Reversing the CBS-Inequality for Sequences of Vectors in Hilbert Spaces with Applications (I)

REVERSING THE CBS-INEQUALITY FOR SEQUENCES OF VECTORS IN HILBERT SPACES WITH APPLICATIONS (I)

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Abstract. Several reverses for the Cauchy-Bunyakovsky-Schwarz (CBS) inequality for sequences of vectors in Hilbert spaces are obtained. Applications for bounding the distance to a finite-dimensional subspace, in reversing the generalised triangle inequality and for Fourier coefficients are also given.

1. Introduction

Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. One of the most important inequalities in inner product spaces with numerous applications, is the *Schwarz inequality*

$$|\langle x,y\rangle|^2 \le \|x\|^2 \|y\|^2, \quad x,y\in H \tag{1.1}$$

or, equivalently,

$$|\langle x,y\rangle| \le \|x\| \|y\|, \quad x,y\in H. \tag{1.2}$$

The case of equality holds iff there exists a scalar $\alpha\in\mathbb{K}$ such that $x=\alpha y$.

By a *multiplicative reverse* of the Schwarz inequality we understand an inequality of the form

$$(1\le)\,\frac{\|x\|\,\|y\|}{|\langle x,y\rangle|}\le k_1 \quad\text{or}\quad (1\le)\,\frac{\|x\|^2\,\|y\|^2}{|\langle x,y\rangle|^2}\le k_2 \tag{1.3}$$

with appropriate k_1 and k_2 and under various assumptions for the vectors x and y, while by an *additive reverse* we understand an inequality of the form

$$(0\le)\,\|x\|\,\|y\|-|\langle x,y\rangle|\le h_1 \quad\text{or}\quad (0\le)\,\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\le h_2. \tag{1.4}$$

Similar definition apply when $|\langle x,y\rangle|$ is replaced by $\operatorname{Re}\langle x,y\rangle$ or $|\operatorname{Re}\langle x,y\rangle|$.

The following recent reverses for the Schwarz inequality hold (see for instance the monograph on line [3, p. 20]):

Theorem 1. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. If $x,y\in H$ and $r>0$ are such that*

$$\|x-y\|\le r<\|y\|, \tag{1.5}$$

then we have the following multiplicative reverse of the Schwarz inequality

$$(1\le)\,\frac{\|x\|\,\|y\|}{|\langle x,y\rangle|}\le\frac{\|x\|\,\|y\|}{\operatorname{Re}\langle x,y\rangle}\le\frac{\|y\|}{\sqrt{\|y\|^2-r^2}} \tag{1.6}$$

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and the subsequent additive reverses

$$(0\le)\,\|x\|\,\|y\|-|\langle x,y\rangle|\le\|x\|\,\|y\|-\operatorname{Re}\langle x,y\rangle \le\frac{r^2}{\sqrt{\|y\|^2-r^2}\left(\|y\|+\sqrt{\|y\|^2-r^2}\right)}\operatorname{Re}\langle x,y\rangle \tag{1.7}$$

and

$$(0\le)\,\|x\|^2\,\|y\|^2-|\langle x,y\rangle|^2\le\|x\|^2\,\|y\|^2-[\operatorname{Re}\langle x,y\rangle]^2 \le r^2\,\|x\|^2 . \tag{1.8}$$

All the above inequalities are sharp.

Other additive reverses of the quadratic Schwarz's inequality are incorporated in the following result [3, p. 18-19]:

Theorem 2. *Let $x,y\in H$ and $a,A\in\mathbb{K}$. If*

$$\operatorname{Re}\langle Ay-x,x-ay\rangle\ge 0 \tag{1.9}$$

or, equivalently,

$$\left\|x-\frac{a+A}{2}\cdot y\right\|\le\frac{1}{2}\,|A-a|\,\|y\|, \tag{1.10}$$

then

$$\begin{aligned}(0\le)\,&\|x\|^2\,\|y\|^2-|\langle x,y\rangle|^2\\ &\le\frac{1}{4}\,|A-a|^2\,\|y\|^4-\begin{cases}\left|\frac{A+a}{2}\,\|y\|^2-\langle x,y\rangle\right|^2\\ \|y\|^2\operatorname{Re}\langle Ay-x,x-ay\rangle\end{cases}\\ &\le\frac{1}{4}\,|A-a|^2\,\|y\|^4 .\end{aligned} \tag{1.11}$$

The constant $\frac{1}{4}$ is best possible in all inequalities.

If one were to assume more about the complex numbers A and a, then one may state the following result as well [3, p. 21-23].

Theorem 3. *With the assumptions of Theorem 2 and, if in addition,* $\operatorname{Re}(A\bar{a})>0$, *then*

$$\|x\|\,\|y\|\le\frac{1}{2}\cdot\frac{\operatorname{Re}\left[\left(\bar{A}+\bar{a}\right)\langle x,y\rangle\right]}{\sqrt{\operatorname{Re}(A\bar{a})}}\le\frac{1}{2}\cdot\frac{|A+a|}{\sqrt{\operatorname{Re}(A\bar{a})}}\,|\langle x,y\rangle|, \tag{1.12}$$

$$(0\le)\,\|x\|\,\|y\|-\operatorname{Re}\langle x,y\rangle\le\frac{1}{2}\cdot\frac{\operatorname{Re}\left[\left(\bar{A}+\bar{a}-2\sqrt{\operatorname{Re}(A\bar{a})}\right)\langle x,y\rangle\right]}{\sqrt{\operatorname{Re}(A\bar{a})}} \tag{1.13}$$

and

$$(0\le)\,\|x\|^2\,\|y\|^2-|\langle x,y\rangle|^2\le\frac{1}{4}\cdot\frac{|A-a|^2}{\operatorname{Re}(A\bar{a})}\,|\langle x,y\rangle|^2 . \tag{1.14}$$

The constants $\frac{1}{2}$ and $\frac{1}{4}$ are best possible.

Remark 1. *If $A = M$, $a = m$ and $M \geq m > 0$, then (1.12) and (1.13) may be written in a more convenient form as*

$$\|x\| \, \|y\| \leq \frac{M+m}{2\sqrt{mM}} \operatorname{Re} \langle x, y \rangle \tag{1.15}$$

and

$$(0 \leq) \, \|x\| \, \|y\| - \operatorname{Re} \langle x, y \rangle \leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re} \langle x, y \rangle . \tag{1.16}$$

Here the constant $\frac{1}{2}$ is sharp in both inequalities.

In this paper several reverses for the Cauchy-Bunyakovsky-Schwarz (CBS) inequality for sequences of vectors in Hilbert spaces are obtained. Applications for bounding the distance to a finite-dimensional subspace and in reversing the generalised triangle inequality are also given.

