Complete Monotonicities of Functions Involving the Gamma and Digamma Functions

COMPLETE MONOTONICITIES OF FUNCTIONS INVOLVING THE GAMMA AND DIGAMMA FUNCTIONS

FENG QI AND BAI-NI GUO

Abstract. In the article, the completely monotonic results of the functions $[\Gamma(x+1)]^{1/x}$, $\frac{[\Gamma(x+\alpha+1)]^{1/(x+\alpha)}}{[\Gamma(x+1)]^{1/x}}$, $\frac{[\Gamma(x+1)]^{1/x}}{(x+1)^\alpha}$ and $\frac{[\Gamma(x+1)]^{1/x}}{x^\alpha}$ in $x \in (-1,\infty)$ for $\alpha \in \mathbb{R}$ are obtained. In the final, three open problems are posed.

1. Introduction

The classical gamma function is usually defined for $\operatorname{Re} z > 0$ by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \,\mathrm{d}t. \tag{1}$$

The psi or digamma function $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, the logarithmic derivative of the gamma function, and the polygamma functions can be expressed (See [1, 8] and [12, p. 16]) for $x > 0$ and $k \in \mathbb{N}$ as

$$\psi(x) = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{1+n} - \frac{1}{x+n} \right), \tag{2}$$

$$\psi^{(k)}(x) = (-1)^{k+1} k! \sum_{i=0}^{\infty} \frac{1}{(x+i)^{k+1}}, \tag{3}$$

where $\gamma = 0.57721566490153286\cdots$ is the Euler-Mascheroni constant.

A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I which alternate successively in sign, that is

$$(-1)^n f^{(n)}(x) \geq 0 \tag{4}$$

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for $x \in I$ and $n \geq 0$. If inequality (4) is strict for all $x \in I$ and for all $n \geq 0$, then f is said to be strictly completely monotonic. For more information, please refer to [14, 15, 18, 23, 25] and references therein.

A function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^k[\ln f(x)]^{(k)} \geq 0 \tag{5}$$

for $k \in \mathbb{N}$ on I. If inequality (5) is strict for all $x \in I$ and for all $k \geq 1$, then f is said to be strictly logarithmically completely monotonic.

In this article, using Leibnitz's formula and the formulas (2) and (3), the complete monotonicity properties of the functions $[\Gamma(x+1)]^{1/x}$, $\frac{[\Gamma(x+\alpha+1)]^{1/(x+\alpha)}}{[\Gamma(x+1)]^{1/x}}$, $\frac{[\Gamma(x+1)]^{1/x}}{(x+1)^\alpha}$ and $\frac{[\Gamma(x+1)]^{1/x}}{x^\alpha}$ in $x \in (-1, \infty)$ for $\alpha \in \mathbb{R}$ are obtained. From these, some well known results are deduced, extended and generalized. The main results of this paper are as follows.

Theorem 1. *The function $[\Gamma(x+1)]^{1/x}$ is strictly increasing in $(-1, \infty)$. The function $\frac{\psi(x+1)}{x} - \frac{\ln\Gamma(x+1)}{x^2}$, the logarithmic derivative of $[\Gamma(x+1)]^{1/x}$, is strictly completely monotonic in $(-1, \infty)$. The function $\frac{[\Gamma(x+\alpha+1)]^{1/(x+\alpha)}}{[\Gamma(x+1)]^{1/x}}$ is logarithmically strictly completely monotonic with $x \in (-1, \infty)$ for $\alpha > 0$.*

Theorem 2. *For $\alpha \geq 1$, the function $\frac{[\Gamma(x+1)]^{1/x}}{(x+1)^\alpha}$ is strictly decreasing and the function $\frac{\ln\Gamma(x+1)}{x^2} - \frac{\psi(x+1)}{x} + \frac{\alpha}{x+1}$, the logarithmic derivative of $\frac{(x+1)^\alpha}{[\Gamma(x+1)]^{1/x}}$, is strictly completely monotonic with $x \in (-1, \infty)$.*

