Monotonicity and Logarithmic Convexity for a Class of Elementary Functions Involving the Exponential Function

MONOTONICITY AND LOGARITHMIC CONVEXITY FOR A CLASS OF ELEMENTARY FUNCTIONS INVOLVING THE EXPONENTIAL FUNCTION

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Abstract. In this paper, the monotonicity and logarithmically convexity of the function $\frac{e^{-\alpha t}-e^{-\beta t}}{1-e^{-t}}$ are obtained, where $t \in \mathbb{R}$ and α and β are real numbers such that $\alpha \ne \beta$, $(\alpha,\beta) \ne (0,1)$ and $(\alpha,\beta) \ne (1,0)$.

1. Introduction

For real numbers α and β with $\alpha \ne \beta$, $(\alpha,\beta) \ne (0,1)$ and $(\alpha,\beta) \ne (1,0)$ and for $t \in \mathbb{R}$, let

$$q_{\alpha,\beta}(t) = \begin{cases} \dfrac{e^{-\alpha t}-e^{-\beta t}}{1-e^{-t}}, & t \ne 0, \\ \beta-\alpha, & t = 0. \end{cases} \tag{1}$$

In order to obtain the best bounds in Gautschi-Kershaw's inequalities, it was proved in [9] that the function $q_{\alpha,\beta}(t)$ is logarithmically convex in $(0,\infty)$ and logarithmically concave in $(-\infty,0)$ if $\beta-\alpha > 1$ and is logarithmically concave in $(0,\infty)$ and logarithmically convex in $(-\infty,0)$ if $0 < \beta-\alpha < 1$.

When ones study the logarithmically completely monotonic property of some functions involving Euler's gamma Γ function, the psi function ψ and the polygamma functions $\psi^{(i)}$ for $i \in \mathbb{N}$, the elementary function $q_{\alpha,\beta}(t)$ is encountered now and then. The so-called logarithmically completely monotonic function on an interval $I \subset \mathbb{R}$ is a positive function f which has derivatives of all orders on I and whose logarithm $\ln f$ satisfies $0 \le (-1)^k[\ln f(x)]^{(k)} < \infty$ for $k \in \mathbb{N}$ on I. The set of the logarithmically completely monotonic functions on I is denoted by $\mathcal{L}[I]$. For more information on the class $\mathcal{L}[I]$, please refer to [1, 2, 3, 5, 6, 7, 8, 9] and the references therein.

The first aim of this paper is to research the monotonicity of the function $q_{\alpha,\beta}(t)$. The first main result of ours is the following Theorem 1 or Corollary 1.

Theorem 1. *The following conclusions present the monotonic properties of* $q_{\alpha,\beta}(t)$.

(1) *The function* $q_{\alpha,\beta}(t)$ *is increasing in* $(0,\infty)$ *if either* $1 \ge \alpha+\beta > 2\alpha+1$ *or* $1 \le \alpha+\beta < 2\alpha < \alpha+\beta+1$ *holds.*
(2) *The function* $q_{\alpha,\beta}(t)$ *is decreasing in* $(0,\infty)$ *if either* $1 \ge \alpha+\beta > 2\beta+1$ *or* $1 \le \alpha+\beta < 2\beta < \alpha+\beta+1$ *is valid.*

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(3) *The function* $q_{\alpha,\beta}(t)$ *is increasing in* $(-\infty,0)$ *if either* $2\alpha>\alpha+\beta+1\ge 2$ *or* $\alpha+\beta<2\beta<\alpha+\beta+1\le 2$ *validates.*
(4) *The function* $q_{\alpha,\beta}(t)$ *is decreasing in* $(-\infty,0)$ *if either* $2\beta>\alpha+\beta+1\ge 2$ *or* $\alpha+\beta<2\alpha<\alpha+\beta+1\le 2$ *sounds.*
(5) *The function* $q_{\alpha,\beta}(t)$ *is increasing in* $(-\infty,\infty)$ *if and only if one of the following conditions holds:*
  (a) $\alpha=\beta+1>1$,
  (b) $\alpha>\beta+1\ge 1$,
  (c) $\beta=\alpha+1<1$,
  (d) $1\ge\beta>\alpha+1$,
  (e) $\alpha<\beta<\alpha+1\le 1$,
  (f) $\beta+1\le\alpha+\beta<2\alpha<\alpha+\beta+1$.
(6) *The function* $q_{\alpha,\beta}(t)$ *is decreasing in* $(-\infty,\infty)$ *if and only if one of the following conditions holds:*
  (a) $\beta=\alpha+1>1$,
  (b) $\beta>\alpha+1\ge 1$,
  (c) $\beta<\alpha<\beta+1\le 1$,
  (d) $1>\alpha=\beta+1$,
  (e) $1\ge\alpha>\beta+1$,
  (f) $\alpha+1\le\alpha+\beta<2\beta<\alpha+\beta+1$.

Remark 1. The (α,β)-domain where the function $q_{\alpha,\beta}(t)$ is monotonic in Theorem 1 can be described respectively by Figure 1 to Figure 6 below.

