A Potpourri of Schwarz Related Inequalities in Inner Product Spaces (II)

A POTPOURRI OF SCHWARZ RELATED INEQUALITIES IN INNER PRODUCT SPACES (II)

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Abstract. Further inequalities related to the Schwarz inequality in real or complex inner product spaces are given.

1. Introduction

Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. One of the most important inequalities in inner product spaces with numerous applications is the Schwarz inequality, that may be written in two forms:

$$|\langle x,y\rangle|^2 \le \|x\|^2\|y\|^2, \quad x,y\in H \quad \text{(quadratic form)} \tag{1.1}$$

or, equivalently,

$$|\langle x,y\rangle| \le \|x\|\|y\|, \quad x,y\in H \quad \text{(simple form).} \tag{1.2}$$

The case of equality holds in (1.1) or in (1.2) if and only if the vectors x and y are linearly dependent.

In the previous paper [6], several results related to Schwarz inequalities have been established. We recall few of them below:

1. If $x,y\in H\backslash\{0\}$ and $\|x\|\ge\|y\|$, then

$$\|x\|\|y\| - \operatorname{Re}\langle x,y\rangle \le \begin{cases} \frac{1}{2}r^2\left(\frac{\|x\|}{\|y\|}\right)^{r-1}\|x-y\|^2 & \text{if } r\ge 1 \\ \\ \frac{1}{2}\left(\frac{\|x\|}{\|y\|}\right)^{1-r}\|x-y\|^2 & \text{if } r<1. \end{cases} \tag{1.3}$$

2. If $(H;\langle\cdot,\cdot\rangle)$ is complex, $\alpha\in\mathbb{C}$ with $\operatorname{Re}\alpha, \operatorname{Im}\alpha>0$ and $x,y\in H$ are such that

$$\left\|x-\frac{\operatorname{Im}\alpha}{\operatorname{Re}\alpha}\cdot y\right\|\le r \tag{1.4}$$

then

$$\|x\|\|y\| - \operatorname{Re}\langle x,y\rangle \le \frac{1}{2}\cdot\frac{\operatorname{Re}\alpha}{\operatorname{Im}\alpha}r^2. \tag{1.5}$$

3. If $\alpha\in\mathbb{K}\backslash\{0\}$, then for any $x,y\in H$

$$\|x\|\|y\| - \operatorname{Re}\left[\frac{\alpha^2}{|\alpha|^2}\langle x,y\rangle\right] \le \frac{1}{2}\cdot\frac{\left[|\operatorname{Re}\alpha|\|x-y\|+|\operatorname{Im}\alpha|\|x+y\|\right]^2}{|\alpha|^2}. \tag{1.6}$$

Date: May 25, 2005.
2000 *Mathematics Subject Classification.* 46C05, 26D15.
Key words and phrases. Schwarz inequality, Inner product spaces, Reverse inequalities.

4. If $p \geq 1$, then for any $x, y \in H$ one has

$$\|x\| \|y\| - \operatorname{Re}\langle x, y\rangle \leq \frac{1}{2} \times \begin{cases} \left[(\|x\| + \|y\|)^{2p} - \|x + y\|^{2p}\right]^{\frac{1}{p}}, \\ \\ \left[\|x - y\|^{2p} - \left|\|x\| - \|y\|\right|^{2p}\right]^{\frac{1}{p}}. \end{cases} \tag{1.7}$$

5. If $\alpha, \gamma > 0$ and $\beta \in \mathbb{K}$ with $|\beta|^2 \geq \alpha\gamma$ then for $x, a \in H$ with $a \neq 0$ and

$$\left\| x - \frac{\beta}{\alpha} a \right\| \leq \frac{\left(|\beta|^2 - \alpha\gamma\right)^{\frac{1}{2}}}{\alpha} \|a\|, \tag{1.8}$$

one has

$$\begin{aligned} \|x\| \|a\| &\leq \frac{\operatorname{Re}\beta \operatorname{Re}\langle x, a\rangle + \operatorname{Im}\beta \operatorname{Im}\langle x, a\rangle}{\sqrt{\alpha\gamma}} \\ &\leq \frac{|\beta| \left|\langle x, a\rangle\right|}{\sqrt{\alpha\gamma}} \end{aligned} \tag{1.9}$$

and

$$\|x\|^2 \|a\|^2 - \left|\langle x, a\rangle\right|^2 \leq \frac{|\beta|^2 - \alpha\gamma}{\alpha\gamma} \left|\langle x, a\rangle\right|^2. \tag{1.10}$$

The aim of this paper is to provide other results related to the Schwarz inequality. Applications for reversing the generalised triangle inequality are also given.

2. Quadratic Schwarz Related Inequalities

The following result holds.