Let $\tau(s,t) = \frac{1}{s}\left[t - (t+s+1)\left(\frac{t}{t+1}\right)^{s+1}\right] > 0$ for $(s,t) \in \mathbb{N} \times (0, \infty)$ and $\tau_0 = \tau(s_0, t_0) > 0$ be the maximum of $\tau(s,t)$ on the set $\mathbb{N} \times (0, \infty)$. For a given real number α satisfying $\alpha \leq \frac{1}{1+\tau_0} < 1$, the function $\frac{[\Gamma(x+1)]^{1/x}}{(x+1)^\alpha}$ is strictly increasing and the function $\frac{\psi(x+1)}{x} - \frac{\ln\Gamma(x+1)}{x^2} - \frac{\alpha}{x+1}$ is strictly completely monotonic in $x \in (-1, \infty)$.

Theorem 3. *For $\alpha \leq 0$, the function $\frac{[\Gamma(x+1)]^{1/x}}{x^\alpha}$ is strictly increasing and the function $\frac{\psi(x+1)}{x} - \frac{\ln\Gamma(x+1)}{x^2} - \frac{\alpha}{x}$, the logarithmic derivative of $\frac{[\Gamma(x+1)]^{1/x}}{x^\alpha}$, is strictly completely monotonic in $(0, \infty)$. For $\alpha \geq 1$, the function $\frac{[\Gamma(x+1)]^{1/x}}{x^\alpha}$ is strictly decreasing and the function $\frac{\ln\Gamma(x+1)}{x^2} - \frac{\psi(x+1)}{x} + \frac{\alpha}{x}$ is strictly completely monotonic in $(0, \infty)$.*

For $\alpha \leq 0$ such that x^α is real in $(-1, 0)$, the function $\frac{[\Gamma(x+1)]^{1/x}}{x^\alpha}$ is strictly decreasing and the function $\frac{\ln\Gamma(x+1)}{x^2} - \frac{\psi(x+1)}{x} + \frac{\alpha}{x}$ is strictly completely monotonic *in* $(-1,0)$*. For* $\alpha \geq 1$ *such that* x^α *is real in* $(-1,0)$*, the function* $\frac{[\Gamma(x+1)]^{1/x}}{x^\alpha}$ *is strictly increasing and the function* $\frac{\psi(x+1)}{x} - \frac{\ln\Gamma(x+1)}{x^2} - \frac{\alpha}{x}$ *is strictly completely monotonic in* $(-1,0)$*.*

Theorem 4. *A* (*strictly*) *logarithmically completely monotonic function is also* (*strictly*) *completely monotonic.*

As a direct consequence of combining Theorem 1 with Theorem 4, we have the following corollary.

Corollary 1. *The function* $\frac{[\Gamma(x+\alpha+1)]^{1/(x+\alpha)}}{[\Gamma(x+1)]^{1/x}}$ *is strictly completely monotonic with* $x \in (-1,\infty)$ *for* $\alpha > 0$*.*

In [3] and [4, p. 83], the following result was given: Let f and g be functions such that $f \circ g$ is defined. If f and g' are completely monotonic, then $f \circ g$ is also completely monotonic. Thus, from Theorem 1 and Theorem 2 and the fact that the exponential function e^{-x} is strictly completely monotonic in $(-\infty,\infty)$, the following corollary can be deduced.

Corollary 2. *The following complete monotonicity properties holds:*

(1) *The function* $\frac{1}{[\Gamma(x+1)]^{1/x}}$ *is strictly completely monotonic in* $(-1,\infty)$*.*

(2) *For* $\alpha \geq 1$*, the function* $\frac{[\Gamma(x+1)]^{1/x}}{(x+1)^\alpha}$ *is strictly completely monotonic in* $(-1,\infty)$*. For a given real number* α *with* $\alpha \leq \frac{1}{1+\tau_0} < 1$*, the function* $\frac{(x+1)^\alpha}{[\Gamma(x+1)]^{1/x}}$ *is strictly completely monotonic in* $(-1,\infty)$*.*

