CUBATURE REDUCTION USING THE THEORY OF INEQUALITIES

VICTORIA UNIVERSITY OF TECHNOLOGY

Securing your Future

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by

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Abstract

This dissertation is a detailed analysis of two-dimensional integration providing *a priori* error bounds in a variety of measures of integrand derivatives. Cubature formulae involving both function evaluations and one-dimensional integration are furnished and numerical experiments to investigate the efficacy of the error formulae are performed. Product (and singular) double integration is investigated.

Two-dimensional rectangular integral inequalities are constructed via embedding two one dimensional Peano kernels. In one dimension, linear kernels with a parametric discontinuity furnish "three point" rules where sampling occurs at the boundary and an interior point. The error is bounded in terms of the Lebesgue norms of the first derivative of the integrand. In two dimensions for a rectangular region, we find that the rule generalises to three "three point" rules in each dimension. That is nine sample points and six one dimensional integrals. The error bound is expressed in terms of norms of the first mixed partial derivative of the integrand.

These results are further generalised to provide error bounds in terms an arbitrary order mixed partial derivative of the integrand. That is, error bounds in measures of $\frac{\partial f^{n+m}}{\partial t^n \partial s^m}$ for some integers $n, m > 0$ where the integrand is f. In this case, we find that the rule involves both sample points and one-dimensional integrals involving all the partial derivatives of the integrand up to the stated order.

Finally, we explore product integrands, where the weight $w(\cdot, \cdot)$ is positive and integrable. In this case, the rule and the error bound involve moments of the weight. Particular attention is applied to identifying *a priori* two dimensional grids for which the error bound is minimized. Various weights and weight null spaces are explored and cubature formulae providing "optimal" grids are given.

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Dedication

To the memory of my *father.*

To my *mother* *" MARY".*

إليك يا أعز أم

هذا

حصاد مازرعته

Last but not least, I praise **GOD** for leading me to pursue this dream, answering all my prayers and fulfilling all my needs during this study

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Papers published during the author's candidature

rom the material in this thesis there are, at the time of submission, three papers which have been published in refereed publications. As well as other which are submitted for publication.

- Part of the work from Chapter 3 has been published in *SCIRF'99 Collaborative Research Forum*, (1999), pp. 138-143.
- Part of the work from Chapter 3 has been published in *ANZIAM J.*, 42(E), (2000), pp. C671-C689.
- The work from Chapter 3 and Chapter 4 has been published as a chapter in S. S. Dragomir and T. M. Rassias (Eds), *Ostrowski Type Inequalities and Application in Numerical Integration*, in press, Kluwer Academic Publishers.
- Part of Chapter 4 has been accepted for publication in the *Tamakng Journal Of Mathematics* [Pre-Print available online]*RGMIA RES. Rep. Coll.*, **3**(2), Article 10, 2000.
- Part of the work from Chapter 5 has been published in *ANZIAM J.*, 42(E), (2000), pp. C340-C361.
- Part of Chapter 6 has been submitted for publication in the *Journal of the Computational Analysis and Applications, JCAAA*[Pre-Print available online]*RGMIA RES. Rep. Coll.*, **4**(3), Article 11, 2001.

In addition, the following paper submitted for refereed publication during the author's candidature is listed below.

Hanna, G., S. S. Dragomir and P. Cerone. A Taylor Like Formula for Mappings of Two Variables Defined on a Rectangle in the Plane. (has been accepted for publication in the *TAMSUI OXFORD Journal of Mathematical Sciences*)[Pre-Print available online]*RGMIA RES. Rep. Coll.*, **3**(4), Article 5, 2001.

Chapter 1

Introduction

1.1 History and Integration

We begin by briefly examining approximating the definite integral $\int_a^b f(x)dx$. There are many reason why it may be important or desirable to perform this approximation. For example, it may be either difficult or impossible to find a mathematical formula for the integral or, if the problem can be solved analytically, the function concerned may be too complicated for efficient computation. Also, an integration program for a computer library may be required which could be used for a general function without special mathematical analysis on each occasion.

The early period was overcome by the contriving of functions in the modern sense to scrutinize general properties and to treat problems such as interpolation or approximation as general concepts. Newton interpolated functions at equidistant points and integrated the interpolants, thus becoming the father of the famous Newton-Cotes Quadratures. The simplest two quadratures of this class are the trapezoidal rule and Simpson rule. In the years that followed, a large number of quadratures of this type were developed involving sundry corrections and levelheaded combinations of subsistent formulas.

Later Gauss was the first to notice that a suitable variation of the points led to better accuracy in general. A specialist text for those interested in Gaussian quadratures is the book by Stroud and Secrest (1966). Numerous new quadratures were subsequently found which sluggishly tended towards optimality, properties for certain class of functions or problems (see, for example, Romberg (1955)).

It is believed that, the accurate and efficient evaluation of the single dimensional integral is very well established (see, for example, Abramowitz and Stegun (1972), Engels (1980), Davis and Robinowitz (1984), Press *et al.* (1986) and Atkinson (1988)).

1.1.1 Multiple Integration

Multiple integration was first used by Newton, but his arguments were geometrical and somewhat obscure. In the first half of the eighteenth century Leonhard Euler used repeated integrations in order to integrate over a bounded domain. Joseph Louis Lagrange used a triple integral in a work on gravitation involving an ellipsoid at around 1775. Followed by Mikhail Spirogyras who wrote integrals of n-forms over an n-dimensional "hypersurface". By the nineteenth century the use of multiple integrals had become fairly common. In fact, the first good theory was developed fairly recently, by Henri Lebesgue (1902) and Guido Fubini (1910).

In moving from the problem of computing one-dimensional integrals to the multidimensional (two-dimensional in our thesis) case leads to a series of new problems. While in one dimension we may encounter three possible types of integration intervals - finite, semi-infinite and infinite, now a wide variety of domains have to be accommodated. Also, many cubature problems do not have a unique solution or a real solution at all. These complications make the multidimensional case considerably more difficult than the univariate one, and accounts for the fact that the theory of multidimensional cubature is by no means as complete as the one-dimensional case.

One technique for evaluating multiple integrals numerically is to treat them as single integrals in each of the directions. This approach is costly with regards to the work required to achieve a particular accuracy. Another approach is to aim at reducing the dimension through utilizing the symmetry of the boundary of the function by using appropriate co-ordinate systems. However, this approach is, in practice, rarely used since symmetry is not always present.

Our proposed method in the current work, aims at reducing the dimensions by approximating a multiple integral by the evaluation of lower dimensional integrals for general functions, that do not necessarily contain symmetry.

The Monte Carlo Method (MCM) is one of the most popular methods used in the evaluation of multidimensional integrals. The basic idea in MCM is to replace an analytic problem by a probabilistic problem with the same solution and then investigate the latter problem by statistical simulation. These are useful for functions whose convergence is slow and also when integrating over irregular regions.

Other methods have been stated for decreasing the error in the MCM. All such approximations are called Quasi-Monte Carlo Methods. Many different Quasi-Monte Carlo Methods were developed by Haber (1967), Haber (1970). An extensive theory of Number-Theoretic-Methods (NTM) is given by Korobov (1963). Recently, new references for NTM have been given by Fang and Wang (1994) and Fang and Zhang (1999).

A research monograph and reference work is the book by Stroud (1971), in which, the best introductions to the area of multivariable quadrature can be found and numerical methods for the approximate calculation of multiple integrals have been discussed.

Some other numerical methods and techniques that have been used for multidimensional integrations, are for example, adaptive quadrature, lattice rules and the use of parallel implementation of more traditional methods.

Adaptive quadrature, Rice (1973), is an automatic procedure for increasing the accuracy of numerical approximation to an integral by increasing the number of samples of the integrand. It should be noted that,

- When an adaptive algorithm is used, the nodes at which the integrand is evaluated cannot be determined beforehand. Therefore, adaptive techniques are inappropriate for tabulated integrands.
- The subdivision procedure used in most adaptive quadrature codes is a simple bisection of the chosen interval. Bertsen *et al.* (1991), present an algorithm in which a

subdivision strategy results in three differently sized subintervals.

Another important application to one-dimensional and multidimensional integrals on the unit cube is the lattice rules. It uses all the nodes on a lattice that lie within and on the boundary of the unit cube.
Sloan and Lyness (1989), consider quadrature rules for the S-dimensional hypercube. It has been noticed that for the one-dimensional integration of a periodic function, the trapezoidal rule is an efficient choice. However, for S-dimensional integration of periodic functions over a hypercube, the S-dimensional product trapezoidal rule is not generally cost effective. Other lattice rules can be more effective as shown by Sloan and Walsh (1990), such as lattice rules of rank 2. Also, Worlet (1991), introduced some new families of integration lattices. They have a better order of convergence than previously known constructions.
Another approach is the use of parallel computer methods, which can sometimes speed up the numerical computation of an integral. There are many procedures that can be used depending on the type of parallel computer under consideration. The most common use of a parallel computer is to partition an integration interval into many sub-intervals and have the integration on each sub-interval performed in parallel. Software such as QUADPACK (IMSL,NAG) Piessens *et al.* (1983), has been around for some time, which is very robust and it uses adaptive algorithms taking into account the function behaviour. While these methods are robust, they are generally the least efficient.

More recently, Cools *et al.* (1997) of the Numerical Integration, Nonlinear Equation and Software (NINES) group, have developed CUBPACK++ which can handle double integrals over a variety of regions.

Cools (1999), also has presented an article on cubature rules as an extension to the work of Stroud (1971). Thus he has presented both theoretical and practical aspects of multidimensional integrations, a comprehensive bibliography and presentation of multiple integration or cubature rules for different shaped regions.

Utilizing the theory of orthogonal polynomials of several variables, Dunkl and Xu (2001) have developed cubature rules using some classical types of polynomials whose weight functions are supported on standard domains. A variety of domains have been investigated such as the simplex, the ball, or domains of Gaussian type, which satisfy differential difference equations, and for which fairly explicit formulae exist.

1.2 Why Inequalities?

From a practical point of view, sometimes definite integrals cannot be evaluated explicitly, that is because the function may not be known at all points in the given domain, or may possess a complicated antiderivative. Thus it is often faster and easier to perform the integration using approximations to as high an accuracy as desired. Of course, numerical integration (or quadrature) by its very nature is an approximation and this introduces the concept of numerical error, and it is important to understand and to be able to estimate or bound the resulting error. Thus, as Richard Bellman has said, as mentioned in Mitrinovič *et al.* (1994), "There are reasons for the study of inequalities, practical, theoretical, and aesthetic. In many practical investigations, it is necessary to bound one quantity by another. From the theoretical point of view we use the principle that every inequality should come from an equality which makes the inequality obvious". Inequalities will be obtained in this thesis to provide *a priori* bounds on quadrature rules. Bounds are obtained from identities procured by the use of a Peano kernel methodology.

1.2.1 Peano Kernel

From an estimation or error analysis point of view, we observe that a method like the Peano kernel formula for quadrature rule errors is more general and can be applied in other cases besides interpolation. Further, it can be used for error bounds as well as for study of the behavior of the error itself. Consider all the functions $f \in C^{n+1}[a,b]$, then the error $E[f]$ can be represented by the formula $E[f] = \int_a^b f^{(n+1)}(t)K(t)dt$ where $K(t)$ is the Peano kernel for the error and is defined by

$$K(t) = \frac{1}{n!}E[g(x;t)], \tag{1.1}$$

$$g(x;t) = (x-t)_+^n = \begin{cases} (x-t)^n & if \quad x \geq t, \\ 0 \quad if & x \leq t. \end{cases}$$

where t is just a parameter in the g function and the E operates only with respect to the x variable. The fruitful thing about the Peano kernel, is that it can be used to determine the error in integration rules explicitly, as well as being applied for the case when the function has only a low order of differentiability.

1.3 The boundary integration

Chapter XV of Mitrinoviċ *et al.* (1994) deals with integral inequalities involving functions with bounded derivatives or **Ostrowski Type Inequalities**, which is now itself a special domain of the theory of inequalities with many powerful results and a large number of applications to numerical integration, probability theory and statistics, information theory and integral operator theory.

The main aim of this thesis involves utilizing the result of one-dimensional Ostrowski inequalities to develop cubature rules for the two-dimensional problem over almost rectangular regions.
By combining the results of the Ostrowski inequality and the three point rules (Cerone and Dragomir 1999) and applying them in two dimension we obtain *a priori* error bounds for functions whose first partial derivatives exist and are bounded. In particular the methodology to be adopted involves the following :

- Determine a particular quadrature rule for one-dimensional integrals using a Peano kernel approach to produce an identity.

- Utilize the one-dimensional integral identity to obtain identities for higher dimensional integrals.

- Use the Modern Theory of Inequalities to obtain bounds on the approximation by estimating the bound on the error.

- Determine the partition required in a composite rule that will achieve a desired accuracy.

- Extend the Peano kernel to cater for singular or product integrands which appear quite naturally in practical problems.
- Compare the quadrature routines as developed above with standard approaches for specific test integrands.

We develop two-dimensional three point integral inequalities for functions with bounded first derivatives for different types of norms. In each case applications in numerical integration of two-dimensional integrals are investigated. We also develop some generalizations of an Ostrowski type inequality in two-dimensions for n-time differentiable mappings. An extension of the Ostrowski result to one-dimensional weighted integrals is considered, where the integrand may posses some singularity structure, or the integrand may be perfectly analytic, but the region of integration is infinite or semi-infinite. This is accomplished in the manner outlined below.