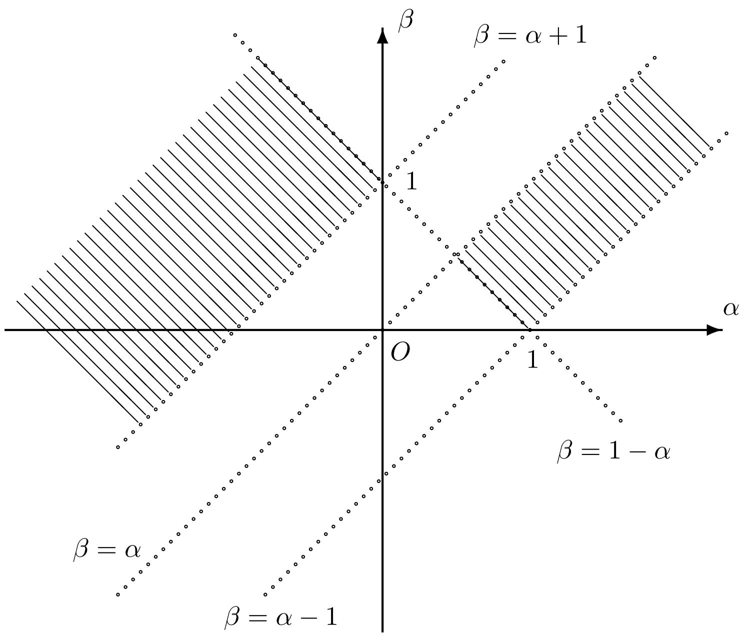

FIGURE 1. (α,β)-domain where the function $q_{\alpha,\beta}(t)$ is increasing in $(0,\infty)$ in Theorem 1

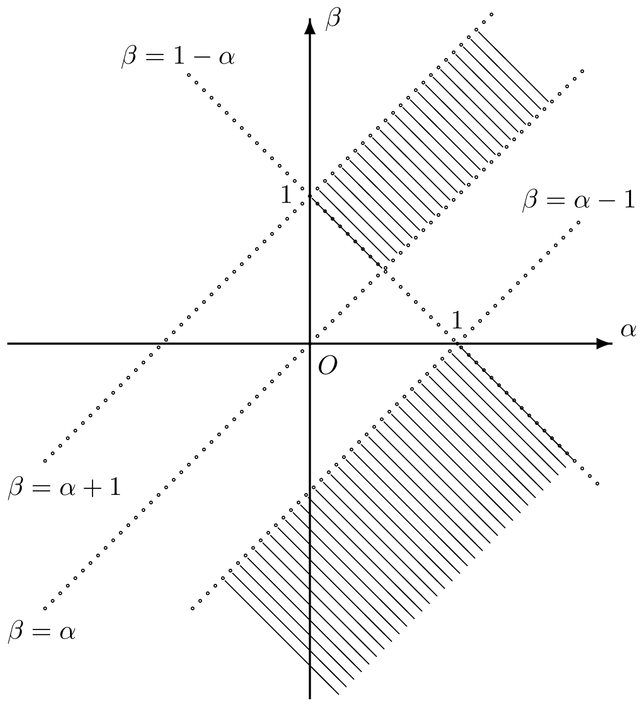

FIGURE 2. (α, β)-domain where the function $q_{\alpha,\beta}(t)$ is decreasing in $(0, \infty)$ in Theorem 1

Remark 2. Note that the (α, β)-domain where the function $q_{\alpha,\beta}(t)$ is increasing (or decreasing) in $(0, \infty)$ (or in $(-\infty, 0)$) is an union where the function $q_{\alpha,\beta}(t)$ increases (or decreases) in either $(0, \infty)$ (or $(-\infty, 0)$) or $(-\infty, \infty)$. Therefore, Theorem 1 can be restated as the following Corollary 1.

Corollary 1. *The following conclusions describe the monotonic properties of* $q_{\alpha,\beta}(t)$.

(1) *The function* $q_{\alpha,\beta}(t)$ *is increasing in* $(-\infty, \infty)$ *if and only if* $(\alpha, \beta) \in \{(\alpha, \beta) : \alpha > \beta \ge 0, \alpha \ge 1\} \cup \{(\alpha, \beta) : \alpha < \beta \le 0\} \cup \{(\alpha, \beta) : \alpha \le \beta - 1, 0 \le \beta \le 1\} \setminus \{(1, 0), (0, 1)\}$.
(2) *The function* $q_{\alpha,\beta}(t)$ *is decreasing in* $(-\infty, \infty)$ *if and only if* $(\alpha, \beta) \in \{(\alpha, \beta) : \beta > \alpha \ge 0, \beta \ge 1\} \cup \{(\alpha, \beta) : \beta < \alpha \le 0\} \cup \{(\alpha, \beta) : \beta \le \alpha - 1, 0 \le \alpha \le 1\} \setminus \{(1, 0), (0, 1)\}$.
(3) *The function* $q_{\alpha,\beta}(t)$ *is increasing in* $(0, \infty)$ *if and only if* $(\alpha, \beta) \in \{(\alpha, \beta) : \alpha > \beta \ge \frac{1}{2}\} \cup \{(\alpha, \beta) : \alpha \ge 1 - \beta, 0 \le \beta < \frac{1}{2}\} \cup \{(\alpha, \beta) : \alpha + 1 \le \beta \le 1 - \alpha, \alpha < 0\} \cup \{(\alpha, \beta) : \beta - 1 \le \alpha < \beta \le 0\} \setminus \{(1, 0)\}$.
(4) *The function* $q_{\alpha,\beta}(t)$ *is decreasing in* $(0, \infty)$ *if and only if* $(\alpha, \beta) \in \{(\alpha, \beta) : \beta \ge 1 - \alpha, \frac{1}{2} > \alpha \ge 0\} \cup \{(\alpha, \beta) : \beta > \alpha \ge \frac{1}{2}\} \cup \{(\alpha, \beta) : \beta < \alpha \le 0\} \cup \{(\alpha, \beta) : \beta \le \alpha - 1, 0 \le \alpha \le 1\} \cup \{(\alpha, \beta) : 1 \le \alpha \le 1 - \beta\} \setminus \{(1, 0), (0, 1)\}$.
(5) *The function* $q_{\alpha,\beta}(t)$ *is increasing in* $(-\infty, 0)$ *if and only if* $(\alpha, \beta) \in \{(\alpha, \beta) : 1 - \alpha \le \beta < \alpha, \alpha \ge 1\} \cup \{(\alpha, \beta) : \alpha < \beta \le 1, \alpha \le 0\} \cup \{(\alpha, \beta) : \alpha < \beta \le 1 - \alpha, 0 \le \alpha < \frac{1}{2}\} \setminus \{(1, 0), (0, 1)\}$.