Theorem 1. *Let $(H; \langle\cdot,\cdot\rangle)$ be a complex inner product space and $x, y \in H$, $\alpha \in [0, 1]$. Then*

$$\begin{aligned} &\left[\alpha \|ty - x\|^2 + (1 - \alpha) \|ity - x\|^2\right] \|y\|^2 \\ &\qquad \geq \|x\|^2 \|y\|^2 - \left[(1 - \alpha) \operatorname{Im}\langle x, y\rangle + \alpha \operatorname{Re}\langle x, y\rangle\right]^2 \geq 0 \end{aligned} \tag{2.1}$$

for any $t \in \mathbb{R}$.

Proof. Firstly, recall that for a quadratic polynomial $P : \mathbb{R} \to \mathbb{R}$, $P(t) = at^2 + 2bt + c$, $a > 0$, we have that

$$\inf_{t\in\mathbb{R}} P(t) = P\left(-\frac{b}{a}\right) = \frac{ac - b^2}{a}. \tag{2.2}$$

Now, consider the polynomial $P : \mathbb{R} \to \mathbb{R}$ given by

$$P(t) := \alpha \|ty - x\|^2 + (1 - \alpha) \|ity - x\|^2. \tag{2.3}$$

Since

$$\|ty - x\|^2 = t^2 \|y\|^2 - 2t \operatorname{Re}\langle x, y\rangle + \|x\|^2$$

and

$$\|ity - x\|^2 = t^2 \|y\|^2 - 2t \operatorname{Im}\langle x, y\rangle + \|x\|^2,$$

hence

$$P(t) = t^2 \|y\|^2 - 2t \left[\alpha \operatorname{Re}\langle x, y\rangle + (1 - \alpha) \operatorname{Im}\langle x, y\rangle\right] + \|x\|^2.$$

By the definition of P (see (2.3)), we observe that $P(t) \geq 0$ for every $t \in \mathbb{R}$, therefore $\frac{1}{4}\Delta \leq 0$, i.e.,

$$[(1-\alpha)\operatorname{Im}\langle x,y\rangle + \alpha \operatorname{Re}\langle x,y\rangle]^2 - \|x\|^2\|y\|^2 \leq 0,$$

proving the second inequality in (2.1).

The first inequality follows by (2.2) and the theorem is proved. ∎

The following particular cases are of interest.

Corollary 1. *For any $x, y \in H$ one has the inequalities:*

$$\|ty - x\|^2 \|y\|^2 \geq \|\alpha\|^2 \|y\|^2 - [\operatorname{Re}\langle x,y\rangle]^2 \geq 0; \tag{2.4}$$

$$\|ity - x\|^2 \|y\|^2 \geq \|\alpha\|^2 \|y\|^2 - [\operatorname{Im}\langle x,y\rangle]^2 \geq 0; \tag{2.5}$$

and

$$\frac{1}{2}\left[\|ty-x\|^2 + \|ity-x\|^2\right]\|y\|^2 \geq \|x\|^2\|y\|^2 - \left(\frac{\operatorname{Im}\langle x,y\rangle + \operatorname{Re}\langle x,y\rangle}{2}\right)^2 \geq 0, \tag{2.6}$$

for any $t \in \mathbb{R}$.

The following corollary may be stated as well:

Corollary 2. *Let $x, y \in H$ and $M_i, m_i \in \mathbb{R}$, $i \in \{1,2\}$ such that $M_i \geq m_i > 0$, $i \in \{1,2\}$. If either*

$$\operatorname{Re}\langle M_1 y - x, x - m_1 y\rangle \geq 0 \quad \textit{and} \quad \operatorname{Re}\langle M_2 iy - x, x - im_2 y\rangle \geq 0, \tag{2.7}$$

or, equivalently,

$$\left\|x - \frac{M_1+m_1}{2}y\right\| \leq \frac{1}{2}(M_1 - m_1)\|y\| \quad \textit{and} \tag{2.8}$$

$$\left\|x - \frac{M_2+m_2}{2}iy\right\| \leq \frac{1}{2}(M_2 - m_2)\|y\|$$

hold, then

$$\begin{aligned}(0 \leq)\, &\|x\|^2\|y\|^2 - [(1-\alpha)\operatorname{Im}\langle x,y\rangle + \alpha\operatorname{Re}\langle x,y\rangle]^2 \\ &\leq \frac{1}{4}\|y\|^4\left[\alpha(M_1-m_1)^2 + (1-\alpha)(M_2-m_2)^2\right]\end{aligned} \tag{2.9}$$

for any $\alpha \in [0,1]$.

Proof. It is easy to see that, if $x, z, Z \in H$, then the following statements are equivalent:

(i) $\operatorname{Re}\langle Z - x, x - z\rangle \geq 0$,
(ii) $\left\|x - \frac{z+Z}{2}\right\| \leq \frac{1}{2}\|Z - z\|$.

Utilising this property one may simply realize that the statements (2.7) and (2.8) are equivalent.

