Logarithmically Completely Monotonic Functions Involving Gamma and Polygamma Functions

LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS INVOLVING GAMMA AND POLYGAMMA FUNCTIONS

FENG QI, DA-WEI NIU, AND JIAN CAO

Abstract. There exists a constant b^* such that the function $[\Gamma(x+1)]^{1/x}(1+1/x)^{x+b}/x$ is logarithmically completely monotonic and the function $\psi''(x)+[x^4+5x^3+(7+3b)x^2+(7+b)x+2]/x^3(x+1)^3$ is completely monotonic in $(0,\infty)$ if $b\ge b^*\ge -1$. The function $(1+a/x)^{x+b}$ for $a>0$ and $b\in\mathbb{R}$ is logarithmically completely monotonic in $(0,\infty)$ if and only if $2b\ge a$.

1. Introduction

A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x)\ge 0 \tag{1}$$

for $x\in I$ and $n\ge 0$. The set of completely monotonic functions is denoted by $\mathcal{C}[I]$.

A positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^k[\ln f(x)]^{(k)}\ge 0 \tag{2}$$

for $k\in\mathbb{N}$ on I. The set of logarithmically completely monotonic functions is denoted by $\mathcal{L}[I]$.

A function f is called a Stieltjes transform if it can be of the form

$$f(x)=a+\int_0^\infty\frac{\mathrm{d}\mu(s)}{s+x}, \tag{3}$$

where a is a nonnegative number and μ is a nonnegative measure on $[0,\infty)$ satisfying $\int_0^\infty 1/(1+s)\,\mathrm{d}\mu(s)<\infty$. The set of Stieltjes transforms is denoted by $\mathcal{S}$.

The notion "logarithmically completely monotonic function" was explicitly coined in [2] and recovered in [10, 12, 14] and the following much useful, important and key conclusion was proved: $\mathcal{L}[I]\subset\mathcal{C}[I]$. Stimulated by the papers [12, 14], among other things, it is proved in [3] that $\mathcal{S}\setminus\{0\}\subset\mathcal{L}[(0,\infty)]\subset\mathcal{C}[(0,\infty)]$.

One motivation for studying completely monotonic functions is the connection between infinitely divisible probability distributions (measure) on $\mathbb{R}$ and $[0,\infty)$ and completely monotonic functions related to their Fourier and Laplace transforms. See [16, p. 161]. The class of logarithmically completely monotonic functions is characterized as the infinitely divisible completely monotonic functions established by Horn in [6, Theorem 4.4] and restated in [3, Theorem 1.1].

It is well-known that the sequence $\{(1+1/n)^n\}_{n\in\mathbb{N}}$ is increasing. A theorem of I. Schur [8, pp. 30 and 186] states that the sequence $\{(1+1/n)^{n+b}\}_{n\in\mathbb{N}}$ decreases if and only if $b\ge 1/2$. Furthermore, all parameters $a>0$ and $b\in\mathbb{R}$ such that the sequence $\left\{(1+a/n)^{n+b}\right\}_{n\in\mathbb{N}}$ or the corresponding function $(1+a/x)^{x+b}$ in $(0,\infty)$

2000 *Mathematics Subject Classification.* primary 33B15; secondary 26D07.

Key words and phrases. completely monotonic function, logarithmically completely monotonic function, gamma function, polygamma function.

The first author was supported in part by the Science Foundation of Project for Fostering Innovation Talents at Universities of Henan Province, China.

This paper was typeset using $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX.

are monotonic are determined in [11]. Let $F_n(x) = P_n(x)[e - (1 + 1/x)^x]$ and $G_n(x) = P_n(x)[(1+1/x)^{x+1} - e]$, where $P_n(x) = x^n + \sum_{i=0}^{n-1} c_i x^i$ is a polynomial of degree $n \ge 1$ with real coefficients. Then the following conclusions were obtained in [1]: F_n is completely monotonic if and only if $n = 1$ and $c_0 \ge 11/12$, and G_n is completely monotonic if and only if $n = 1$ and $c_0 \ge 1/12$; The function $e-(1+1/x)^x$ and $(1 + 1/x)^{x+1} - e$ are Stieltjes transforms; Let $a > 0$ and b be real numbers, then the function $(1 + a/x)^{x+b} - e^a$ is completely monotonic if and only if $a \le 2b$. In Theorem 2 below, the sufficient and necessary conditions in order the function $(1 + a/x)^{x+b}$ to be logarithmically completely monotonic in $(0, \infty)$ are given.

