Identities by L-Summing Method and Maple

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Abstract

Let a_{ij} be a $n \times n$ array. $L-$Summing Method is the following rearrange:

$$\sum_{1\leq i,j\leq n} a_{ij} = \sum_{k=1}^{n}\left(\sum_{i=1}^{k} a_{ik} + \sum_{j=1}^{k} a_{kj} - a_{kk}\right).$$

By using Maple software and this method, we get some new identities.

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1 Introduction

In [3] we introduced $L-$Summing Method for proving and yielding some identities. In this method we sum the numbers in array a_{ij} by two ways; first by an arbitrary method, for example line by line and then by summing L by L. The $k-$th L in array a_{ij} is:

$$L_k = \sum_{i=1}^{k} a_{ik} + \sum_{j=1}^{k} a_{kj} - a_{kk},$$

and $L-$Summing Method is:

$$\sum_{1\leq i,j\leq n} a_{ij} = \sum_{k=1}^{n} L_k.$$

If we apply this method on $a_{ij} = ij$ (i.e. $n \times n$ Multiplication Table), then we have [3]:

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Also, applying this method on $a_{ij} = (ij)^{-s}$ we yield [3]:

$$\sum_{k=1}^{n} \frac{\zeta_k(s)}{k^s} = \frac{\zeta_n^2(s) + \zeta_n(2s)}{2}, \qquad (s \in \mathbb{C}).$$