A continuation of this work for different classes of reverse inequalities is planed to be considered in the subsequent paper [5].

2. Reverses of the (CBS) −Inequality for Two Sequences in $\ell_{\mathbf{p}}^2(K)$

Let $(K, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{K}$, $p_i \geq 0$, $i \in \mathbb{N}$ with $\sum_{i=1}^{\infty} p_i = 1$. Consider $\ell_{\mathbf{p}}^2(K)$ as the space

$$\ell_{\mathbf{p}}^2(K) := \left\{ x = (x_i)_{i \in \mathbb{N}} \left| x_i \in K, \ i \in \mathbb{N} \text{ and } \sum_{i=1}^{\infty} p_i \|x_i\|^2 < \infty \right. \right\}.$$

It is well known that $\ell_{\mathbf{p}}^2(K)$ endowed with the inner product

$$\langle x, y \rangle_{\mathbf{p}} := \sum_{i=1}^{\infty} p_i \langle x_i, y_i \rangle$$

is a Hilbert space over $\mathbb{K}$. The norm $\|\cdot\|_{\mathbf{p}}$ of $\ell_{\mathbf{p}}^2(K)$ is given by

$$\|x\|_{\mathbf{p}} := \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}}.$$

If $x, y \in \ell_{\mathbf{p}}^2(K)$, then the following Cauchy-Bunyakovsky-Schwarz (CBS) inequality holds true

$$\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2 \geq \left| \sum_{i=1}^{\infty} p_i \langle x_i, y_i \rangle \right|^2 \tag{2.1}$$

with equality iff there exists a $\lambda \in \mathbb{K}$ such that $x_i = \lambda y_i$ for each $i \in \mathbb{N}$.

This is an obvious consequence of the Schwarz inequality (1.1) written for the inner product $\langle \cdot, \cdot \rangle_{\mathbf{p}}$ defined on $\ell_{\mathbf{p}}^2(K)$.

The following proposition may be stated.

Proposition 1. *Let $x, y \in \ell_{\mathbf{p}}^2(K)$ and $r > 0$. Assume that*

$$\|x_i - y_i\| \leq r < \|y_i\| \quad \text{for each } i \in \mathbb{N}. \tag{2.2}$$

Then we have the inequality

$$(1\leq)\frac{\left(\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2\right)^{\frac{1}{2}}}{\left|\sum_{i=1}^{\infty}p_i\left\langle x_i,y_i\right\rangle\right|}\leq\frac{\left(\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2\right)^{\frac{1}{2}}}{\sum_{i=1}^{\infty}p_i\operatorname{Re}\left\langle x_i,y_i\right\rangle}\leq\frac{\left(\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2\right)^{\frac{1}{2}}}{\sqrt{\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2-r^2}},\tag{2.3}$$

$$\begin{aligned}(0\leq)&\left(\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2\right)^{\frac{1}{2}}-\left|\sum_{i=1}^{\infty}p_i\left\langle x_i,y_i\right\rangle\right|\\&\leq\left(\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2\right)^{\frac{1}{2}}-\sum_{i=1}^{\infty}p_i\operatorname{Re}\left\langle x_i,y_i\right\rangle\\&\leq\frac{r^2\cdot\sum_{i=1}^{\infty}p_i\operatorname{Re}\left\langle x_i,y_i\right\rangle}{\sqrt{\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2-r^2}\left[\left(\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2\right)^{\frac{1}{2}}+\sqrt{\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2-r^2}\right]}\end{aligned}\tag{2.4}$$

and

$$\begin{aligned}(0\leq)&\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2-\left|\sum_{i=1}^{\infty}p_i\left\langle x_i,y_i\right\rangle\right|^2\\&\leq\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2-\left[\sum_{i=1}^{\infty}p_i\operatorname{Re}\left\langle x_i,y_i\right\rangle\right]^2\\&\leq r^2\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2.\end{aligned}\tag{2.5}$$

Proof. From (2.2), we have

$$\left\|x-y\right\|_{\mathbf{p}}^2=\sum_{i=1}^{\infty}p_i\left\|x_i-y_i\right\|^2\leq r^2\sum_{i=1}^{\infty}p_i\leq\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2=\left\|y\right\|_{\mathbf{p}}^2,$$

giving $\left\|x-y\right\|_{\mathbf{p}}\leq r\leq\left\|y\right\|_{\mathbf{p}}$. Applying Theorem 1 for $\ell_{\mathbf{p}}^2(K)$ and $\langle\cdot,\cdot\rangle_{\mathbf{p}}$, we deduce the desired inequality. ∎

The following proposition holds.

Proposition 2. *Let $x,y\in\ell_{\mathbf{p}}^2(K)$ and $a,A\in\mathbb{K}$. If*

$$\operatorname{Re}\left\langle Ay_i-x_i,x_i-ay_i\right\rangle\geq0\quad\text{for each } i\in\mathbb{N}\tag{2.6}$$

or, equivalently,

$$\left\|x_i-\frac{a+A}{2}y_i\right\|\leq\frac{1}{2}\left|A-a\right|\left\|y_i\right\|\quad\text{for each } i\in\mathbb{N}\tag{2.7}$$

then

$$(2.8)\qquad (0 \le) \sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2 - \left| \sum_{i=1}^{\infty} p_i \langle x_i, y_i \rangle \right|^2$$

$$\le \frac{1}{4} |A-a|^2 \left(\sum_{i=1}^{\infty} p_i \|y_i\|^2 \right)^2$$

$$- \begin{cases} \left| \frac{A+a}{2} \sum_{i=1}^{\infty} p_i \|y_i\|^2 - \sum_{i=1}^{\infty} p_i \langle x_i, y_i \rangle \right|^2 \\ \\ \sum_{i=1}^{\infty} p_i \|y_i\|^2 \sum_{i=1}^{\infty} p_i \operatorname{Re} \langle A y_i - x_i, x_i - a y_i \rangle \end{cases}$$

$$\le \frac{1}{4} |A-a|^2 \left(\sum_{i=1}^{\infty} p_i \|y_i\|^2 \right)^2.$$

The proof follows by Theorem 2, we omit the details.