It is well known that $\Gamma(x)$ denotes the classical Euler gamma function defined for $\operatorname{Re} z > 0$ by $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, \mathrm{d} t$ which is one of the most important special functions and has much extensive applications in many branches, for example, statistics, physics, engineering, and other mathematical sciences. In the recent past, various authors showed that numerous functions, which are defined in terms of gamma, polygamma, and other special functions, are (logarithmically) completely monotonic and used this fact to derive many interesting new inequalities. See [1, 3, 4, 5, 7, 10, 12, 13, 14] and the references therein.

In various papers complete monotonicity for special functions has been established by proving the stronger statement that the function is logarithmically completely monotonic or is a Stieltjes transform. In concrete cases it is often easier to establish that a function is logarithmically completely monotonic or a Stieltjes transform than to verify its complete monotonicity. See [3, 5, 10, 12, 13, 14] and the references therein.

In [12, 13], it is proved that for $\alpha \ge 1$ the function $[\Gamma(x + 1)]^{1/x}/x^\alpha$ is logarithmically completely monotonic in $(0, \infty)$. In [14], the authors found that the function

$$\Phi(x) = \frac{[\Gamma(x+1)]^{1/x}}{x} \left(1 + \frac{1}{x}\right)^x \tag{4}$$

is logarithmically completely monotonic in $(0, \infty)$. Motivated by [14], among other things, the author in [3] showed that $\Phi(x)$ and $\ln \Phi(x)$ are both Stieltjes transforms.

Define for $x \in (0, \infty)$

$$\Phi_c(x) = \frac{[\Gamma(x+1)]^{1/x}}{x^c} \left(1 + \frac{1}{x}\right)^x. \tag{5}$$

It is clear that $\Phi_1(x) = \Phi(x)$. In [5] recently, the following sufficient and necessary conditions are established: The function $\Phi_c(x)$ is logarithmically completely monotonic in $(0, \infty)$ if and only if $c \ge 1$ and its reciprocal $1/\Phi_c(x)$ is logarithmically completely monotonic in $(0, \infty)$ if and only if $c \le 0$; The function

$$\psi''(x) + \frac{2 + (6 + c)x + (4 + 3c)x^2 + (2 + 3c)x^3 + cx^4}{x^3(x + 1)^3}, \tag{6}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the psi or digamma function, is completely monotonic in $(0, \infty)$ if and only if $c \ge 1$ and its negative is completely monotonic in $(0, \infty)$ if and only if $c \le 0$.

Define for $x \in (0, \infty)$

$$\Psi_b(x) = \frac{[\Gamma(x+1)]^{1/x}}{x} \left(1 + \frac{1}{x}\right)^{x+b}. \tag{7}$$

It is clear that $\Psi_0(x) = \Phi_0(x) = \Phi(x)$.

The main purpose of this paper is to confirm the range of b such that $\Psi_b(x)$ is logarithmically completely monotonic in $(0, \infty)$.

Theorem 1. *There exists a constant $b^* \ge -1$ such that the function $\Psi_b(x)$ is logarithmically completely monotonic in $(0,\infty)$ if $b \ge b^*$.*

As a by-product of the proof of Theorem 1, we have

Corollary 1. *There exists a constant $b^* \ge -1$ such that the function*

$$\psi''(x) + \frac{x^4 + 5x^3 + (7+3b)x^2 + (7+b)x + 2}{x^3(x+1)^3} \tag{8}$$

is completely monotonic in $(0,\infty)$ if $b \ge b^*$.

By the way, we establish the following

Theorem 2. *The function $(1+a/x)^{x+b}$ for $a>0$ and $b\in\mathbb{R}$ is logarithmically completely monotonic in $(0,\infty)$ if and only if $2b \ge a$.*

Since $\mathcal{L}\subset\mathcal{C}$, it is straightforward to obtain the following

Corollary 2 ([1])**.** *Let $a>0$ and b be real numbers. The function $(1+a/x)^{x+b}-e^a$ is completely monotonic if and only if $a\le 2b$.*

Remark 1. The upper bound of the constant b^* is estimated in [9, 15].