(3) *For* $\alpha \leq 0$*, the function* $\frac{x^\alpha}{[\Gamma(x+1)]^{1/x}}$ *is strictly completely monotonic in* $(0,\infty)$*. For* $\alpha \geq 1$*, the function* $\frac{[\Gamma(x+1)]^{1/x}}{x^\alpha}$ *is strictly completely monotonic in* $(0,\infty)$*. For* $\alpha \leq 0$ *such that* x^α *is real in* $(-1,0)$*, the function* $\frac{[\Gamma(x+1)]^{1/x}}{x^\alpha}$ *is strictly completely monotonic in* $(-1,0)$*. For* $\alpha \geq 1$ *such that* x^α *is real in* $(-1,0)$*, the function* $\frac{x^\alpha}{[\Gamma(x+1)]^{1/x}}$ *strictly completely monotonic in* $(-1,0)$*.*

2. Proofs of theorems

Proof of Theorem 1. For $\alpha > 0$, let

$$f_\alpha(x) = \frac{[\Gamma(x+\alpha+1)]^{1/(x+\alpha)}}{[\Gamma(x+1)]^{1/x}} \tag{6}$$

for $x > -1$.

By direct calculation and using Leibnitz's formula and formulas (2) and (3), we obtain for $n \in \mathbb{N}$,

$$\begin{aligned} \ln f_\alpha(x) &= \frac{\ln\Gamma(x+\alpha+1)}{x+\alpha} - \frac{\ln\Gamma(x+1)}{x} \triangleq g(x+\alpha) - g(x), \\ g^{(n)}(x) &= \frac{1}{x^{n+1}} \sum_{k=0}^{n} \frac{(-1)^{n-k} n! x^k \psi^{(k-1)}(x+1)}{k!} \triangleq \frac{h_n(x)}{x^{n+1}}, \end{aligned} \tag{7}$$

$$\begin{aligned} h_n'(x) &= x^n \psi^{(n)}(x+1) \\ &\begin{cases} > 0, & \text{if } n \text{ is odd and } x \in (0,\infty), \\ \le 0, & \text{if } n \text{ is odd and } x \in (-1,0] \text{ or } n \text{ is even and } x \in (-1,\infty), \end{cases} \end{aligned} \tag{8}$$

where $\psi^{(-1)}(x+1) = \ln\Gamma(x+1)$ and $\psi^{(0)}(x+1) = \psi(x+1)$. Hence, the function $h_n(x)$ increases if n is odd and $x \in (0,\infty)$ and decreases if n is odd and $x \in (-1,0)$ or n is even and $x \in (-1,\infty)$. Since $h_n(0) = 0$, it is easy to see that $h_n(x) \ge 0$ if n is odd and $x \in (-1,\infty)$ or n is even and $x \in (-1,0)$ and $h_n(x) \le 0$ if n is even and $x \in (0,\infty)$. Then, for $x \in (-1,\infty)$, we have $g^{(n)}(x) \ge 0$ if n is odd and $g^{(n)}(x) \le 0$ if n is even. Since $\lim_{x\to\infty} \frac{\psi^{(k)}(x+1)}{x^{n+1}} = 0$ for $-1 \le k \le n$, it is easy to see that $\lim_{x\to\infty} g^{(n)}(x) = \lim_{x\to\infty} \frac{h_n(x)}{x^{n+1}} = 0$. Therefore $(-1)^{n+1} g^{(n)}(x) > 0$ with $x \in (-1,\infty)$ for $n \in \mathbb{N}$. Then the function $g'(x)$ is strictly completely monotonic and $[\Gamma(x+1)]^{1/x} = \exp(g(x))$ is strictly increasing in $(-1,\infty)$.