Now, on making use of (2.8) and (2.1), one may deduce the desired inequality (2.9). ∎

Remark 1. *If one assumes that $M_1 = M_2 = M$, $m_1 = m_2 = m$ in either (2.7) or (2.8), then*

$$\begin{aligned}(0 \leq)\, &\|x\|^2\|y\|^2 - [(1-\alpha)\operatorname{Im}\langle x,y\rangle + \alpha\operatorname{Re}\langle x,y\rangle]^2 \\ &\leq \frac{1}{4}\|y\|^4(M-m)^2\end{aligned} \tag{2.10}$$

for each $\alpha \in [0,1]$.

Remark 2. *Corollary 2 may be seen as a potential source of some reverse results for the Schwarz inequality. For instance, if* $x, y \in H$ *and* $M \geq m > 0$ *are such that either*

$$\operatorname{Re}\langle My - x, x - my\rangle \geq 0 \quad \text{or} \quad \left\| x - \frac{M+m}{2} y \right\| \leq \frac{1}{2}(M-m)\|y\| \tag{2.11}$$

hold, then

$$(0 \leq)\ \|x\|^2 \|y\|^2 - [\operatorname{Re}\langle x, y\rangle]^2 \leq \frac{1}{4}(M-m)^2 \|y\|^4. \tag{2.12}$$

If $x, y \in H$ *and* $N \geq n > 0$ *are such that either*

$$\operatorname{Re}\langle Niy - x, x - niy\rangle \geq 0 \quad \text{or} \quad \left\| x - \frac{N+n}{2} iy \right\| \leq \frac{1}{2}(N-n)\|y\| \tag{2.13}$$

hold, then

$$(0 \leq)\ \|x\|^2 \|y\|^2 - [\operatorname{Im}\langle x, y\rangle]^2 \leq \frac{1}{4}(N-n)^2 \|y\|^4. \tag{2.14}$$

We notice that (2.12) is an improvement of the inequality

$$(0 \leq)\ \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{4}(M-m)^2 \|y\|^4$$

that has been established in [4] *under the same condition (2.11) given above.*

The following result may be stated as well.

Theorem 2. *Let* $(H; \langle\cdot,\cdot\rangle)$ *be a real or complex inner product space and* $x, y \in H$, $\alpha \in [0,1]$. *Then*

$$\begin{aligned} &\left[\alpha \|ty - x\|^2 + (1-\alpha)\|y - tx\|^2\right]\left[\alpha \|y\|^2 + (1-\alpha)\|x\|^2\right] \\ &\geq \left[(1-\alpha)\|x\|^2 + \alpha\|y\|^2\right]\left[\alpha\|x\|^2 + (1-\alpha)\|y\|^2\right] - [\operatorname{Re}\langle x, y\rangle]^2 \geq 0 \end{aligned} \tag{2.15}$$

for any $t \in \mathbb{R}$.

Proof. Consider the polynomial $P : \mathbb{R} \to \mathbb{R}$ given by

$$P(t) := \alpha\|ty - x\|^2 + (1-\alpha)\|y - tx\|^2. \tag{2.16}$$

Since

$$\|ty - x\|^2 = t^2\|y\|^2 - 2t\operatorname{Re}\langle x, y\rangle + \|x\|^2$$

and

$$\|y - tx\|^2 = t^2\|x\|^2 - 2t\operatorname{Re}\langle x, y\rangle + \|y\|^2,$$

hence

$$P(t) = \left[\alpha\|y\|^2 + (1-\alpha)\|x\|^2\right]t^2 - 2t\operatorname{Re}\langle x, y\rangle + \left[\alpha\|x\|^2 + (1-\alpha)\|y\|^2\right]$$

for any $t \in \mathbb{R}$.

By the definition of P (see (2.16)), we observe that $P(t) \geq 0$ for every $t \in \mathbb{R}$, therefore $\frac{1}{4}\Delta \leq 0$, i.e., the second inequality in (2.15) holds true.

The first inequality follows by (2.2) and the theorem is proved. ∎

Remark 3. *We observe that if either* $\alpha = 0$ *or* $\alpha = 1$, *then (2.15) will generate the same reverse of the Schwarz inequality as (2.4) does.*

Corollary 3. *If $x, y \in H$, then*

$$\frac{\|ty - x\|^2 + \|y - tx\|^2}{2} \cdot \frac{\|x\|^2 + \|y\|^2}{2} \geq \left(\frac{\|x\|^2 + \|y\|^2}{2}\right)^2 - [\operatorname{Re}\langle x, y\rangle]^2 \geq 0 \tag{2.17}$$

for any $t \in \mathbb{R}$ and

$$\begin{aligned} \|x \pm y\|^2 &\left[\alpha \|y\|^2 + (1-\alpha)\|x\|^2\right] \\ &\geq \left[(1-\alpha)\|x\|^2 + \alpha\|y\|^2\right]\left[\alpha\|x\|^2 + (1-\alpha)\|y\|^2\right] - [\operatorname{Re}\langle x, y\rangle]^2 \geq 0 \end{aligned} \tag{2.18}$$

for any $\alpha \in [0, 1]$.