Finally, on using Theorem 3, we may state:

Proposition 3. *Assume that x, y, a and A are as in Proposition 2. Moreover, if* $\operatorname{Re}(A\bar{a}) > 0$, *then we have the inequality:*

$$(2.9)\qquad \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2 \right)^{\frac{1}{2}} \le \frac{1}{2} \cdot \frac{\operatorname{Re}\left[\left(\bar{A} + \bar{a} \right) \sum_{i=1}^{\infty} p_i \langle x_i, y_i \rangle \right]}{\sqrt{\operatorname{Re}(A\bar{a})}}$$

$$\le \frac{1}{2} \cdot \frac{|A-a|}{\sqrt{\operatorname{Re}(A\bar{a})}} \left| \sum_{i=1}^{\infty} p_i \langle x_i, y_i \rangle \right|,$$

$$(2.10)\qquad (0 \le) \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2 \right)^{\frac{1}{2}} - \sum_{i=1}^{\infty} p_i \operatorname{Re} \langle x_i, y_i \rangle$$

$$\le \frac{1}{2} \cdot \frac{\operatorname{Re}\left[\left(\bar{A} + \bar{a} - 2\sqrt{\operatorname{Re}(A\bar{a})} \right) \sum_{i=1}^{\infty} p_i \langle x_i, y_i \rangle \right]}{\sqrt{\operatorname{Re}(A\bar{a})}}$$

and

$$(2.11)\qquad (0 \le) \sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2 - \left| \sum_{i=1}^{\infty} p_i \langle x_i, y_i \rangle \right|^2$$

$$\le \frac{1}{4} \cdot \frac{|A-a|^2}{\operatorname{Re}(A\bar{a})} \left| \sum_{i=1}^{\infty} p_i \langle x_i, y_i \rangle \right|^2.$$

3. Reverses of the (CBS) –Inequality for Mixed Sequences

Let $(K, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{K}$ and for $p_i \ge 0$, $i \in \mathbb{N}$ with $\sum_{i=1}^{\infty} p_i = 1$, and $\ell_{\mathbf{p}}^2(K)$ the Hilbert space defined in the previous section.

If

$$\alpha \in \ell_{\mathbf{p}}^2(\mathbb{K}) := \left\{ \alpha = (\alpha_i)_{i \in \mathbb{N}} \middle| \alpha_i \in \mathbb{K},\ i \in \mathbb{N} \text{ and } \sum_{i=1}^{\infty} p_i |\alpha_i|^2 < \infty \right\}$$

and $x \in \ell_{\mathbf{p}}^2(K)$, then the following Cauchy-Bunyakovsky-Schwarz (CBS) inequality holds true:

$$\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \geq \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|^2, \tag{3.1}$$

with equality if and only if there exists a vector $v \in K$ such that $x_i = \overline{\alpha_i} v$ for any $i \in \mathbb{N}$.

The inequality (3.1) follows by the obvious identity

$$\begin{aligned} &\sum_{i=1}^{n} p_i |\alpha_i|^2 \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i \alpha_i x_i \right\|^2 \\ = &\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j \|\overline{\alpha_i} x_j - \overline{\alpha_j} x_i\|^2, \end{aligned}$$

for any $n \in \mathbb{N}$, $n \geq 1$.

In the following we establish some reverses of the (CBS) −inequality in some of its various equivalent forms that will be specified where they occur.

Theorem 4. *Let $\alpha \in \ell_{\mathbf{p}}^2(\mathbb{K})$, $x \in \ell_{\mathbf{p}}^2(K)$ and $a \in K$, $r > 0$ such that $\|a\| > r$. If the following condition holds*

$$\|x_i - \overline{\alpha_i} a\| \leq r |\alpha_i| \quad \text{for each } i \in \mathbb{N}, \tag{3.2}$$

(note that if $\alpha_i \neq 0$ for any $i \in \mathbb{N}$, then the condition (3.2) is equivalent to

$$\left\| \frac{x_i}{\overline{\alpha_i}} - a \right\| \leq r \quad \text{for each } i \in \mathbb{N}), \tag{3.3}$$

then we have the following inequalities

$$\begin{aligned} \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} &\leq \frac{1}{\sqrt{\|a\|^2 - r^2}} \operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, a \right\rangle \\ &\leq \frac{\|a\|}{\sqrt{\|a\|^2 - r^2}} \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|; \end{aligned} \tag{3.4}$$

$$\begin{aligned} 0 &\leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} - \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\| \\ &\leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} - \operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, \frac{a}{\|a\|} \right\rangle \\ &\leq \frac{r^2}{\sqrt{\|a\|^2 - r^2} \left(\|a\| + \sqrt{\|a\|^2 - r^2} \right)} \operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, \frac{a}{\|a\|} \right\rangle \\ &\leq \frac{r^2}{\sqrt{\|a\|^2 - r^2} \left(\|a\| + \sqrt{\|a\|^2 - r^2} \right)} \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|; \end{aligned} \tag{3.5}$$

$$\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \leq \frac{1}{\|a\|^2 - r^2} \left[\operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, a \right\rangle \right]^2 \leq \frac{\|a\|^2}{\|a\|^2 - r^2} \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|^2 \tag{3.6}$$

and

$$\begin{aligned} 0 &\leq \sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|^2 \\ &\leq \sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 - \left[\operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, \frac{a}{\|a\|} \right\rangle \right]^2 \\ &\leq \frac{r^2}{\|a\|^2 \left(\|a\|^2 - r^2 \right)} \left[\operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, a \right\rangle \right]^2 \\ &\leq \frac{r^2}{\|a\|^2 - r^2} \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|^2 . \end{aligned} \tag{3.7}$$

All the inequalities in (3.4) – (3.7) are sharp.

Proof. From (3.2) we deduce

$$\|x_i\|^2 - 2 \operatorname{Re} \langle x_i, \overline{\alpha_i} a \rangle + |\alpha_i|^2 \|a\|^2 \leq |\alpha_i|^2 r^2$$

for any $i \in \mathbb{N}$, which is clearly equivalent to

$$\|x_i\|^2 + \left(\|a\|^2 - r^2 \right) |\alpha_i|^2 \leq 2 \operatorname{Re} \langle \alpha_i x_i, a \rangle \tag{3.8}$$

for each $i \in \mathbb{N}$.

If we multiply (3.8) by $p_i \geq 0$ and sum over $i \in \mathbb{N}$, then we deduce

$$\sum_{i=1}^{\infty} p_i \|x_i\|^2 + \left(\|a\|^2 - r^2 \right) \sum_{i=1}^{\infty} p_i |\alpha_i|^2 \leq 2 \operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, a \right\rangle . \tag{3.9}$$

Now, dividing (3.9) by $\sqrt{\|a\|^2 - r^2} > 0$ we get

$$\frac{1}{\sqrt{\|a\|^2 - r^2}} \sum_{i=1}^{\infty} p_i \|x_i\|^2 + \sqrt{\|a\|^2 - r^2} \sum_{i=1}^{\infty} p_i |\alpha_i|^2 \leq \frac{2}{\sqrt{\|a\|^2 - r^2}} \operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, a \right\rangle . \tag{3.10}$$

On the other hand, by the elementary inequality

$$\frac{1}{\alpha} p + \alpha q \geq 2\sqrt{pq}, \qquad \alpha > 0, \; p, q \geq 0,$$

we can state that:

$$2\left[\sum_{i=1}^{\infty}p_i\left|\alpha_i\right|^2\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2\right]^{\frac{1}{2}} \leq \frac{1}{\sqrt{\left\|a\right\|^2-r^2}}\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2+\sqrt{\left\|a\right\|^2-r^2}\sum_{i=1}^{\infty}p_i\left|\alpha_i\right|^2. \tag{3.11}$$

Making use of (3.10) and (3.11), we deduce the first part of (3.4).