1.4 Outline of the thesis

A review of the one-dimensional Ostrowski type inequality is investigated, and some recent results relating to it are given in Chapter 2. In Chapter 3, we utilize the three point technique from the previous chapter to obtain two-dimensional Ostrowski inequalities in terms of L_∞, L_p and L_1 norms, where at most the first derivatives are involved in the bound. Applications of the cubature formulas are produced and some related numerical results are demonstrated.

Chapter 4 is reserved for some generalizations of Ostrowski type inequalities in two-dimensions for $n-$times differentiable functions. The results involve integral inequalities with bounds in terms of the n^{th} derivative of the integrand. This is employed to approximate double integrals using one-dimensional integrals and function evaluated at the interior points. In Chapter 5, we consider the extension of the Ostrowski result to one-dimensional weighted integrals. Some fruitful weighted (or product) integral inequalities using the Ostrowski approach are demonstrated. These inequalities furnish an error estimate for weighted integrals where both the quadrature rule and error bound are given in terms of (at most) the first three moments of the weight. Also, the upper bound is a function of the first few derivatives of the mapping.

This analysis is then taken up in Chapter 6 where we again focus on two-dimensional integral inequalities. We develop weighted first and second order double integral inequalities. We focus in particular to minimizing the bound for different weights and weight null-spaces. Finally, we develop a method for calculating cubature grids that rely only on the first few moments of the weight.

Chapter 2

One Dimensional Integral Inequalities

Many of the techniques used for developing multiple integral inequalities are based on analogous one dimensional results. With this in mind this chapter will focus on one dimensional integral inequalities and we review some recent results. Generalizations of the Ostrowski inequality (Ostrowski 1938) are employed to obtain a variety of integral inequalities involving one and three points and/or weighted integrals. The inequalities thus obtained are then employed to produce one dimensional quadrature rules with an estimate of the error in a variety of norms.

In the subsequent two chapters, the techniques used here are generalized to obtain two dimensional integral inequalities.

2.1 The Ostrowski Inequality

The classical Ostrowski integral inequality in one dimension stipulates a bound between a function evaluated at an interior point x and the average of the function of over an interval (see for example, Mitrinovič *et al.* (1994, p.468)). That is,

THEOREM 2.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, (I° is the interior of I) and let $a, b \in I^\circ$ with $a < b$. If $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), that is,*

$$||f'||_\infty := \sup_{t\in[a,b]} |f'(t)| < \infty,$$

is the $L_\infty[a,b]$ norm, then we have the inequality:

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\,\|f'\|_\infty \tag{2.1}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

For completeness, we include the proof of Theorem 2.1. The appropriate construction and manipulation of the Peano kernel is an underlying theme which underpins many of the results in this dissertation.

Ostrowski himself did not use a Peano theorem in his proof, but as will be evident this approach, in conjunction with Hölder and other inequalities, leads to integral inequalities with upper bounds expressed in a variety of norms.

Proof. (of Theorem 2.1) Consider the Peano kernel

$$K(x,t) := \begin{cases} t-a, & t \in [a,x], \\ t-b, & t \in (x,b]. \end{cases} \tag{2.2}$$

See Figure 2.1(a) for a diagrammatic representation of (2.2).

We notice that this kernel produces sampling only at an interior point and does not at the boundary. This is because K vanishes at the boundary and is discontinuous at the interior point x.

Consider the integral

$$\int_a^b K(x,t)f'(t)dt.$$

Integrating by parts over the given intervals in (2.2) and simplifying produces the identity

$$f(x) - \frac{1}{b-a}\int_a^b f(t)dt = \frac{1}{b-a}\int_a^b K(x,t)f'(t)dt. \tag{2.3}$$

Now, utilizing (2.3) we have, using well known properties of the modulus and integral,

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{b-a}\int_a^b |K(x,t)||f'(t)|dt \tag{2.4}$$

from which a simple calculation gives (2.1). □

Of course it would be normal to use Hölder inequality in the p and /or 1-norms. This was done by Dragomir and Wang (1998a) and Dragomir and Wang (1997). These results appear below.

THEOREM 2.2. *Let f as be in Theorem 2.1 and let $f' \in L_p[a,b], (p > 1,\ \frac{1}{p}+\frac{1}{q}=1)$, then the following inequality exists*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\ dt\right| \leq \frac{1}{b-a}\left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}} \|f'\|_p \tag{2.5}$$

where

$$\|f'\|_p := \left(\int_a^b |f(t)|^p\ dt\right)^{\frac{1}{p}},$$

is the $L_p[a,b]$-norm.

THEOREM 2.3. *Let f be defined as in (2.1). Further, let $f' \in L_1[a,b]$. The following inequality holds*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\ dt\right| \leq \left|\frac{1}{2} + \frac{|x-\frac{a+b}{2}|}{(b-a)}\right| \|f'\|_1 \tag{2.6}$$

for all $x \in [a,b]$ and $\|f'\|_1 := \int_a^b |f'(t)|\ dt$.

Since Ostrowski first produced his inequality in 1938, there has been an *explosion* of related results. See for example, the well known book by Mitrinoviċ *et al.* (1994). Extensions to other norms, and higher derivatives have been considered by Anastassiou (1995), Cerone *et al.* (1999a), Sofo and Dragomir (2001) and Matić and Pečarić (2001). See the recent book edited by Dragomir and Rassias (2001), and the pre-print archive of the Research Group in Mathematical Inequalities and Applications (http://rgmia.vu.edu.au). Below we expand on a few Ostrowski-like results as they impact on the main work in this thesis. We will highlight the role of the kernel and other techniques as appropriate.

Further, Milovanović and Pečarić (1976) increased the order of the derivative in (2.1) to an arbitrary n by considering n-times differentiable mappings as shown in the following theorem.

THEOREM 2.4. *Let $f(x)$ be an $n(\geq 1)$ times differentiable function such that $f^{(n)} \in L_\infty[a,b]$ for $x \in (a,b)$. Then, for every $x \in [a,b]$*

$$\left|\frac{1}{n}\left(f(x) + \sum_{k=1}^{n-1} F_k\right) - \frac{1}{b-a}\int_a^b f(y)dy\right| \leq \frac{\|f^{(n)}\|_\infty}{n(n+1)!}\left[\frac{(x-a)^{n+1}+(b-x)^{n+1}}{b-a}\right], \tag{2.7}$$

where F_k is defined by

$$F_k = F_k(f;n;x,a,b) = \frac{n-k}{k!}\frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a}. \tag{2.8}$$

Equation (2.7) was proved by employing Taylor's formula

$$f(x) = f(y) + \sum_{k=1}^{n-1} \frac{1}{k!} f^{(k)}(y)(x-y)^k + \frac{1}{n!} f^{(n)}(\xi)(x-y)^n \tag{2.9}$$

and integration by parts, (see Mitrinoviċ *et al.* (1994) for the complete proof).

Remark 2.5. *Substituting $n = 1$ in (2.7) produces (2.1).*

Fink (1992) used the integral remainder form of a Taylor series to generalize the Milovanović and Pečarić (1976) result (Theorem 2.4) to include functions in L_p spaces.

THEOREM 2.6. *Let $f^{(n-1)}$ be absolutely continuous on $[a,b]$ with $f^{(n)} \in L_p[a,b]$ then*

$$\left| \frac{1}{n} \left(f(x) + \sum_{k=1}^{n-1} F_k \right) - \frac{1}{b-a} \int_a^b f(y)dy \right| \leq K(n,p,x) \|f^{(n)}\|_p \tag{2.10}$$

where

$$K(n,p,x) = \frac{1}{n!} \frac{[(x-a)^{n+\frac{1}{q}} + (b-x)^{n+\frac{1}{q}}]^{1/q}}{b-a} B((n-1)q+1, q+1)^{1/q}$$

,

for $1 < p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $K(n,1,x) = \frac{(n-1)^{n-1}}{n^n n!} \frac{\max\{(x-a)^n, (b-x)^n\}}{b-a}$ *with $B(x,y)$ is the beta function of Euler, that is*

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad x,y > 0.$$

Remark 2.7. *It is easily observed that for $n = 1$, the result is as in Theorem 2.2.*

Anastassiou (1995) established an optimal upper bound on the deviation of a function from its average. He gave a different proof to Theorem 2.1 and from that of Ostrowski's initial proof of 1938 (Ostrowski 1938).

In the same paper, he has been motivated by the important work of Fink (1992) and obtained more general Ostrowski type inequalities as follows below.

THEOREM 2.8. *Let $f \in C^{n+1}([a,b]), n \in \mathbb{N}$ and $x \in [a,b]$ be fixed, such that $f^{(k)}(x) = 0$, $k = 1, \cdots, n$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(y)dy - f(x) \right| \leq \frac{\|f^{(n+1)}\|_\infty}{(n+2)!} \cdot \left(\frac{(x-a)^{n+2} + (b-x)^{n+2}}{(b-a)} \right). \tag{2.11}$$

THEOREM 2.9. *Let $f \in C^{n+1}([a,b]), n \in \mathbb{N}$ such that $f^{(k)}((a+b)/2) = 0$, all k even $\in \{1, \cdots, n\}$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(y)dy - f(\frac{a+b}{2})\right| \leq \frac{\|f^{(n+1)}\|_\infty}{(n+2)!}\cdot\left(\frac{(b-a)^{n+1}}{2^{n+1}}\right). \tag{2.12}$$

Through the use of a Peano kernel approach Cerone *et al.* (1999a) established another generalization of the Ostrowski inequality for n-time differentiable mappings, as illustrated in the theorem below.

THEOREM 2.10. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $f^{(n)} \in L_\infty[a,b]$, then for all $x \in [a,b]$, the following inequality holds:*

$$\begin{aligned}\left|\int_a^b f(t)dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \\ \leq \frac{\|f^{(n)}\|_\infty}{(n+1)!}\left[(x-a)^{n+1} + (b-x)^{n+1}\right] \leq \frac{\|f^{(n)}\|_\infty(b-a)^{(n+1)}}{(n+1)!}.\end{aligned} \tag{2.13}$$

The theorem is proved utilizing mathematical induction and using the Peano kernel mapping $K(.,.) : [a,b]^2 \to \mathbb{R}$,

$$K(x,t) := \begin{cases} \frac{(t-a)^n}{n!}, & \text{if } t \in [a,x], \\ \frac{(t-b)^n}{n!}, & \text{if } t \in (x,b]. \end{cases} \tag{2.14}$$

The kernel (2.14) is similar in sense to that of (2.2). It vanishes at the boundary points and is discontinuous at the interior point, thus producing a rule that provides sampling at the interior point and not at the end points. Since (2.14) is a polynomial of order n, an integral inequality in the n^{th} derivative will result (2.13). We can compare this to (2.1) which has a bound in the first derivative due to the linear Peano kernel (2.2). Equation (2.14) is sketched in Figure 2.1 (c) and 2.1 (d).

Higher order derivative norms are not the only extensions to Theorem 2.1. Introducing more branches of the Peano kernel; that is extending the number of discontinuities will produce an integral inequality with many sampling points. This avenue has been explored by Dragomir with bounds involving the first derivative and by A. Sofo (see Dragomir and Rassias 2001, Chapter 2) involving the n^{th} derivative. Sofo used the Peano kernel

$$K_{n,k}(t) := \begin{cases} \frac{(t-\alpha_1)^n}{n!}, & t \in [a, x_1) \\ \frac{(t-\alpha_2)^n}{n!}, & t \in [x_1, x_2) \\ & \vdots \\ \frac{(t-\alpha_{k-1})^n}{n!}, & t \in [x_{k-2}, x_{k-1}) \\ \frac{(t-\alpha_k)^n}{n!}, & t \in [x_{k-1}, b]. \end{cases} \tag{2.15}$$

To begin, it is immediately evident that $K_{n,k}(t)$ is of order n, thus the integral inequality will be bounded by a measure of $f^{(n)}$. In addition, (2.15) has discontinuities at $x_1, x_2, \cdots, x_{k-1}$ and does not vanish at the boundary, thus we would expect sampling at the points $\{a, x_1, x_2, \cdots, x_{k-1}, b\}$. The kernel (2.15) is sketched in Figures 2.1 (e) and 2.1 (f). The integral inequality furnished for this kernel is

$$\left| \int_a^b f(t)dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!} \left[\sum_{i=0}^{k} \{(x_i - \alpha_i)^j - (x_i - \alpha_{i+1})^j\} f^{(j-1)}(x_i) \right] \right| \tag{2.16}$$

$$\leq \frac{\|f^{(n)}\|_\infty}{(n+1)!} \sum_{i=0}^{k-1} \{(\alpha_{i+1} - x_i)^{n+1} + (x_{i+1} - \alpha_{i+1})^{n+1}\}$$

$$\leq \frac{\|f^{(n)}\|_\infty}{(n+1)!} \sum_{i=0}^{k-1} h_i^{n+1}$$

$$\leq \frac{\|f^{(n)}\|_\infty}{(n+1)!} (b-a)\nu^n(h) \quad if \quad f^{(n)} \in L_\infty[a, b],$$

where $h_i := x_{i+1} - x_i$ and $\nu(h) := \max\{h_i \,|i = 0, \cdots, k-1\}$.

A unique extension was proposed and explored by Cerone and Dragomir (1999) wherein the constants 'a' and 'b' in the kernel (2.2) were replaced by linear parametric functions- the zero's and discontinuity of the kernel were themselves functions whose positions were allowed to change.

The kernel is

$$K(x,t) := \begin{cases} t - \alpha(x), & \text{if } t \in [a, x], \\ t - \beta(x), & \text{if } t \in (x, b], \end{cases} \tag{2.17}$$

where

$$\alpha(x) = \gamma x + (1-\gamma)a \qquad \text{and} \qquad \beta(x) = \gamma x + (1-\gamma)b \tag{2.18}$$

$\gamma \in [0, 1]$ and $x \in [a, b]$. Hence the sampling occurs at three points; the boundary 'a' and 'b' and the point x. The sampling is controlled by the parameter γ. This is further explored in the next section.

