C\|, \|B-C\|\right\} \leq w_e\left(B,C\right) \leq \frac{\sqrt{2}}{2}\left[\|B+C\|^2 + \|B-C\|^2\right]^{1/2}. \tag{2.11}$$

The constant $\frac{\sqrt{2}}{2}$ is best possible in both inequalities.

Corollary 4. *Let A be a bounded linear operator on H. Then*

$$\begin{aligned} &\frac{\sqrt{2}}{2}\max\left\{\left\|\frac{(1-i)A+(1+i)A^*}{2}\right\|, \left\|\frac{(1+i)A+(1-i)A^*}{2}\right\|\right\} \\ &\leq w\left(A\right) \\ &\leq \frac{\sqrt{2}}{2}\left[\left\|\frac{(1-i)A+(1+i)A^*}{2}\right\|^2 + \left\|\frac{(1+i)A+(1-i)A^*}{2}\right\|^2\right]^{1/2}. \end{aligned} \tag{2.12}$$

Proof. Follows from (2.11) applied for the Cartesian decomposition of A. ∎

The following result may be stated as well:

Corollary 5. *For any A a bounded linear operator on H and $\alpha, \beta \in \mathbb{C}$, we have:*

$$\begin{aligned} &\frac{\sqrt{2}}{2}\max\left\{w\left(\alpha A+\beta A^{*}\right), w\left(\alpha A-\beta A^{*}\right)\right\} \\ &\leq \left(|\alpha|^{2}+|\beta|^{2}\right)^{1/2} w(A) \\ &\leq \frac{\sqrt{2}}{2}\left[w^{2}\left(\alpha A+\beta A^{*}\right)+w^{2}\left(\alpha A-\beta A^{*}\right)\right]^{1/2}. \end{aligned} \tag{2.13}$$

Remark 3. *The above inequality (2.13) contains some particular cases of interest. For instance, if $\alpha=\beta\neq 0$, then by (2.13) we get*

$$\begin{aligned} &\frac{1}{2}\max\left\{\left\|A+A^{*}\right\|, \left\|A-A^{*}\right\|\right\} \\ &\qquad\leq w(A) \leq \frac{1}{2}\left[\left\|A+A^{*}\right\|^{2}+\left\|A-A^{*}\right\|^{2}\right]^{1/2}, \end{aligned} \tag{2.14}$$

since, obviously $w(A+A^{*})=\|A+A^{*}\|$ and $w(A-A^{*})=\|A-A^{*}\|$, $A-A^{*}$ being a normal operator.

Now, if we choose in (2.13), $\alpha=1$ and $\beta=i$, and taking into account that $A+iA^{*}$ and $A-iA^{*}$ are normal operators, then we get

$$\begin{aligned} &\frac{1}{2}\max\left\{\left\|A+iA^{*}\right\|, \left\|A-iA^{*}\right\|\right\} \\ &\qquad\leq w(A) \leq \frac{1}{2}\left[\left\|A+iA^{*}\right\|^{2}+\left\|A-iA^{*}\right\|^{2}\right]^{1/2}. \end{aligned} \tag{2.15}$$

The constant $\frac{1}{2}$ is best possible in both inequalities (2.14) and (2.15).

The following simple result may be stated as well.

Proposition 1. *For any two bounded linear operators B and C on H, we have the inequality:*

$$w_{e}(B, C) \leq \left[w^{2}(C-B)+2w(B)\,w(C)\right]^{1/2}. \tag{2.16}$$

Proof. For any $x\in H$, $\|x\|=1$, we have

$$\begin{aligned} &|\langle Cx, x\rangle|^{2}-2\operatorname{Re}\left[\langle Cx, x\rangle\overline{\langle Bx, x\rangle}\right]+|\langle Bx, x\rangle|^{2} \\ &\qquad = |\langle Cx, x\rangle-\langle Bx, x\rangle|^{2} \leq w^{2}(C-B), \end{aligned}$$

giving

$$\begin{aligned} |\langle Cx, x\rangle|^{2}+|\langle Bx, x\rangle|^{2} &\leq w^{2}(C-B)+2\operatorname{Re}\left[\langle Cx, x\rangle\overline{\langle Bx, x\rangle}\right] \\ &\leq w^{2}(C-B)+2|\langle Cx, x\rangle||\langle Bx, x\rangle| \end{aligned} \tag{2.17}$$

for any $x\in H$, $\|x\|=1$.