From $(-1)^{n+1} g^{(n)}(x) \ge 0$ with $x \in (-1,\infty)$ for $n \in \mathbb{N}$, it follows that $g^{2k}(x)$ increases and $g^{2k-1}(x)$ decreases with $x \in (-1,\infty)$ for all $k \in \mathbb{N}$. This implies that $(-1)^n [\ln f_\alpha(x)]^{(n)} \ge 0$, and then the function $\frac{[\Gamma(x+\alpha+1)]^{1/(x+\alpha)}}{[\Gamma(x+1)]^{1/x}}$ is logarithmically completely monotonic with $x \in (-1,\infty)$. □

Proof of Theorem 2. Let

$$\nu_\alpha(x) = \frac{[\Gamma(x+1)]^{1/x}}{(x+1)^\alpha} \tag{9}$$

for $x \in (-1,\infty)$. Then for $n \in \mathbb{N}$,

$$\ln \nu_\alpha(x) = \frac{\ln\Gamma(x+1)}{x} - \alpha \ln(x+1), \tag{10}$$

$$[\ln \nu_\alpha(x)]^{(n)} = \frac{1}{x^{n+1}} \left[h_n(x) + \frac{(-1)^n (n-1)! \alpha x^{n+1}}{(x+1)^n} \right] \triangleq \frac{\mu_{\alpha,n}(x)}{x^{n+1}}, \tag{11}$$

$$\mu_{\alpha,n}'(x) = x^n \psi^{(n)}(x+1) + \frac{(-1)^n (n-1)! \alpha x^n (x+n+1)}{(x+1)^{n+1}}$$

$$
\begin{aligned}
&= x^n\left[\psi^{(n)}(x+1)+\frac{(-1)^n(n-1)!\alpha}{(x+1)^n}+\frac{(-1)^n n!\alpha}{(x+1)^{n+1}}\right] \\
&= x^n\Biggl\{(-1)^{n+1}n!\sum_{i=1}^{\infty}\frac{1}{(x+i)^{n+1}} \\
&\quad+(-1)^n(n-1)!\alpha\sum_{i=1}^{\infty}\left[\frac{1}{(x+i)^n}-\frac{1}{(x+i+1)^n}\right] \\
&\quad+(-1)^n n!\alpha\sum_{i=1}^{\infty}\left[\frac{1}{(x+i)^{n+1}}-\frac{1}{(x+i+1)^{n+1}}\right]\Biggr\} \qquad (12) \\
&= (-1)^n(n-1)!x^n\sum_{i=1}^{\infty}\biggl[\frac{\alpha}{(x+i)^n}-\frac{\alpha}{(x+i+1)^n} \\
&\quad-\frac{n\alpha}{(x+i+1)^{n+1}}+\frac{n(\alpha-1)}{(x+i)^{n+1}}\biggr]
\end{aligned}
$$

$$
\begin{aligned}
&= (n-1)!(-x)^n\sum_{i=1}^{\infty}\frac{[\alpha y+n(\alpha-1)](y+1)^{n+1}-\alpha(y+n+1)y^{n+1}}{y^{n+1}(y+1)^{n+1}} \\
&= (n-1)!(-x)^n\sum_{i=1}^{\infty}\frac{\alpha[(y+n)(y+1)^{n+1}-(y+n+1)y^{n+1}]-n(y+1)^{n+1}}{y^{n+1}(y+1)^{n+1}} \\
&= n!(-x)^n\sum_{i=1}^{\infty}\frac{1}{y^{n+1}}\left\{\alpha\left[1+\frac{1}{n}\left\langle y-(y+n+1)\left(\frac{y}{y+1}\right)^{n+1}\right\rangle\right]-1\right\},
\end{aligned}
$$

where $y=x+i>0$.

In [5, p. 28] and [11, p. 154], the Bernoulli's inequality states that if $x\ge -1$ and $x\ne 0$ and if $\alpha>1$ or if $\alpha<0$ then $(1+x)^\alpha>1+\alpha x$. This means that $1+\frac{s+1}{t}<\bigl(1+\frac{1}{t}\bigr)^{s+1}$ for $t>0$, which is equivalent to $t-(t+s+1)\bigl(\frac{t}{t+1}\bigr)^{s+1}>0$ for $t>0$, and then $\tau(s,t)>0$ for $s\ge 1$ and $t>0$ and $\tau(s,0)=0$.