Recently, the Research Group in Mathematical Inequalities and Applications (RGMIA) has carried out a considerable amount of work in the application of the Modern Theory of Inequalities to obtain *a priori* bounds for a variety of Newton-Cotes rules. The classical rules of mid-point, trapezoidal and Simpson's in particular have been investigated, giving error bounds in terms of a variety of norms (see Cerone and Dragomir (2000a) and Cerone and Dragomir (2000b)).

The investigations were carried out for both Riemann and Riemann-Stieltjes integrals in which the bounds involved the behaviour of the integrand. As we mentioned before this was done through a Peano kernel development and so the order or accuracy of the approximation does not depend on the order of the highest polynomial that the rule integrates exactly.

For more results related to the Ostrowski inequality see Dragomir and Rassias (2001), Anastassiou (1995), Milovanović and Pečarić (1976), Cerone *et al.* (1999b), Cerone *et al.* (1999c), Cerone *et al.* (1998), Cerone and Dragomir (2000b), Dragomir and Wang (1998b), Dragomir *et al.* (2000), Dragomir (1998) and Dragomir (1999).

2.2 Three point Quadrature rules

The inequality, which combines and generalizes the interior point (mid-point type) and boundary point (trapezoidal type) inequalities via a parameterization for distinguishing the types, has been investigated by P. Cerone. This new inequality has been called the "three point rule". Cerone and Dragomir (1999) examined the three point quadrature rule of Newton-Cotes type where the error involved the behaviour of, at most, a first derivative. Simpson type rules are recaptured as particular cases. Moreover, Riemann integrals are approximated for the derivative of the integrand belonging to a variety of norms.

The following inequality involves a three-point rule whose bound may be obtained in terms of the first derivative, $f' \in L_\infty[a,b]$.

THEOREM 2.11. *Let f and f' be as in Theorem 2.1. Further, let $\alpha : [a,b] \in \mathbb{R}$ and*

$\beta : [a, b] \in \mathbb{R}$. *with* $\alpha \leq x \leq \beta$ *Then, for all* $x \in [a, b]$ *we have the inequality*

$$\left| \int_a^b f(t)dt - [(\beta(x) - \alpha(x))f(x) + (b - \beta(x))f(b) + (\alpha(x) - a)f(a)] \right|$$
$$\leq \|f'\|_\infty \left\{ \frac{1}{2}\left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{b-a}{2}\right)^2 \right] + \left(\alpha(x) - \frac{a+x}{2}\right)^2 + \left(\beta(x) - \frac{b+x}{2}\right)^2 \right\}. \quad (2.19)$$

Proof. Let $K(.,.) : [a, b]^2 \to \mathbb{R}$, where $K(x, t)$ is the kernel (2.17) and consider the integral

$$\int_a^b K(x,t)f'(t)dt.$$

Integrating by parts over the given intervals in (2.17) and simplifing produces an identity from which, taking the modulus and using well known properties of the modulus and integral, gives the result. □

Inspection af the bound in (2.19) reveals that α and β should take on linear profiles for the bound to be minimized. Thus the motivation is to prescribe a linear parameterization in (2.18). Utilizing equation (2.18), we get the following theorem,

THEOREM 2.12. *Let the conditions of Theorem.2.11 hold, then*

$$\left| \int_a^b f(t)dt - (b-a)\left\{ (1-\gamma)f(x) + \gamma \left[\left(\frac{x-a}{b-a}\right) f(a) + \left(\frac{b-x}{b-a}\right) f(b) \right] \right\} \right|$$
$$\leq 2\|f'\|_\infty \left[\frac{1}{4} + \left(\gamma - \frac{1}{2}\right)^2 \right] \left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2 \right] \quad (2.20)$$

Remark 2.13. $\gamma = 0$ *in (2.20) reproduces Ostrowski's inequality equation (2.1) whose bound is sharpest where* $x = \frac{a+b}{2}$, *giving the mid-point inequality.*

Remark 2.14. $\gamma = 1$ *produces the generalized trapezoidal inequality for which again the best bound occurs when* $x = \frac{a+b}{2}$ *giving the standard trapezoidal-type inequality.*

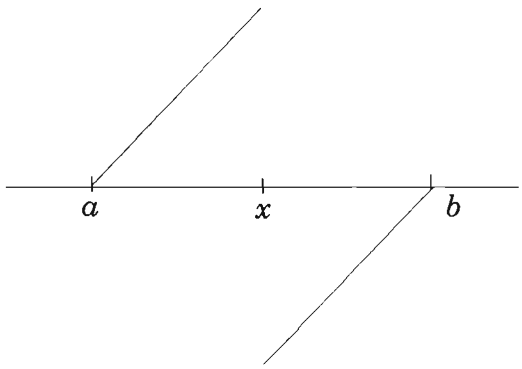
Remark 2.15. $\gamma = \frac{1}{3}$ *gives a Simpson-type rule for which the optimal value* $x = \frac{a+b}{2}$, *giving the optimal bound when only the assumption of a bounded first derivative is used.*

Further, the stated three-point rules when $f' \in L_p[a, b]$, as represented below.

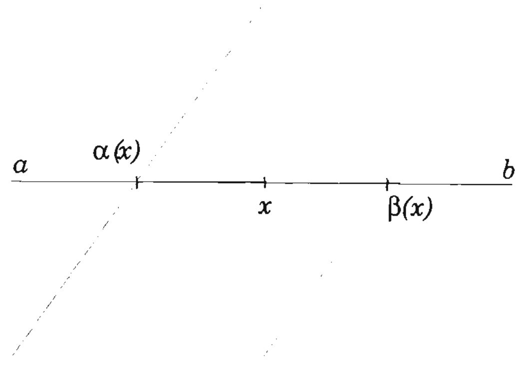
THEOREM 2.16. *Let $f : [a,b] \in \mathbb{R}$ be a differentiable mapping on (a,b) and $f' \in L_p(a,b)$ where $p > 1$ and $\frac{1}{p}+\frac{1}{q}=1$. Then the following inequality holds for all $x \in [a,b]$, $\alpha(x) \in [a,x]$ and $\beta(x) \in (x,b]$,*

$$\begin{aligned}\left|\int_a^b f(t)dt - [(\beta(x)-\alpha(x))f(x) + (b-\beta(x))f(b) + (\alpha(x)-a)f(a)]\right| \\ \leq [(\alpha(x)-a)^{q+1} + (x-\alpha(x))^{q+1} + (\beta(x)-x)^{q+1} \\ + (b-\beta(x))^{q+1}]^{\frac{1}{q}}(q+1)^{\frac{1}{q}}||f'||_p \\ \leq \left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}}||f'||_p \\ \leq (b-a)\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}||f'||_p. \qquad (2.21)\end{aligned}$$

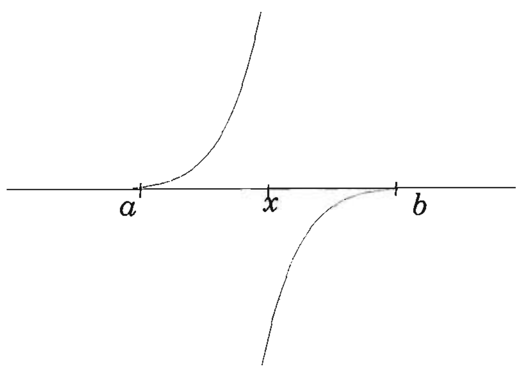
In the next chapter, the three point technique used here is generalized to obtain two-dimensional integral inequalities involving the L_∞, L_p and L_1 norms in terms of the first derivatives of the function in order to produce cubature rules. Three point integral inequalities in which, at most the first derivative is involved, are derived for two-dimentional integrlas.



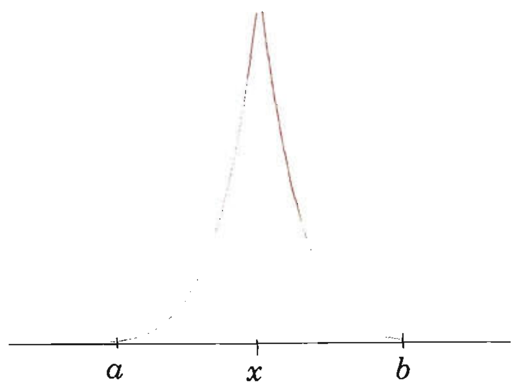
(a) The Peano kernel for equation (2.2)



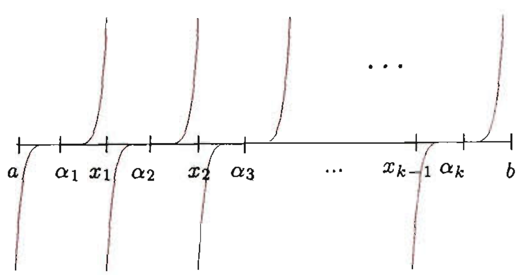
(b) The three point Peano kernel of equation (2.17)



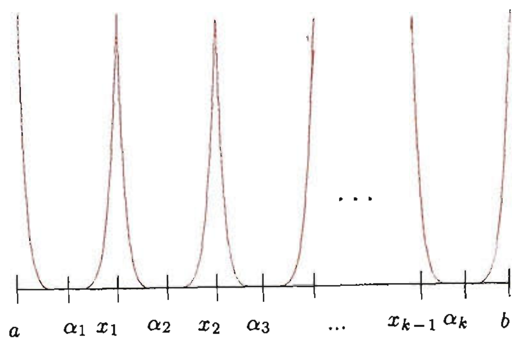
(c) The Peano kernel for the mid-point rule in equation (2.14) for odd n



(d) The Peano kernel for the mid-point rule in equation (2.14) for even n



(e) The multivariate Peano kernel which represents equation (2.15) for odd n



(f) The multivariate Peano kernel which represents equation (2.15) for even n

Figure 2.1: Sketch of different Peano kernels.

Chapter 3

Techniques for two-dimensional Integrals

3.1 Introduction

Moving from the problem of computing one-dimensional integrals to the multidimensional case leads to a series of new problems. While in one dimension one may encounter three possible types of integration intervals - finite, semi-infinite and infinite, now we have to deal with a wide variety of domains. In addition, as is already evident in two dimensions, the functions being integrated can have singularities not only at a point, but even on an entire manifold. These complications make the multidimensional case considerably more difficult than the univariate one, and accounts for the fact that the theory of multidimensional cubature is by no means as complete as the one-dimensional case. Indeed, cubature formulae are most often evaluated as iterated one-dimensional integrals. The approach is straightforward but has some disadvantages, two of which, are that the error estimates are unnecessarily large, since they too rely on embedding the one-dimensional error results, and it is often difficult to discretize regions that are other than ideal. That is, regions whose boundaries lie on coordinate lines of some orthogonal system.

In this chapter we employ the Peano kernel techniques of Chapter 2 to produce two-dimensional integral inequalities. Specifically we will combine and extend the work of Cerone and Dragomir (1999) and Barnett and Dragomir (2001). In Cerone and Dragomir (1999), a one-dimensional three point inequality was investigated, while in Cerone and Dragomir (1999) a two-dimensional version of the Ostrowski result was produced. Here we will develop a two-dimensional three point integral inequality for functions with bounded first derivatives for different types of norms. In each case applications in the numerical integration of a two-dimensional integral is investigated. An *a priori* error bound is obtained for functions whose first partial derivatives exist and are bounded. The rule presented here approximates a two-dimensional integral via application of function evaluations and one-dimensional integrals at the boundary and interior points. A parameterization, similar to that of Cerone and Dragomir (1999) and reviewed in Theorem 2.11, is employed to distinguish rule type. If the one-dimensional integrals are not known, they themselves can be approximated to produce a cubature rule consisting only of sampling points. An additional three point rule, as in Cerone and Dragomir (1999), may be subsequently used, or indeed any other desired quadrature rule. For example, the optimal rules of Golomb and Weinberger (1959) and Traub and Wozniakowski (1980). As a result the error bound will be larger.

The Chapter is arranged in the following manner. In Section 3.1, an inequality for double integrals is obtained in terms of first derivatives where $\frac{\partial^2 f}{\partial t_1 \partial t_2} \in L_\infty \left[[a_1, b_1] \times [a_2, b_2] \right]$. Some numerical results are computed in Section 3.7. An application for the cubature formula is illustrated in Section 3.3. In Section 3.3, an inequality is developed for mappings whose first derivatives $\frac{\partial^2 f}{\partial t_1 \partial t_2} \in L_p \left[[a_1, b_1] \times [a_2, b_2] \right]$ and an application is demonstrated through numerical results in Section 3.5. Section 3.5 is reserved for results involving mappings whose first derivatives belong to the $\|.\|_1$-norm.

The method presented here is based on Ostrowski's integral inequality, and as such is amenable to the production of error bounds for a variety of norms. In addition smoother and product integrands may also be considered as has been done for one-dimensional integrals, see for example (Cerone and Dragomir 1999; Cerone *et al.* 1999a; Roumeliotis *et al.* 1999).

3.2 Mappings Whose First Derivative Belongs to $L_\infty[[a_1, b_1] \times [a_2, b_2]]$.

Here we consider a function whose first partial derivatives exist and are bounded over a given rectangular region. We state the following theorem (see also, Hanna *et al.* (2000)).

