On Integral Version of Alzer's Inequality and Martins' Inequality

ON INTEGRAL VERSION OF ALZER'S INEQUALITY AND MARTINS' INEQUALITY

CHAO-PING CHEN AND FENG QI

ABSTRACT. Let $c > b > a$ and r be real numbers, and let f be a positive, twice differentiable function and satisfy $f'(t) > 0$ and $(\ln f(t))'' \geq 0$ on $(a, +\infty)$. Then

$$\frac{\sup_{x\in[a,b]} f(x)}{\sup_{x\in[a,c]} f(x)} < \left(\frac{\frac{1}{b-a}\int_a^b f^r(x)\,\mathrm{d}x}{\frac{1}{c-a}\int_a^c f^r(x)\,\mathrm{d}x}\right)^{1/r} < 1 \text{ for all real } r,$$

$$\left(\frac{\frac{1}{b-a}\int_a^b f^r(x)\,\mathrm{d}x}{\frac{1}{c-a}\int_a^c f^r(x)\,\mathrm{d}x}\right)^{1/r} \lesseqgtr \frac{\exp\left(\frac{1}{b-a}\int_a^b \ln f(x)\,\mathrm{d}x\right)}{\exp\left(\frac{1}{c-a}\int_a^c \ln f(x)\,\mathrm{d}x\right)} \quad \text{according as } r \gtreqless 0.$$

This solves a recently open problem of B.-N. Guo and F. Qi.

1. Introduction

It was shown in [1, 2, 8, 13, 17] that let n be a positive integer, then for $r > 0$,

$$\frac{n}{n+1} < \left(\frac{1}{n}\sum_{i=1}^{n} i^r \Big/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \tag{1}$$

We call the left-hand side of (1) H. Alzer's inequality [1], and the right-hand side of (1) J. S. Martins' inequality [8]. In [3, 14] Alzer's inequality is extended to all real r. In [5] it was proved that Martins' inequality is reversed for $r < 0$.

F. Qi and B.-N. Guo [10, 11] presented an integral version of inequality (1) as follows: Let $b > a > 0$ and $\delta > 0$, then for $r > 0$,

$$\frac{b}{b+\delta} < \left(\frac{\frac{1}{b-a}\int_a^b x^r\,\mathrm{d}x}{\frac{1}{b+\delta-a}\int_a^{b+\delta} x^r\,\mathrm{d}x}\right)^{1/r} < \frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}}. \tag{2}$$

We note that the inequality (4) can be written for $r > 0$ as

$$\frac{b}{b+\delta} < \frac{L_r(a,b)}{L_r(a,b+\delta)} < \frac{I(a,b)}{I(a,b+\delta)}, \tag{3}$$

where $L_r(a,b)$ and $I(a,b)$ are respectively the generalized logarithmic mean and the exponential mean of two positive numbers a, b, defined in [6, 15, 16] by, for $a = b$ by $L_r(a,b) = a$ and for $a \neq b$ by

$$L_r(a,b) = \left(\frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}\right)^{1/r}, \quad r \neq -1, 0;$$

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$$L_{-1}(a,b)=\frac{b-a}{\ln b-\ln a}=L(a,b);$$
$$L_0(a,b)=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}=I(a,b).$$

$L(a,b)$ is the logarithmic mean of two positive numbers a,b. When $a\neq b$, $L_r(a,b)$ is a strictly increasing function of r. In particular,

$$\lim_{r\to-\infty}L_r(a,b)=\min\{a,b\},\qquad \lim_{r\to+\infty}L_r(a,b)=\max\{a,b\}.$$

In [4], it was indirectly shown that the function $r\mapsto L_r(a,b)/L_r(a,b+\delta)$ is strictly decreasing with $r\in(-\infty,+\infty)$. This yields that

$$\frac{b}{b+\delta}<\left(\frac{\frac{1}{b-a}\int_a^b x^r\,\mathrm{d}x}{\frac{1}{b+\delta-a}\int_a^{b+\delta}x^r\,\mathrm{d}x}\right)^{1/r}\quad\text{for all real}\quad r,\tag{4}$$

$$\left(\frac{\frac{1}{b-a}\int_a^b x^r\,\mathrm{d}x}{\frac{1}{b+\delta-a}\int_a^{b+\delta}x^r\,\mathrm{d}x}\right)^{1/r}\lesseqgtr\frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}}\quad\text{according as}\quad r\gtreqless 0.\tag{5}$$