2. Proofs of main results

Proof of Theorem 1 and Corollary 1. Taking the logarithm of $\Psi_b(x)$ gives

$$\ln\Psi_b(x) = (x+b)\ln\left(1+\frac{1}{x}\right) + \frac{\ln\Gamma(x+1)}{x} - \ln x.$$

Differentiating yields

$$[\ln\Psi_b(x)]' = \ln\left(1+\frac{1}{x}\right) - \frac{x+b}{x(x+1)} + \frac{x\psi(x+1)-\ln\Gamma(x+1)}{x^2} - \frac{1}{x} \tag{9}$$

and, for $n\ge 2$,

$$\begin{aligned}
[\ln\Psi_b(x)]^{(n)} &= (-1)^{(n-1)}(n-1)!(x+b)\left[\frac{1}{(x+1)^n}-\frac{1}{x^n}\right]\\
&\quad+(-1)^n(n-2)!n\left[\frac{1}{(x+1)^{n-1}}-\frac{1}{x^{n-1}}\right]\\
&\quad+\frac{h_n(x)}{x^{n+1}}+(-1)^n(n-1)!\frac{1}{x^n}\\
&=(-1)^n(n-2)!\left[\frac{(b+1)(n-1)-x}{x^n}+\frac{x+n-b(n-1)}{(x+1)^n}\right]+\frac{h_n(x)}{x^{n+1}},
\end{aligned}$$

where $\psi^{(-1)}(x+1)=\ln\Gamma(x+1)$, $\psi^{(0)}(x+1)=\psi(x+1)$, and

$$h_n(x)=\sum_{k=0}^{n}\frac{(-1)^{n-k}n!x^k\psi^{(k-1)}(x+1)}{k!}, \tag{10}$$

$$h_n'(x)=x^n\psi^{(n)}(x+1)\begin{cases}>0 & \text{if } n \text{ is odd},\\ <0 & \text{if } n \text{ is even}.\end{cases} \tag{11}$$

Therefore, we have

$$\begin{aligned}
&(-1)^n x^{n+1}[\ln\Psi_b(x)]^{(n)}+(-1)^{n+1}h_n(x)\\
&=(n-2)!\left\{(b+1)(n-1)-x+\frac{x^n[x+n-b(n-1)]}{(x+1)^n}\right\}x
\end{aligned}$$

and, by

$$\psi^{(i-1)}(x+1)=\psi^{(i-1)}(x)+\frac{(-1)^{i-1}(i-1)!}{x^i},\quad i\in\mathbb{N}, \tag{12}$$

$$
\begin{aligned}
&\frac{\mathrm{d}\big\{(-1)^n x^{n+1}[\ln \Psi_b(x)]^{(n)}\big\}}{\mathrm{d}x}\\
&= (-1)^n x^n \psi^{(n)}(x+1) + (n-2)!\bigg\{(b+1)(n-1) - 2x\\
&\quad + \frac{x^n[b+n+(1-b)n^2+(2+b+2n-bn)x+2x^2]}{(x+1)^{n+1}}\bigg\}\\
&= x^n\bigg\{(-1)^n\psi^{(n)}(x+1) + (n-2)!\bigg[\frac{(b+1)(n-1)-2x}{x^n}\\
&\quad + \frac{b+n+(1-b)n^2+(2+b+2n-bn)x+2x^2}{(x+1)^{n+1}}\bigg]\bigg\}\\
&= x^n\bigg\{(-1)^n\psi^{(n)}(x) + \frac{n!}{x^{n+1}} + (n-2)!\bigg[\frac{(b+1)(n-1)-2x}{x^n}\\
&\quad + \frac{b+n+(1-b)n^2+(2+b+2n-bn)x+2x^2}{(x+1)^{n+1}}\bigg]\bigg\}\\
&= x^n\bigg\{(-1)^n\psi^{(n)}(x) + (n-2)!\bigg[\frac{n(n-1)+(b+1)(n-1)x-2x^2}{x^{n+1}}\\
&\quad + \frac{b+n+(1-b)n^2+(2+b+2n-bn)x+2x^2}{(x+1)^{n+1}}\bigg]\bigg\}\\
&\triangleq x^n\big\{(-1)^n\psi^{(n)}(x) + (n-2)![g_n(x)+h_n(x)]\big\}
\end{aligned}
$$

with

$$g_n'(x) = -(n-1)g_{n+1}(x) \quad \text{and} \quad h_n'(x) = -(n-1)h_{n+1}(x)$$

which implies

$$g_2^{(n-2)}(x) = (-1)^n(n-2)!g_n(x)$$

and

$$h_2^{(n-2)}(x) = (-1)^n(n-2)!h_n(x)$$

by induction. Hence

$$\frac{\mathrm{d}\big\{(-1)^n x^{n+1}[\ln \Psi_b(x)]^{(n)}\big\}}{\mathrm{d}x} = (-1)^n x^n[\psi''(x)+g_2(x)+h_2(x)]^{(n-2)}.$$

