# *Completely Monotonic Functions Related to the Gamma Functions*

# COMPLETELY MONOTONIC FUNCTIONS RELATED TO THE GAMMA FUNCTIONS

CHAO-PING CHEN AND FENG QI

ABSTRACT. (i) Let $a, b > 0$ be real numbers, and let

$$f_{a,b}(x) = \frac{1}{x^{b-a}}\left[\frac{\Gamma(bx+1)}{\Gamma(ax+1)}\right]^{1/x}.$$

Then, for $x > 0$ and $n = 1, 2, \dots$, $(-1)^n(\ln f_{a,b}(x))^{(n)} \gtreqless 0$ according as $b \gtreqless a$.

(ii) Let $p > 0$ be a real number, and let $f_p(x) = \theta(px) - p\theta(x)$, where

$$\theta(x) = \int_0^\infty \left(\frac{t}{e^t - 1} - 1 + \frac{t}{2}\right)\frac{e^{-xt}}{t^2}\,\mathrm{d}t, x > 0$$

is remainder of Binet's formula. Then, for $x > 0$ and $n = 0, 1, 2, \dots$,

$$(-1)^n f_p^{(n)}(x) \gtreqless 0 \quad \text{according as} \quad p \lesseqgtr 1.$$

## 1. INTRODUCTION

The Euler gamma function $\Gamma$ and its logarithmic derivative $\psi$, the so-called digamma function, are defined for $\operatorname{Re} z > 0$ by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t \quad \text{and} \quad \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

There exists a very extensive literature on these functions. In particular, inequalities, monotonicity and complete monotonicity properties for these functions have been published, we refer to the paper [1] and [2], and the references given therein. We recall that a function $f$ is said to be completely monotonic on an interval $I$, if $f$ has derivatives of all orders on $I$ and satisfies

$$(-1)^n f^{(n)}(x) \geq 0 \quad (x \in I; n = 0, 1, 2, \dots). \tag{1}$$

If the inequality (1) is strict, then $f$ is said to be strictly completely monotonic on $I$. Completely monotonic functions have remarkable applications in different branches. For instance, they play a role in potential theory [4], probability theory [6, 8, 10], physics [7], numerical and asymptotic analysis [9, 15], and combinatorics [3]. A detailed collection of the most important properties of completely monotonic functions can be found in [14, Chapter IV], and in an abstract in [5].

In a recent paper [12], the terminology "(strictly) logarithmically completely monotonic function" was introduced. It was also shown in this paper that a (strictly) logarithmically completely monotonic function is also (strictly) completely

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monotonic. For convenience, we recall that a positive function $f$ is said to be logarithmically completely monotonic on an interval $I$ if its logarithm $\ln f$ satisfies

$$(-1)^n(\ln f(x))^{(n)} \geq 0 \quad (x \in I; n = 1, 2, \ldots). \tag{2}$$

If inequality (2) is strict, then $f$ is said to be strictly logarithmically completely monotonic.

In 2003, J. Sándor [13] showed that

$$\lim_{x\to\infty} \frac{1}{x^{b-a}} \left[\frac{\Gamma(bx+1)}{\Gamma(ax+1)}\right]^{1/x} = \frac{b^b}{a^a} e^{b-a}. \tag{3}$$

Our first theorem considers logarithmically complete monotonicity property of the function in (3).

**Theorem 1.** *Let $a, b > 0$ be real numbers, and let*

$$f_{a,b}(x) = \frac{1}{x^{b-a}} \left[\frac{\Gamma(bx+1)}{\Gamma(ax+1)}\right]^{1/x}.$$

*Then, for $x > 0$ and $n = 1, 2, \ldots$, $(-1)^n(\ln f_{a,b}(x))^{(n)} \gtrless 0$ according as $b \gtrless a$.*

If we denote by

$$I(a, b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, \quad a > 0, b > 0, a \neq b,$$

the so-called identric mean, then, we yield from (3) and the monotonicity of the function $f_{a,b}$ that, for $x > 0$,

$$\frac{1}{x^{b-a}} \left[\frac{\Gamma(bx+1)}{\Gamma(ax+1)}\right]^{1/x} \gtrless [e^2 I(a, b)]^{b-a} \quad \text{according as} \quad b \gtrless a. \tag{4}$$

Binet's formula [16, p. 103] states that for $x > 0$,

$$\ln\Gamma(x) = \left(x - \frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \theta(x),$$

where

$$\theta(x) = \int_0^\infty \left(\frac{t}{e^t - 1} - 1 + \frac{t}{2}\right)\frac{e^{-xt}}{t^2}\,\mathrm{d}t. \tag{5}$$

Let $p > 0$ be a real number. Our second theorem considers complete monotonicity property of the function $x \mapsto \theta(px) - p\theta(x)$ on $(0, \infty)$.