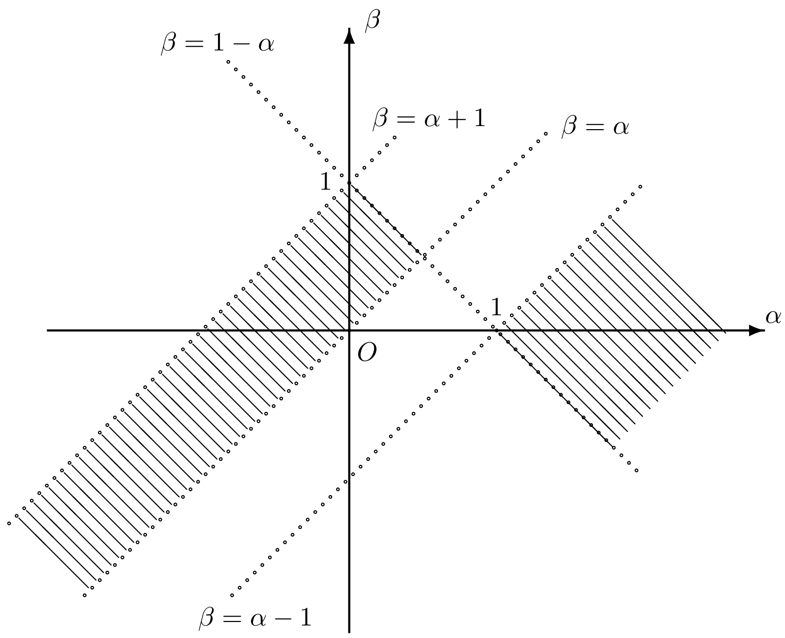

FIGURE 3. (α,β)-domain where the function $q_{\alpha,\beta}(t)$ is increasing in $(-\infty,0)$ in Theorem 1

(6) *The function $q_{\alpha,\beta}(t)$ is decreasing in $(-\infty,0)$ if and only if* $(\alpha,\beta)\in\{(\alpha,\beta): 1-\beta\le\alpha<\beta,\beta\ge1\}\cup\{(\alpha,\beta):\beta<\alpha\le\frac12\}\cup\{(\alpha,\beta):\beta\le1-\alpha,\frac12<\alpha\le 1\}\setminus\{(1,0),(0,1)\}$.

Remark 3. The corresponding (α,β)-domains where the function $q_{\alpha,\beta}(t)$ is monotonic in Corollary 1 can be described respectively by Figure 5 to Figure 10 below.

The second aim of this paper is to reconsider the logarithmically convexity of the function $q_{\alpha,\beta}(t)$ by a very simpler approach than that in [9]. The second main result of ours is the following Theorem 2.

Theorem 2. *The function $q_{\alpha,\beta}(t)$ in $(-\infty,\infty)$ is logarithmically convex if $\beta-\alpha>1$ and logarithmically concave if $0<\beta-\alpha<1$.*

Remark 4. Theorem 2 shows that the logarithmically convexity and logarithmically concavity in the interval $(-\infty,0)$ of $q_{\alpha,\beta}(t)$ presented in [9] and mentioned at the beginning of this paper are wrong. However, this does not affect the correctness of the main results established in [9], since the wrong properties about $q_{\alpha,\beta}(t)$ in the interval $(-\infty,0)$ are unuseful there luckily.

Remark 5. Recall that a r-times differentiable function $f(x)>0$ is said to be r-log-convex (or r-log-concave) on an interval I with $r\ge2$ if and only if $[\ln f(x)]^{(r)}$ exists and $[\ln f(x)]^{(r)}\ge0$ (or $[\ln f(x)]^{(r)}\le0$) on I. In [4], the following conclusions are obtained: If $1>\beta-\alpha>0$, then $q_{\alpha,\beta}(t)$ is 3-log-convex in $(0,\infty)$ and 3-log-concave in $(-\infty,0)$; if $\beta-\alpha>1$, then $q_{\alpha,\beta}(t)$ is 3-log-concave in $(0,\infty)$ and 3-log-convex in $(-\infty,0)$.