From $\tau(s,t)>0$, it is deduced that $[\alpha y+n(\alpha-1)](y+1)^{n+1}-\alpha(y+n+1)y^{n+1}>0$ for $y=x+i>0$ and $n\in\mathbb{N}$ if $\alpha\ge 1$. Therefore, for $\alpha\ge 1$, we have

$$
\mu'_{\alpha,n}(x)\begin{cases}>0, & \text{if } n \text{ is even and } x\in(-1,0)\cup(0,\infty) \text{ or } n \text{ is odd and } x\in(-1,0),\\ <0, & \text{if } n \text{ is odd and } x\in(0,\infty),\end{cases}
$$

and then $\mu_{\alpha,n}(x)$ is strictly increasing with $x\in(-1,\infty)$ if n is even or with $x\in(-1,0)$ if n is odd and $\mu_{\alpha,n}(x)$ is strictly decreasing with $x\in(0,\infty)$ if n is odd. Since $\mu_{\alpha,n}(0)=0$, thus $\mu_{\alpha,n}(x)<0$ with $x>-1$ and $x\ne 0$ if n is odd or with $x\in(-1,0)$ if n is even and $\mu_{\alpha,n}(x)>0$ with $x\in(0,\infty)$ if n is even. Therefore, from $\lim_{x\to\infty}[\ln\nu_\alpha(x)]^{(n)}=0$, it is deduced that $[\ln\nu_\alpha(x)]^{(n)}>0$ if n is even and $[\ln\nu_\alpha(x)]^{(n)}<0$ if n is odd, which is equivalent to $(-1)^n[\ln\nu_\alpha(x)]^{(n)}>0$ in $x\in(-1,\infty)$ for $n\in\mathbb{N}$ and $\alpha\ge-1$. Hence, if $\alpha\ge1$, then the function $\frac{[\Gamma(x+1)]^{1/x}}{(x+1)^\alpha}$ is strictly decreasing and the function $\frac{\ln\Gamma(x+1)}{x^2}-\frac{\psi(x+1)}{x}+\frac{\alpha}{x+1}$ is strictly completely monotonic in $x\in(-1,\infty)$.

It is clear that $\tau_0>0$. When $\alpha\le\frac{1}{1+\tau_0}<1$, it follows that $\mu'_{\alpha,n}(x)<0$ and $\mu_{\alpha,n}(x)$ is decreasing with $x\in(-1,\infty)$ and $x\ne0$ for n an even integer or with $x\in(-1,0)$ for n an odd integer, and $\mu'_{\alpha,n}(x)>0$ and $\mu_{\alpha,n}(x)$ is increasing with $x\in(0,\infty)$ for n an odd integer. Since $\mu_{\alpha,n}(0)=0$ and $\lim_{x\to\infty}[\ln\nu_\alpha(x)]^{(n)}=0$, we have $[\ln\nu_\alpha(x)]^{(n)}<0$ for n an even and $[\ln\nu_\alpha(x)]^{(n)}>0$ for n an odd in $x\in(-1,\infty)$, this implies that $(-1)^{n+1}[\ln\nu_\alpha(x)]^{(n)}>0$ in $x\in(-1,\infty)$ for $n\in\mathbb{N}$. Therefore $\nu_\alpha(x)$ is strictly increasing and $(-1)^{n-1}\{[\ln\nu_\alpha(x)]'\}^{(n-1)}>0$ in $(-1,\infty)$ for $n\in\mathbb{N}$. Hence, if $\alpha\le\frac{1}{1+\tau_0}$, then the function $\frac{[\Gamma(x+1)]^{1/x}}{(x+1)^\alpha}$ is strictly increasing and the function $\frac{\psi(x+1)}{x}-\frac{\ln\Gamma(x+1)}{x^2}-\frac{\alpha}{x+1}$ is strictly completely monotonic in $(-1,\infty)$. □

Proof of Theorem 3. The procedure is same as the one of Theorem 2. Hence, we leave it to the readers. □

Proof of Theorem 4. It is clear that $\exp\phi(x)\ge0$. Further, it is easy to see that $[\exp\phi(x)]'=\phi'(x)\exp\phi(x)\le0$ and $[\exp\phi(x)]''=\{\phi''(x)+[f'(x)]^2\}\exp\phi(x)\ge0$.