In other hand we have Maple software, which can compute summations. So, we can use this useful software and $L-$summing method to get some new identities. Our Maple program and its out put, for example for $a_{ij} = ij$, is:

```
restart:
a[ij]:=i*j;
L:=sum(eval(a[ij],i=k),j=1..k):
C:=sum(eval(a[ij],j=k),i=1..k):
L[k]:=simplify(C+L-eval(eval(a[ij],i=k),j=k)):
S(A):=factor(simplify(sum(sum(a[ij],i=1..n),j=1..n))):
Sum(L[k],k=1..n)=S(A);
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$$a_{ij} := ij$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2 (n+1)^2}{4}$$

By LSMMP ($L-$Summing Method's Maple Program), we call above program. Also by $\mathsf{LSMI}(a_{ij})$, we call the identity related by a_{ij} and generated by LSMMP. So, we have proven:

$$\mathsf{LSMI}(ij) : \sum_{k=1}^{n} k^3 = \frac{n^2 (n+1)^2}{4}.$$

Now, we are ready to use LSMMP on some arrays to get some new identities or reprove some of know identities.

2 Identities by LSMMP

All propositions here has their proofs in their heart. They yield by using LSMMP on their related array which mentioned in the left hand side of them by $\mathsf{LSMI}(a_{ij})$.

2.1 Reproving Some Well-known Identities

Proposition 1 *For every $n \in \mathbb{N}$, we have*

$$\mathit{LSMI}(1) : \sum_{k=1}^{n} (2k-1) = n^2.$$

Corollary 2 *For every $n \in \mathbb{N}$, we have*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Proposition 3 *For every $n \in \mathbb{N}$, we have*

$$\mathit{LSMI}(i+j+\frac{1}{2}) : \sum_{k=1}^{n}(3k^2-\frac{1}{2}) = \frac{n^2(2n+3)}{2}.$$

Corollary 4 *For every $n \in \mathbb{N}$, we have*

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

2.2 Some New Other Identities

Proposition 5 *Let $\Psi(x) = \frac{d}{dx}\Gamma(x)$, where $\Gamma(x)$ is well-known gamma function [1]. For every $n \in \mathbb{N}$, we have*

$$\mathit{LSMI}(\frac{1}{ij}) : \sum_{k=1}^{n} \frac{2\,\Psi\,(k+1)\,k + 2\,\gamma\,k - 1}{k^2} = \left(\Psi\,(n+1) + \gamma\right)^2,$$

in which γ is Euler constant.

Proposition 6 *For every $n \in \mathbb{N}$, we have*

$$\mathit{LSMI}(\log(i)) : \sum_{k=1}^{n} \ln\left(\Gamma\,(k+1)\,\right) + k\ln(k) - \ln(k) = n\ln\left(\Gamma\,(n+1)\,\right).$$

Proposition 7 *For every $n \in \mathbb{N}$, we have*

$$\sum_{k=1}^{n}\left(\sum_{i=1}^{k}\Gamma\,(i+1,-1) + k\Gamma\,(k+1,-1) - \Gamma\,(k+1,-1)\right) = n\sum_{i=1}^{n}\Gamma\,(i+1,-1)\,.$$

Proof: This is $e\mathsf{LSMI}(e^{-1}\Gamma\,(i+1,-1))$; i.e. both sides of the identity multiplied by e. This completes the proof. $\square$

Remark 8 *Note that $d(n) = e^{-1}\Gamma\,(i+1,-1)$ is the number of derangements; i.e. permutations of $\mathbb{N}_n = \{1,2,3,\cdots,n\}$ that has no fixed points [2]. Considering this notation and $\mathit{LSMI}(e^{-1}\Gamma\,(i+1,-1))$, we have*

$$\sum_{k=1}^{n}\left(\sum_{i=1}^{k} d(i) + kd(k) - d(k)\right) = n\sum_{i=1}^{n} d(i).$$

Proposition 9 *For every $n \in \mathbb{N}$, we have*

$$\sum_{k=1}^{n}\binom{2\,k+1}{k} + \binom{2\,k+1}{k+1} - \binom{2\,k}{k} = \binom{2\,n+2}{n+1} - 2.$$

Proof: This is $2n + \mathsf{LSMI}(\binom{i+j}{i})$. This completes the proof. $\square$

Proposition 10 *For every $n \in \mathbb{N}$, we have*

$$\mathit{LSMI}(e^{\frac{i}{j}}) : \sum_{k=1}^{n} \frac{-e^{k^{-1}} + \sum_{j=1}^{k} e^{\frac{k}{j}}\, e^{k^{-1}} - \sum_{j=1}^{k} e^{\frac{k}{j}} + e^1}{e^{k^{-1}} - 1} = \sum_{j=1}^{n} \frac{e^{\frac{n+1}{j}} - e^{j^{-1}}}{e^{j^{-1}} - 1}.$$

Proposition 11 *For every $n \in \mathbb{N}$, we have*

$$\sum_{k=1}^{n} \Big((2k-1)\tan(k) + 2\sum_{i=1}^{k}\tan(i)\Big) = 2n\sum_{i=1}^{n}\tan(i),$$

and

$$\sum_{k=1}^{n} \Big((2k-1)\cot(k) + 2\sum_{i=1}^{k}\cot(i)\Big) = 2n\sum_{i=1}^{n}\cot(i).$$

Proof: These are easy reformation of $\mathsf{LSMI}(\tan(i))$ and $\mathsf{LSMI}(\cot(i))$, respectively. $\square$

3 Identities by LSMMP and Maple Software

We can compute $\sum L_k$ by using Maple. The representation of this computation some times is differ by the representation which has yield by $L-$Summing Method. In this case, comparing these two representations led us to some other identities.

Corollary 12 *For every $n \in \mathbb{N}$, we have*

$$\sum_{k=1}^{n} \frac{\Psi(k)}{k} = \frac{\big(\Psi(n+1)+\gamma\big)^2 + \Psi(1,n+1)}{2} - \frac{\pi^2}{12} - \Psi(n+1)\gamma - \gamma^2$$

Proof: Consider $\mathsf{LSMI}(\frac{1}{ij})$ and the following identity by Maple:

$$\sum_{k=1}^{n} \frac{2\Psi(k+1)k + 2\gamma k - 1}{k^2} = -\Psi(1,n+1) + \frac{\pi^2}{6} + 2\Psi(n+1)\gamma + 2\gamma^2 + 2\sum_{k=1}^{n}\frac{\Psi(k)}{k}.$$

This completes the proof. $\square$

Corollary 13 *For every $n \in \mathbb{N}$, we have*

$$\sum_{k=1}^{n} \ln(\Gamma(k+1)) + k\ln(k) = (n+1)\ln\big(\Gamma(n+1)\big).$$

Proof: Consider $\mathsf{LSMI}(\log(i))$ and the following identity by Maple:

$$\sum_{k=1}^{n} \ln(\Gamma(k+1)) + k\ln(k) - \ln(k) = -\ln(\Gamma(n+1)) + \sum_{k=1}^{n}\ln(\Gamma(k+1)) + k\ln(k).$$

This completes the proof. $\square$

Remark 14 *Using Maple and comparing its result by $\mathsf{LSMI}(\binom{i+j}{i})$, we yield a nice and huge identity which you can see it in Maple by running the following program:*

```
restart:
a[ij]:=binomial(i+j,j);
L:=sum(eval(a[ij],i=k),j=1..k):
C:=sum(eval(a[ij],j=k),i=1..k):
L[k]:=simplify(C+L-eval(eval(a[ij],i=k),j=k)):
S(A):=factor(simplify(sum(sum(a[ij],i=1..n),j=1..n))):
Sum(L[k],k=1..n)=S(A);
Sum(L[k],k=1..n)=simplify(sum(L[k],k=1..n));
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