THEOREM 3.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable mapping on $[a_1, b_1] \times [a_2, b_2]$ and let $f''_{t_1,t_2} = \frac{\partial^2 f}{\partial t_1 \partial t_2}$ be bounded on $(a_1, b_1) \times (a_2, b_2)$. That is,*

$$\left\| f''_{t_1,t_2} \right\|_\infty := \sup_{(x_1,x_2)\in(a_1,b_1)\times(a_2,b_2)} \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right| < \infty.$$

Furthermore, let $x_i \in (a_i, b_i)$ and introduce the parameterization α_i, β_i defined by

$$\begin{aligned} \alpha_i &= (1-\gamma_i)\, a_i + \gamma_i x_i, \\ \beta_i &= (1-\gamma_i)\, b_i + \gamma_i x_i, \end{aligned} \tag{3.1}$$

where $\gamma_i \in [0,1]$, for $i = 1, 2$. Then the following inequality holds

$$\begin{aligned} |G(x_1,t_1,x_2,t_2)| \le \frac{\left\| f''_{t_1,t_2} \right\|_\infty}{4} \left(1 + (2\gamma_1 - 1)^2\right) \left[\left(\frac{b_1 - a_1}{2}\right)^2 + \left(x_1 - \frac{a_1 + b_1}{2}\right)^2 \right] \\ \times \left(1 + (2\gamma_2 - 1)^2\right) \left[\left(\frac{b_2 - a_2}{2}\right)^2 + \left(x_2 - \frac{a_2 + b_2}{2}\right)^2 \right], \end{aligned} \tag{3.2}$$

given that

$$\begin{aligned} G(x_1,t_1,x_2,t_2) = \sum_{k=1}^{3}\sum_{j=1}^{3} C_{k1} C_{j2} f_{jk} - \sum_{j=1}^{3} \left(C_{j1} I_{j2} + C_{j2} I_{j1}\right) \\ + \int_{a_2}^{b_2}\int_{a_1}^{b_1} f\left(t_1, t_2\right) dt_1 dt_2 \end{aligned} \tag{3.3}$$

$$(f_{jk}) = \begin{pmatrix} f\left(a_1, a_2\right) & f\left(x_1, a_2\right) & f\left(b_1, a_2\right) \\ f\left(a_1, x_2\right) & f\left(x_1, x_2\right) & f\left(b_1, x_2\right) \\ f\left(a_1, b_2\right) & f\left(x_1, b_2\right) & f\left(b_1, b_2\right) \end{pmatrix}, \tag{3.4}$$

$$(C_{jk}) = \begin{pmatrix} \gamma_1(x_1 - a_1) & \gamma_2(x_2 - a_2) \\ (1-\gamma_1)\left(b_1 - a_1\right) & (1-\gamma_2)\left(b_2 - a_2\right) \\ \gamma_1\left(b_1 - x_1\right) & \gamma_2(b_2 - a_2) \end{pmatrix}, \tag{3.5}$$

$$(I_{jk}) = \begin{pmatrix} \int_{a_1}^{b_1} f(t_1, a_2)\, dt_1 & \int_{a_2}^{b_2} f(a_1, t_2)\, dt_2 \\ \int_{a_1}^{b_1} f(t_1, x_2)\, dt_1 & \int_{a_1}^{b_1} f(x_1, t_2)\, dt_2 \\ \int_{a_1}^{b_1} f(t_1, b_2)\, dt_1 & \int_{a_1}^{b_1} f(b_1, t_2)\, dt_2 \end{pmatrix}. \tag{3.6}$$

Proof. Define the kernel

$$p(x,t) = \begin{cases} t-\alpha, & t \in [a,x], \\ t-\beta, & t \in (x,b], \end{cases} \tag{3.7}$$

where, $\alpha = (1-\gamma)\,a + \gamma x$, and $\beta = (1-\gamma)\,b + \gamma x$. Using (3.7) and integrating by parts we obtain, after some simplification, the identity

$$\int_a^b p(x,t)\,F'(t)\,dt$$
$$= (1-\gamma)(b-a)\,F(x) + \gamma\left[(x-a)\,F(a) + (b-x)\,F(b)\right] - \int_a^b F(t)\,dt. \tag{3.8}$$

A two-dimensional identity can be developed via repeated application of (3.8). To this end, we define the mapping

$$p_i(x_i,t_i) = \begin{cases} t_i-\alpha_i, & a_i \le t_i \le x_i, \\ t_i-\beta_i, & x_i < t_i \le b_i, \end{cases} \quad \text{for } i = 1,2. \tag{3.9}$$

Substituting p_1 for p and $f(t_1,\cdot)$ for $F(t)$ into (3.8) gives

$$\int_{a_1}^{b_1} p_1(x_1,t_1)\,\frac{\partial f}{\partial t_1}dt_1 = (1-\gamma_1)(b_1-a_1)\,f(x_1,t_2)$$
$$+\gamma_1(x_1-a_1)\,f(a_1,t_2) + \gamma_1(b_1-x_1)\,f(b_1,t_2) - \int_{a_1}^{b_1} f(t_1,t_2)\,dt_1. \tag{3.10}$$

Employing (3.8) again with p_2 as the kernel, $F(t_2) = \int_{a_1}^{b_1} p_1(x_1,t_1)\,\frac{\partial f}{\partial t_1}dt_1$ as the integrand and expanding with (3.10) produces,

$$\begin{aligned}
\int_{a_2}^{b_2} p_2(x_2,t_2)F'(t_2)\,dt_2 &= \int_{a_2}^{b_2}\int_{a_1}^{b_1} p_2(x_2,t_2)\,p_1(x_1,t_1)\,\frac{\partial^2 f}{\partial t_1\partial t_2}dt_1dt_2 \\
&= (1-\gamma_2)(b_2-a_2)\,F(x_2) + \gamma_2\left[(2-x_2)\,F_2(b_2) + (x_2-a_2)\,F_2(a_2)\right] \\
&\quad - \int_{a_2}^{b_2} F_2(t_2)\,dt_2 \\
&= (1-\gamma_1)(1-\gamma_2)(b_1-a_1)(b_2-a_2)\,f(x_1,x_2) \\
&\quad + \gamma_1(1-\gamma_2)(b_2-a_2)(b_1-x_1)\,f(b_1,x_2) \\
&\quad + \gamma_1(1-\gamma_2)(b_2-a_2)(x_1-a_1)\,f(a_1,x_2) \\
&\quad + \gamma_2(1-\gamma_1)(b_1-a_1)(b_2-x_2)\,f(x_1,b_2) \\
&\quad + \gamma_1\gamma_2(b_2-x_2)(b_1-x_1)\,f(b_1,b_2) \\
&\quad + \gamma_1\gamma_2(b_2-x_2)(x_1-a_1)\,f(a_1,b_2) \\
&\quad + \gamma_2(1-\gamma_1)(x_2-a_2)(b_1-a_1)\,f(x_1,a_2)
\end{aligned}$$

$$
\begin{aligned}
&+ \gamma_1\gamma_2 (x_2 - a_2)(b_1 - x_1) f(b_1, a_2) \\
&+ \gamma_1\gamma_2 (x_2 - a_2)(x_1 - a_1) f(a_1, a_2) \\
&- (1-\gamma_2)(b_2 - a_2) \int_{a_1}^{b_1} f(t_1, x_2)\, dt_1 - \gamma_2 (b_2 - x_2) \int_{a_1}^{b_1} f(t_1, b_2)\, dt_1 \\
&- \gamma_2 (x_2 - a_2) \int_{a_1}^{b_1} f(t_1, a_2)\, dt_1 - (1-\gamma_1)(b_1 - a_1) \int_{a_2}^{b_2} f(x_1, t_2)\, dt_2 \\
&- \gamma_1 (b_1 - x_1) \int_{a_2}^{b_2} f(b_1, t_2)\, dt_2 - \gamma_1 (x_1 - a_1) \int_{a_2}^{b_2} f(a_1, t_2)\, dt_2 \\
&+ \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(t_1, t_2)\, dt_1 dt_2. \\
=& \sum_{k=1}^{3} \sum_{j=1}^{3} C_{k1} C_{j2} f_{jk} - \sum_{j=1}^{3} (C_{j1} I_{j2} + C_{j2} I_{j1}) + \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(t_1, t_2)\, dt_1 dt_2.
\end{aligned}
$$

so that

$$
G(x_1, t_1, x_2, t_2) = \int_{a_2}^{b_2} \int_{a_1}^{b_1} p_2(x_2, t_2)\, p_1(x_1, t_1) \frac{\partial^2 f}{\partial t_1 \partial t_2} dt_1 dt_2. \tag{3.11}
$$

Assuming that both first partial derivatives of f are bounded, we can use Hölder's inequality to give

$$
\begin{aligned}
&\left| \int_{a_2}^{b_2} \int_{a_1}^{b_1} p_2(x_2, t_2)\, p_1(x_1, t_1) \frac{\partial^2 f}{\partial t_1 \partial t_2} dt_1 dt_2 \right| \\
&\qquad \leq \left\| f''_{t_1,t_2} \right\|_\infty \left(\int_{a_2}^{b_2} |p_2(x_2, t_2)|\, dt_2 \right) \left(\int_{a_1}^{b_1} |p_1(x_1, t_1)|\, dt_1 \right).
\end{aligned} \tag{3.12}
$$

Now, consider

$$
\begin{aligned}
G_1(x_1) &= \int_{a_1}^{b_1} |p_1(x_1, t_1)|\, dt_1 \\
&= -\int_{a_1}^{\alpha_1} (t_1 - \alpha_1)\, dt_1 + \int_{\alpha_1}^{x_1} (t_1 - \alpha_1)\, dt_1 \\
&\quad - \int_{x_1}^{\beta_1} (t_1 - \beta_1)\, dt_1 + \int_{\beta_1}^{b_1} (t_1 - \beta_1)\, dt_1 \\
&= \frac{1}{2} \left[(\alpha_1 - a_1)^2 + (x_1 - \alpha_1)^2 + (\beta_1 - x_1)^2 + (b_1 - \beta_1)^2 \right] \\
&= \frac{1}{2} \left[1 + (2\gamma_1 - 1)^2 \right] \left[\left(\frac{b_1 - a_1}{2} \right)^2 + \left(x_1 - \frac{a_1 + b_1}{2} \right)^2 \right].
\end{aligned} \tag{3.13}
$$

Similarly, with $G_2(x_2) = \int_{a_2}^{b_2} |p_2(x_2, t_2)|\, dt_2$, we have

$$
G_2(x) = \frac{1}{2} \left[1 + (2\gamma_2 - 1)^2 \right] \left[\left(\frac{b_2 - a_2}{2} \right)^2 + \left(x_2 - \frac{a_2 + b_2}{2} \right)^2 \right]. \tag{3.14}
$$

Using (3.4), (3.5) and (3.6) and substituting (3.11), (3.13) and (3.14) into (3.12) will produce the result (3.2) and thus the theorem is proved. □

The following result gives an Ostrowski type inequality for double integrals. It involves double and single integrals together with a function evaluation at an interior point.

Corollary 3.2. *With the conditions as in Theorem 3.1, then*

$$\begin{aligned}\Big|(b_1-a_1)(b_2-a_2)f(x_1,x_2)&-(b_2-a_2)\int_{a_1}^{b_1}f(t_1,x_2)\,dt_1\\&-(b_1-a_1)\int_{a_2}^{b_2}f(x_1,t_2)\,dt_2+\int_{a_2}^{b_2}\int_{a_1}^{b_1}f(t_1,t_2)\,dt_1dt_2\Big|\\\leq \left\|f''_{t_1,t_2}\right\|_\infty&\left[\left(\frac{b_1-a_1}{2}\right)^2+\left(x_1-\frac{a_1+b_1}{2}\right)^2\right]\left[\left(\frac{b_2-a_2}{2}\right)^2+\left(x_2-\frac{a_2+b_2}{2}\right)^2\right].\end{aligned}\tag{3.15}$$

Proof. Place $\gamma_1=\gamma_2=0$ into equation (3.2). □

Thus, the earlier results of Barnett and Dragomir (2001) and Mitrinovič *et al.* (1994, p. 468) are reproduced as a special case of Theorem 3.1. We note that unlike Barnett and Dragomir (2001), the proof for Theorem 3.1 can be readily extended to more than two dimensions. Different values of the parameters γ_1, γ_2, x_1 and x_2 give rise to Newton-Cotes type inequalities for functions with bounded derivatives. For example $\gamma_1=\gamma_2=0$, $x_1=\frac{a_1+b_1}{2}$ and $x_2=\frac{a_2+b_2}{2}$ produces the two-dimensional mid-point inequality; $\gamma_1=\gamma_2=1$ a two-dimensional trapezoid-like inequality (a similar result has been obtained by Pachpatte (2001)) and $\gamma_1=\gamma_2=\frac{1}{3}$ a two-dimensional Simpson's like inequality.

From Theorem 3.1 it is a simple matter to show that the tightest bound is obtained when $\gamma_1=\gamma_2=\frac{1}{2}$ and x_1 and x_2 are at their mid-points. That is the average of the mid-point and trapezoid inequalities.

Remark 3.3. *Let $f(t_1,t_2)=g(t_1)g(t_2)$ where $g:[a,b]\to\mathbb{R}$. If g is differentiable and satisfies the condition that $\|g'\|_\infty<\infty$, then, for $x_1=x_2=x$ and $\gamma_1=\gamma_2=\gamma$, we obtain a result from Theorem 3.1 which may be factored to recover the three point rule of Theorem 2.11, namely*

$$\begin{aligned}\left|\int_a^b g(t)\,dt-\gamma\big((x-a)g(a)+(b-x)g(b)\big)-(1-\gamma)(b-a)g(x)\right|\\\leq\frac{\|g'\|_\infty}{2}\left(1+(2\gamma-1)^2\right)\left(\left(\frac{b-a}{2}\right)^2+\left(x-\frac{a+b}{2}\right)^2\right).\end{aligned}\tag{3.16}$$

In general, cubature formulae are written only in terms of function evaluations, but Theorem 3.1 approximates a double integral in terms of single integrals and function evaluations. Therefore we write down the following corollary which eliminates the one-dimensional integrals by approximating them using the 3-point rule in equation (3.16). The resulting inequality has a coarser bound than equation (3.2).