The second part is obvious by Schwarz's inequality

$$\operatorname{Re}\left\langle\sum_{i=1}^{\infty}p_i\alpha_i x_i, a\right\rangle \leq \left\|\sum_{i=1}^{\infty}p_i\alpha_i x_i\right\|\left\|a\right\|.$$

If $p_1=1$, $x_1=x$, $\alpha_1=1$ and $p_i=0$, $\alpha_i=0$, $x_i=0$ for $i\geq 2$, then from (3.4) we deduce the inequality

$$\left\|x\right\| \leq \frac{1}{\sqrt{\left\|a\right\|^2-r^2}}\operatorname{Re}\left\langle x,a\right\rangle \leq \frac{\left\|x\right\|\left\|a\right\|}{\sqrt{\left\|a\right\|^2-r^2}}$$

provided $\left\|x-a\right\|\leq r<\left\|a\right\|$, $x,a\in K$. The sharpness of this inequality has been shown in [3, p. 20], and we omit the details.

The other inequalities are obvious consequences of (3.4) and we omit the details. ∎

The following corollary may be stated.

Corollary 1. *Let* $\alpha\in\ell_{\mathbf{p}}^2(\mathbb{K})$, $x\in\ell_{\mathbf{p}}^2(K)$, $e\in H$, $\left\|e\right\|=1$ *and* $\varphi,\phi\in\mathbb{K}$ *with* $\operatorname{Re}(\phi\bar{\varphi})>0$. *If*

$$\left\|x_i-\overline{\alpha_i}\cdot\frac{\varphi+\phi}{2}\cdot e\right\|\leq\frac{1}{2}\left|\phi-\varphi\right|\left|\alpha_i\right| \tag{3.12}$$

for each $i\in\mathbb{N}$, *or, equivalently*

$$\operatorname{Re}\left\langle\phi\overline{\alpha_i}e-x_i, x_i-\varphi\overline{\alpha_i}e\right\rangle\geq 0 \tag{3.13}$$

for each $i\in\mathbb{N}$, *(note that, if* $\alpha_i\neq 0$ *for any* $i\in\mathbb{N}$, *then (3.12) is equivalent to*

$$\left\|\frac{x_i}{\overline{\alpha_i}}-\frac{\varphi+\phi}{2}\cdot e\right\|\leq\frac{1}{2}\left|\phi-\varphi\right| \tag{3.14}$$

for each $i\in\mathbb{N}$ *and (3.13) is equivalent to*

$$\operatorname{Re}\left\langle\phi e-\frac{x_i}{\overline{\alpha_i}},\frac{x_i}{\overline{\alpha_i}}-\varphi e\right\rangle\geq 0$$

for each $i\in\mathbb{N}$*), then the following reverses of the* (CBS) *−inequality are valid:*

$$\left(\sum_{i=1}^{\infty}p_i\left|\alpha_i\right|^2\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2\right)^{\frac{1}{2}} \leq \frac{\operatorname{Re}\left[\left(\bar{\phi}+\bar{\varphi}\right)\left\langle\sum_{i=1}^{\infty}p_i\alpha_i x_i,e\right\rangle\right]}{2\left[\operatorname{Re}\left(\phi\overline{\varphi}\right)\right]^{\frac{1}{2}}} \leq \frac{1}{2}\cdot\frac{\left|\varphi+\phi\right|}{\left[\operatorname{Re}\left(\phi\overline{\varphi}\right)\right]^{\frac{1}{2}}}\left\|\sum_{i=1}^{\infty}p_i\alpha_i x_i\right\|; \tag{3.15}$$

$$
\begin{aligned}
(3.16)\qquad 0 &\leq \left(\sum_{i=1}^{\infty} p_i \left|\alpha_i\right|^2 \sum_{i=1}^{\infty} p_i \left\|x_i\right\|^2\right)^{\frac{1}{2}} - \left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\| \\
&\leq \left(\sum_{i=1}^{\infty} p_i \left|\alpha_i\right|^2 \sum_{i=1}^{\infty} p_i \left\|x_i\right\|^2\right)^{\frac{1}{2}} - \operatorname{Re}\left[\frac{\bar{\phi}+\bar{\varphi}}{|\varphi+\phi|}\left\langle\sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle\right] \\
&\leq \frac{|\phi-\varphi|^2}{2\sqrt{\operatorname{Re}(\phi\varphi)}\left(|\varphi+\phi|+2\sqrt{\operatorname{Re}(\phi\overline{\varphi})}\right)} \operatorname{Re}\left[\frac{\bar{\phi}+\bar{\varphi}}{|\varphi+\phi|}\left\langle\sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle\right] \\
&\leq \frac{|\phi-\varphi|^2}{2\sqrt{\operatorname{Re}(\phi\varphi)}\left(|\varphi+\phi|+2\sqrt{\operatorname{Re}(\phi\overline{\varphi})}\right)} \left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\|;
\end{aligned}
$$

$$
\begin{aligned}
(3.17)\qquad \sum_{i=1}^{\infty} p_i \left|\alpha_i\right|^2 \sum_{i=1}^{\infty} p_i \left\|x_i\right\|^2 &\leq \frac{1}{4\operatorname{Re}(\phi\bar{\varphi})}\left[\operatorname{Re}\left\{\left(\bar{\phi}+\bar{\varphi}\right)\left\langle\sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle\right\}\right]^2 \\
&\leq \frac{1}{4}\cdot\frac{|\varphi+\phi|^2}{\operatorname{Re}(\phi\bar{\varphi})}\left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\|^2
\end{aligned}
$$

and

$$
\begin{aligned}
(3.18)\qquad 0 &\leq \sum_{i=1}^{\infty} p_i \left|\alpha_i\right|^2 \sum_{i=1}^{\infty} p_i \left\|x_i\right\|^2 - \left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\|^2 \\
&\leq \sum_{i=1}^{\infty} p_i \left|\alpha_i\right|^2 \sum_{i=1}^{\infty} p_i \left\|x_i\right\|^2 - \left[\operatorname{Re}\left\{\frac{\bar{\phi}+\bar{\varphi}}{|\varphi+\phi|}\left\langle\sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle\right\}\right]^2 \\
&\leq \frac{|\phi-\varphi|^2}{4|\phi+\varphi|^2\operatorname{Re}(\phi\bar{\varphi})}\left\{\operatorname{Re}\left[\left(\bar{\phi}+\bar{\varphi}\right)\left\langle\sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle\right]\right\}^2 \\
&\leq \frac{|\phi-\varphi|^2}{4\operatorname{Re}(\phi\bar{\varphi})}\left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\|^2.
\end{aligned}
$$

All the inequalities in (3.15) – (3.18) are sharp.