Taking the supremum in (2.17) over $\|x\|=1$, we deduce the desired inequality (2.16). ∎

In particular, if B and C are self-adjoint operators, then

$$w_{e}(B, C) \leq \left(\|B-C\|^{2}+2\|B\|\|C\|\right)^{1/2}. \tag{2.18}$$

Now, if we apply the inequality (2.18) for $B=\frac{A+A^{*}}{2}$ and $C=\frac{A-A^{*}}{2i}$, where $A\in B(H)$, then we deduce:

$$w(A) \leq \left[\left\| \frac{(1+i)A + (1-i)A^*}{2} \right\|^2 + 2 \cdot \left\| \frac{A+A^*}{2} \right\| \left\| \frac{A-A^*}{2} \right\| \right]^{1/2}.$$

The following result provides a different upper bound for the Euclidean operator radius than (2.16).

Proposition 2. *For any two bounded linear operators B and C on H, we have*

$$w_e(B,C) \leq \left[2 \min \left\{ w^2(B), w^2(C) \right\} + w(B-C)\, w(B+C) \right]^{1/2}. \tag{2.19}$$

Proof. Utilising the parallelogram identity (2.10), we have, by taking the supremum over $x \in H$, $\|x\| = 1$, that

$$2 w_e^2(B,C) = w_e^2(B-C, B+C). \tag{2.20}$$

Now, if we apply Proposition 1 for $B-C, B+C$ instead of B and C, then we can state

$$w_e^2(B-C, B+C) \leq 4w^2(C) + 2w(B-C)\, w(B+C)$$

giving

$$w_e^2(B,C) \leq 2w^2(C) + w(B-C)\, w(B+C). \tag{2.21}$$

Now, if in (2.21) we swap the C with B then we also have

$$w_e^2(B,C) \leq 2w^2(B) + w(B-C)\, w(B+C). \tag{2.22}$$

The conclusion follows now by (2.21) and (2.22). ∎

A different upper bound for the Euclidean operator radius is incorporated in the following

Theorem 3. *Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space and B, C two bounded linear operators on H. Then*

$$w_e^2(B,C) \leq \max \left\{ \|B\|^2, \|C\|^2 \right\} + w(C^* B). \tag{2.23}$$

The inequality (2.23) is sharp.

Proof. Firstly, let us observe that for any $y, u, v \in H$ we have successively

$$\begin{aligned} & \|\langle y,u\rangle u + \langle y,v\rangle v\|^2 \\ &= |\langle y,u\rangle|^2 \|u\|^2 + |\langle y,v\rangle|^2 \|v\|^2 + 2\operatorname{Re}\left[\langle y,u\rangle \overline{\langle y,v\rangle} \langle u,v\rangle \right] \\ &\leq |\langle y,u\rangle|^2 \|u\|^2 + |\langle y,v\rangle|^2 \|v\|^2 + 2|\langle y,u\rangle| |\langle y,v\rangle| |\langle u,v\rangle| \\ &\leq |\langle y,u\rangle|^2 \|u\|^2 + |\langle y,v\rangle|^2 \|v\|^2 + \left(|\langle y,u\rangle|^2 + |\langle y,v\rangle|^2 \right) |\langle u,v\rangle| \\ &\leq \left(|\langle y,u\rangle|^2 + |\langle y,v\rangle|^2 \right) \left(\max\left\{ \|u\|^2, \|v\|^2 \right\} + |\langle u,v\rangle| \right). \end{aligned} \tag{2.24}$$

On the other hand,

$$\begin{aligned} \left(|\langle y,u\rangle|^2 + |\langle y,v\rangle|^2 \right)^2 &= \left[\langle y,u\rangle \langle u,y\rangle + \langle y,v\rangle \langle v,y\rangle \right]^2 \\ &= \left[\langle y, \langle y,u\rangle u + \langle y,v\rangle v\rangle \right]^2 \\ &\leq \|y\|^2 \|\langle y,u\rangle u + \langle y,v\rangle v\|^2 \end{aligned} \tag{2.25}$$

for any $y, u, v \in H$.