In [7], B.-N. Guo and F. Qi ask under which conditions the inequality

$$\frac{\sup_{x\in[a,b]}f(x)}{\sup_{x\in[a,b+\delta]}f(x)}<\left(\frac{\frac{1}{b-a}\int_a^b f^r(x)\,\mathrm{d}x}{\frac{1}{b+\delta-a}\int_a^{b+\delta}f^r(x)\,\mathrm{d}x}\right)^{1/r}<\frac{\exp\left(\frac{1}{b-a}\int_a^b\ln f(x)\,\mathrm{d}x\right)}{\exp\left(\frac{1}{b+\delta-a}\int_a^{b+\delta}\ln f(x)\,\mathrm{d}x\right)}\tag{6}$$

holds for $b>a>0$, $\delta>0$ and $r>0$.

V. Mascioni [9] found the sufficient conditions on the function f, and proved the right-hand inequality of (6) for $r>0$. Motivated by the paper of Mascioni [9], we establish the following

Theorem. *Let $c>b>a$ and r be real numbers, and let f be a positive, twice differentiable function and satisfy $f'(t)>0$ and $(\ln f(t))''\geq 0$ on $(a,+\infty)$. Then*

$$\frac{\sup_{x\in[a,b]}f(x)}{\sup_{x\in[a,c]}f(x)}<\left(\frac{\frac{1}{b-a}\int_a^b f^r(x)\,\mathrm{d}x}{\frac{1}{c-a}\int_a^c f^r(x)\,\mathrm{d}x}\right)^{1/r}<1\text{ for all real }r,\tag{7}$$

$$\left(\frac{\frac{1}{b-a}\int_a^b f^r(x)\,\mathrm{d}x}{\frac{1}{c-a}\int_a^c f^r(x)\,\mathrm{d}x}\right)^{1/r}\lesseqgtr\frac{\exp\left(\frac{1}{b-a}\int_a^b\ln f(x)\,\mathrm{d}x\right)}{\exp\left(\frac{1}{c-a}\int_a^c\ln f(x)\,\mathrm{d}x\right)}\quad\text{according as } r\gtreqless 0.\tag{8}$$

Both bounds in (7) are best possible.

2. Lemmas

Lemma 1. *Let the function f be a positive and twice differentiable on $(a,+\infty)$, where a is a given real number, and let*

$$G(t)=\frac{\frac{1}{t-a}\int_a^t f(x)\,\mathrm{d}x}{f(t)},\quad t>a.$$

Then we have

(i) If $f'(t)>0$ and $(\ln f(t))''\geq 0$, then the function G is strictly decreasing on $(a,+\infty)$.

(ii) If $f'(t) < 0$ and $(\ln f(t))'' \le 0$, then the function G is strictly increasing on $(a, +\infty)$.

Proof. Easy calculation reveals that

$$\begin{aligned}\frac{[(t-a)f(t)]^2 G'(t)}{f(t)+(t-a)f'(t)} &= \frac{(t-a)f^2(t)}{f(t)+(t-a)f'(t)} - \int_a^t f(x) \triangleq H(t),\\ \frac{[f(t)+(t-a)f'(t)]^2 H'(t)}{(t-a)f^3(t)} &= -(t-a)\frac{f''(t)f(t)-[f'(t)]^2}{f^2(t)} - \frac{f'(t)}{f(t)}\\ &= -[(t-a)(\ln(f(t))'' + (\ln f(t))'].\end{aligned}$$

If $(\ln f(t))' > (<)0$ and $(\ln(f(t))'' \ge (\le)0$ for $t > a$, then $H'(t) < (>)0$ for $t > a$, and then, $H(t) < (>)H(a) = 0$ and $G'(t) < (>)0$ for $t > a$. The proof is complete. □

Lemma 2 ([12]). *If $\mathcal{F}(t)$ is a strictly increasing (decreasing) integrable function on an interval $I \subseteq \mathbb{R}$, then the arithmetic mean $\mathcal{G}(r, s)$ of function $\mathcal{F}(t)$,*

$$\mathcal{G}(r,s) = \begin{cases} \dfrac{1}{s-r}\displaystyle\int_r^s \mathcal{F}(t)\,\mathrm{d}t, & r \ne s,\\ \mathcal{F}(r), & r = s,\end{cases}$$

is also strictly increasing (decreasing) with both r and s on I.