Corollary 3.4. *Let f be given as in Theorem 3.1. Then*

$$\begin{aligned}
&\left|\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(t_1,t_2)\,dt_1dt_2 - \sum_{k=1}^{3}\sum_{j=1}^{3} C_{k1}C_{j2}f_{jk}\right| \\
&\quad\le \frac{\left\|f''_{t_1,t_2}\right\|_\infty}{4}\left(1+(2\gamma_1-1)^2\right)\left(1+(2\gamma_2-1)^2\right) \\
&\quad\times\left[\left(\frac{b_1-a_1}{2}\right)^2+\left(x_1-\frac{a_1+b_1}{2}\right)^2\right]\left[\left(\frac{b_2-a_2}{2}\right)^2+\left(x_2-\frac{a_2+b_2}{2}\right)^2\right] \\
&\quad+\frac{1}{2}\left(1+(2\gamma_1-1)^2\right)\left[\left(\frac{b_1-a_1}{2}\right)^2+\left(x_1-\frac{a_1+b_1}{2}\right)^2\right] \\
&\quad\times\left\{\gamma_2(x_2-a_2)\left\|f'_{t_1,a_2}\right\|_\infty+(1-\gamma_2)(b_2-a_2)\left\|f'_{t_1,x_2}\right\|_\infty+\gamma_2(b_2-x_2)\left\|f'_{t_1,b_2}\right\|_\infty\right\} \\
&\quad+\frac{1}{2}\left(1+(2\gamma_2-1)^2\right)\left[\left(\frac{b_2-a_2}{2}\right)^2+\left(x_2-\frac{a_2+b_2}{2}\right)^2\right] \\
&\quad\times\left\{\gamma_1(x_1-a_1)\left\|f'_{a_1,t_2}\right\|_\infty+(1-\gamma_1)(b_1-a_1)\left\|f'_{x_1,t_2}\right\|_\infty+\gamma_1(b_1-x_1)\left\|f'_{b_1,t_2}\right\|_\infty\right\}
\end{aligned}\tag{3.17}$$

Proof. Approximating each single integral in (3.2) by (3.16) and applying the triangle inequality produces the desired result. □

Remark 3.5. *If $\gamma_1=\gamma_2=0$ and $x_i=\frac{a_i+b_i}{2}$, then*

$$\begin{aligned}
&\left|\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(t_1,t_2)\,dt_1dt_2-(b_1-a_1)(b_2-a_2)f\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2}\right)\right| \\
&\quad\le\frac{\left\|f''_{t_1,t_2}\right\|_\infty}{16}(b_1-a_1)^2(b_2-a_2)^2+\frac{\left\|f'_{t_1,\frac{a_2+b_2}{2}}\right\|_\infty}{4}(b_2-a_2)(b_1-a_1)^2 \\
&\quad+\frac{\left\|f'_{\frac{a_1+b_1}{2},t_2}\right\|_\infty}{4}(b_1-a_1)(b_2-a_2)^2.
\end{aligned}\tag{3.18}$$

We can apply any other rule rather than the one in (3.16) to approximate each single integral in (3.2)

3.3 Application to Cubature Formulae

To illustrate the use of a cubature formula, we form a composite rule from the inequality (3.15).

Let us consider the arbitrary division:

$$I_n : a_1 = \xi_0 < \xi_1 < ... < \xi_n = b_1$$

on the interval $[a_1, b_1]$ with $x_i \in [\xi_i, \xi_{i+1}]$ for $i = 0, 1, ..., n-1$ and $J_m : a_2 = \tau_0 < \tau_1 < ... < \tau_m = b_2$ on the interval $[a_2, b_2]$ with $y_j \in [\tau_j, \tau_{j+1}]$ for $j = 0, 1, ..., m-1$. Consider the sum

$$\begin{aligned} A(f, I_n, J_m, x, y) := \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i v_j f(x_i, y_j) - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i \int_{\tau_j}^{\tau_{j+1}} f(x_i, t_2)\, dt_2 \\ - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} v_j \int_{\xi_i}^{\xi_{i+1}} f(t_1, y_j)\, dt_1, \quad (3.19) \end{aligned}$$

where

$h_i = \xi_{i+1} - \xi_i$ $(i = 0, 1, ..., n-1)$ and $v_j = \tau_{j+1} - \tau_j$ $(j = 0, 1, ..., m-1)$ and $\gamma_1 = \gamma_2 = 0$. Under the above assumptions the following theorem holds.

THEOREM 3.6. *Let $f : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ be a differentiable mapping on $[a_1, b_1] \times [a_2, b_2]$, let $f''_{t_1 t_2} \in L_\infty(a, b) \times (c, d)$ and I_n, J_m, x, y be as above. Then we have the cubature formula*

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(t_1, t_2)\, dt_1 dt_2 = A(f, I_n, J_m, x, y) + R(f, I_n, J_m, x, y), \quad (3.20)$$

where the remainder term $R(f, I_n, J_m, x, y)$ satisfies the inequality

$$\begin{aligned} &|R(f, I_n, J_m, x, y)| \\ &\leq \left\| f''_{t_1, t_2} \right\|_\infty \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} \left[\frac{1}{4}h_i^2 + \left(x_i - \frac{\xi_i + \xi_{i+1}}{2}\right)^2\right]\left[\frac{1}{4}v_j^2 + \left(y_j - \frac{\tau_j + \tau_{j+1}}{2}\right)^2\right]. \end{aligned}$$

Proof. Apply Corollary 3.15 on the interval $[\xi_i, \xi_{i+1}] \times [\tau_j, \tau_{j+1}]$, $(i = 0, 1, ..., n-1)$,

$(j = 0, 1, ..., m-1)$ to get

$$\left| \underbrace{(\xi_{i+1} - \xi_i)}_{h_i} \underbrace{(\tau_{j+1} - \tau_j)}_{v_j} f(x_i, y_j) - v_j \int_{\xi_i}^{\xi_{i+1}} f(t_1, y_j)\, dt_1 \right.$$
$$\left. -h_i \int_{\tau_j}^{\tau_{j+1}} f(x_i, t_2)\, dt_2 + \int_{\xi_i}^{\xi_{i+1}} \int_{\tau_j}^{\tau_{j+1}} f(t_1, t_2)\, dt_1 dt_2 \right|$$
$$\leq \left\| f'_{t_1,t_2} \right\|_\infty \left[\frac{1}{4} h_i^2 + \left(x_i - \frac{\xi_i + \xi_{i+1}}{2} \right)^2 \right] \left[\frac{1}{4} v_j^2 + \left(y_j - \frac{\tau_j + \tau_{j+1}}{2} \right)^2 \right] \tag{3.21}$$

for all $(i = 0, 1, ..., n-1), (j = 0, 1, ..., m-1)$.

Now, summing over i from 0 to $n-1$ and over j from 0 to $m-1$, and using the generalized triangle inequality, we deduce (3.21). □

Corollary 3.7. *We know that* $\left| x_i - \frac{\xi_i + \xi_{i+1}}{2} \right| \leq \frac{1}{2} h_i$ *and* $\left| y_j - \frac{\tau_j + \tau_{j+1}}{2} \right| \leq \frac{1}{2} v_j$. *Applying these to (3.21), we find that*

$$|R(f, I_n, J_m, x, y)| \leq \left\| f'_{t_1,t_2} \right\|_\infty \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left[\frac{1}{4} h_i^2 + \frac{1}{4} h_i^2 \right] \left[\frac{1}{4} v_j^2 + \frac{1}{4} v_j^2 \right]$$
$$\leq \frac{\left\| f'_{t_1,t_2} \right\|_\infty}{4} \sum_{i=0}^{n-1} h_i^2 \sum_{j=0}^{m-1} v_j^2. \tag{3.22}$$

Corollary 3.8. *Now, consider the case where x_i and y_i are the mid-points. At the mid-point we have*

$$\int_{a_2}^{b_2} \int_{a_1}^{b_1} f(t_1, t_2)\, dt_1 dt_2 = A(f, I_n, J_m) + R(f, I_n, J_m), \tag{3.23}$$

where the remainder term $R(f, I_n, J_m)$ satisfies

$$|R(f, I_n, J_m)| \leq \frac{\left\| f''_{t_1,t_2} \right\|_\infty}{16} \sum_{i=0}^{n-1} h_i^2 \sum_{j=0}^{m-1} v_j^2.$$

Corollary 3.9. *Let the conditions of Theorem 3.6 hold. In addition, let I_n be the equidistant partition of $[a_1, b_1]$, $I_n : x_i = a_1 + \left(\frac{b_1 - a_1}{n}\right) i$, $i = 0, 1, ..., n-1$, and J_m be the equidistant partition of*

$[a_2, b_2]$, $J_m : y_j = a_2 + \left(\frac{b_2 - a_2}{m}\right) j$, $j = 0, 1, ..., m-1$, *then*

$$\left| \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(t_1, t_2) dt_1 dt_2 - A(f, I_n, J_m) \right| \leq \frac{\left\| f''_{t_1,t_2} \right\|_\infty (b_1 - a_1)^2 (b_2 - a_2)^2}{16nm} \tag{3.24}$$

Proof. From Theorem 3.6 with $h_i = \frac{b_1 - a_1}{n}$ for all i so that

$$
\begin{aligned}
|R(f, I_n, J_m)| &\leq \frac{\left\| f''_{t_1, t_2} \right\|_\infty}{16} \sum_{i=0}^{n-1} \left(\frac{b_1 - a_1}{n} \right)^2 \sum_{j=0}^{m-1} \left(\frac{b_2 - a_2}{m} \right)^2 \\
&= \frac{\left\| f''_{t_1, t_2} \right\|_\infty (b_1 - a_1)^2 (b_2 - a_2)^2}{16nm}
\end{aligned}
$$

and hence the result is proved. □

Remark 3.10. *If we were to use (3.19) to approximate the integral $\int_{a_2}^{b_2} \int_{a_1}^{b_1} f(t_1, t_2)\, dt_1 dt_2$ with a uniform grid and sampling at each mid-point, then the remainder R is bounded by*

$$
|R(f, I_n, J_m, \mathbf{x}, \mathbf{y})| \leq \frac{\left\| f''_{t_1, t_2} \right\|_\infty (b_1 - a_1)^2 (b_2 - a_2)^2}{16nm}. \tag{3.25}
$$

3.4 Mappings Whose First Derivative Belongs to $L_p\left[[a_1, b_1] \times [a_2, b_2]\right]$.

For this section we will refer to Dragomir and Wang (1998b) where the authors considered an inequality of Ostrowski type for $\|\cdot\|_p$ −norms as in Theorem 2.2. Also we will utilize the result of Dragomir *et al.* (1998) wherein the authors acquired a double integral in terms of $\|\cdot\|_p$ −norms. Utilizing Theorem 2.11 and amalgamating the above two results we point out a three point inequality of Ostrowski type for double integrals in terms of the $\|\cdot\|_p$ −norm of the first derivatives.

THEOREM 3.11. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable mapping on $[a_1, b_1] \times [a_2, b_2]$ and let $f''_{t_1,t_2} = \frac{\partial^2 f}{\partial t_1 \partial t_2}$ be L_p bounded on $(a_1, b_1) \times (a_2, b_2)$, that is,*

$$
\left\| f''_{t_1,t_2} \right\|_p := \left(\int_{a_2}^{b_2} \int_{a_1}^{b_1} \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right|^p dt_1 dt_2 \right)^{\frac{1}{p}}, \quad 1 < p < \infty.
$$

We obtain the following inequality

$$
\begin{aligned}
|G(x_1, t_1, x_2, t_2)| \leq{}& \frac{\left\| f''_{t_1,t_2} \right\|_p}{(q+1)^{\frac{2}{q}}} \left[\gamma_1^{q+1} + (1 - \gamma_1)^{q+1} \right]^{\frac{1}{q}} \left[(x_1 - a_1)^{q+1} + (b_1 - x_1)^{q+1} \right]^{\frac{1}{q}} \\
&\times \left[\gamma_2^{q+1} + (1 - \gamma_2)^{q+1} \right]^{\frac{1}{q}} \left[(x_2 - a_2)^{q+1} + (b_2 - x_2)^{q+1} \right]^{\frac{1}{q}}
\end{aligned} \tag{3.26}
$$

where G, x_1, x_2, γ_1, γ_2 are as in Theorem 3.1 and $\dfrac{1}{p} + \dfrac{1}{q} = 1$

Proof. We proceed as in the proof Theorem (3.1) by applying Hölder's inequality for double integrals, that is,

$$\left|\int_{a_2}^{b_2}\int_{a_1}^{b_1} p_2(x_2,t_2)\,p_1(x_1,t_1)\,\frac{\partial^2 f}{\partial t_1\partial t_2}dt_1dt_2\right|$$
$$\le\left(\int_{a_2}^{b_2}\int_{a_1}^{b_1}|p_2(x_2,t_2)\,p_1(x_1,t_1)|^q\,dt_1dt_2\right)^{\frac{1}{q}}\left(\int_{a_2}^{b_2}\int_{a_1}^{b_1}\left|\frac{\partial^2 f}{\partial t_1\partial t_2}\right|^p dt_1dt_2\right)^{\frac{1}{p}}$$
$$=\left(\int_{a_1}^{b_1}|p_1(x_1,t_1)|^q\,dt_1\right)^{\frac{1}{q}}\left(\int_{a_2}^{b_2}|p_2(x_2,t_2)|^q\,dt_2\right)^{\frac{1}{q}}\left\|f''_{t_1,t_2}\right\|_p. \tag{3.27}$$

