# *Approximation of the Dilogarithm Function*

# APPROXIMATION OF THE DILOGARITHM FUNCTION

MEHDI HASSANI

ABSTRACT. In this short note, we approximate Dilogarithm function, defined by $\mathrm{dilog}(x)=\int_1^x \frac{\log t}{1-t}dt$. Letting

$$\mathcal{D}(x,N)=-\frac{1}{2}\log^2 x-\frac{\pi^2}{6}+\sum_{n=1}^{N}\frac{\frac{1}{n^2}+\frac{1}{n}\log x}{x^n},$$

we show that for every $x>1$, the inequalities

$$\mathcal{D}(x,N)<\mathrm{dilog}(x)<\mathcal{D}(x,N)+\frac{1}{x^N}$$

holds true for all $N\in\mathbb{N}$.

***Definition.*** The Dilogarithm function $\mathrm{dilog}(x)$ is defined for every $x>0$ as follows [5]:

$$\mathrm{dilog}(x)=\int_1^x \frac{\log t}{1-t}dt.$$

***Expansion.*** The following expansion holds true when $x$ tends to infinity:

$$\mathrm{dilog}(x)=\mathcal{D}(x,N)+O\left(\frac{1}{x^{N+1}}\right),$$

where

$$\mathcal{D}(x,N)=-\frac{1}{2}\log^2 x-\frac{\pi^2}{6}+\sum_{n=1}^{N}\frac{\frac{1}{n^2}+\frac{1}{n}\log x}{x^n}.$$

***Aim of Present Work.*** The aim of this note is to prove that:

$$0<\mathrm{dilog}(x)-\mathcal{D}(x,N)<\frac{1}{x^N}\qquad (x>1,\ N\in\mathbb{N}).$$

***Lower Bound.*** For every $x>0$ and $N\in\mathbb{N}$, let:

$$\mathcal{L}(x,N)=\mathrm{dilog}(x)-\mathcal{D}(x,N).$$

A simple computation, yields that:

$$\frac{d}{dx}\mathcal{L}(x,N)=\log x\left(\frac{x}{1-x}+\sum_{n=0}^{N+1}\frac{1}{x^n}\right)<\log x\left(\frac{x}{1-x}+\sum_{n=0}^{\infty}\frac{1}{x^n}\right)=0.$$

So, $\mathcal{L}(x,N)$ is a strictly decreasing function of the variable $x$, for every $N\in\mathbb{N}$. Considering $\mathcal{L}(x,N)=O\left(\frac{1}{x^{N+1}}\right)$, we obtain desired lower bound for the Dilogarithm function, as follow:

$$\mathcal{L}(x,N)>\lim_{x\to+\infty}\mathcal{L}(x,N)=0.$$

---

1991 *Mathematics Subject Classification.* 33E20.

*Key words and phrases.* Special function, Dilogarithm function, digamma function, polygamma function, polylogarithm function, Lerch zeta function.

***Upper Bound.*** For every $x > 0$ and $N \in \mathbb{N}$, let:

$$\mathcal{U}(x, N) = \mathrm{dilog}(x) - \mathcal{D}(x, N) - \frac{1}{x^N}.$$

First, we observe that

$$\mathcal{U}(1, N) = \frac{\pi^2}{6} - \sum_{n=1}^{N} \frac{1}{n^2} - 1 = \Psi(1, N+1) - 1 \leq \frac{\pi^2}{6} - 2 < 0,$$

in which $\Psi(m, x)$ is the $m$-th polygamma function, the $m$-th derivative of the digamma function, $\Psi(x) = \frac{d}{dx} \log \Gamma(x)$, with $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ (see [1, 2]). A simple computation, yields that:

$$\frac{d}{dx}\mathcal{U}(x, N) = \log x \left( \frac{x}{1-x} + \sum_{n=0}^{N+1} \frac{1}{x^n} \right) + \frac{N}{x^{N+1}}.$$