In particular, we have

$$\|x \pm y\|^2 \cdot \left(\frac{\|x\|^2 + \|y\|^2}{2}\right) \geq \left(\frac{\|x\|^2 + \|y\|^2}{2}\right)^2 - [\operatorname{Re}\langle x, y\rangle]^2 \geq 0. \tag{2.19}$$

In [7, p. 210], C.S. Lin has proved the following reverse of the Schwarz inequality in real or complex inner product spaces $(H; \langle\cdot,\cdot\rangle)$:

$$(0 \leq) \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{r^2}\|x\|^2 \|x - ry\|^2$$

for any $r \in \mathbb{R}$, $r \neq 0$ and $x, y \in H$.

The following slightly more general result may be stated:

Theorem 3. *Let $(H; \langle\cdot,\cdot\rangle)$ be a real or complex inner product space. Then for any $x, y \in H$ and $\alpha \in \mathbb{K}$ $(\mathbb{C}, \mathbb{R})$ with $\alpha \neq 0$ we have*

$$(0 \leq) \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{|\alpha|^2}\|x\|^2 \|x - \alpha y\|^2. \tag{2.20}$$

The case of equality holds in (2.20) if and only if

$$\operatorname{Re}\langle x, \alpha y\rangle = \|x\|^2. \tag{2.21}$$

Proof. Observe that

$$\begin{aligned} I(\alpha) &:= \|x\|^2 \|x - \alpha y\|^2 - |\alpha|^2 \left[\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2\right] \\ &= \|x\|^2 \left[\|x\|^2 - 2\operatorname{Re}[\bar{\alpha}\langle x, y\rangle] + |\alpha|^2 \|y\|^2\right] \\ &\quad - |\alpha|^2 \|x\|^2 \|y\|^2 + |\alpha|^2 |\langle x, y\rangle|^2 \\ &= \|x\|^4 - 2\|x\|^2 \operatorname{Re}[\bar{\alpha}\langle x, y\rangle] + |\alpha|^2 |\langle x, y\rangle|^2. \end{aligned}$$

Since

$$\operatorname{Re}[\bar{\alpha}\langle x, y\rangle] \leq |\alpha| |\langle x, y\rangle|, \tag{2.22}$$

hence

$$\begin{aligned} I(\alpha) &\geq \|x\|^4 - 2\|x\|^2 |\alpha| |\langle x, y\rangle| + |\alpha|^2 |\langle x, y\rangle|^2 \\ &= \left(\|x\|^2 - |\alpha| |\langle x, y\rangle|\right)^2 \geq 0. \end{aligned} \tag{2.23}$$

Conversely, if (2.21) holds true, then $I(\alpha) = 0$, showing that the equality case holds in (2.20).

Now, if the equality case holds in (2.20), then we must have equality in (2.22) and in (2.23) implying that

$$\operatorname{Re}\left[\langle x, \alpha y\rangle\right] = |\alpha| \left|\langle x, y\rangle\right| \quad \text{and} \quad |\alpha| \left|\langle x, y\rangle\right| = \|x\|^2$$

which imply (2.21). ∎

The following result may be stated.

Corollary 4. *Let $(H; \langle\cdot,\cdot\rangle)$ be as above and $x, y \in H$, $\alpha \in \mathbb{K}\setminus\{0\}$ and $r > 0$ such that $|\alpha| \geq r$. If*

$$\|x - \alpha y\| \leq r\|y\|, \tag{2.24}$$

then

$$\frac{\sqrt{|\alpha|^2 - r^2}}{|\alpha|} \leq \frac{|\langle x, y\rangle|}{\|x\|\|y\|}\ (\leq 1). \tag{2.25}$$

Proof. From (2.24) and (2.20) we have

$$\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{r^2}{|\alpha|^2}\|x\|^2\|y\|^2,$$

that is,

$$\frac{\left(|\alpha|^2 - r^2\right)}{|\alpha|^2}\|x\|^2\|y\|^2 \leq |\langle x, y\rangle|^2,$$

which is clearly equivalent to (2.25). ∎

Remark 4. *Since for $\Gamma, \gamma \in \mathbb{K}$ the following statements are equivalent*

(i) $\operatorname{Re}\langle\Gamma y - x, x - \gamma y\rangle \geq 0,$

(ii) $\left\|x - \frac{\gamma+\Gamma}{2}\cdot y\right\| \leq \frac{1}{2}|\Gamma - \gamma|\|y\|,$

hence by Corollary 4 we deduce

$$\frac{2\left[\operatorname{Re}\left(\Gamma\bar{\gamma}\right)\right]^{\frac{1}{2}}}{|\Gamma + \gamma|} \leq \frac{|\langle x, y\rangle|}{\|x\|\|y\|}, \tag{2.26}$$

provided $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$, *an inequality that has been obtained in a different way in* [3].