Making use of (2.24) and (2.25) we deduce that

$$|\langle y,u\rangle|^2+|\langle y,v\rangle|^2\le\|y\|^2\left[\max\left\{\|u\|^2,\|v\|^2\right\}+|\langle u,v\rangle|\right] \tag{2.26}$$

for any $y,u,v\in H$, which is a vector inequality of interest in itself.

Now, if we apply the inequality (2.26) for $y=x$, $u=Bx$, $v=Cx$, $x\in H$, $\|x\|=1$, then we can state that

$$|\langle Bx,x\rangle|^2+|\langle Cx,x\rangle|^2\le\max\left\{\|Bx\|^2,\|Cx\|^2\right\}+|\langle Bx,Cx\rangle| \tag{2.27}$$

for any $x\in H$, $\|x\|=1$, which is of interest in itself.

Taking the supremum over $x\in H$, $\|x\|=1$, we deduce the desired result (2.23).

To prove the sharpness of the inequality (2.23) we choose $C=B$, B a self-adjoint operator on H. In this case, both sides of (2.23) become $2\|B\|^2$. ∎

If information about the sum and the difference of the operators B and C are available, then one may use the following result:

Corollary 6. *For any two operators $B,C\in B(H)$ we have*

$$w_e^2(B,C)\le\frac{1}{2}\left\{\max\left\{\|B-C\|^2,\|B+C\|^2\right\}+w\left[(B^*-C^*)(B+C)\right]\right\}. \tag{2.28}$$

The constant $\frac{1}{2}$ is best possible in (2.28).

Proof. Follows by the inequality (2.23) written for $B+C$ and $B-C$ instead of B and C and by utilising the identity (2.20).

The fact that $\frac{1}{2}$ is best possible in (2.28) follows by the fact that for $C=B$, B a self-adjoint operator, we get in both sides of the inequality (2.28) the quantity $2\|B\|^2$. ∎

Corollary 7. *Let $A:H\to H$ be a bounded linear operator on the Hilbert space H. Then:*

$$w^2(A)\le\frac{1}{4}\left[\max\left\{\|A+A^*\|^2,\|A-A^*\|^2\right\}+w\left[(A^*-A)(A+A^*)\right]\right]. \tag{2.29}$$

The constant $\frac{1}{4}$ is best possible.

Proof. If $B=\frac{A+A^*}{2}$, $C=\frac{A-A^*}{2i}$ is the Cartesian decomposition of A, then

$$w_e^2(B,C)=w^2(A)$$

and

$$w(C^*B)=\frac{1}{4}w\left[(A^*-A)(A+A^*)\right].$$

Utilising (2.23) we deduce (2.29). ∎

Remark 4. *If we choose in (2.23), $B=A$ and $C=A^*, A\in B(H)$ then we can state that*

$$w^2(A)\le\frac{1}{2}\left[\|A\|^2+w\left(A^2\right)\right]. \tag{2.30}$$

The constant $\frac{1}{2}$ is best possible in (2.30).

Note that this inequality has been obtained in [1] *by the use of a different argument based on the Buzano's inequality.*

Finally, the following upper bound for the Euclidean radius involving different composite operators also holds:

Theorem 4. *With the assumptions of Theorem 3, we have*

$$w_e^2(B,C) \leq \frac{1}{2}\left[\|B^*B + C^*C\| + \|B^*B - C^*C\|\right] + w(C^*B). \tag{2.31}$$

The inequality (2.31) is sharp.

Proof. We use (2.27) to write that

$$\begin{aligned} & |\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2 \\ \leq\ & \frac{1}{2}\left[\|Bx\|^2 + \|Cx\|^2 + \left|\|Bx\|^2 - \|Cx\|^2\right|\right] + |\langle Bx, Cx\rangle| \end{aligned} \tag{2.32}$$

for any $x \in H$, $\|x\| = 1$.