Remark 2. *We remark that if* $M \geq m > 0$ *and for* $\alpha \in \ell_{\mathbf{p}}^2(\mathbb{K})$, $x \in \ell_{\mathbf{p}}^2(K)$, $e \in H$ *with* $\|e\| = 1$, *one would assume that either*

$$
(3.19)\qquad \left\|\frac{x_i}{\alpha_i} - \frac{M+m}{2}\cdot e\right\| \leq \frac{1}{2}(M-m)
$$

for each $i \in \mathbb{N}$, *or, equivalently*

$$
(3.20)\qquad \operatorname{Re}\left\langle Me - \frac{x_i}{\alpha_i}, \frac{x_i}{\alpha_i} - me\right\rangle \geq 0
$$

for each $i \in \mathbb{N}$, then the following, much simpler reverses of the (CBS) −inequality may be stated:

$$\left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} \leq \frac{M+m}{2\sqrt{mM}} \operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle \leq \frac{M+m}{2\sqrt{mM}} \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|; \tag{3.21}$$

$$\begin{aligned} 0 &\leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} - \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\| \\ &\leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} - \operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle \\ &\leq \frac{(M-m)^2}{2\left(\sqrt{M}+\sqrt{m}\right)^2 \sqrt{mM}} \operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle \\ &\leq \frac{(M-m)^2}{2\left(\sqrt{M}+\sqrt{m}\right)^2 \sqrt{mM}} \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|; \end{aligned} \tag{3.22}$$

$$\begin{aligned} &\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|^2 \\ &\leq \frac{(M+m)^2}{4mM} \left[\operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle\right]^2 \\ &\leq \frac{(M+m)^2}{4mM} \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|^2 \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} 0 &\leq \sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|^2 \\ &\leq \sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 - \left[\operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle\right]^2 \\ &\leq \frac{(M-m)^2}{4mM} \left[\operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle\right]^2 \\ &\leq \frac{(M-m)^2}{4mM} \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|^2. \end{aligned} \tag{3.24}$$

4. Reverses for the Generalised Triangle Inequality

In 1966, J.B. Diaz and F.T. Metcalf [2] proved the following reverse of the generalised triangle inequality holding in an inner product space $(H; \langle \cdot, \cdot \rangle)$ over the real or complex number field $\mathbb{K}$:

$$r\sum_{i=1}^{n}\|x_i\| \le \left\|\sum_{i=1}^{n}x_i\right\| \tag{4.1}$$

provided the vectors $x_1,\dots,x_n \in H\backslash\{0\}$ satisfy the assumption

$$0 \le r \le \frac{\operatorname{Re}\langle x_i, a\rangle}{\|x_i\|}, \tag{4.2}$$

where $a \in H$ and $\|a\| = 1$.

In an attempt to diversify the assumptions for which such reverse results hold, the author pointed out in [4] that

$$\sqrt{1-\rho^2}\sum_{i=1}^{n}\|x_i\| \le \left\|\sum_{i=1}^{n}x_i\right\|, \tag{4.3}$$

where the vectors $x_i, i \in \{1,\dots,n\}$ satisfy the condition

$$\|x_i - a\| \le \rho, \qquad i \in \{1,\dots,n\} \tag{4.4}$$

where $a \in H$, $\|a\| = 1$ and $\rho \in (0,1)$.

If, for $M \ge m > 0$, the vectors $x_i \in H$, $i \in \{1,\dots,n\}$ verify either

$$\operatorname{Re}\langle Ma - x_i, x_i - ma\rangle \ge 0, \qquad i \in \{1,\dots,n\}, \tag{4.5}$$

or, equivalently,

$$\left\|x_i - \frac{M+m}{2}\cdot a\right\| \le \frac{1}{2}(M-m), \qquad i \in \{1,\dots,n\}, \tag{4.6}$$

where $a \in H$, $\|a\| = 1$, then the following reverse of the generalised triangle inequality may be stated as well [4]

$$\frac{2\sqrt{mM}}{M+m}\sum_{i=1}^{n}\|x_i\| \le \left\|\sum_{i=1}^{n}x_i\right\|. \tag{4.7}$$

Note that the inequalities (4.1), (4.3), and (4.7) are sharp; necessary and sufficient equality conditions were provided (see [2] and [4]).

It is obvious, from Theorem 4, that, if

$$\|x_i - a\| \le r, \quad \text{for} \quad i \in \{1,\dots,n\}, \tag{4.8}$$

where $\|a\| > r$, $a \in H$ and $x_i \in H$, $i \in \{1,\dots,n\}$, then one can state the inequalities

$$\begin{aligned} \sum_{i=1}^{n}\|x_i\| &\le \sqrt{n}\left(\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}} \\ &\le \frac{1}{\sqrt{\|a\|^2 - r^2}}\operatorname{Re}\left\langle\sum_{i=1}^{n}x_i, a\right\rangle \\ &\le \frac{\|a\|}{\sqrt{\|a\|^2 - r^2}}\left\|\sum_{i=1}^{n}x_i\right\|; \end{aligned} \tag{4.9}$$

and

$$0 \leq \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \leq \sqrt{n} \left(\sum_{i=1}^{n} \|x_i\|^2 \right)^{\frac{1}{2}} - \left\| \sum_{i=1}^{n} x_i \right\| \leq \sqrt{n} \left(\sum_{i=1}^{n} \|x_i\|^2 \right)^{\frac{1}{2}} - \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, \frac{a}{\|a\|} \right\rangle \leq \frac{r^2}{\sqrt{\|a\|^2 - r^2} \left(\|a\| + \sqrt{\|a\|^2 - r^2} \right)} \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, \frac{a}{\|a\|} \right\rangle \leq \frac{r^2}{\sqrt{\|a\|^2 - r^2} \left(\|a\| + \sqrt{\|a\|^2 - r^2} \right)} \left\| \sum_{i=1}^{n} x_i \right\|. \tag{4.10}$$

We note that for $\|a\| = 1$ and $r \in (0,1)$, the inequality (3.9) becomes

$$\sqrt{1-r^2} \sum_{i=1}^{n} \|x_i\| \leq \sqrt{(1-r^2)\, n} \left(\sum_{i=1}^{n} \|x_i\|^2 \right)^{\frac{1}{2}} \leq \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, a \right\rangle \leq \left\| \sum_{i=1}^{n} x_i \right\|, \tag{4.11}$$

which is a refinement of (4.3).

