A Note on Short Intervals Containing Primes

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Abstract

In 1999, P. Dusart showed that for $x \geq 3275$, there exists at least a prime number in the interval $\big(x, x(1+\frac{1}{2\ln^2 x})\big]$ and in 2003, O. Ramaré and Y. Saouter showed that for $x \geq 10726905041$ there exists at least a prime number in the interval $\big(x(1-\Delta^{-1}), x\big]$, in which $\Delta = 28314000$. In this note, we show that for $x \geq 1.17 \times 10^{1634}$, we can yield Ramaré-Saouter's result from Dusart's result.

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As usual, suppose $\mathbb{P}$ be the set of all prime numbers. In 1999, P. Dusart [1] showed that for every $x \geq 3275$, we have

$$\mathbb{P} \cap \left(x, x(1+\frac{1}{2\ln^2 x})\right] \neq \phi.$$

In 2003, O. Ramaré and Y. Saouter [2] showed that for $x \geq 10726905041$, we have

$$\mathbb{P} \cap \big(x(1-\Delta^{-1}), x\big] \neq \phi,$$

in which $\Delta = 28314000$. If $L_D(x)$ denotes the length of Dusart's interval, $\big(x, x(1+\frac{1}{2\ln^2 x})\big]$, and $L_{RS}(x)$ denotes the length of Ramaré-Saouter's interval, $\big(x(1-\Delta^{-1}), x\big]$, then clearly we have:

$$L_D(x) = \frac{x}{2\ln^2 x} = O\left(\frac{x}{\ln^2 x}\right),$$

and

$$L_{RS}(x) = \frac{x}{\Delta} = O(x).$$

Also, we observe that

$$\lim_{x\to\infty}\frac{L_D(x)}{L_{RS}(x)}=0,$$

and this suggest that for sufficiently large values of x's, Dusart's interval become shorter than Ramaré-Saouter's interval. In this research report, we show that for $x\geq 1.17\times 10^{1634}$, we can yield Ramaré-Saouter's interval from Dusart's interval.

Let $d(x)=x(1+\frac{1}{2\ln^2 x})$. For $x>0$, $d'(x)>0$; so, $d^{-1}(x)$, the inverse of the function $d(x)$, is well-define. According to Dusart, for $x\geq 3275$, $\mathbb{P}\cap(x,d(x)]\neq\phi$, and so for such x's that $d^{-1}(x)\geq 3275$, we have

$$\mathbb{P}\cap\big(d^{-1}(x),x\big]\neq\phi.$$

Therefore, $\mathbb{P}\cap\big(d^{-1}(x),x\big]\neq\phi$ holds for $x\geq d(3275)$ or for $x\geq 3300$. Now, we search such x's that $x(1-\Delta^{-1})\leq d^{-1}(x)$ or $d(x(1-\Delta^{-1}))\leq x$ and this is equivalent to

$$x(1-\Delta^{-1})\left(1+\frac{1}{2\ln^2\big(x(1-\Delta^{-1})\big)}\right)\leq x,$$

and since $x>0$, we yield that for $x\geq\frac{e^{\sqrt{\frac{\Delta-1}{2}}}}{1-\frac{1}{\Delta}}\approx 1.167417545\times 10^{1634}$, Dusart's interval yields Ramaré-Saouter's interval. This prove our claim at above.

We end this short note with a question about Dusart's interval:

Question. For every $x\in\mathbb{R}$, let

$$n(x):=\#\mathbb{P}\cap\left(x,x(1+\frac{1}{2\ln^2 x})\right].$$

Is there some elementary function $f(x)$ such that $n(x)\sim f(x)$, when $x\to\infty$?

More generally study of $n(x)$ is a nice subject.

Note. All computations in this note done by Maple software.

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References

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[2] O. Ramaré and Y. Saouter, Short effective intervals containing primes, *Journal of Number Theory*, **98** (2003), 10-33.