**Theorem 2.** *Let $p > 0$ be a real number, and let $f_p(x) = \theta(px) - p\theta(x)$, where $\theta(x)$ is defined by (5). Then, for $x > 0$ and $n = 0, 1, 2, \ldots$,*

$$(-1)^n f_p^{(n)}(x) \gtrless 0 \quad \textit{according as} \quad p \lessgtr 1.$$

## 2. Proofs of Theorems

*Proof of Theorem 1.* Using Leibniz' rule

$$[u(x)v(x)]^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)}(x)v^{(n-k)}(x),$$

we obtain

$$(\ln f_{a,b}(x))^{(n)} = -\frac{(b-a)(-1)^{n-1}(n-1)!}{x^n}$$

$$
\begin{aligned}
&+\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{x}\right)^{(n-k)}[\ln\Gamma(bx+1)-\ln\Gamma(ax+1)]^{(k)}\\
&=-\frac{(b-a)(-1)^{n-1}(n-1)!}{x^n}+\frac{(-1)^n n!}{x^{n+1}}[\ln\Gamma(bx+1)-\ln\Gamma(ax+1)]\\
&+\frac{(-1)^n n!}{x^{n+1}}\sum_{k=1}^{n}\frac{(-1)^k}{k!}x^k[b^k\psi^{(k-1)}(bx+1)-a^k\psi^{(k-1)}(ax+1)].
\end{aligned}
$$

Define for $x>0$,

$$
\begin{aligned}
g_{a,b}(x)&=\frac{(-1)^n x^{n+1}}{n!}(\ln f(x))^{(n)}\\
&=\frac{(b-a)x}{n}+\ln\Gamma(bx+1)-\ln\Gamma(ax+1)\\
&\quad+\sum_{k=1}^{n}\frac{(-1)^k}{k!}x^k[b^k\psi^{(k-1)}(bx+1)-a^k\psi^{(k-1)}(ax+1)].
\end{aligned}
$$

Using the representations

$$
\begin{aligned}
\frac{(n-1)!}{x^n}&=\int_0^\infty t^{n-1}e^{-xt}\,\mathrm{d}t,\ (x>0),\\
\psi^{(n)}(x)&=(-1)^{n+1}\int_0^\infty \frac{t^n}{1-e^{-t}}e^{-xt}\,\mathrm{d}t,\ (x>0, n=1,2,\ldots),
\end{aligned}
$$

see [11, p. 16], we imply

$$
\begin{aligned}
\frac{n!}{x^n}g'_{a,b}(x)&=\frac{(b-a)(n-1)!}{x^n}+(-1)^n[b^{n+1}\psi^{(n)}(bx+1)-a^{n+1}\psi^{(n)}(ax+1)]\\
&=(b-a)\int_0^\infty t^{n-1}e^{-xt}\,\mathrm{d}t-\int_0^\infty\frac{b^{n+1}t^n}{e^t-1}e^{-bxt}\,\mathrm{d}t+\int_0^\infty\frac{a^{n+1}t^n}{e^t-1}e^{-axt}\,\mathrm{d}t\\
&=(b-a)\int_0^\infty t^{n-1}e^{-xt}\,\mathrm{d}t-\int_0^\infty\frac{t^n}{e^{t/b}-1}e^{-xt}\,\mathrm{d}t+\int_0^\infty\frac{t^n}{e^{t/a}-1}e^{-xt}\,\mathrm{d}t\\
&=\int_0^\infty\left[\left(\frac{t}{e^{t/a}-1}-a\right)-\left(\frac{t}{e^{t/b}-1}-b\right)\right]t^{n-1}e^{-xt}\,\mathrm{d}t.
\end{aligned}
$$