It is well known that, for $x>0$ and $r>0$,

$$\frac{1}{x^r} = \frac{1}{\Gamma(r)}\int_0^\infty t^{r-1}e^{-xt}\,\mathrm{d}t, \tag{13}$$

and the polygamma functions $\psi^{(k)}(x)$ can be expressed for $x>0$ and $k\in\mathbb{N}$ as

$$\psi^{(k)}(x) = (-1)^{k+1}\int_0^\infty \frac{t^k e^{-xt}}{1-e^{-t}}\,\mathrm{d}t, \tag{14}$$

where $\gamma = 0.57721566\dots$ is the Euler-Mascheroni constant. From these two formulas, for $x\in(0,\infty)$ and any nonnegative integer i, it follows that

$$
\begin{aligned}
\phi(x) &\triangleq \psi''(x) + g_2(x) + h_2(x)\\
&= \psi''(x) + \frac{2+(b+1)x-2x^2}{x^3} + \frac{3(2-b)+(6-b)x+2x^2}{(x+1)^3}\\
&= \psi''(x) + \frac{2}{x^3} + \frac{b+1}{x^2} - \frac{2}{x} + \frac{2(1-b)}{(1+x)^3} + \frac{2-b}{(1+x)^2} + \frac{2}{1+x}\\
&= -\int_0^\infty \frac{t^2e^{-xt}}{1-e^{-t}}\,\mathrm{d}t + \int_0^\infty t^2e^{-xt}\,\mathrm{d}t + \int_0^\infty (b+1)te^{-xt}\,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&-2\int_0^\infty e^{-xt}\,\mathrm{d}t+2\int_0^\infty e^{-(x+1)t}\,\mathrm{d}t+\int_0^\infty (2-b)te^{-(x+1)t}\,\mathrm{d}t\\
&+\int_0^\infty (1-b)t^2e^{-(1+x)t}\,\mathrm{d}t\\
=&\int_0^\infty \big[(t-2)e^{2t}+(t+4)e^t-(t^2+2t+2)\\
&-bt(e^t-1)(1+t-e^t)\big]\frac{e^{-(x+1)t}}{e^t-1}\,\mathrm{d}t\\
\triangleq&\int_0^\infty q(t)\frac{e^{-(x+1)t}}{e^t-1}\,\mathrm{d}t
\end{aligned}
$$

and

$$
\phi^{(i)}(x)=(-1)^i\int_0^\infty t^iq(t)\frac{e^{-(x+1)t}}{e^t-1}\,\mathrm{d}t. \tag{15}
$$

Standard argument shows that $q(t)\lesseqgtr 0$ is equivalent to

$$
b\lesseqgtr\frac{(t-2)e^{2t}+(t+4)e^t-(t^2+2t+2)}{t(e^t-1)(1+t-e^t)}=\varphi(t) \tag{16}
$$

for $t\ge 0$.

By L'Hospital rule, it is straightforward to obtain that $\lim_{t\to0}\varphi(t)=-\infty$ and $\lim_{t\to\infty}\varphi(t)=-1$. This implies that $q(t)$ keeps constantly non-positive in $(0,\infty)$ is impossible for any $b\in\mathbb{R}$. Further, since $\varphi(t)$ is differentiable in $(0,\infty)$, there must exist a maximum $b^*\ge-1$ of $\varphi(t)$ in $(0,\infty)$ such that $q(t)\ge0$ in $(0,\infty)$ if $b\ge b^*$. This means that the function $\psi''(x)+g_2(x)+h_2(x)$ is completely monotonic in $(0,\infty)$ for $b\ge b^*$.

For $n\ge2$, if $b\ge b^*$, then

$$
\frac{\mathrm{d}\big\{(-1)^nx^{n+1}[\ln\Psi_b(x)]^{(n)}\big\}}{\mathrm{d}x}=x^n\int_0^\infty t^{n-2}q(t)\frac{e^{-(x+2)t}}{1-e^{-t}}\,\mathrm{d}t\ge0,
$$

and the function $(-1)^nx^{n+1}[\ln\Psi_b(x)]^{(n)}$ is increasing in $(0,\infty)$. From