Consider

$$G_1(x_1)=\left(\int_{a_1}^{b_1}|p_1(x_1,t_1)|^q\,dt_1\right)^{\frac{1}{q}}$$
$$=\left[\left(\int_{a_1}^{\alpha_1}(\alpha_1-t_1)^q\,dt_1\right)+\left(\int_{\alpha_1}^{x_1}(t_1-\alpha_1)^q\,dt_1\right)\right.$$
$$\left.+\left(\int_{x_1}^{\beta_1}(\beta_1-t_1)^q\,dt_1\right)+\left(\int_{\beta_1}^{b_1}(t_1-\beta_1)^q\,dt_1\right)\right]^{\frac{1}{q}}$$
$$=\left[\frac{(\alpha_1-a_1)^{q+1}+(x_1-\alpha_1)^{q+1}+(\beta_1-x_1)^{q+1}+(b_1-\beta_1)^{q+1}}{q+1}\right]^{\frac{1}{q}} \tag{3.28}$$

and we get on using the parametric equations(3.1)

$$G_1(x_1)=\left[\frac{\left[\gamma_1^{q+1}+(1-\gamma_1)^{q+1}\right]\left[(x_1-a_1)^{q+1}+(b_1-x_1)^{q+1}\right]}{q+1}\right]^{\frac{1}{q}}.$$

Similarly,

$$G_2(x_2)=\left[\frac{\left[\gamma_2^{q+1}+(1-\gamma_2)^{q+1}\right]\left[(x_2-a_2)^{q+1}+(b_2-x_2)^{q+1}\right]}{q+1}\right]^{\frac{1}{q}}.$$

Substituting $G_1(x_1)$ and $G_2(x_2)$ into (3.27) will produce the result (3.26) and thus the theorem is proved. □

Remark 3.12. *We notice from (3.26) that the bound is convex in $\gamma_i\in[0,1]$ and $x_i\in[a_i,b_i]$ for $i=1,2$. The sharpest bound occurs on taking $\gamma_i=\frac{1}{2}$ and $x_i=\frac{a_i+b_i}{2}$ for $i=1,2$. The coarsed bound is obtained when the γ_i and x_i are taken at either of their boundary point for their respective intervals. The following results investigate this relationship further.*

Corollary 3.13. *With the conditions as in Theorem 3.11, then*

$$\left|G(\frac{a_1+b_1}{2},t_1,\frac{a_2+b_2}{2},t_2)\right|\le\frac{\left\|f''_{t_1,t_2}\right\|_p}{4(q+1)^{\frac{2}{q}}}\left[\gamma_1^{q+1}+(1-\gamma_1)^{q+1}\right]^{\frac{1}{q}}\left[(b_1-a_1)\right]^{\frac{q+1}{q}}$$
$$\times\left[\gamma_2^{q+1}+(1-\gamma_2)^{q+1}\right]^{\frac{1}{q}}\left[(b_2-a_2)\right]^{\frac{q+1}{q}} \tag{3.29}$$

where $G(.,t_1,.,t_2)$ is defined in (3.3).

Proof. Place $x_i = \frac{a_i+b_i}{2}$ in the right hand side of equation (3.26). □

Remark 3.14. *If $p=q=2$, then (3.29) becomes a mid-point type rule*

$$\left|G(\frac{a_1+b_1}{2},t_1,\frac{a_2+b_2}{2},t_2)\right| \leq \frac{\left\|f''_{t_1,t_2}\right\|_2}{12}\left[\gamma_1^3+(1-\gamma_1)^3\right]^{\frac{1}{2}}[b_1-a_1]^{\frac{3}{2}} \times\left[\gamma_2^3+(1-\gamma_2)^3\right]^{\frac{1}{q}}[b_2-a_2]^{\frac{3}{2}}. \quad (3.30)$$

Remark 3.15. *If $\gamma_1=\gamma_2=0$, then (3.30) becomes a trapezoidal type rule*

$$\left|(b_1-a_1)(b_2-a_2)f\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2}\right) - (b_2-a_2)\int_{a_1}^{b_1} f\left(t_1,\frac{a_2+b_2}{2}\right)dt_1-(b_1-a_1)\int_{a_2}^{b_2} f\left(\frac{a_1+b_1}{2},t_2\right)dt_2 + \int_{a_2}^{b_2}\int_{a_1}^{b_1} f(t_1,t_2)\,dt_1dt_2\right| \leq \frac{\left\|f''_{t_1,t_2}\right\|_2}{12}\left[(b_1-a_1)(b_2-a_2)\right]^{\frac{3}{2}}. \quad (3.31)$$

Remark 3.16. *If $\gamma_1=\gamma_2=1$,(3.30) becomes*

$$\left|\frac{(b_1-a_1)(b_2-a_2)}{4}\left[f(b_1,b_2)+f(a_1,b_2)+f(b_1,a_2)+f(a_1,a_2)\right] - \frac{1}{2}\left[(b_2-a_2)\int_{a_1}^{b_1} f(t_1,b_2)\,dt_1+(b_2-a_2)\int_{a_1}^{b_1} f(t_1,a_2)\,dt_1 + (b_1-a_1)\int_{a_2}^{b_2} f(b_1,t_2)\,dt_2+(b_1-a_1)\int_{a_2}^{b_2} f(a_1,t_2)\,dt_2\right] + \int_{a_2}^{b_2}\int_{a_1}^{b_1} f(t_1,t_2)\,dt_1dt_2\right| \leq \frac{\left\|f''_{t_1,t_2}\right\|_2}{12}\left[(b_1-a_1)(b_2-a_2)\right]^{\frac{3}{2}}. \quad (3.32)$$

Remark 3.17. *If $\gamma_1 = \gamma_2 = \frac{1}{2}$, then (3.30) becomes*

$$\left| \frac{(b_1 - a_1)(b_2 - a_2)}{4} f\left(\frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{2}\right) \right.$$
$$+ \frac{(b_1 - a_1)(b_2 - a_2)}{8} \left[f\left(b_1, \frac{a_2 + b_2}{2}\right) + f\left(a_1, \frac{a_2 + b_2}{2}\right) \right.$$
$$\left. + f\left(\frac{a_1 + b_1}{2}, b_2\right) + f\left(\frac{a_1 + b_1}{2}, a_2\right) \right]$$
$$+ \frac{(b_1 - a_1)(b_2 - a_2)}{16} \left[f(b_1, b_2) + f(a_1, b_2) + f(b_1, a_2) + f(a_1, a_2) \right]$$
$$- \frac{(b_1 - a_1)}{4} \left[2 \int_{a_2}^{b_2} f\left(\frac{a_1 + b_1}{2}, t_2\right) dt_2 + \int_{a_2}^{b_2} f(b_1, t_2)\, dt_2 + \int_{a_2}^{b_2} f(a_1, t_2)\, dt_2 \right]$$
$$- \frac{(b_2 - a_2)}{4} \left[2 \int_{a_1}^{b_1} f\left(t_1, \frac{a_2 + b_2}{2}\right) dt_1 + \int_{a_1}^{b_1} f(t_1, b_2)\, dt_1 + \int_{a_1}^{b_1} f(t_1, a_2)\, dt_1 \right]$$
$$\left. + \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(t_1, t_2)\, dt_1 dt_2 \right|$$
$$\leq \frac{\left\| f''_{t_1,t_2} \right\|_2}{48} \left[(b_1 - a_1)(b_2 - a_2) \right]^{\frac{3}{2}}. \quad (3.33)$$

Remark 3.18. *Let $f(t_1, t_2) = g(t_1)\, g(t_2)$ where $g : [a, b] \to \mathbb{R}$. If g is differentiable and satisfies the condition that $g' \in L_p[a, b]$ then, for $x_1 = x_2 = x$ and $\gamma_1 = \gamma_2 = \gamma$ we get*

$$\left| (b - a)^2 g(x)\, g(x) - g(x)(b - a) \int_a^b g(t)\, dt \right.$$
$$\left. - g(x)(b - a) \int_a^b g(t)\, dt + \int_a^b \int_a^b g(t)\, g(t)\, dt\, dt \right|$$
$$\leq \left(\frac{\|g'\|_p}{(q + 1)^{\frac{1}{q}}} \right)^2 \left[(x - a)^{q+1} + (b - x)^{q+1} \right]^{\frac{2}{q}}. \quad (3.34)$$

Therefore,

$$\left| \int_a^b g(t)\, dt - (b - a)\, g(x) \right|^2 \leq \left(\frac{\|g'\|_p}{(q + 1)^{\frac{1}{q}}} \right)^2 \left((x - a)^{q+1} + (b - x)^{q+1} \right)^{\frac{2}{q}}.$$

This gives

$$\left| \int_a^b g(t)\, dt - (b - a)\, g(x) \right| \leq \|g'\|_p \left[\frac{(x - a)^{q+1} + (b - x)^{q+1}}{q + 1} \right]^{\frac{1}{q}}$$

which is an Ostrowski type inequality for the $\|\cdot\|_p$ −norm obtained by Dragomir and Wang (1998a). Thus, (3.26) is a generalization for two dimensional integrals of the Ostrowski type for $\|\cdot\|_p$ −norms.

The following corollary provides a coarser upper limit for $|G(x_1, t_1, x_2, t_2)|$

Corollary 3.19. *With the conditions as in Theorem 3.11 holding, then*

$$|G(x_1,t_1,x_2,t_2)| \le \frac{\left\|f''_{t_1,t_2}\right\|_p}{(q+1)^{\frac{2}{q}}}\left[(b_1-a_1)(b_2-a_2)\right]^{1+\frac{1}{q}}. \tag{3.35}$$

where $G(x_1,t_1,x_2,t_2)$ is as given in (3.11).

Proof. $(x_i-a_i)^{q+1}+(b_i-x_i)^{q+1} \le (b_i-a_i)^{q+1}$ and $\gamma_i^{q+1}+(1-\gamma_i)^{q+1} \le 1$. ☐

3.5 Application to Cubature Formulae

Consider the arbitrary division:

$$I_n : a_1 = \xi_0 < \xi_1 < \ldots < \xi_n = b_1$$

on the interval $[a_1,b_1]$ with $x_i \in [\xi_i,\xi_{i+1}]$ for $i=0,1,\ldots,n-1$ and $J_m : a_2 = \tau_0 < \tau_1 < \ldots < \tau_m = b_2$ on the interval $[a_2,b_2]$ with $y_j \in [\tau_j,\tau_{j+1}]$ for $j=0,1,\ldots,m-1$. Consider the sum

$$\begin{aligned} A(f,I_n,J_m,x,y) := \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i v_j f(x_i,y_j) - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i \int_{\tau_j}^{\tau_{j+1}} f(x_i,t_2)\,dt_2 \\ - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} v_j \int_{\xi_i}^{\xi_{i+1}} f(t_1,y_j)\,dt_1, \end{aligned} \tag{3.36}$$

where

$h_i = \xi_{i+1}-\xi_i$ $(i=0,1,\ldots,n-1)$ and $v_j=\tau_{j+1}-\tau_j$ $(j=0,1,\ldots,m-1)$ and $\gamma_1=\gamma_2=0$.

THEOREM 3.20. *Let $f:[a_1,b_1]\times[a_2,b_2]\to\mathbb{R}$ be a differentiable mapping on $[a_1,b_1]\times[a_2,b_2]$, let $f''_{t_1t_2}\in L_p(a,b)\times(c,d)$ and I_n, J_m, x,y be as above. Then we have the cubature formula*

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(t_1,t_2)\,dt_1dt_2 = A(f,I_n,J_m,x,y)+R(f,I_n,J_m,x,y), \tag{3.37}$$

and the remainder term $R(f,I_n,J_m,x,y)$ satisfies the inequality

$$\begin{aligned} |R(f,I_n,J_m,x,y)| &\le \frac{\left\|f''_{t_1,t_2}\right\|_p}{(q+1)^{\frac{2}{q}}}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\Big(\left[(x_i-\xi_i)^{q+1}+(\xi_{i+1}-x_i)^{q+1}\right]^{\frac{1}{q}} \\ &\quad\times\left[(y_i-\tau_i)^{q+1}+(\tau_{i+1}-y_i)^{q+1}\right]^{\frac{1}{q}}\Big) \qquad (3.38) \\ &\le \frac{\left\|f''_{t_1,t_2}\right\|_p}{(q+1)^{\frac{2}{q}}}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(h_i\,v_j)^{1+\frac{1}{q}}. \qquad (3.39) \end{aligned}$$