With the same assumptions for a and r, we have from (4.10) the following additive reverse of the generalised triangle inequality:

$$0 \leq \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \leq \frac{r^2}{\sqrt{1-r^2}\left(1+\sqrt{1-r^2}\right)} \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, a \right\rangle \leq \frac{r^2}{\sqrt{1-r^2}\left(1+\sqrt{1-r^2}\right)} \left\| \sum_{i=1}^{n} x_i \right\|. \tag{4.12}$$

We can obtain the following reverses of the generalised triangle inequality from Corollary 1 when the assumptions are in terms of complex numbers ϕ and φ :

If $\varphi, \phi \in \mathbb{K}$ with $\operatorname{Re}(\phi\bar{\varphi}) > 0$ and $x_i \in H$, $i \in \{1, \dots, n\}$, $e \in H$, $\|e\| = 1$ are such that

$$\left\| x_i - \frac{\varphi + \phi}{2} e \right\| \leq \frac{1}{2} |\phi - \varphi| \quad \text{for each} \quad i \in \{1, \dots, n\}, \tag{4.13}$$

or, equivalently,

$$\operatorname{Re} \langle \phi e - x_i, x_i - \varphi e \rangle \geq 0 \quad \text{for each} \quad i \in \{1, \dots, n\},$$

then we have the following reverses of the generalised triangle inequality:

$$\sum_{i=1}^{n}\|x_i\| \le \sqrt{n}\left(\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}} \le \frac{\operatorname{Re}\left[\left(\bar{\phi}+\bar{\varphi}\right)\left\langle\sum_{i=1}^{n}x_i,e\right\rangle\right]}{2\sqrt{\operatorname{Re}\left(\phi\bar{\varphi}\right)}} \le \frac{1}{2}\cdot\frac{\left|\bar{\phi}+\bar{\varphi}\right|}{\sqrt{\operatorname{Re}\left(\phi\bar{\varphi}\right)}}\left\|\sum_{i=1}^{n}x_i\right\| \tag{4.14}$$

and

$$\begin{aligned}0 &\le \sum_{i=1}^{n}\|x_i\| - \left\|\sum_{i=1}^{n}x_i\right\| \\ &\le \sqrt{n}\left(\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}} - \left\|\sum_{i=1}^{n}x_i\right\| \\ &\le \sqrt{n}\left(\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}} - \operatorname{Re}\left[\frac{\left|\bar{\phi}+\bar{\varphi}\right|}{\sqrt{\operatorname{Re}\left(\bar{\phi}\bar{\varphi}\right)}}\left\langle\sum_{i=1}^{n}x_i,e\right\rangle\right] \\ &\le \frac{|\phi-\varphi|^2}{2\sqrt{\operatorname{Re}\left(\phi\bar{\varphi}\right)}\left(|\phi+\varphi|+2\sqrt{\operatorname{Re}\left(\phi\bar{\varphi}\right)}\right)}\operatorname{Re}\left[\frac{\bar{\phi}+\bar{\varphi}}{\left|\bar{\phi}+\bar{\varphi}\right|}\left\langle\sum_{i=1}^{n}x_i,e\right\rangle\right] \\ &\le \frac{|\phi-\varphi|^2}{2\sqrt{\operatorname{Re}\left(\phi\bar{\varphi}\right)}\left(|\phi+\varphi|+2\sqrt{\operatorname{Re}\left(\phi\bar{\varphi}\right)}\right)}\left\|\sum_{i=1}^{n}x_i\right\|.\end{aligned} \tag{4.15}$$

Obviously (4.14) for $\phi = M$, $\varphi = m$, $M \ge m > 0$ provides a refinement for (4.7).

5. Lower Bounds for the Distance to Finite-Dimensional Subspaces

Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $\{y_1,\dots,y_n\}$ a subset of H and $G(y_1,\dots,y_n)$ the *gram matrix* of $\{y_1,\dots,y_n\}$ where $(i,j)-$entry is $\langle y_i,y_j\rangle$. The determinant of $G(y_1,\dots,y_n)$ is called the *Gram determinant* of $\{y_1,\dots,y_n\}$ and is denoted by $\Gamma(y_1,\dots,y_n)$. Thus,

$$\Gamma(y_1,\dots,y_n) = \left|\begin{array}{cccc}\langle y_1,y_1\rangle & \langle y_1,y_2\rangle & \cdots & \langle y_1,y_n\rangle \\ \langle y_2,y_1\rangle & \langle y_2,y_2\rangle & \cdots & \langle y_2,y_n\rangle \\ \multicolumn{4}{c}{\dots\dots\dots\dots\dots} \\ \langle y_n,y_1\rangle & \langle y_n,y_2\rangle & \cdots & \langle y_n,y_n\rangle\end{array}\right|. \tag{5.1}$$

Following [1, p. 129 − 133], we state here some general results for the Gram determinant that will be used in the sequel.

1. Let $\{x_1,\dots,x_n\}\subset H$. Then $\Gamma(x_1,\dots,x_n)\ne 0$ if and only if $\{x_1,\dots,x_n\}$ is linearly independent;
2. Let $M = span\{x_1,\dots,x_n\}$ be $n-$dimensional in H, i.e., $\{x_1,\dots,x_n\}$ is linearly independent. Then for each $x\in H$, the distance $d(x,M)$ from x to the linear subspace H has the representations

$$d^2(x,M) = \frac{\Gamma(x_1,\dots,x_n,x)}{\Gamma(x_1,\dots,x_n)} \tag{5.2}$$

and

$$d^2(x,M)=\begin{cases}\|x\|^2-\frac{\left(\sum_{i=1}^n|\langle x,x_i\rangle|^2\right)^2}{\left\|\sum_{i=1}^n\langle x,x_i\rangle x_i\right\|^2} & \text{if } x\notin M^{\perp},\\ \\ \|x\|^2 & \text{if } x\in M^{\perp},\end{cases}\tag{5.3}$$

where $M^{\perp}$ denotes the orthogonal complement of M.

(3) If $\{x_1,\dots,x_n\}$ is an orthornormal set in H, i.e., $\langle x_i,x_j\rangle=\delta_{ij}$, $i,j\in\{1,\dots,n\}$, where δ_{ij} is Kronecker's delta, then

$$d^2(x,M)=\|x\|^2-\sum_{i=1}^n|\langle x,x_i\rangle|^2.\tag{5.4}$$

(4) Let $\{x_1,\dots,x_n\}$ be a set of nonzero vectors in H. Then

$$0\le\Gamma(x_1,\dots,x_n)\le\|x_1\|^2\|x_2\|^2\cdots\|x_n\|^2.\tag{5.5}$$

The equality holds on the left (respectively right) side of (5.5) if and only if $\{x_1,\dots,x_n\}$ is linearly dependent (respectively orthogonal). The first inequality in (5.5) is known in the literature as *Gram's inequality* while the second one is known as *Hadamard's inequality.*

The following result may be stated.