$$
\lim_{x\to0}\big\{(-1)^nx^{n+1}[\ln\Psi_b(x)]^{(n)}\big\}=0
$$

by L'Hospital rule, it is clear for $n\ge2$ that $(-1)^nx^{n+1}[\ln\Psi_b(x)]^{(n)}\ge0$ and $(-1)^n[\ln\Psi_b(x)]^{(n)}\ge0$ in $(0,\infty)$. This implies the function $[\ln\Psi_b(x)]'$ is increasing in $(0,\infty)$. It is ready to obtain $\lim_{x\to\infty}[\ln\Psi_b(x)]'=0$ by using

$$
\ln\Gamma(x)=\left(x-\frac12\right)\ln x-x+\frac{\ln(2\pi)}{2}+\frac{1}{12x}+O\left(\frac1x\right), \tag{17}
$$

$$
\psi(x)=\ln x-\frac{1}{2x}-\frac{1}{12x^2}+O\left(\frac{1}{x^2}\right), \tag{18}
$$

$$
(-1)^{n+1}\psi^{(n)}(x)=\frac{(n-1)!}{x^n}+\frac{n!}{2x^{n+1}}+\frac{(n+1)!}{12x^{n+2}}+O\left(\frac{1}{x^{n+2}}\right) \tag{19}
$$

as $x\to\infty$, so $[\ln\Psi_b(x)]'\le0$ and $\ln\Psi_b(x)$ is decreasing in $(0,\infty)$. In conclusion, the function $\ln\Psi_b(x)$ for $b\ge b^*$ is completely monotonic in $(0,\infty)$.

If $\Psi_b(x)$ is logarithmically completely monotonic in $(0,\infty)$, then from the fact $(-1)^n[\ln\Psi_b(x)]^{(n)}\ge0$ for $n\in\mathbb{N}$ and by utilizing L'Hospital rule and formulas (17),

(18) and (19), one finds that

$$
\begin{aligned}
b &\ge x(x+1)\left[\ln\left(1+\frac{1}{x}\right)+\frac{x\psi(x+1)-\ln\Gamma(x+1)}{x^2}-\frac{1}{x}\right]-x \\
&\to \begin{cases} -1 & \text{as } x\to 0 \\ -\infty & \text{as } x\to\infty \end{cases}
\end{aligned}
\tag{20}
$$

and

$$
\begin{aligned}
b &\ge -\frac{1}{n-1}\frac{x^n(x+1)^n}{(x+1)^n-x^n}\left[\frac{n-1-x}{x^n}+\frac{x+n}{(x+1)^n}+\frac{(-1)^n h_n(x)}{(n-2)!x^{n+1}}\right] \\
&\to \begin{cases} -1 & \text{as } x\to 0 \\ -\infty & \text{as } x\to\infty \end{cases}
\end{aligned}
\tag{21}
$$

for $x\in(0,\infty)$. Hence $b^*\ge -1$. The proof is complete. □

Proof of Theorem 2. Let $f_{a,b}(x)=(1+a/x)^{x+b}$. Then directly calculating and employing (13) yields

$$
\begin{aligned}
\ln f_{a,b}(x) &= (x+b)[\ln(x+a)-\ln x], \\
[\ln f_{a,b}(x)]' &= \ln(x+a)-\ln x+\frac{b-a}{x+a}-\frac{b}{x}, \\
[\ln f_{a,b}(x)]'' &= \frac{1}{x+a}-\frac{1}{x}-\frac{b-a}{(x+a)^2}+\frac{b}{x^2} \\
&= \int_0^\infty \left[bt\left(e^{at}-1\right)-\left(e^{at}-at-1\right)\right]e^{-(x+a)t}\,\mathrm{d}t \\
&\triangleq \int_0^\infty p_{a,b}(t)e^{-(x+a)t}\,\mathrm{d}t.
\end{aligned}
$$

The function $p_{a,b}(t)\ge 0$ is equivalent to

$$
b\ge\frac{e^{at}-at-1}{t\left(e^{at}-1\right)}=\frac{a\left(e^u-u-1\right)}{u\left(e^u-1\right)}\triangleq a\omega(u)
$$

for $u=at$ with $\lim_{u\to0}\omega(u)=1/2$ and $\lim_{u\to\infty}\omega(u)=0$. Since

$$
\begin{aligned}
\omega'(u) &= \frac{u^2e^u-(e^u-1)^2}{u^2(e^u-1)^2}\triangleq\frac{\mu(u)}{u^2(e^u-1)^2}, \\
\mu'(u) &= 2\left(1+u+\frac{u^2}{2}-e^u\right)e^u<0,
\end{aligned}
$$

the function $\mu(u)$ is decreasing with $\mu(0)=0$, which means $\mu(u)<0$ and $\omega'(u)<0$. Hence, the function $\omega(u)$ is decreasing strictly in $u\in(0,\infty)$. This tells us that the function $p_{a,b}(t)\ge 0$ is equivalent to $b\ge a/2$.