Corollary 5. *If $x, y \in H$, $\alpha \in \mathbb{K}\setminus\{0\}$ and $\rho > 0$ such that $\|x - \alpha y\| \leq \rho$, then*

$$(0 \leq)\ \|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{\rho^2}{|\alpha|^2}\|x\|^2. \tag{2.27}$$

3. Other Inequalities

The following result holds.

Proposition 1. *Let $x, y \in H\setminus\{0\}$ and $\varepsilon \in (0, \frac{1}{2}]$. If*

$$(0 \leq)\ 1 - \varepsilon - \sqrt{1 - 2\varepsilon} \leq \frac{\|x\|}{\|y\|} \leq 1 - \varepsilon + \sqrt{1 - 2\varepsilon}, \tag{3.1}$$

then

$$(0 \leq)\ \|x\|\|y\| - \operatorname{Re}\langle x, y\rangle \leq \varepsilon\|x - y\|^2. \tag{3.2}$$

Proof. If $x = y$, then (3.2) is trivial.

Suppose $x \neq y$. Utilising the inequality (2.5) from [6], we can state that

$$\frac{\|x\| \|y\| - \operatorname{Re}\langle x, y\rangle}{\|x-y\|^2} \leq \frac{2\|x\| \|y\|}{(\|x\|+\|y\|)^2}$$

for any $x, y \in H \setminus \{0\}, x \neq y$.

Now, if we assume that

$$\frac{2\|x\| \|y\|}{(\|x\|+\|y\|)^2} \leq \varepsilon,$$

then, after some manipulation, we get that

$$\varepsilon \|x\|^2 + 2(\varepsilon - 1)\|x\| \|y\| + \varepsilon \|y\|^2 \geq 0,$$

which, for $\varepsilon \in (0, \frac{1}{2}]$ and $y \neq 0$, is clearly equivalent to (3.1).

The proof is complete. ∎

The following result may be stated:

Proposition 2. *Let $(H; \langle\cdot,\cdot\rangle)$ be a real or complex inner product space. Then for any $x, y \in H$ and $\varphi \in \mathbb{R}$ one has:*

$$\|x\| \|y\| - [\cos 2\varphi \cdot \operatorname{Re}\langle x, y\rangle + \sin 2\varphi \cdot \operatorname{Im}\langle x, y\rangle] \leq \frac{1}{2}\left[|\cos\varphi| \|x-y\| + |\sin\varphi| \|x+y\|\right]^2.$$

Proof. For $\varphi \in \mathbb{R}$, consider the complex number $\alpha = \cos\varphi - i\sin\varphi$. Then $\alpha^2 = \cos 2\varphi - i \sin 2\varphi$, $|\alpha| = 1$ and by the inequality (1.6) we deduce the desired result. ∎

From a different perspective, we may consider the following results as well:

Theorem 4. *Let $(H; \langle\cdot,\cdot\rangle)$ be a real or complex inner product space, $\alpha \in \mathbb{K}$ with $|\alpha - 1| = 1$. Then for any $e \in H$ with $\|e\| = 1$ and $x, y \in H$, we have*

$$|\langle x, y\rangle - \alpha\langle x, e\rangle\langle e, y\rangle| \leq \|x\| \|y\|. \tag{3.3}$$

The equality holds in (3.3) if and only if there exists a $\lambda \in \mathbb{K}$ such that

$$\alpha\langle x, e\rangle e = x + \lambda y. \tag{3.4}$$

Proof. It is known that for $u, v \in H$, we have equality in the Schwarz inequality

$$|\langle u, v\rangle| \leq \|u\| \|v\| \tag{3.5}$$

if and only if there exists a $\lambda \in \mathbb{K}$ such that $u = \lambda v$.

If we apply (3.5) for $u = \alpha\langle x, e\rangle e - x$, $v = y$, we get

$$|\langle \alpha\langle x, e\rangle e - x, y\rangle| \leq \|\alpha\langle x, e\rangle e - x\| \|y\| \tag{3.6}$$

with equality iff there exists a $\lambda \in \mathbb{K}$ such that

$$\alpha\langle x, e\rangle e = x + \lambda y.$$

Since

$$\begin{aligned}
\|\alpha\langle x, e\rangle e - x\|^2 &= |\alpha|^2 |\langle x, e\rangle|^2 - 2\operatorname{Re}[\alpha] |\langle x, e\rangle|^2 + \|x\|^2 \\
&= \left(|\alpha|^2 - 2\operatorname{Re}[\alpha]\right) |\langle x, e\rangle|^2 + \|x\|^2 \\
&= \left(|\alpha - 1|^2 - 1\right) |\langle x, e\rangle|^2 + \|x\|^2 \\
&= \|x\|^2
\end{aligned}$$