Proof. Apply inequality (3.26) on the interval $[\xi_i, \xi_{i+1}] \times [\tau_j, \tau_{j+1}]$, $(i = 0, 1, ..., n-1)$, $(j = 0, 1, ..., m-1)$ to get

$$\begin{aligned}
&\Bigg| (\xi_{i+1} - \xi_i)(\tau_{j+1} - \tau_j) f(x_i, y_j) - v_j \int_{\xi_i}^{\xi_{i+1}} f(t_1, y_j)\, dt_1 \\
&\qquad - h_i \int_{\tau_j}^{\tau_{j+1}} f(x_i, t_2)\, dt_2 + \int_{\xi_i}^{\xi_{i+1}} \int_{\tau_j}^{\tau_{j+1}} f(t_1, t_2)\, dt_1 dt_2 \Bigg| \\
&\leq \frac{1}{(q+1)^{\frac{2}{q}}} \left[(x_i - \xi_i)^{q+1} + (\xi_{i+1} - x_i)^{q+1}\right]^{\frac{1}{q}} \left[(y_i - \tau_i)^{q+1} + (\tau_{i+1} - y_i)^{q+1}\right]^{\frac{1}{q}} \\
&\qquad \times \left(\int_{\tau_j}^{\tau_{j+1}} \int_{\xi_i}^{\xi_{i+1}} \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right|^p dt_1 dt_2 \right)^{\frac{1}{p}}
\end{aligned} \tag{3.40}$$

for all $(i = 0, 1, ..., n-1)$, $(j = 0, 1, ..., m-1)$. Summing over i from 0 to $n-1$ and over j from 0 to $m-1$, and using the generalized triangle inequality and Hölder's discrete inequality, we obtain

$$\begin{aligned}
|R(f, I_n, J_m, x, y)| &\leq \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Bigg| \frac{1}{(q+1)^{\frac{2}{q}}} \Big[\left[(x_i - \xi_i)^{q+1} + (\xi_{i+1} - x_i)^{q+1}\right]^{\frac{1}{q}} \\
&\qquad \times \left[(y_i - \tau_i)^{q+1} + (\tau_{i+1} - y_i)^{q+1}\right]^{\frac{1}{q}} \Big] \\
&\qquad \times \left(\int_{\tau_j}^{\tau_{j+1}} \int_{\xi_i}^{\xi_{i+1}} \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right|^p dt_1 dt_2 \right)^{\frac{1}{p}} \Bigg| \\
&\leq \frac{1}{(q+1)^{\frac{2}{q}}} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Big[\Big(\left[(x_i - \xi_i)^{q+1} + (\xi_{i+1} - x_i)^{q+1}\right]^{\frac{1}{q}} \\
&\qquad \times \left[(y_i - \tau_i)^{q+1} + (\tau_{i+1} - y_i)^{q+1}\right]^{\frac{1}{q}} \Big) \\
&\qquad \times \left(\int_{\tau_j}^{\tau_{j+1}} \int_{\xi_i}^{\xi_{i+1}} \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right|^p dt_1 dt_2 \right)^{\frac{1}{p}} \Big] \\
&\leq \frac{1}{(q+1)^{\frac{2}{q}}} \left(\sum_{i=0}^{n-1} \left(\left[(x_i - \xi_i)^{q+1} + (\xi_{i+1} - x_i)^{q+1}\right]^{\frac{1}{q}} \right)^q \right)^{\frac{1}{q}} \\
&\qquad \times \left(\sum_{j=0}^{m-1} \left(\left[(y_i - \tau_i)^{q+1} + (\tau_{i+1} - y_i)^{q+1}\right]^{\frac{1}{q}} \right)^q \right)^{\frac{1}{q}} \\
&\qquad \times \left(\sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left(\left(\int_{\tau_j}^{\tau_{j+1}} \int_{\xi_i}^{\xi_{i+1}} \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right|^p dt_1 dt_2 \right)^{\frac{1}{p}} \right)^p \right)^{\frac{1}{p}}
\end{aligned}$$

$$= \frac{1}{(q+1)^{\frac{2}{q}}} \left[\sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left(\left[(x_i - \xi_i)^{q+1} + (\xi_{i+1} - x_i)^{q+1} \right]^{\frac{1}{q}} \right. \right.$$

$$\left. \left. \times \left[(y_i - \tau_i)^{q+1} + (\tau_{i+1} - y_i)^{q+1} \right]^{\frac{1}{q}} \right) \right] \left\| f''_{t_1,t_2} \right\|_p . \quad (3.41)$$

and the first inequality in (3.38) is proved.

The second part follows directly from the fact that

$$(x_i - \xi_i)^{q+1} + (\xi_{i+1} - x_i)^{q+1} \leq h_i^{q+1} \text{ and } (y_i - \tau_i)^{q+1} + (\tau_{i+1} - y_i)^{q+1} \leq v_j^{q+1}.$$

□

3.6 Mappings Whose First Derivative Belongs to $L_1[[a_1, b_1] \times [a_2, b_2]]$.

In this section an inequality of Ostrowski type involving two-dimensional integrals for functions whose first derivatives belong to L_1 can be produced as shown in the following theorem.

THEOREM 3.21. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable mapping on $[a_1, b_1] \times [a_2, b_2]$ and let $f''_{t_1,t_2} = \frac{\partial^2 f}{\partial t_1 \partial t_2}$ be bounded on $(a_1, b_1) \times (a_2, b_2)$, that is,*

$$\left\| f''_{t_1,t_2} \right\|_1 := \int_{a_2}^{b_2} \int_{a_1}^{b_1} \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right| dt_1 dt_2 < \infty \ .$$

We obtain the following inequality

$$|G(x_1, t_1, x_2, t_2)| \leq \left\| f''_{t_1,t_2} \right\|_1 M_1 M_2 \quad (3.42)$$

where

$$M_i = \frac{(b_i - a_i)}{4} \left[1 + |2\gamma_i - 1| \right] + 2 \left| (x_i - \frac{a_i + b_i}{2})(1 + |2\gamma_i - 1|) \right|$$

.

and $G, x_1, x_2, \gamma_1, \gamma_2$ are defined in Theorem 3.1.

Proof. The proof follows that of Theorem 3.1. we have,

$$\begin{aligned}
&\left|\int_{a_2}^{b_2}\int_{a_1}^{b_1} p_2(x_2,t_2)\,p_1(x_1,t_1)\,\frac{\partial^2 f}{\partial t_1\partial t_2}dt_1dt_2\right| \\
&\quad\le \left(\int_{a_2}^{b_2}\int_{a_1}^{b_1} |p_2(x_2,t_2)\,p_1(x_1,t_1)|\,dt_1dt_2\right)\left(\int_{a_2}^{b_2}\int_{a_1}^{b_1}\left|\frac{\partial^2 f}{\partial t_1\partial t_2}\right|dt_1dt_2\right) \\
&\quad= \sup_{(t_1,t_2)\in[a_1,b_1]\times[a_2,b_2]}\left|p_2(x_2,t_2)p_1(x_1,t_1)\right|\left\|f''_{t_1,t_2}\right\|_1 \\
&\quad= \sup_{t_2\in[a_2,b_2]}\left|p_2(x_2,t_2)\right|\sup_{t_1\in[a_1,b_1]}\left|p_1(x_1,t_1)\right|\left\|f''_{t_1,t_2}\right\|_1 .
\end{aligned}$$

Now, consider

$$\begin{aligned}
\mathcal{G}_1(x_1) &= \sup_{t_1\in[a_1,b_1]}\left|p_1(x_1,t_1)\right| \\
&= \max\{\alpha_1-a_1, x_1-\alpha_1, \beta_1-x_1, b_1-\beta_1\}.
\end{aligned} \tag{3.43}$$

Let

$$\begin{aligned}
\mathfrak{M}_1(x_1) &= \max\{\alpha_1-a_1, x_1-\alpha_1\} \\
&= \frac{x_1-a_1}{2}+\left|\alpha_1-\frac{a_1+x_1}{2}\right| \\
&= \frac{x_1-a_1}{2}\times[1+|2\gamma_1-1|]
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{M}_2(x_1) &= \max\{\beta_1-x_1, b_1-\beta_1\} \\
&= \frac{b_1-x_1}{2}+\left|\beta_1-\frac{b_1+x_1}{2}\right| \\
&= \frac{b_1-x_1}{2}\times[1+|2\gamma_1-1|]
\end{aligned}$$

then

$$\begin{aligned}
\mathcal{G}_1(x_1) &= \max\{\mathfrak{M}_1(x_1), \mathfrak{M}_2(x_1)\} \\
&= \frac{b_1-a_1}{4}[1+|2\gamma_1-1|]+2\left|(x_1-\frac{a_1+b_1}{2})(1+|2\gamma_1-1|)\right|
\end{aligned}$$

and similarly

$$\mathcal{G}_2(x_2) = \frac{b_2-a_2}{4}[1+|2\gamma_2-1|]+2\left|(x_2-\frac{a_2+b_2}{2})(1+|2\gamma_2-1|)\right|.$$

Substituting into (3.43) will produce the result in (3.42) and thus the proof completed. We note here that the discussion in Remark 3.12 continues to be valid here also. □

Corollary 3.22. *With the conditions as in Theorem 3.21, then*

$$|G(x_1,t_1,x_2,t_2)| \le \frac{\left\|f''_{t_1,t_2}\right\|_1}{16}\prod_{i=1}^{2}(b_i-a_i)\left[1+|2\gamma_i-1|\right] \tag{3.44}$$

Proof. Put $x_i = \frac{a_i+b_i}{2}$ in equation (3.42). □

Remark 3.23. *If $\gamma_1=\gamma_2=0$, then (3.44) becomes*

$$\begin{aligned}
&\Bigg|(b_1-a_1)(b_2-a_2)f\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2}\right)\\
&\quad-(b_2-a_2)\int_{a_1}^{b_1}f\left(t_1,\frac{a_2+b_2}{2}\right)dt_1-(b_1-a_1)\int_{a_2}^{b_2}f\left(\frac{a_1+b_1}{2},t_2\right)dt_2\\
&\qquad-\int_{a_2}^{b_2}\int_{a_1}^{b_1}f(t_1,t_2)\,dt_1dt_2\Bigg|\\
&\qquad\qquad\le\frac{\left\|f''_{t_1,t_2}\right\|_1}{4}\left[(b_1-a_1)(b_2-a_2)\right].
\end{aligned}\tag{3.45}$$

Remark 3.24. *If $\gamma_1=\gamma_2=1$, then (3.44) becomes*

$$\begin{aligned}
&\Bigg|\frac{(b_1-a_1)(b_2-a_2)}{4}\left[f(b_1,b_2)+f(a_1,b_2)+f(b_1,a_2)+f(a_1,a_2)\right]\\
&\quad-\frac{1}{2}\Bigg[(b_2-a_2)\int_{a_1}^{b_1}f(t_1,b_2)\,dt_1+(b_2-a_2)\int_{a_1}^{b_1}f(t_1,a_2)\,dt_1\\
&\quad+(b_1-a_1)\int_{a_2}^{b_2}f(b_1,t_2)\,dt_2+(b_1-a_1)\int_{a_2}^{b_2}f(a_1,t_2)\,dt_2\Bigg]\\
&\qquad+\int_{a_2}^{b_2}\int_{a_1}^{b_1}f(t_1,t_2)\,dt_1dt_2\Bigg|\\
&\qquad\qquad\le\frac{\left\|f''_{t_1,t_2}\right\|_1}{4}\left[(b_1-a_1)(b_2-a_2)\right].
\end{aligned}\tag{3.46}$$

Remark 3.25. *If $\gamma_1=\gamma_2=\frac{1}{2}$, then (3.44) becomes*

$$\begin{aligned}
&\Bigg|\frac{(b_1-a_1)(b_2-a_2)}{4}f\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2}\right)\\
&\quad+\frac{(b_1-a_1)(b_2-a_2)}{8}\left[f\left(b_1,\frac{a_2+b_2}{2}\right)+f\left(a_1,\frac{a_2+b_2}{2}\right)\right.\\
&\qquad\left.+f\left(\frac{a_1+b_1}{2},a_2\right)+f\left(\frac{a_1+b_1}{2},b_2\right)\right]\\
&\qquad+\frac{(b_1-a_1)(b_2-a_2)}{16}\left[f(b_1,b_2)+f(a_1,b_2)+f(b_1,a_2)+f(a_1,a_2)\right]
\end{aligned}$$

$$
\begin{aligned}
&-\frac{(b_1-a_1)}{4}\left[2\int_{a_2}^{b_2} f\left(\frac{a_1+b_1}{2},t_2\right)dt_2+\int_{a_2}^{b_2} f(b_1,t_2)\,dt_2+\int_{a_2}^{b_2} f(a_1,t_2)\,dt_2\right]\\
&\quad-\frac{(b_2-a_2)}{4}\left[2\int_{a_1}^{b_1} f\left(t_1,\frac{a_2+b_2}{2}\right)dt_1+\int_{a_1}^{b_1} f(t_1,b_2)\,dt_1+\int_{a_1}^{b_1} f(t_1,a_2)\,dt_1\right]\\
&\qquad+\left.\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(t_1,t_2)\,dt_1dt_2\right|\\
&\qquad\qquad\le\frac{\left\|f''_{t_1,t_2}\right\|_1}{16}\left[(b_1-a_1)(b_2-a_2)\right] \quad (3.47)
\end{aligned}
$$

Remark 3.26. *Let $f(t_1,t_2)=g(t_1)g(t_2)$ where $g:[a,b]\to\mathbb{R}$. If g is differentiable and satisfies the condition that $g'\in L_1[a,b]$ then, for $x_1=x_2=x$ and $\gamma_1=\gamma_2=\gamma$ we get*

$$
\begin{aligned}
&\left|(b-a)^2 g(x)g(x)-g(x)(b-a)\int_a^b g(t)\,dt\right.\\
&\qquad\left.-g(x)(b-a)\int_a^b g(t)\,dt+\int_a^b\int_a^b g(t)g(t)\,dt\,dt\right|\\
&\qquad\qquad\le\left(\|g'\|_1\right)^2\left[\frac{(b-a)}{2}+4\left|x-\frac{a+b}{2}\right|\right]^2. \quad (3.48)
\end{aligned}
$$

Therefore,

$$\left|\int_a^b g(t)\,dt-(b-a)g(x)\right|^2\le\left(\|g'\|_1\right)^2\left[\frac{(b-a)}{2}+4\left|x-\frac{a+b}{2}\right|\right]^2.$$

This gives

$$\left|\int_a^b g(t)\,dt-(b-a)g(x)\right|\le\|g'\|_1\left[\frac{(b-a)}{2}+4\left|x-\frac{a+b}{2}\right|\right]$$

which is Ostrowski type inequality for the $\|\cdot\|_1$ −norm. Thus, (3.42) is a generalization for two dimensional integrals of the Ostrowski type inequality for $\|\cdot\|_1$ −norms (see Dragomir and Wang (1997) and Dragomir (2001)).