Proposition 4. *Let $\{x_1,\dots,x_n\}$ be a system of linearly independent vectors, $M=span\{x_1,\dots,x_n\}$, $x\in H\backslash M^{\perp}$, $a\in H$, $r>0$ and $\|a\|>r$. If*

$$\left\|x_i-\overline{\langle x,x_i\rangle}a\right\|\le|\langle x,x_i\rangle|\,r\quad\textit{for each}\quad i\in\{1,\dots,n\},\tag{5.6}$$

(note that if $\langle x,x_i\rangle\neq0$ for each $i\in\{1,\dots,n\}$, then (5.6) can be written as

$$\left\|\frac{x_i}{\langle x,x_i\rangle}-a\right\|\le r\quad\textit{for each}\quad i\in\{1,\dots,n\}),\tag{5.7}$$

then we have the inequality

$$\begin{aligned}d^2(x,M)&\ge\|x\|^2-\frac{\|a\|^2}{\|a\|^2-r^2}\cdot\frac{\sum_{i=1}^n|\langle x,x_i\rangle|^2}{\sum_{i=1}^n\|x_i\|^2}\\&\ge0.\end{aligned}\tag{5.8}$$

Proof. Utilising (5.3) we can state that

$$d^2(x,M)=\|x\|^2-\frac{\sum_{i=1}^n|\langle x,x_i\rangle|^2}{\left\|\sum_{i=1}^n\langle x,x_i\rangle x_i\right\|^2}\cdot\sum_{i=1}^n|\langle x,x_i\rangle|^2.\tag{5.9}$$

Also, by the inequality (3.6) applied for $\alpha_i=\langle x,x_i\rangle$, $p_i=\frac{1}{n}$, $i\in\{1,\dots,n\}$, we can state that

$$\frac{\sum_{i=1}^n|\langle x,x_i\rangle|^2}{\left\|\sum_{i=1}^n\langle x,x_i\rangle x_i\right\|^2}\le\frac{\|a\|^2}{\|a\|^2-r^2}\cdot\frac{1}{\sum_{i=1}^n\|x_i\|^2}\tag{5.10}$$

provided the condition (5.7) holds true.

Combining (5.9) with (5.10) we deduce the first inequality in (5.8).

The last inequality is obvious since, by Schwarz's inequality

$$\|x\|^2\sum_{i=1}^n\|x_i\|^2\ge\sum_{i=1}^n|\langle x,x_i\rangle|^2\ge\frac{\|a\|^2}{\|a\|^2-r^2}\sum_{i=1}^n|\langle x,x_i\rangle|^2.$$

∎

Remark 3. *Utilising (5.2), we can state the following result for Gram determinants*

$$\Gamma(x_1,\dots,x_n,x) \geq \left[\|x\|^2 - \frac{\|a\|^2}{\|a\|^2 - r^2} \cdot \frac{\sum_{i=1}^n |\langle x, x_i\rangle|^2}{\sum_{i=1}^n \|x_i\|^2}\right] \Gamma(x_1,\dots,x_n) \geq 0 \tag{5.11}$$

for $x \notin M^{\perp}$ *and* x, x_i, a *and* r *are as in Proposition 4.*

The following corollary of Proposition 4 may be stated as well.

Corollary 2. *Let* $\{x_1,\dots,x_n\}$ *be a system of linearly independent vectors,* $M = span\{x_1,\dots,x_n\}$, $x \in H \backslash M^{\perp}$ *and* $\phi, \varphi \in K$ *with* $\operatorname{Re}(\phi\bar{\varphi}) > 0$. *If* $e \in H$, $\|e\| = 1$ *and*

$$\left\| x_i - \overline{\langle x, x_i\rangle} \cdot \frac{\varphi + \phi}{2} e \right\| \leq \frac{1}{2} |\phi - \varphi| \, |\langle x, x_i\rangle| \tag{5.12}$$

or, equivalently,

$$\operatorname{Re}\left\langle \phi \cdot \overline{\langle x, x_i\rangle} e - x_i, x_i - \varphi \cdot \overline{\langle x, x_i\rangle} e \right\rangle \geq 0,$$

for each $i \in \{1,\dots,n\}$, *then*

$$d^2(x, M) \geq \|x\|^2 - \frac{1}{4} \cdot \frac{|\varphi + \phi|^2}{\operatorname{Re}(\phi\bar{\varphi})} \cdot \frac{\sum_{i=1}^n |\langle x, x_i\rangle|^2}{\sum_{i=1}^n \|x_i\|^2} \geq 0, \tag{5.13}$$

or, equivalently,

$$\begin{aligned} &\Gamma(x_1,\dots,x_n,x) \\ &\geq \left[\|x\|^2 - \frac{1}{4} \cdot \frac{|\varphi + \phi|^2}{\operatorname{Re}(\phi\bar{\varphi})} \cdot \frac{\sum_{i=1}^n |\langle x, x_i\rangle|^2}{\sum_{i=1}^n \|x_i\|^2}\right] \Gamma(x_1,\dots,x_n) \geq 0. \end{aligned} \tag{5.14}$$

6. Applications for Fourier Coefficients

Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space over the real or complex number field $\mathbb{K}$ and $\{e_i\}_{i \in I}$ an *orthornormal basis* for H. Then (see for instance [1, p. 54 – 61]):

(i) Every element $x \in H$ can be expanded in a *Fourier series,* i.e.,

$$x = \sum_{i \in I} \langle x, e_i\rangle e_i,$$

where $\langle x, e_i\rangle$, $i \in I$ are the *Fourier coefficients* of x;

(ii) (Parseval identity)

$$\|x\|^2 = \sum_{i \in I} \langle x, e_i\rangle e_i, \qquad x \in H;$$

(iii) (Extended Parseval identity)

$$\langle x, y\rangle = \sum_{i \in I} \langle x, e_i\rangle \langle e_i, y\rangle, \qquad x, y \in H;$$

(iv) (Elements are uniquely determined by their Fourier coefficients)

$$\langle x, e_i\rangle = \langle y, e_i\rangle \quad \text{for every } i \in I \text{ implies that } x = y.$$

Now, we must remark that all the results from the second and third sections can be stated for $K=\mathbb{K}$ where $\mathbb{K}$ is the Hilbert space of complex (real) numbers endowed with the usual norm and inner product.

Therefore, we can state the following proposition.