When $2b\ge a$, the fact that the function $p_{a,b}(t)\ge 0$ implies $[\ln f_{a,b}(x)]''$ is completely monotonic in $(0,\infty)$. Therefore, the function $[\ln f_{a,b}(x)]'$ is increasing with $\lim_{x\to\infty}[\ln f_{a,b}(x)]'=0$, and then $[\ln f_{a,b}(x)]'<0$. By definition, it follows that $f_{a,b}(x)$ is logarithmically completely monotonic in $(0,\infty)$ for $2b\ge a$.

If $f_{a,b}(x)$ is logarithmically completely monotonic in $(0,\infty)$, then $[\ln f_{a,b}(x)]'<0$ can be rewritten as

$$
ab\ge x\left[(x+a)\ln\left(1+\frac{a}{x}\right)-a\right]\to\frac{a^2}{2}
$$

as $x\to\infty$. The proof is complete. □

References

[1] H. Alzer and C. Berg, *Some classes of completely monotonic functions*, Ann. Acad. Sci. Fenn. Math. **27** (2002), no. 2, 445–460.

[2] R. D. Atanassov and U. V. Tsoukrovski, *Some properties of a class of logarithmically completely monotonic functions*, C. R. Acad. Bulgare Sci. **41** (1988), no. 2, 21–23.

[3] C. Berg, *Integral representation of some functions related to the gamma function*, Mediterr. J. Math. **1** (2004), no. 4, 433–439.

[4] W. E. Clark and M. E. H. Ismail, *Inequalities involving gamma and psi functions*, Anal. Appl. **1** (2003), no. 1, 129–140.

[5] B.-N. Guo and F. Qi, *Two classes of completely monotonic functions involving gamma and polygamma functions* (2005), submitted.

[6] R. A. Horn, *On infinitely divisible matrices, kernels and functions*, Z. Wahrscheinlichkeitstheorie und Verw. Geb **8** (1967), 219–230.

[7] K. S. Miller and S. G. Samko, *Completely monotonic functions*, Integr. Transf. Spec. Funct. **12** (2001), no. 4, 389–402.

[8] G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis I*, Springer, Berlin-Heidelberg-New York, 1970.

[9] F. Qi, J. Cao, D.-W. Niu and N. Ujevic, *An upper bound of a function with two independent variables*, Appl. Math. E-Notes **6** (2006), in press.

[10] F. Qi and Ch.-P. Chen, *A complete monotonicity property of the gamma function*, J. Math. Anal. Appl. **296** (2004), no. 2, 603–607.

[11] F. Qi and Ch.-P. Chen, *Monotonicities of two sequences*, Mathematics and Informatics Quarterly **9** (1999), no. 4, 136–139.

[12] F. Qi and B.-N. Guo, *Complete monotonicities of functions involving the gamma and digamma functions*, RGMIA Res. Rep. Coll. **7** (2004), no. 1, Art. 8, 63–72. Available online at `http://rgmia.vu.edu.au/v7n1.html`.

[13] F. Qi and B.-N. Guo, *Some classes of logarithmically completely monotonic functions involving gamma function*, submitted.

[14] F. Qi, B.-N. Guo, and Ch.-P. Chen, *Some completely monotonic functions involving the gamma and polygamma functions*, J. Austral. Math. Soc. **79** (2006), no. 1, in press. RGMIA Res. Rep. Coll. **7** (2004), no. 1, Art. 5, 31–36. Available online at `http://rgmia.vu.edu.au/v7n1.html`.

[15] F. Qi, D.-W. Niu and J. Cao, *An infimum and an upper bound of a function with two independent variables*, Octogon Math. Mag. **14** (2006), no. 1, in press.

[16] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, 1941.

(F. Qi) Research Institute of Mathematical Inequality Theory, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

E-mail address: `qifeng@hpu.edu.cn, fengqi618@member.ams.org, qifeng618@hotmail.com, qifeng618@msn.com`

URL: `http://rgmia.vu.edu.au/qi.html`

(D.-W. Niu) School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

E-mail address: `nnddww@tom.com`

(J. Cao) School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

E-mail address: `goodfriendforeve@163.com`