Since $\|Bx\|^2 = \langle B^*Bx, x\rangle$, $\|Cx\|^2 = \langle C^*Cx, x\rangle$, then (2.32) can be written as

$$\begin{aligned} & |\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2 \\ \leq\ & \frac{1}{2}\left[\langle (B^*B + C^*C)x, x\rangle + |\langle (B^*B - C^*C)x, x\rangle|\right] + |\langle Bx, Cx\rangle| \end{aligned} \tag{2.33}$$

$x \in H$, $\|x\| = 1$.

Taking the supremum in (2.33) over $x \in H$, $\|x\| = 1$ and noticing that the operators $B^*B \pm C^*C$ are self-adjoint, we deduce the desired result (2.31).

The sharpness of the constant will follow from the one of (2.36) pointed out below. ∎

Corollary 8. *For any two operators $B, C \in B(H)$, we have*

$$\begin{aligned} & w_e^2(B,C) \\ \leq\ & \frac{1}{2}\left\{\|B^*B + C^*C\| + \|B^*C + C^*B\| + w\left[(B^* - C^*)(B + C)\right]\right\}. \end{aligned} \tag{2.34}$$

The constant $\frac{1}{2}$ is best possible.

Proof. If we write (2.31) for $B+C, B-C$ instead of B, C and perform the required calculations then we get

$$\begin{aligned} & w_e^2(B+C, B-C) \\ \leq\ & \frac{1}{2}\left[2\|B^*B + C^*C\| + 2\|B^*C + C^*B\|\right] + w\left[(B^* - C^*)(B + C)\right], \end{aligned}$$

which, by the identity (2.20) is clearly equivalent with (2.34).

Now, if we choose in (2.34) $B = C$, then we get the inequality $w(B) \leq \|B\|$, which is a sharp inequality. ∎

Corollary 9. *If B, C are self-adjoint operators on H then*

$$w_e^2(B,C) \leq \frac{1}{2}\left[\left\|B^2 + C^2\right\| + \left\|B^2 - C^2\right\|\right] + w(CB). \tag{2.35}$$

We observe that, if B and C are chosen to be the Cartesian decomposition for the bounded linear operator A, then we can get from (2.35) that

$$w^2(A) \leq \frac{1}{4}\left\{\|A^*A + AA^*\| + \left\|A^2 + (A^*)^2\right\| + w\left[(A^* - A)(A + A^*)\right]\right\}. \tag{2.36}$$

The constant $\frac{1}{4}$ is best possible. This follows by the fact that for A a self-adjoint operator, we obtain in both sides of (2.36) the same quantity $\|A\|^2$.

Now, if we choose in (2.31) $B = A$ and $C = A^*$, $A \in B(H)$, then we get

$$w^2(A) \leq \frac{1}{4}\{\|A^*A + AA^*\| + \|A^*A - AA^*\|\} + \frac{1}{2}w\left(A^2\right). \tag{2.37}$$

This inequality is sharp. The equality holds if, for instance, we assume that A is normal, i.e., $A^*A = AA^*$. In this case we get in both sides of (2.37) the quantity $\|A\|^2$, since for normal operators, $w\left(A^2\right) = w^2(A) = \|A\|^2$.

Acknowledgement. The author would like to thank the anonymous referee for his/her comments that have been implemented in the final version of the paper.

References

[1] S.S. DRAGOMIR, Inequalities for the norm and the numerical radius of linear operators in Hilbert spaces, *RGMIA Res. Rep. Coll.*, **8**(2005), Supplement, Article 10. [ONLINE http://rgmia.vu.edu.au/v8(E).html]

[2] K.E. GUSTAFSON and D.K.M. RAO, *Numerical Range,* Springer-Verlag, New York, Inc., 1997.

[3] P.R. HALMOS, *A Hilbert Space Problem Book,* Springer-Verlag, New York, Heidelberg, Berlin, Second edition, 1982.

[4] F. KITTANEH, Numerical radius inequalities for Hilbert space operators, *Studia Math.,* **168**(1) (2005), 73–80.

[5] G. POPESCU, Unitary invariants in multivariable operator theory, Preprint, Arχiv.math.OA/0410492.

School of Computer Science and Mathematics, Victoria University, PO Box 14428, Melbourne City, VIC, 8001, Australia.

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://rgmia.vu.edu.au/dragomir`