To determine the sign of $\frac{d}{dx}\mathcal{U}(x, N)$, we distinguish two cases:
◇ Suppose $x > 1$. Since, $\frac{\log x}{x-1}$ is strictly decreasing, we have

$$N \geq 1 = \lim_{x \to 1} \frac{\log x}{x-1} > \frac{\log x}{x-1},$$

which is $\frac{N}{\log x} > \frac{1}{x-1}$ or equivalently $\frac{N}{x^{N+1} \log x} > \sum_{n=N+2}^{\infty} \frac{1}{x^n}$, and this yields that $\frac{d}{dx}\mathcal{U}(x, N) > 0$. So, $\mathcal{U}(x, N)$ is strictly increasing for every $N \in \mathbb{N}$. Thus, $\mathcal{U}(x, N) < \lim_{x \to +\infty} \mathcal{U}(x, N) = 0$; as desired in this case. Also, note that in this case we obtain

$$\mathcal{U}(x, N) > \mathcal{U}(1, N) = \Psi(1, N+1) - 1.$$

◇ Suppose $0 < x < 1$ and $N - \frac{\log x}{x-1} \geq 0$. We observe that $1 < \frac{\log x}{x-1} < +\infty$ and $\sum_{n=0}^{N+1} \frac{1}{x^n} = \frac{1-x^{N+2}}{x^{N+1}(1-x)}$. Considering these facts, we see that $\frac{d}{dx}\mathcal{U}(x, N)$ and $N - \frac{\log x}{x-1}$ have same sign; i.e.

$$\mathrm{sgn}\left( \frac{d}{dx}\mathcal{U}(x, N) \right) = \mathrm{sgn}\left( N - \frac{\log x}{x-1} \right).$$

Thus, $\mathcal{U}(x, N)$ is increasing and so,

$$\mathcal{U}(x, N) \leq \lim_{x \to 1^-} \mathcal{U}(x, N) = \Psi(1, N+1) - 1 \leq \frac{\pi^2}{6} - 2 < 0.$$

***Connection with Other Functions.*** Using Maple software, we have:

$$\begin{aligned} \mathcal{D}(x, N) &= -\frac{1}{2}\log^2 x - \frac{\pi^2}{6} + \frac{1}{N^2 x^N} + \frac{\log x}{N x^N} - \log\left(\frac{x-1}{x}\right)\log x \\ &+ \mathrm{polylog}\left(2, \frac{1}{x}\right) - \frac{\log x}{x^N}\Phi\left(\frac{1}{x}, 1, N\right) - \frac{1}{x^N}\Phi\left(\frac{1}{x}, 2, N\right), \end{aligned}$$

in which

$$\mathrm{polylog}(a, z) = \sum_{n=1}^{\infty} \frac{z^n}{n^a},$$

is the polylogarithm function of index $a$ at the point $z$ and defined by above series if $|z| < 1$, and by analytic continuation otherwise [4]. Also,

$$\Phi(z, a, v) = \sum_{n=1}^{\infty} \frac{z^n}{(v+n)^a},$$

is the Lerch zeta (or Lerch-$\Phi$) function defined by above series for $|z| < 1$, with $v \neq 0, -1, -2, \cdots$, and by analytic continuation, it is extended to the whole complex $z$-plane for each value of $a$ and $v$ (see [3, 6]).

## References


[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions: with Formulas, Graphs, and Mthematical Tables*, Dover Publications, 1972.

[2] N.N. Lebedev, *Special Functions and their Applications*, Translated and edited by Richard A. Silverman, Dover Publications, New York, 1972.

[3] L. Lewin, Dilogarithms and associated functions, Foreword by J. C. P. Miller, *Macdonald, London*, 1958.

[4] Leonard Lewin, Polylogarithms and associated functions, With a foreword by A. J. Van der Poorten, *North-Holland Publishing Co., New York-Amsterdam*, 1981.

[5] Eric W. Weisstein. "Dilogarithm." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/Dilogarithm.html

[6] Eric W. Weisstein. "Lerch Transcendent." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/LerchTranscendent.html



Institute for Advanced, Studies in Basic Sciences, P.O. Box 45195-1159, Zanjan, Iran.
*E-mail address*: `mmhassany@srttu.edu`