Suppose $(-1)^k[\exp\phi(x)]^{(k)}\ge0$ for all nonnegative integers $k\le n$, where $n\in\mathbb{N}$ is a given positive integer. By Leibnitz's formula, we have

$$\begin{aligned}(-1)^{n+1}[\exp\phi(x)]^{(n+1)}&=(-1)^{n+1}\{[\exp\phi(x)]'\}^{(n)}\\&=(-1)^{n+1}[\phi'(x)\exp\phi(x)]^{(n)}\\&=(-1)^{n+1}\sum_{i=0}^n\binom{n}{i}\phi^{(i+1)}(x)[\exp\phi(x)]^{(n-i)}\\&=\sum_{i=0}^n\binom{n}{i}\left[(-1)^{i+1}\phi^{(i+1)}(x)\right]\left\{(-1)^{n-i}[\exp\phi(x)]^{(n-i)}\right\}\\&\ge0.\end{aligned}\tag{13}$$

By induction, it is proved that the function $\exp\phi(x)$ is completely monotonic. □

3. Remarks

Remark 1. In [10, 13], among other things, the following monotonicity results were obtained

$$[\Gamma(1+k)]^{1/k} < [\Gamma(2+k)]^{1/(k+1)}, \quad k \in \mathbb{N};$$
$$\left[\Gamma\left(1+\frac{1}{x}\right)\right]^x \text{ decreases with } x > 0.$$

These are extended and generalized in [16]: The function $[\Gamma(r)]^{1/(r-1)}$ is increasing in $r > 0$. Clearly, Theorem 1 generalizes and extends these results for the range of the argument.

Remark 2. It is proved in [19] that the function $\frac{1}{x}\ln\Gamma(x+1) - \ln x + 1$ is strictly completely monotonic on $(0,\infty)$ and tends to $+\infty$ as $x \to 0$ and to 0 as $x \to \infty$. A similar result was found in [24]: The function $1 + \frac{1}{x}\ln\Gamma(x+1) - \ln(x+1)$ is strictly completely monotonic on $(-1,\infty)$ and tends to 1 as $x \to -1$ and to 0 as $x \to \infty$. Our main results generalize these ones.

Remark 3. From our main results, the following can be deduced: Let n be natural number. Then the sequence $\frac{\sqrt[n]{n!}}{\sqrt[n+k]{(n+k+1)!}}$ are increasing with $n \in \mathbb{N}$.

Remark 4. A function f is logarithmic convex on an interval I if f is positive and $\ln f$ is convex on I. Since $f(x) = \exp[\ln f(x)]$, it follows that a logarithmic convex function is convex.

Remark 5. Straightforward computation shows that the maximum τ_2 of $\tau(2,t)$ in $(0,\infty)$ is

$$\tau\left(2,\frac{2+\sqrt{7}}{3}\right) = \frac{1}{2}\left[\frac{2+\sqrt{7}}{3} - \frac{\left(2+\sqrt{7}\right)^3\left(3+\frac{2+\sqrt{7}}{3}\right)}{27\left(1+\frac{2+\sqrt{7}}{3}\right)^3}\right] = 0.264076\cdots \tag{14}$$

and the maximum τ_3 of $\tau(3,t)$ in $(0,\infty)$ is

$$\tau\left(3,\frac{5}{9} + \frac{\sqrt[3]{2836-54\sqrt{406}}}{18} + \frac{\sqrt[3]{1418+27\sqrt{406}}}{9\sqrt[3]{4}}\right) = 0.271807\cdots. \tag{15}$$