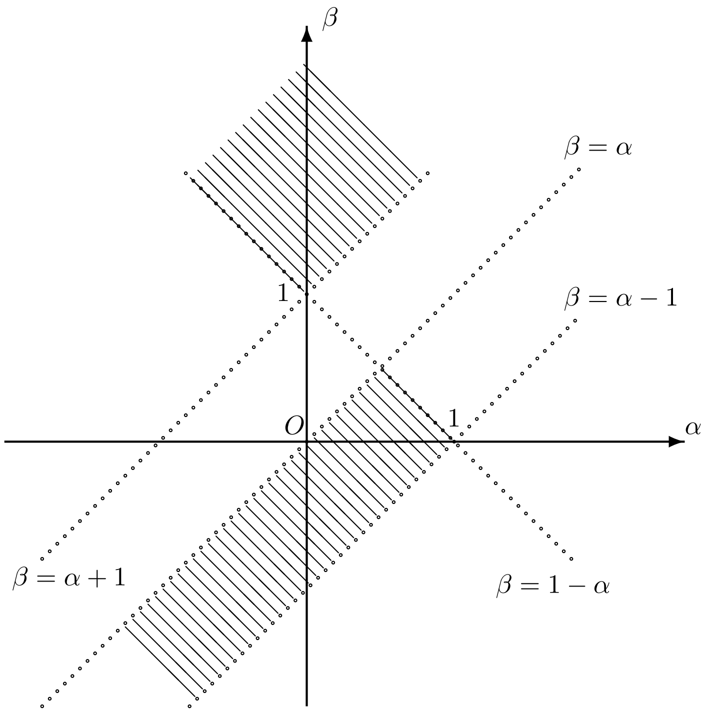

FIGURE 4. (α, β)-domain where the function $q_{\alpha,\beta}(t)$ is decreasing in $(-\infty, 0)$ in Theorem 1

2. Proofs of theorems

Proof of Theorem 1. It is clear that the function $q_{\alpha,\beta}(t)$ can be rewritten as

$$q_{\alpha,\beta}(t) = \frac{\sinh\frac{(\beta-\alpha)t}{2}}{\sinh\frac{t}{2}} \exp\frac{(1-\alpha-\beta)t}{2} \triangleq p_{\alpha,\beta}\left(\frac{t}{2}\right). \tag{2}$$

If $\alpha = \beta + 1$, then $q_{\alpha,\beta}(t) = -e^{-\beta t}$ is increasing for $\beta > 0$ and decreasing for $\beta < 0$ in $(-\infty, \infty)$. If $\alpha = \beta - 1$, then $q_{\alpha,\beta}(t) = e^{-\alpha t}$ is decreasing for $\alpha > 0$ and increasing for $\alpha < 0$ in $(-\infty, \infty)$.

For $|\alpha - \beta| \ne 1$, direct differentiation shows

$$p'_{\alpha,\beta}(t) = \frac{\sinh((\beta-\alpha)t)}{\sinh t} e^{(1-\alpha-\beta)t} \varphi_{\alpha,\beta}(t),$$

where

$$\varphi_{\alpha,\beta}(t) = (\beta-\alpha)\coth((\beta-\alpha)t) - \coth t - \alpha - \beta + 1 \tag{3}$$

and

$$\varphi'_{\alpha,\beta}(t) = \left(\frac{1}{\sinh t}\right)^2 - \left[\frac{\beta-\alpha}{\sinh((\beta-\alpha)t)}\right]^2 = \frac{1}{t^2}\left\{\left(\frac{t}{\sinh t}\right)^2 - \left[\frac{(\beta-\alpha)t}{\sinh((\beta-\alpha)t)}\right]^2\right\}. \tag{4}$$

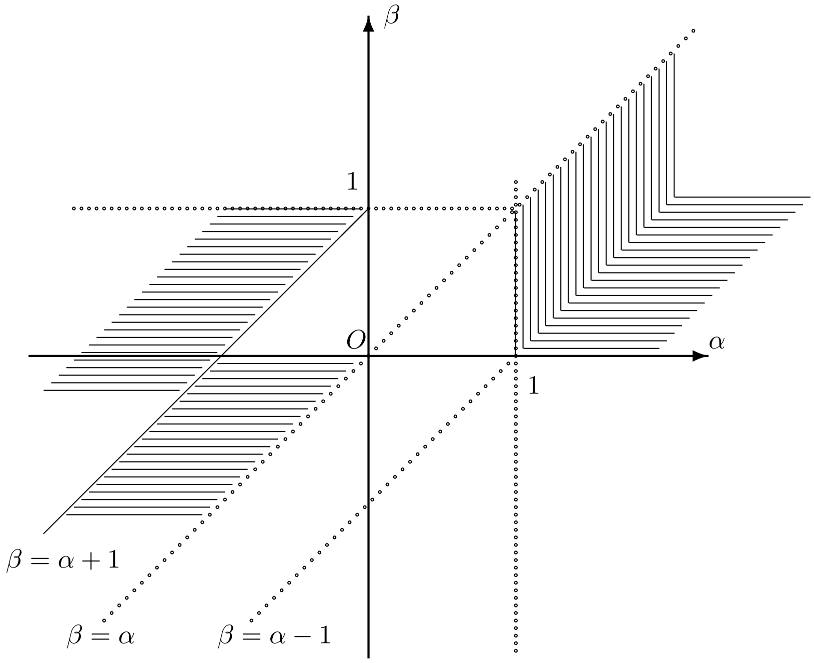

FIGURE 5. (α, β)-domain where the function $q_{\alpha,\beta}(t)$ is increasing in $(-\infty, \infty)$ in Theorem 1 and Corollary 1

Since $\varphi'_{\alpha,\beta}(t) = \varphi'_{\alpha,\beta}(-t)$ and the function $\frac{t}{\sinh t} > 0$ is decreasing in $(0, \infty)$ and increasing in $(-\infty, 0)$, then $\varphi'_{\alpha,\beta}(t) \ge 0$ for $|\alpha - \beta| > 1$ and $\varphi'_{\alpha,\beta}(t) \le 0$ for $0 < |\alpha - \beta| < 1$ in $(-\infty, \infty)$. This means that the function $\varphi_{\alpha,\beta}(t)$ is increasing for $|\alpha - \beta| > 1$ and decreasing for $0 < |\alpha - \beta| < 1$ in $(-\infty, \infty)$. It is not difficult to obtain $\lim_{t\to-\infty} \varphi_{\alpha,\beta}(t) = 2 - \alpha - \beta - |\alpha - \beta|$, $\lim_{t\to 0} \varphi_{\alpha,\beta}(t) = 1 - \alpha - \beta$ and $\lim_{t\to\infty} \varphi_{\alpha,\beta}(t) = |\alpha - \beta| - \alpha - \beta$.