For fixed $t>0$, we define the function

$$h_t(a)=\frac{t}{e^{t/a}-1}-a\quad(a>0).$$

Differentiation yields

$$h'_t(a)=\frac{(t/a)^2e^{t/a}-(e^{t/a}-1)^2}{(e^{t/a}-1)^2}.$$

Now we are in a position to prove $h'_t(a)<0$ for $a>0$, which is equivalent to

$$(t/a)e^{t/(2a)}<e^{t/a}-1,$$

i.e.,

$$(t/a)<e^{t/(2a)}-e^{-t/(2a)}.$$

Using power series expansion, we have

$$e^{t/(2a)} - e^{-t/(2a)} - (t/a) = 2\sum_{n=2}^{\infty} \frac{1}{(2n-1)!}\left(\frac{t}{2a}\right)^{2n-1} > 0$$

for $a > 0$. Hence $h_t'(a) < 0$ for $a > 0$, and then, for $x > 0$, $g_{a,b}'(x) \gtreqless 0$ and $g_{a,b}(x) \gtreqless g_{a,b}(0) = 0$ according as $b \gtreqless a$. This implies that for $x > 0$ and $n = 1, 2, \dots$, $(-1)^n(\ln f_{a,b}(x))^{(n)} \gtreqless 0$ according as $b \gtreqless a$. The proof is complete. □

*Proof of Theorem 2.* By (5), we imply

$$\begin{aligned}
f_p(x) &= \int_0^\infty \left(\frac{u}{e^u - 1} - 1 + \frac{u}{2}\right)\frac{e^{-pxu}}{u^2}\,\mathrm{d}u - p\int_0^\infty \left(\frac{t}{e^t - 1} - 1 + \frac{t}{2}\right)\frac{e^{-xt}}{t^2}\,\mathrm{d}t \\
&= p\int_0^\infty \left[\frac{t}{p(e^{t/p} - 1)} - 1 + \frac{t}{2p}\right]\frac{e^{-xt}}{t^2}\,\mathrm{d}t - p\int_0^\infty \left(\frac{t}{e^t - 1} - 1 + \frac{t}{2}\right)\frac{e^{-xt}}{t^2}\,\mathrm{d}t \\
&= p\int_0^\infty \left[\frac{t}{p(e^{t/p} - 1)} - \frac{1}{e^t - 1} + \frac{1-p}{p}\right]\frac{e^{-xt}}{t^2}\,\mathrm{d}t \\
&= \int_0^\infty \frac{\delta_p(t)}{2(e^{t/p} - 1)(e^t - 1)t} e^{-xt}\,\mathrm{d}t
\end{aligned}$$

and therefore,

$$(-1)^n f_p^{(n)}(x) = \int_0^\infty \frac{t^{n-1}\delta_p(t)}{2(e^{t/p} - 1)(e^t - 1)} e^{-xt}\,\mathrm{d}t.$$

where

$$\begin{aligned}
\delta_p(t) &= (1+p)e^t - (1+p)e^{t/p} + (1-p)e^{[(1+p)/p]t} + p - 1 \\
&= \sum_{k=3}^{\infty} [p^k - 1 + (1-p)(1+p)^{k-1}]\frac{(1+p)t^k}{p^k \cdot k!}.
\end{aligned}$$

It is easy to see that

$$p^k - 1 + (1-p)(1+p)^{k-1} = (p-1)\sum_{m=0}^{k-1} p^m + (1-p)\sum_{m=0}^{k-1}\binom{k-1}{m}p^m$$

$$= (p-1)\sum_{m=1}^{k-2}\left[1 - \binom{k-1}{m}\right]p^m \gtreqless 0 \quad \text{according as} \quad p \lesseqgtr 1.$$

This implies for $x > 0$ and $n \geq 0$,

$$(-1)^n f_p^{(n)}(x) \gtreqless 0 \quad \text{according as} \quad p \lesseqgtr 1.$$

The proof is complete. □

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(Ch.-P. Chen) Department of Applied Mathematics and Informatics, Research Institute of Applied Mathematics, Henan Polytechnic University, Jiaozuo City, Henan 454010, China

*E-mail address*: chenchaoping@hpu.edu.cn, chenchaoping@sohu.com

(F. Qi) Department of Applied Mathematics and Informatics, Research Institute of Applied Mathematics, Henan Polytechnic University, Jiaozuo City, Henan 454010, China

*E-mail address*: qifeng@hpu.edu.cn, fengqi618@member.ams.org

*URL*: http://rgmia.vu.edu.au/qi.html