and

$$\langle \alpha \langle x, e \rangle e - x, y \rangle = \alpha \langle x, e \rangle \langle e, y \rangle - \langle x, y \rangle$$

hence by (3.6) we deduce the desired inequality (3.3). ∎

Remark 5. *If $\alpha = 0$ in (3.3), then it reduces to the Schwarz inequality.*

Remark 6. *If $\alpha \neq 0$, then (3.3) is equivalent to*

$$\left| \langle x, e \rangle \langle e, y \rangle - \frac{1}{\alpha} \langle x, y \rangle \right| \leq \frac{1}{|\alpha|} \|x\| \|y\| . \tag{3.7}$$

Utilising the continuity property of modulus for complex numbers, i.e., $|z - w| \geq ||z| - |w||$ we then obtain

$$\left| |\langle x, e \rangle \langle e, y \rangle| - \frac{1}{|\alpha|} |\langle x, y \rangle| \right| \leq \frac{1}{|\alpha|} \|x\| \|y\| ,$$

which implies that

$$|\langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{|\alpha|} [|\langle x, y \rangle| + \|x\| \|y\|] . \tag{3.8}$$

For $e = \frac{z}{\|z\|}$, $z \neq 0$, we get from (3.8) that

$$|\langle x, z \rangle \langle z, y \rangle| \leq \frac{1}{|\alpha|} [|\langle x, y \rangle| + \|x\| \|y\|] \|z\|^2 \tag{3.9}$$

for any $\alpha \in \mathbb{K} \setminus \{0\}$ with $|\alpha - 1| = 1$ and $x, y, z \in H$.

For $\alpha = 2$, we get from (3.9) the *Buzano inequality* [1]

$$|\langle x, z \rangle \langle z, y \rangle| \leq \frac{1}{2} [|\langle x, y \rangle| + \|x\| \|y\|] \|z\|^2 \tag{3.10}$$

for any $x, y, z \in H$.

Remark 7. *In the case of real spaces the condition $|\alpha - 1| = 1$ is equivalent to either $\alpha = 0$ or $\alpha = 2$. For $\alpha = 2$ we deduce from (3.7) that*

$$\frac{1}{2} [\langle x, y \rangle - \|x\| \|y\|] \leq \langle x, e \rangle \langle e, y \rangle \leq \frac{1}{2} [\langle x, y \rangle + \|x\| \|y\|] \tag{3.11}$$

for any $x, y \in H$ and $e \in H$ with $\|e\| = 1$, which implies Richard's inequality [8]*:*

$$\frac{1}{2} [\langle x, y \rangle - \|x\| \|y\|] \|z\|^2 \leq \langle x, z \rangle \langle z, y \rangle \leq \frac{1}{2} [\langle x, y \rangle + \|x\| \|y\|] \|z\|^2 , \tag{3.12}$$

for any $x, y, z \in H$.

The following result concerning a generalisation for orthornormal families of the inequality (3.3) may be stated.

Theorem 5. *Let $(H; \langle \cdot, \cdot \rangle)$ be a real or complex inner product space, $\{e_i\}_{i \in F}$ a finite orthonormal family, i.e., $\langle e_i, e_j \rangle = \delta_{ij}$ for $i, j \in F$, where δ_{ij} is Kronecker's delta and $\alpha_i \in \mathbb{K}$, $i \in F$ such that $|\alpha_i - 1| = 1$ for each $i \in F$. Then*

$$\left| \langle x, y \rangle - \sum_{i \in F} \alpha_i \langle x, e_i \rangle \langle e_i, y \rangle \right| \leq \|x\| \|y\| . \tag{3.13}$$

The equality holds in (3.13) if and only if there exists a constant $\lambda \in \mathbb{K}$ such that

$$\sum_{i \in F} \alpha_i \langle x, e_i \rangle e_i = x + \lambda y. \tag{3.14}$$

Proof. As above, by Schwarz's inequality, we have

$$\left|\left\langle \sum_{i\in F}\alpha_i \langle x, e_i\rangle e_i - x, y\right\rangle\right| \leq \left\|\sum_{i\in F}\alpha_i \langle x, e_i\rangle e_i - x\right\| \|y\| \tag{3.15}$$

with equality if and only if there exists a $\lambda \in \mathbb{K}$ such that $\sum_{i\in F}\alpha_i \langle x, e_i\rangle e_i = x+\lambda y$.