1. If $\beta > \alpha + 1$, then $\beta - \alpha > 0$, $|\alpha - \beta| > 1$, $\lim_{t\to-\infty} \varphi_{\alpha,\beta}(t) = 2(1 - \beta)$ and $\lim_{t\to\infty} \varphi_{\alpha,\beta}(t) = -2\alpha$. Further, if $\alpha \ge 0$, then $\varphi_{\alpha,\beta}(t) < 0$ and $p'_{\alpha,\beta}(t) < 0$ in $(-\infty, \infty)$, and then $p_{\alpha,\beta}(t)$ and $q_{\alpha,\beta}(t)$ are decreasing in $(-\infty, \infty)$. Therefore, for $\beta > \alpha + 1 \ge 1$, the function $q_{\alpha,\beta}(t)$ is decreasing in $(-\infty, \infty)$.

If $\beta > \alpha + 1$ and $\beta \le 1$, then $\lim_{t\to-\infty} \varphi_{\alpha,\beta}(t) \ge 0$, $\varphi_{\alpha,\beta}(t) > 0$ and $p'_{\alpha,\beta}(t) > 0$ in $(-\infty, \infty)$, and then $p_{\alpha,\beta}(t)$ and $q_{\alpha,\beta}(t)$ are increasing in $(-\infty, \infty)$. Hence, for $1 \ge \beta > \alpha + 1$, the function $q_{\alpha,\beta}(t)$ is increasing in $(-\infty, \infty)$.

If $\beta > \alpha + 1$ and $\alpha + \beta \le 1$, then $\lim_{t\to 0} \varphi_{\alpha,\beta}(t) \ge 0$, $\varphi_{\alpha,\beta}(t) > 0$ and $p'_{\alpha,\beta}(t) > 0$ in $(0, \infty)$, and then $p_{\alpha,\beta}(t)$ and $q_{\alpha,\beta}(t)$ are increasing in $(0, \infty)$. Consequently, for $2\alpha + 1 < \alpha + \beta \le 1$, the function $q_{\alpha,\beta}(t)$ is increasing in $(0, \infty)$.

If $\beta > \alpha + 1$ and $\alpha + \beta \ge 1$, then $\lim_{t\to 0} \varphi_{\alpha,\beta}(t) \le 0$, $\varphi_{\alpha,\beta}(t) < 0$ and $p'_{\alpha,\beta}(t) < 0$ in $(-\infty, 0)$, and then $p_{\alpha,\beta}(t)$ and $q_{\alpha,\beta}(t)$ are decreasing in $(-\infty, 0)$. Therefore, for $2\beta > \alpha + \beta + 1 \ge 2$, the function $q_{\alpha,\beta}(t)$ is decreasing in $(-\infty, 0)$.

2. If $\alpha < \beta < \alpha + 1$, then $\beta - \alpha > 0$ and $|\alpha - \beta| < 1$. Further, if $\alpha \le 0$, then $\varphi_{\alpha,\beta}(t) > 0$ and $p'_{\alpha,\beta}(t) > 0$ in $(-\infty, \infty)$, and then $p_{\alpha,\beta}(t)$ and $q_{\alpha,\beta}(t)$ are increasing in $(-\infty, \infty)$. Accordingly, for $\alpha < \beta < \alpha + 1 \le 1$, the function $q_{\alpha,\beta}(t)$ is increasing in $(-\infty, \infty)$.

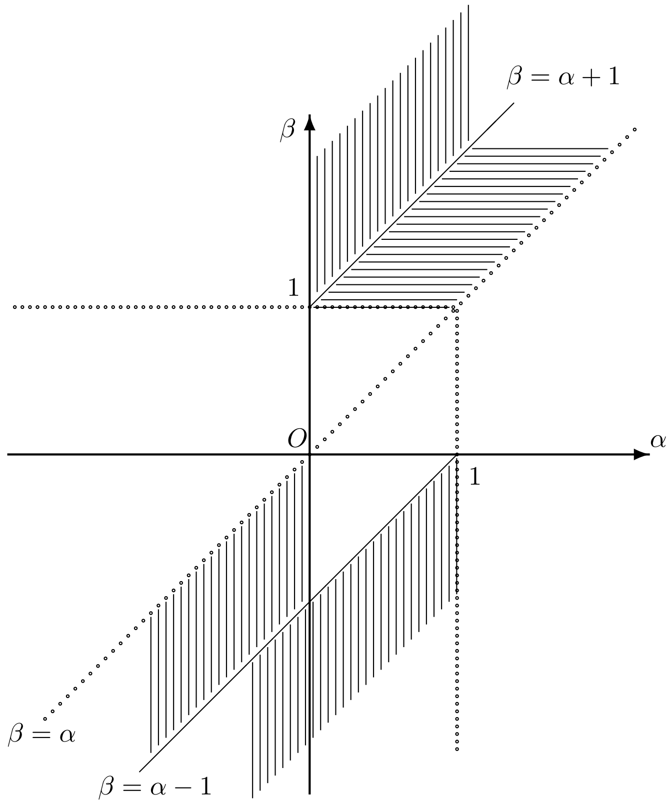

FIGURE 6. (α, β)-domain where the function $q_{\alpha,\beta}(t)$ is decreasing in $(-\infty, \infty)$ in Theorem 1 and Corollary 1

If $\alpha < \beta < \alpha + 1$ and $\beta \ge 1$, then $\lim_{t\to-\infty} \varphi_{\alpha,\beta}(t) \le 0$, $\varphi_{\alpha,\beta}(t) < 0$ and $p'_{\alpha,\beta}(t) < 0$ in $(-\infty, \infty)$, and then $p_{\alpha,\beta}(t)$ and $q_{\alpha,\beta}(t)$ are decreasing in $(-\infty, \infty)$. Therefore, for $\alpha + 1 \le \alpha + \beta < 2\beta < \alpha + \beta + 1$, the function $q_{\alpha,\beta}(t)$ is decreasing in $(-\infty, \infty)$.