3. Proof of Theorem

For $r = 0$, (7) can be interpreted as

$$\frac{f(b)}{f(c)} < \frac{\exp\left(\frac{1}{b-a}\int_a^b \ln f(x)\,\mathrm{d}x\right)}{\exp\left(\frac{1}{c-a}\int_a^c \ln f(x)\,\mathrm{d}x\right)} < 1. \tag{9}$$

Define for $t > a$,

$$P(t) = \frac{\exp\left(\frac{1}{t-a}\int_a^t \ln f(x)\,\mathrm{d}x\right)}{f(t)}.$$

A simple computation yields

$$\begin{aligned}(t-a)^2\frac{P'(t)}{P(t)} &= (t-a)\ln f(t) - \int_a^t \ln f(x)\,\mathrm{d}x - (t-a)(\ln f(t))' \triangleq Q(t),\\ Q'(t) &= -(t-a)\left[(\ln f(t))' + (t-a)(\ln(f(t))''\right] < 0.\end{aligned}$$

Hence, we have $Q(t) < Q(a) = 0$ and $P'(t) < 0$ for $t > a$. This means the left-hand inequality of (9) holds for $c > b > a$. By Lemma 2, the right-hand inequality of (9) holds clearly.

For $r \ne 0$, (7) is equivalent to

$$\frac{f^r(b)}{f^r(c)} \lessgtr \frac{\frac{1}{b-a}\int_a^b f^r(x)\,\mathrm{d}x}{\frac{1}{c-a}\int_a^c f^r(x)\,\mathrm{d}x} \lessgtr 1, \quad \text{according as} \quad r \gtrless 0. \tag{10}$$

Define for $t > a$,

$$G_r(t) = \frac{\frac{1}{t-a}\int_a^t f^r(x)\,\mathrm{d}x}{f^r(t)}.$$

It is easy to see that

$$(\ln f^r(t))'_t \gtrless 0 \quad \text{and} \quad (\ln(f^r(t))''_t \overset{\geq}{\leq} 0, \quad \text{according as} \quad r \gtrless 0. \tag{11}$$

By Lemma 1, the function $t \mapsto G_r(t)$ strictly $\begin{matrix}\text{decreases}\\ \text{increases}\end{matrix}$ with respect to $t \in (a, +\infty)$ according as $r \gtrless 0$. This produces the left-hand inequality of (10). By Lemma 2, the right-hand inequality of (10) holds clearly.

Both bounds in (7) are best possible because of

$$\lim_{r\to+\infty}\left(\frac{\frac{1}{b-a}\int_a^b f^r(x)\,\mathrm{d}x}{\frac{1}{c-a}\int_a^c f^r(x)\,\mathrm{d}x}\right)^{1/r} = \frac{\sup_{x\in[a,b]} f(x)}{\sup_{x\in[a,c]} f(x)},$$

$$\lim_{r\to-\infty}\left(\frac{\frac{1}{b-a}\int_a^b f^r(x)\,\mathrm{d}x}{\frac{1}{c-a}\int_a^c f^r(x)\,\mathrm{d}x}\right)^{1/r} = \frac{\inf_{x\in[a,b]} f(x)}{\inf_{x\in[a,c]} f(x)} = 1.$$

The inequality (8) is equivalent to

$$\frac{\frac{1}{b-a}\int_a^b f^r(x)\,\mathrm{d}x}{\frac{1}{c-a}\int_a^c f^r(x)\,\mathrm{d}x} < \frac{\exp\left(\frac{1}{b-a}\int_a^b \ln f^r(x)\,\mathrm{d}x\right)}{\exp\left(\frac{1}{c-a}\int_a^c \ln f^r(x)\,\mathrm{d}x\right)} \quad \text{for } r \neq 0. \tag{12}$$

Define for $t > a$,

$$F_r(t) = \frac{\frac{1}{t-a}\int_a^t f^r(x)\,\mathrm{d}x}{\exp\left(\frac{1}{t-a}\int_a^t \ln f^r(x)\,\mathrm{d}x\right)}.$$

It is easy to see from the proof of Theorem 1 of [9] that if $f'(t) > 0$ and $(\ln f(t))'' \geq 0$, then the function F_1 is strictly increasing on $(a, +\infty)$; If $f'(t) < 0$ and $(\ln f(t))'' \leq 0$, then the function F_1 is strictly decreasing on $(a, +\infty)$. Applying this result, together with (11), we obviously imply the function $t \mapsto F_r(t)$ strictly $\begin{matrix}\text{increases}\\ \text{decreases}\end{matrix}$ with respect to $t \in (a, +\infty)$ according as $r \gtrless 0$. This produces (12). The proof is complete.

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