Since

$$\begin{aligned}\left\|\sum_{i\in F}\alpha_i \langle x, e_i\rangle e_i - x\right\|^2 &= \|x\|^2 - 2\operatorname{Re}\left\langle x, \sum_{i\in F}\alpha_i \langle x, e_i\rangle e_i\right\rangle + \left\|\sum_{i\in F}\alpha_i \langle x, e_i\rangle e_i\right\|^2 \\ &= \|x\|^2 - 2\sum_{i\in F}\overline{\alpha_i}\langle x, e_i\rangle \overline{\langle x, e_i\rangle} + \sum_{i\in F}|\alpha_i|^2 |\langle x, e_i\rangle|^2 \\ &= \|x\|^2 + \sum_{i\in F}|\langle x, e_i\rangle|^2 \left(|\alpha_i|^2 - 2\operatorname{Re}\alpha_i\right) \\ &= \|x\|^2 + \sum_{i\in F}|\langle x, e_i\rangle|^2 \left(|\alpha_i - 1|^2 - 1\right) \\ &= \|x\|^2,\end{aligned}$$

hence the inequality (3.13) is obtained. ∎

Remark 8. *If the space is real, then the nontrivial case one can get from (3.13) is for all* $\alpha_i = 2$, *obtaining the inequality*

$$\frac{1}{2}\left[\langle x, y\rangle - \|x\|\|y\|\right] \leq \sum_{i\in F}\langle x, e_i\rangle\langle e_i, y\rangle \leq \frac{1}{2}\left[\langle x, y\rangle + \|x\|\|y\|\right] \tag{3.16}$$

that has been obtained in [5].

Corollary 6. *With the above assumptions, we have*

$$\begin{aligned}\left|\sum_{i\in F}\alpha_i \langle x, e_i\rangle\langle e_i, y\rangle\right| &\leq |\langle x, y\rangle| + \left|\langle x, y\rangle - \sum_{i\in F}\alpha_i \langle x, e_i\rangle\langle e_i, y\rangle\right| \\ &\leq |\langle x, y\rangle| + \|x\|\|y\|, \qquad x, y \in H,\end{aligned} \tag{3.17}$$

where $|\alpha_i - 1| = 1$ *for each* $i \in F$ *and* $\{e_i\}_{i\in F}$ *is an orthonormal family in* H.

4. Applications for the Triangle Inequality

In 1966, Diaz and Metcalf [2] proved the following reverse of the triangle inequality:

$$\left\|\sum_{i=1}^{n} x_i\right\| \geq r\sum_{i=1}^{n}\|x_i\|, \tag{4.1}$$

provided the vectors x_i in the inner product space $(H; \langle\cdot,\cdot\rangle)$ over the real or complex number field $\mathbb{K}$ are nonzero and

$$0 \leq r \leq \frac{\operatorname{Re}\langle x_i, a\rangle}{\|x_i\|} \quad \text{for each} \quad i \in \{1, \dots, n\}, \tag{4.2}$$

where $a \in H$, $\|a\| = 1$. The equality holds in (4.2) if and only if

$$\sum_{i=1}^{n} x_i = r\left(\sum_{i=1}^{n}\|x_i\|\right) a. \tag{4.3}$$

The following result may be stated:

Proposition 3. *Let $e \in H$ with $\|e\| = 1$, $\varepsilon \in \left(0, \frac{1}{2}\right]$ and $x_i \in H$, $i \in \{1, \dots, n\}$ with the property that*

$$(0 \leq) \, 1 - \varepsilon - \sqrt{1 - 2\varepsilon} \leq \|x_i\| \leq 1 - \varepsilon + \sqrt{1 - 2\varepsilon} \tag{4.4}$$

for each $i \in \{1, \dots, n\}$. Then

$$\sum_{i=1}^{n} \|x_i\| \leq \left\| \sum_{i=1}^{n} x_i \right\| + \varepsilon \sum_{i=1}^{n} \|x_i - e\|^2 . \tag{4.5}$$

Proof. Utilising Proposition 1 for $x = x_i$ and $y = e$, we can state that

$$\|x_i\| - \operatorname{Re} \langle x_i, e \rangle \leq \varepsilon \|x_i - e\|^2$$

for each $i \in \{1, \dots, n\}$. Summing over i from 1 to n, we deduce that

$$\sum_{i=1}^{n} \|x_i\| \leq \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, e \right\rangle + \varepsilon \sum_{i=1}^{n} \|x_i - e\|^2 . \tag{4.6}$$

By Schwarz's inequality in $(H; \langle \cdot, \cdot \rangle)$, we also have

$$\begin{aligned} \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, e \right\rangle &\leq \left| \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, e \right\rangle \right| \leq \left| \left\langle \sum_{i=1}^{n} x_i, e \right\rangle \right| \\ &\leq \left\| \sum_{i=1}^{n} x_i \right\| \|e\| = \left\| \sum_{i=1}^{n} x_i \right\| . \end{aligned} \tag{4.7}$$

Therefore, by (4.6) and (4.7) we deduce (4.5). ∎

In the same spirit, we can prove the following result as well:

Proposition 4. *Let $(H; \langle \cdot, \cdot \rangle)$ be a real or complex inner product space and $e \in H$ with $\|e\| = 1$. Then for any $\varphi \in \mathbb{R}$ one has the inequality:*

$$\sum_{i=1}^{n} \|x_i\| \leq \left\| \sum_{i=1}^{n} x_i \right\| + \frac{1}{2} \sum_{i=1}^{n} \left[|\cos \varphi| \|x_i - e\| + |\sin \varphi| \|x_i + e\| \right]^2 . \tag{4.8}$$

Proof. Applying Proposition 2 for $x = x_i$ and $y = e$, we have:

$$\begin{aligned} \|x_i\| &\leq \cos 2\varphi \cdot \operatorname{Re} \langle x_i, e \rangle + \sin 2\varphi \cdot \operatorname{Im} \langle x_i, e \rangle \\ &\quad + \frac{1}{2} \left[|\cos \varphi| \|x_i - e\| + |\sin \varphi| \|x_i + e\| \right]^2 \end{aligned} \tag{4.9}$$

for any $i \in \{1, \dots, n\}$.

Summing in (4.5) over i from 1 to n, we have:

$$\begin{aligned} \sum_{i=1}^{n} \|x_i\| &\leq \cos 2\varphi \cdot \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, e \right\rangle + \sin 2\varphi \cdot \operatorname{Im} \left\langle \sum_{i=1}^{n} x_i, e \right\rangle \\ &\quad + \frac{1}{2} \sum_{i=1}^{n} \left[|\cos \varphi| \|x_i - e\| + |\sin \varphi| \|x_i + e\| \right]^2 . \end{aligned} \tag{4.10}$$

Now, by the elementary inequality for the real numbers m, p, M and P,

$$mM + pP \leq \left(m^2 + p^2\right)^{\frac{1}{2}} \left(M^2 + P^2\right)^{\frac{1}{2}} ,$$

we have

$$
\begin{aligned}
(4.11)\qquad & \cos 2\varphi \cdot \operatorname{Re}\left\langle \sum_{i=1}^{n} x_i, e \right\rangle + \sin 2\varphi \cdot \operatorname{Im}\left\langle \sum_{i=1}^{n} x_i, e \right\rangle \\
& \leq \left(\cos^2 2\varphi + \sin^2 2\varphi\right)^{\frac{1}{2}} \left(\left[\operatorname{Re}\left\langle \sum_{i=1}^{n} x_i, e \right\rangle\right]^2 + \left[\operatorname{Im}\left\langle \sum_{i=1}^{n} x_i, e \right\rangle\right]^2 \right)^{\frac{1}{2}} \\
& = \left|\left\langle \sum_{i=1}^{n} x_i, e \right\rangle\right| \leq \left\| \sum_{i=1}^{n} x_i \right\| \|e\| = \left\| \sum_{i=1}^{n} x_i \right\|,
\end{aligned}
$$

where for the last inequality we used Schwarz's inequality in $(H; \langle \cdot, \cdot \rangle)$.

Finally, by (4.10) and (4.11) we deduce the desired result (4.8). ∎

References

[1] M.L. BUZANO, Generalizzazione della disiguaglianza di Cauchy-Schwarz (Italian), *Rend. Sem. Mat. Univ. e Politech. Torino*, **31** (1971/73), 405-409 (1974).

[2] J.B. DIAZ and F.T. METCALF, A complementary inequality in Hilbert and Banach spaces, *Proc. Amer. Math. Soc.*, **17**(1) (1966), 88-97.

[3] S.S. DRAGOMIR, Reverses of Schwarz, triangle and Bessel inequalities in inner product spaces, *J. Inequal. Pure & Appl. Math.,* **5**(2004), No. 3, Article 76. [ONLINE http://jipam.vu.edu.au/article.php?sid=432].

[4] S.S. DRAGOMIR, A counterpart of Schwarz's inequality in inner product spaces, *East Asian Math. J.*, **20**(1) (2004), 1-10.

[5] S.S. DRAGOMIR, Inequalities for orthornormal families of vectors in inner product spaces related to Buzano's, Richard's and Kurepa's results, *RGMIA Res. Rep. Coll.*, **7**(2004), Article 25. [ONLINE http://rgmia.vu.edu.au/v7(E).html].

[6] S.S. DRAGOMIR, A potpourri of Schwarz related inequalities in inner product spaces (I), *J. Inequal. Pure & Appl. Math.*, to appear.

[7] C.S. LIN, Cauchy-Schwarz inequality and generalized Bernstein-type inequalities, in *Inequality Theory and Applications,* Vol. 1, Eds. Y.J. Cho, J.K. Kim and S.S. Dragomir, Nova Science Publishers Inc., N.Y., 2001.

[8] U. RICHARD, Sur des inégalités du type Wirtinger et leurs application aux équationes différentielles ordinaires, Colloquium of Analysis held in Rio de Janeiro, August, 1972, pp. 233-244.

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