If $\alpha < \beta < \alpha + 1$ and $\alpha + \beta \le 1$, then $\lim_{t\to0} \varphi_{\alpha,\beta}(t) \ge 0$, $\varphi_{\alpha,\beta}(t) > 0$ and $p'_{\alpha,\beta}(t) > 0$ in $(-\infty, 0)$, and then $p_{\alpha,\beta}(t)$ and $q_{\alpha,\beta}(t)$ are increasing in $(-\infty, 0)$. As a result, for $\alpha + \beta < 2\beta < \alpha + \beta + 1 \le 2$, the function $q_{\alpha,\beta}(t)$ is increasing in $(-\infty, 0)$.

If $\alpha < \beta < \alpha + 1$ and $\alpha + \beta \ge 1$, then $\lim_{t\to0} \varphi_{\alpha,\beta}(t) \le 0$, $\varphi_{\alpha,\beta}(t) < 0$ and $p'_{\alpha,\beta}(t) < 0$ in $(0, \infty)$, and then $p_{\alpha,\beta}(t)$ and $q_{\alpha,\beta}(t)$ are decreasing in $(0, \infty)$. Consequently, for $1 \le \alpha + \beta < 2\beta < \alpha + \beta + 1$, the function $q_{\alpha,\beta}(t)$ is decreasing in $(0, \infty)$.

3. If $\alpha > \beta + 1$, then $\beta - \alpha < 0$, $|\alpha - \beta| > 1$, $\lim_{t\to-\infty} \varphi_{\alpha,\beta}(t) = 2(1 - \alpha)$ and $\lim_{t\to\infty} \varphi_{\alpha,\beta}(t) = -2\beta$. Further, if $\beta \ge 0$, then $\varphi_{\alpha,\beta}(t) < 0$ and $p'_{\alpha,\beta}(t) > 0$ in $(-\infty, \infty)$, and then $p_{\alpha,\beta}(t)$ and $q_{\alpha,\beta}(t)$ are increasing in $(-\infty, \infty)$. Therefore, for $\alpha > \beta + 1 \ge 1$, the function $q_{\alpha,\beta}(t)$ is increasing in $(-\infty, \infty)$.

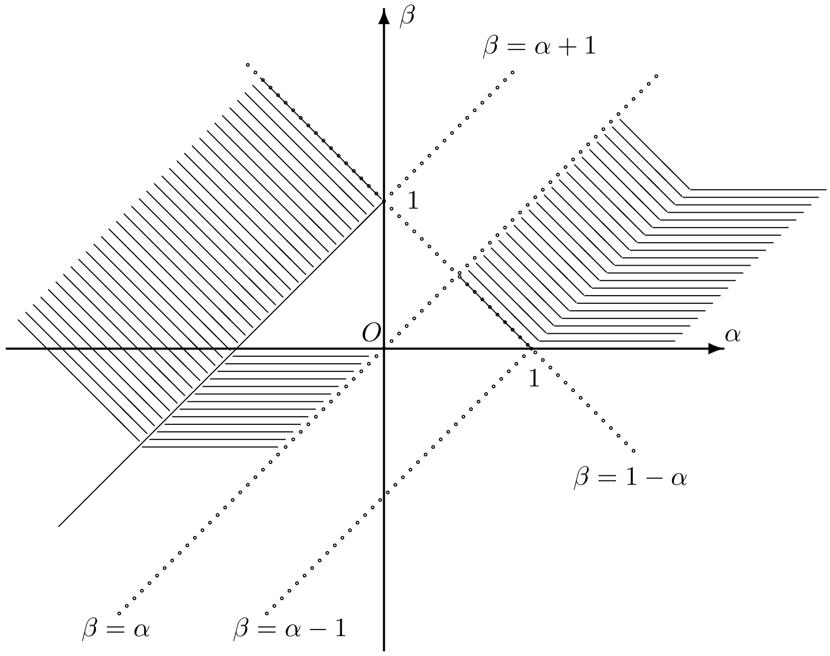

FIGURE 7. (α, β)-domain where the function $q_{\alpha,\beta}(t)$ is increasing in $(0, \infty)$ in Theorem 1

If $\alpha > \beta + 1$ and $\alpha \le 1$, then $\lim_{t\to-\infty} \varphi_{\alpha,\beta}(t) \ge 0$, $\varphi_{\alpha,\beta}(t) > 0$ and $p'_{\alpha,\beta}(t) < 0$ in $(-\infty, \infty)$, and then $p_{\alpha,\beta}(t)$ and $q_{\alpha,\beta}(t)$ are decreasing in $(-\infty, \infty)$. Hence, for $1 \ge \alpha > \beta + 1$, the function $q_{\alpha,\beta}(t)$ is decreasing in $(-\infty, \infty)$.