If $\alpha \le \frac{1}{1+\tau_2} = 0.791091378310519808\cdots$, then $\mu'_{\alpha,2}(x) \le 0$ and $\mu_{\alpha,2}(x)$ decreases in $(-1,\infty)$. Since $\mu_{\alpha,2}(0) = 0$ and $\lim_{x\to\infty}[\ln\nu_\alpha(x)]^{(2)} = 0$, it is obtained that

$[\ln \nu_\alpha(x)]^{(2)} < 0$. Therefore the function $\nu_\alpha(x) = \frac{[\Gamma(x+1)]^{1/x}}{(x+1)^\alpha}$ is strictly increasing and strictly logarithmically concave for $\alpha \le \frac{1}{1+\tau_2}$ in $(-1,\infty)$.

If $\alpha \le \frac{1}{1+\tau_3} = 0.7862824583608\cdots$, then $\mu'_{\alpha,3}(x) < 0$ and $\mu_{\alpha,3}(x)$ decreases in $(-1,0)$ and $\mu'_{\alpha,3}(x) > 0$ and $\mu_{\alpha,3}(x)$ increases in $(0,\infty)$. Thus $\mu_{\alpha,3}(x) \ge 0$ and then $[\ln \nu_\alpha(x)]^{(3)} > 0$ in $(-1,\infty)$. Hence $[\ln \nu_\alpha(x)]^{(2)}$ is strictly increasing in $(-1,\infty)$ if $\alpha \le \frac{1}{1+\tau_3}$.

Mathematica shows that $\tau_0 > 0.2980\cdots$.

Remark 6. The motivation of this paper has been exposited in detail in [21] and a lot of literature is listed therein. Please also refer to [2, 6, 7, 9, 17, 20, 22].

4. Open problems

A function $f(t)$ is said to be absolutely monotonic on an interval I if it has derivatives of all orders and $f^{(k)}(t) \ge 0$ for $t \in I$ and $k \in \mathbb{N}$. A function $f(t)$ is said to be regularly monotonic if it and its derivatives of all orders have constant sign ($+$ or $-$; not all the same) on (a,b). A function $f(t)$ is said to be absolutely convex on (a,b) if it has derivatives of all orders and $f^{(2k)}(t) \ge 0$ for $t \in (a,b)$ and $k \in \mathbb{N}$.

The function $\frac{[\Gamma(x+\alpha+1)]^{1/(x+\alpha)}}{[\Gamma(x+1)]^{1/x}}$ can be expressed as

$$\frac{\sqrt[x+\alpha]{\int_0^\infty t^{x+\alpha} e^{-t}\,\mathrm{d}t}}{\sqrt[x]{\int_0^\infty t^{x} e^{-t}\,\mathrm{d}t}}, \tag{16}$$

where $\int_0^\infty e^{-t}\,\mathrm{d}t = 1$. Then we propose the following

Open Problem 1. *Let $w(x) \ge 0$ be a nonnegative weight defined on a domain Ω with $\int_\Omega w(x)\,\mathrm{d}x = 1$. Find conditions about $w(x)$ and $f(x) \ge 0$ such that the ratio between two power means*

$$\mathcal{Q}(t) = \frac{\left[\int_\Omega w(x) f^{t+\alpha}(x)\,\mathrm{d}x\right]^{1/(t+\alpha)}}{\left[\int_\Omega w(x) f^{t}(x)\,\mathrm{d}x\right]^{1/t}} \tag{17}$$

is completely (absolutely, regularly) monotonic (convex) with $t \in \mathbb{R}$ for a given number $\alpha > 0$.

Open Problem 2. *Find conditions about α and β such that the ratio*

$$\mathcal{F}(x) = \frac{[\Gamma(x+1)]^{1/x}}{(x+\beta)^\alpha} \tag{18}$$

is completely (absolutely, regularly) monotonic (convex) with $x > -1$.

Open Problem 3. *For $(s,t)\in\mathbb{N}\times(0,\infty)$, find the maximum of the following*

$$\tau(s,t)=\frac{1}{s}\left[t-(t+s+1)\left(\frac{t}{t+1}\right)^{s+1}\right]. \tag{19}$$

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