Proposition 5. *Let $(H;\langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{K}$ and $\{e_i\}_{i\in I}$ an orthornormal base for H. If $x,y\in H$ $(y\neq 0)$, $a\in\mathbb{K}$ $(\mathbb{C},\mathbb{R})$ and $r>0$ such that $|a|>r$ and*

$$\left|\frac{\langle x,e_i\rangle}{\langle y,e_i\rangle}-a\right|\leq r \quad \text{for each} \quad i\in I, \tag{6.1}$$

then we have the following reverse of the Schwarz inequality

$$\begin{aligned}\|x\|\,\|y\| &\leq \frac{1}{\sqrt{|a|^2-r^2}}\operatorname{Re}\left[\bar{a}\cdot\langle x,y\rangle\right] \\ &\leq \frac{|a|}{\sqrt{|a|^2-r^2}}\left|\langle x,y\rangle\right|;\end{aligned} \tag{6.2}$$

$$\begin{aligned}(0\leq)\ &\|x\|\,\|y\|-\left|\langle x,y\rangle\right| \\ &\leq \|x\|\,\|y\|-\operatorname{Re}\left[\frac{\bar{a}}{|a|}\cdot\langle x,y\rangle\right] \\ &\leq \frac{r^2}{\sqrt{|a|^2-r^2}\left(|a|+\sqrt{|a|^2-r^2}\right)}\operatorname{Re}\left[\frac{\bar{a}}{|a|}\cdot\langle x,y\rangle\right] \\ &\leq \frac{r^2}{\sqrt{|a|^2-r^2}\left(|a|+\sqrt{|a|^2-r^2}\right)}\left|\langle x,y\rangle\right|;\end{aligned} \tag{6.3}$$

$$\begin{aligned}\|x\|^2\,\|y\|^2 &\leq \frac{1}{|a|^2-r^2}\left(\operatorname{Re}\left[\bar{a}\cdot\langle x,y\rangle\right]\right)^2 \\ &\leq \frac{|a|^2}{|a|^2-r^2}\left|\langle x,y\rangle\right|^2\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}(0\leq)\ &\|x\|^2\,\|y\|^2-\left|\langle x,y\rangle\right|^2 \\ &\leq \|x\|^2\,\|y\|^2-\left(\operatorname{Re}\left[\frac{\bar{a}}{|a|}\cdot\langle x,y\rangle\right]\right)^2 \\ &\leq \frac{r^2}{|a|^2\left(|a|^2-r^2\right)}-\left(\operatorname{Re}\left[\frac{\bar{a}}{|a|}\cdot\langle x,y\rangle\right]\right)^2 \\ &\leq \frac{r^2}{|a|^2-r^2}\left|\langle x,y\rangle\right|.\end{aligned} \tag{6.5}$$

The proof is similar to the one in Theorem 4, where instead of x_i we take $\langle x,e_i\rangle$, instead of α_i we take $\langle e_i,y\rangle$, $\|\cdot\|=|\cdot|$, $p_i=1$, and we use the Parseval identities mentioned above in (ii) and (iii). We omit the details.

The following result may be stated as well.

Proposition 6. *Let $(H;\langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{K}$ and $\{e_i\}_{i\in I}$ an orthornormal base for H. If $x,y\in H$ $(y\neq 0)$, $e,\varphi,\phi\in\mathbb{K}$ with $\operatorname{Re}(\phi\bar{\varphi})>0$, $|e|=1$ and, either*

$$\left|\frac{\langle x,e_i\rangle}{\langle y,e_i\rangle}-\frac{\varphi+\phi}{2}\cdot e\right|\leq\frac{1}{2}|\phi-\varphi| \tag{6.6}$$

or, equivalently,

$$\operatorname{Re}\left[\left(\phi e-\frac{\langle x,e_i\rangle}{\langle y,e_i\rangle}\right)\left(\frac{\langle e_i,x\rangle}{\langle e_i,y\rangle}-\bar{\varphi}\bar{e}\right)\right]\geq 0 \tag{6.7}$$

for each $i\in I$, then the following reverses of the Schwarz inequality hold:

$$\|x\|\,\|y\|\leq\frac{\operatorname{Re}\left[\left(\bar{\phi}+\bar{\varphi}\right)\bar{e}\langle x,y\rangle\right]}{2\sqrt{\operatorname{Re}(\phi\bar{\varphi})}}\leq\frac{1}{2}\cdot\frac{|\varphi+\phi|}{\sqrt{\operatorname{Re}(\phi\bar{\varphi})}}|\langle x,y\rangle|. \tag{6.8}$$

$$\begin{aligned}(0\leq)\,&\|x\|\,\|y\|-|\langle x,y\rangle| \\ &\leq\|x\|\,\|y\|-\operatorname{Re}\left[\frac{\left(\bar{\phi}+\bar{\varphi}\right)\bar{e}}{|\varphi+\phi|}\langle x,y\rangle\right] \\ &\leq\frac{|\phi-\varphi|^2}{2\sqrt{\operatorname{Re}(\phi\bar{\varphi})}\left(|\varphi+\phi|+2\sqrt{\operatorname{Re}(\phi\bar{\varphi})}\right)}\operatorname{Re}\left[\frac{\left(\bar{\phi}+\bar{\varphi}\right)\bar{e}}{|\varphi+\phi|}\langle x,y\rangle\right] \\ &\leq\frac{|\phi-\varphi|^2}{2\sqrt{\operatorname{Re}(\phi\bar{\varphi})}\left(|\varphi+\phi|+2\sqrt{\operatorname{Re}(\phi\bar{\varphi})}\right)}|\langle x,y\rangle|\end{aligned} \tag{6.9}$$

and

$$\begin{aligned}(0\leq)\,&\|x\|^2\|y\|^2-|\langle x,y\rangle|^2 \\ &\leq\|x\|^2\|y\|^2-\left[\operatorname{Re}\left[\frac{\left(\bar{\phi}+\bar{\varphi}\right)\bar{e}}{|\varphi+\phi|}\langle x,y\rangle\right]\right]^2 \\ &\leq\frac{|\phi-\varphi|^2}{4|\phi+\varphi|^2\operatorname{Re}(\phi\bar{\varphi})}\left\{\operatorname{Re}\left[\left(\bar{\phi}+\bar{\varphi}\right)\bar{e}\langle x,y\rangle\right]\right\}^2 \\ &\leq\frac{|\phi-\varphi|^2}{4\operatorname{Re}(\phi\bar{\varphi})}|\langle x,y\rangle|^2.\end{aligned} \tag{6.10}$$

Remark 4. *If $\phi=M\geq m=\varphi>0$, then one may state simpler inequalities from (6.8) – (6.10). We omit the details.*

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