If $\alpha > \beta + 1$ and $\alpha + \beta \le 1$, then $\lim_{t\to 0} \varphi_{\alpha,\beta}(t) \ge 0$, $\varphi_{\alpha,\beta}(t) > 0$ and $p'_{\alpha,\beta}(t) < 0$ in $(0, \infty)$, and then $p_{\alpha,\beta}(t)$ and $q_{\alpha,\beta}(t)$ are decreasing in $(0, \infty)$. Accordingly, for $1 \ge \alpha + \beta > 2\beta + 1$, the function $q_{\alpha,\beta}(t)$ is decreasing in $(0, \infty)$.

If $\alpha > \beta + 1$ and $\alpha + \beta \ge 1$, then $\lim_{t\to 0} \varphi_{\alpha,\beta}(t) \le 0$, $\varphi_{\alpha,\beta}(t) < 0$ and $p'_{\alpha,\beta}(t) > 0$ in $(-\infty, 0)$, and then $p_{\alpha,\beta}(t)$ and $q_{\alpha,\beta}(t)$ are increasing in $(-\infty, 0)$. Hence, for $2\alpha > \alpha + \beta + 1 \ge 2$, the function $q_{\alpha,\beta}(t)$ is increasing in $(-\infty, 0)$.

4. If $\beta < \alpha < \beta + 1$, then $\beta - \alpha < 0$ and $|\alpha - \beta| < 1$. Further, if $\beta \le 0$, then $\varphi_{\alpha,\beta}(t) > 0$ and $p'_{\alpha,\beta}(t) < 0$ in $(-\infty, \infty)$, and then $p_{\alpha,\beta}(t)$ and $q_{\alpha,\beta}(t)$ are decreasing in $(-\infty, \infty)$. Therefore, for $\beta < \alpha < \beta + 1 \le 1$, the function $q_{\alpha,\beta}(t)$ is decreasing in $(-\infty, \infty)$.

If $\beta < \alpha < \beta + 1$ and $\alpha \ge 1$, then $\lim_{t\to-\infty} \varphi_{\alpha,\beta}(t) \le 0$, $\varphi_{\alpha,\beta}(t) < 0$ and $p'_{\alpha,\beta}(t) > 0$ in $(-\infty, \infty)$, and then $p_{\alpha,\beta}(t)$ and $q_{\alpha,\beta}(t)$ are increasing in $(-\infty, \infty)$. Accordingly, for $\beta + 1 \le \alpha + \beta < 2\alpha < \alpha + \beta + 1$, the function $q_{\alpha,\beta}(t)$ is increasing in $(-\infty, \infty)$.

If $\beta < \alpha < \beta + 1$ and $\alpha + \beta \le 1$, then $\lim_{t\to 0} \varphi_{\alpha,\beta}(t) \ge 0$, $\varphi_{\alpha,\beta}(t) > 0$ and $p'_{\alpha,\beta}(t) < 0$ in $(-\infty, 0)$, and then $p_{\alpha,\beta}(t)$ and $q_{\alpha,\beta}(t)$ are decreasing in $(-\infty, 0)$. Consequently, for $\alpha + \beta < 2\alpha < \alpha + \beta + 1 \le 2$, the function $q_{\alpha,\beta}(t)$ is decreasing in $(-\infty, 0)$.

If $\beta < \alpha < \beta + 1$ and $\alpha + \beta \ge 1$, then $\lim_{t\to 0} \varphi_{\alpha,\beta}(t) \le 0$, $\varphi_{\alpha,\beta}(t) < 0$ and $p'_{\alpha,\beta}(t) > 0$ in $(0, \infty)$, and then $p_{\alpha,\beta}(t)$ and $q_{\alpha,\beta}(t)$ are increasing in $(0, \infty)$. As a

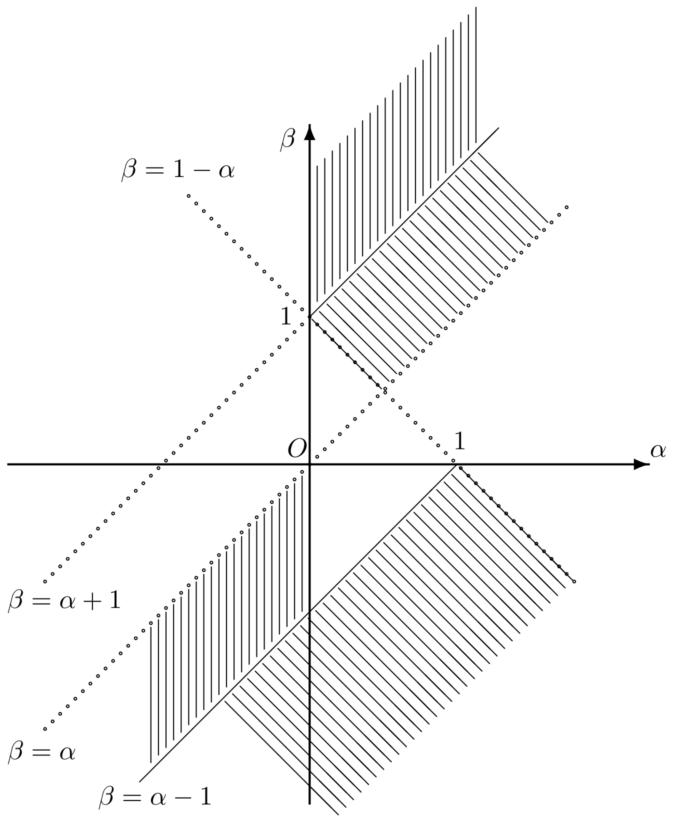

FIGURE 8. (α,β)-domain where the function $q_{\alpha,\beta}(t)$ is decreasing in $(0,\infty)$ in Theorem 1

result, for $1 \le \alpha+\beta < 2\alpha < \alpha+\beta+1$, the function $q_{\alpha,\beta}(t)$ is increasing in $(0,\infty)$. The proof of Theorem 1 is complete. □

Proof of Theorem 2. For $\beta > \alpha$, the functions $q_{\alpha,\beta}(t)$ and $p_{\alpha,\beta}(t)$, related by (2), are positive. Taking logarithm of $p_{\alpha,\beta}(t)$ and differentiating yields

$$\ln p_{\alpha,\beta}(t) = \ln\sinh((\beta-\alpha)t) - \ln\sinh t + (1-\alpha-\beta)t,$$
$$[\ln p_{\alpha,\beta}(t)]' = (\beta-\alpha)\coth((\beta-\alpha)t) - \coth t - \alpha - \beta + 1 = \varphi_{\alpha,\beta}(t),$$

where $\varphi_{\alpha,\beta}(t)$ is defined by (3).

By the same argument as in the proof of Theorem 1 on page 5, it is easy to see that $\varphi'_{\alpha,\beta}(t) = [\ln p_{\alpha,\beta}(t)]'' \ge 0$ for $\beta-\alpha>1$ and $\varphi'_{\alpha,\beta}(t) = [p_{\alpha,\beta}(t)]'' \le 0$ for $0<\beta-\alpha<1$ in $(-\infty,\infty)$. This means that the function $p_{\alpha,\beta}(t) = q_{\alpha,\beta}(2t)$ is logarithmically convex for $\beta-\alpha>1$ and logarithmically concave for $0<\beta-\alpha<1$ in the whole axis $(-\infty,\infty)$. The proof of Theorem 2 is complete. □

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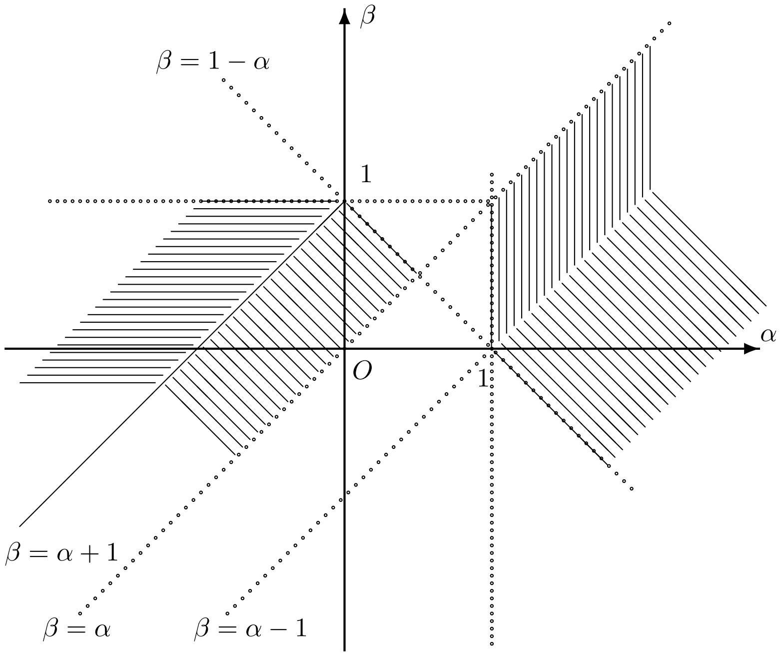

FIGURE 9. (α, β)-domain where the function $q_{\alpha,\beta}(t)$ is increasing in $(-\infty, 0)$ in Theorem 1

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URL: `http://rgmia.vu.edu.au/qi.html`

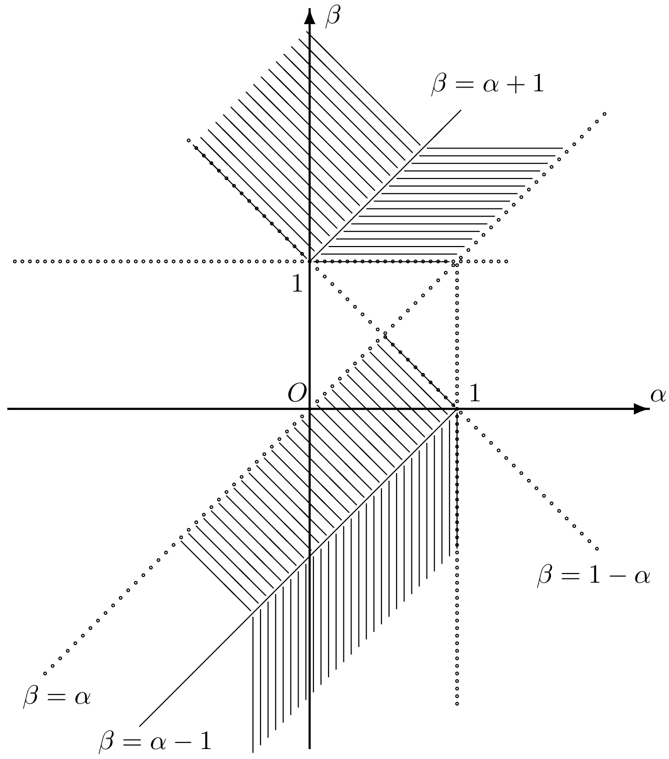

FIGURE 10. (α, β)-domain where the function $q_{\alpha,\beta}(t)$ is decreasing in $(-\infty, 0)$ in Theorem 1