3.7 Numerical Results

In this section the inequalities developed in Section 3.1 are used to approximate the double integral. In the following example we selected the integrand for which integrating in each direction is straightforward, but not so for the double integral.

Example 3.1.

$$\int_0^1\int_0^1 1-e^{-xy}dxdy=0.203400400702947. \quad (3.49)$$

γ_1	γ_2	actual error	L_∞-estimated error	L_2-estimated error	L_1-estimated error
0	0	1.5(-3)	6.3(-2)	5.7(-2)	1.6(-1)
$\frac{1}{3}$	$\frac{1}{3}$	5.4(-7)	1.9(-2)	1.9(-2)	7.1(-2)
0.5	0.5	4.3(-4)	1.6(-2)	1.4(-2)	3.9(-2)
1	1	6.5(-3)	6.3(-2)	5.7(-2)	1.6(-1)

Table 3.1: The actual and estimated errors in computing (3.49) using (3.2), (3.26) and (3.42) with $x_1 = x_2 = 0.5$ and various values of γ_1, γ_2 in the $\|.\|_\infty$ norm, $\|.\|_p$ norm and $\|.\|_1$ norm respectively

.

Namely, $\int_0^1 1 - e^{-xy}\,dx = \frac{y+e^{-y}-1}{y}$ *and* $\int_0^1 1 - e^{-xy}\,dy = \frac{x+e^{-x}-1}{x}$.

Example 3.1 was chosen also because it is infinitely smooth and its ∞-norm becomes smaller with each successive derivative, because

$$
\begin{aligned}
f_x &= ye^{-xy} & f_y &= xe^{-xy} \\
f_{xx} &= -y^2e^{-xy} & f_{yy} &= -x^2e^{-xy} \\
&\vdots & &\vdots \\
\tfrac{\partial^n f}{\partial x^n} &= (-1)^{n+1}y^n e^{-xy} & \tfrac{\partial^n f}{\partial y^n} &= (-1)^{n+1}x^n e^{-xy}
\end{aligned}
$$

as we see, $\forall y \in [0,1)$ the derivative with respect to $x \rightarrow 0$ as $n \rightarrow \infty$, and also, $\forall x \in [0,1)$ the derivative with respect to $y \rightarrow 0$ as $n \rightarrow \infty$. This indicates that the higher order error bound (accompanied by a higher order rule) will give better results.

Example 3.2.

$$\int_0^1 \int_1^2 \frac{y}{x^2} e^{-y/x}\,dx dy = 0.1548181217. \tag{3.50}$$

The integrand in Example 3.2 was chosen because its ∞-norm blows up rapidly with successive derivatives. That is $\forall y \in [0,1)$ the derivative with respect to $x \rightarrow \infty$ as $n \rightarrow \infty$, and also, $\forall x \in [1,2)$ the derivative with respect to $y \rightarrow \infty$ as $n \rightarrow \infty$. This indicates that the higher order error bound (accompanied by a lower order rule) will give better results.

In Table 3.1, results are shown for the approximation of (3.49) using the rule and bound of (3.2), (3.26) and (3.43) respectively. In Table 3.2, results are shown for the approximation

γ_1	γ_2	actual error	L_∞-estimated error	L_2-estimated error	L_1-estimated error
0	0	2.5(-3)	2.2(-2)	3(-2)	3.3(-1)
$\frac{1}{3}$	$\frac{1}{3}$	1.5(-5)	7.1(-3)	1(-2)	1.5(-2)
0.5	0.5	8.6(-4)	5.7(-3)	7.6(-3)	8.3(-3)
1	1	1.9(-2)	2.2(-2)	3(-2)	3.3(-1)

Table 3.2: The actual and estimated errors in computing (3.50) using (3.2), (3.26) and (3.42) with $x_1 = x_2 = 0.5$ and various values of γ_1, γ_2 in the $\|.\|_\infty$ norm, $\|.\|_p$ norm and $\|.\|_1$ norm.

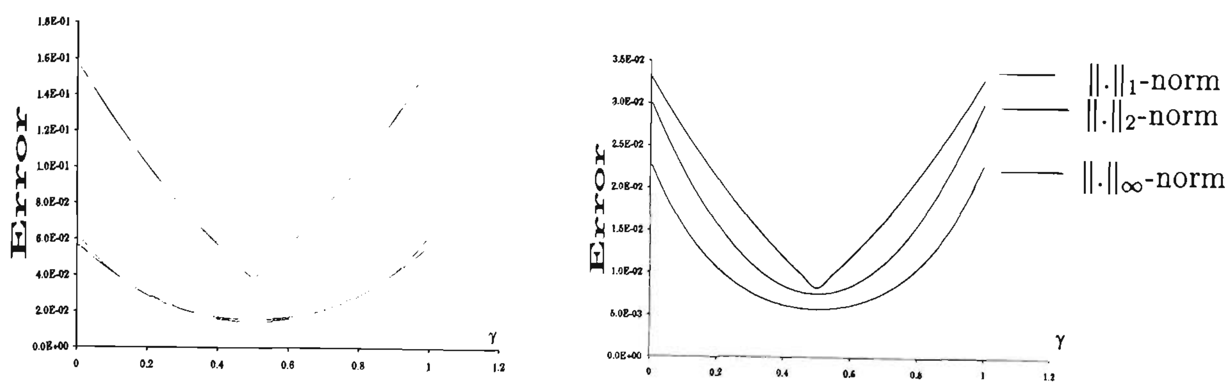
to (3.50) using the rule and bound of (3.2), (3.26) and (3.43). From this point of view we find that the actual error is much smaller than the theoretical one and is smallest when Simpson's rule is applied ($\gamma_1 = \gamma_2 = \frac{1}{3}$). The optimal theoretical bound is attained when $\gamma_1 = \gamma_2 = \frac{1}{2}$. It should be noted that $\gamma_1 = \gamma_2 = 0$ approximates (3.49) and (3.50) with the "mid-point" rule and employs one function evaluation (at the mid-point of the region) and two one-dimensional integrals (along the bi-sectors). The "trapezoidal" rule uses four sample points (the boundary corners) and four one-dimensional integrals (along the boundary). All other values, that is $\gamma_1, \gamma_2 \in (0, 1)$, produce a rule that is a linear combination of the above and results in the use of nine sample points and six one-dimensional integrals.

Furthermore Simpson's rule ($\gamma_1 = \gamma_2 = \frac{1}{3}$, nine sample points) is more accurate than the mid-point rule ($\gamma_1 = \gamma_2 = 0$, one sample point) which in turn is more accurate than the trapezoidal rule ($\gamma_1 = \gamma_2 = 1$, four sample points). We note that the estimated errors are symmetric about $\gamma_1 = \gamma_2 = \frac{1}{2}$ as in the Tables 3.1 and 3.2. Cleary we observe these from Figure 3.1 that the bound is convex in $\gamma_i \in [0, 1]$ for $i = 1, 2$. The sharpest occurs at $\gamma_i = \frac{1}{2}$ for $i = 1, 2$. The harshest bound is achieved when γ_i are taken at either of their boundary points. Next we will employ the composite rules to explore the numerical results for both Example 3.1 and Example 3.2 respectively and produce briefly the actual and estimated errors in applying the mid-point cubature rules to evaluate the double integral (3.49) and (3.50) for an increasing number of intervals for the different norms.

Cleary, we notice that the actual error ratio in both tables suggests that the composite rule in each case has convergence

$$|R| \sim O\left(\frac{1}{m^2 n^2}\right).$$

Also, from Table 3.3 and Table 3.4 we gather that the estimated error predicts a convergence rate of



(a) The estimated error as a function of γ in evaluating (3.49) with $x_1 = x_2 = 0.5$ and various values of γ_1, γ_2 in the $||.||_\infty$ norm (equation (3.2)), $||.||_p$ norm (equation (3.26)) and $||.||_1$ norm (equation (3.43)).

(b) The estimated error as a function of γ in evaluating (3.50) with $x_1 = x_2 = 0.5$ and various values of γ_1, γ_2 in the $||.||_\infty$ norm (equation (3.2)), $||.||_2$ norm (equation (3.26)) and $||.||_1$ norm (equation (3.43)).

Figure 3.1: Diagrammatic representation for the estimated error

- $|R| \leq \frac{\left\|f''_{t_1,t_2}\right\|_\infty}{16mn}$, $\left\|f''_{t_1,t_2}\right\|_\infty = 1$ (Example 3.1) and $\left\|f''_{t_1,t_2}\right\|_\infty = .37$ (Example 3.2) ,
- $|R| \leq \frac{\left\|f''_{t_1,t_2}\right\|_2}{12\sqrt{mn}}$, $\left\|f''_{t_1,t_2}\right\|_2 = .69$ (Example 3.1) and $\left\|f''_{t_1,t_2}\right\|_2 = .36$ (Example 3.2) ,
- $|R| \leq \frac{\left\|f''_{t_1,t_2}\right\|_1}{4mn}$, $\left\|f''_{t_1,t_2}\right\|_1 = .63$ (Example 3.1) and $\left\|f''_{t_1,t_2}\right\|_1 = .13$ (Example 3.2).

In the next chapter generalizations of double integral inequalities for n-times differentiable mappings are obtained. From a general Peano kernel *explicit* bounds for interior point rules are procured, which are used to obtain inequalities for n-times differential mappings for the three norms $||.||_\infty$, $||.||_p$ and $||.||_1$.

n	m	Actual Error	Err ratio	L_∞-estimated error	L_2-estimated error	L_1-estimated error
1	1	1.5(-3)	...	6.3(-2)	5.7(-2)	1.5(-1)
2	2	1.0(-4)	14.51	1.6(-2)	2.9(-2)	4.0(-2)
4	4	6.7(-6)	15.61	3.9(-3)	1.4(-2)	9.9(-3)
8	8	4.2(-7)	15.90	1.0(-3)	7.2(-3)	2.5(-3)
16	16	2.6(-8)	15.98	2.0(-4)	3.6(-3)	6.2(-4)
32	32	1.6(-9)	15.99	6.1(-5)	1.8(-3)	1.5(-4)
64	64	1.0 (-10)	16.00	1.5(-5)	8.9(-4)	3.9(-5)
128	128	6.6 (-12)	16.00	3.8(-6)	4.5(-4)	9.6(-6)

Table 3.3: The actual and estimated errors in evaluating (3.49) using a composite rule, for various values of n, m. Sampling occurs at the mid-point of each region.

n	m	Actual Error	Err ratio	L_∞-estimated error	L_2-estimated error	L_1-estimated error
1	1	2.5(-3)	...	2.2(-2)	1.2(-2)	3.3(-2)
2	2	2.1(-4)	12.32	5.7(-3)	6.1(-2)	8.2(-3)
4	4	1.4(-5)	14.68	1.4(-3)	3.0(-2)	2.1(-3)
8	8	8.9(-7)	15.62	3.5(-4)	1.5(-2)	5.1(-4)
16	16	5.6(-8)	15.90	8.9(-5)	7.6(-3)	1.3(-4)
32	32	3.5(-9)	15.97	2.2(-5)	3.8(-3)	3.2(-5)
64	64	2.2 (-10)	16.00	5.6(-6)	1.9(-3)	8.1(-6)
128	128	1.3 (-11)	16.00	1.4(-6)	9.5(-4)	2.0(-6)

Table 3.4: The actual and estimated errors in evaluating (3.50) using a composite rule, for various values of n, m. Sampling occurs at the mid-point of each region.

Chapter 4

A General Ostrowski Type Inequality for Double Integrals

4.1 Introduction

In the previous chapter, we developed two-dimensional integral inequalities with bounds in terms of the behaviour of the first partial derivatives of the function.
Here, some generalizations of an Ostrowski type inequality in two dimensions for $n-$time differentiable mappings are given. The result is an integral inequality with bounded n^{th} derivatives. This is employed to approximate double integrals using one-dimensional integrals and function evaluations at the boundary and interior points.

4.2 Integral Identities

In Cerone *et al.* (1999a), the authors proved the following Ostrowski type inequality for $n-$time differentiable mappings. For convenience, Theorem 2.10 is reproduced here.

THEOREM 4.1. *Let* $f : [a, b] \to \mathbb{R}$ *be a mapping such that* $f^{(n-1)}$ *is absolutely continuous* *on* $[a,b]$ *and* $f^{(n)} \in L_\infty[a,b]$. *Then for all* $x \in [a,b]$, *we have the inequality:*

$$\begin{aligned}\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \\ \le \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left[(x-a)^{n+1}+(b-x)^{n+1}\right] \\ \le \frac{\left\|f^{(n)}\right\|_\infty (b-a)^{n+1}}{(n+1)!}.\end{aligned} \tag{4.1}$$

For other similar results for $n-$time differentiable mappings, see Chapter 2 and the papers Fink (1992) and Anastassiou (1995) .

In Barnett and Dragomir (2001) and Dragomir *et al.* (1998) the authors proved some inequalities of Ostrowski type for double integrals in terms of different norms.

In this section we combine the above results and develop them in two dimensions to obtain a generalization of the Ostrowski inequality for n-time differentiable mappings using different types of norms. The results presented here approximate a two-dimensional integral for $n-$time differentiable mappings via the application of one-dimensional integrals at the boundary, function evaluations at interior or boundary points and/or its derivatives at a multiple number of points over the given region.

The following result holds.

THEOREM 4.2. *Let* $f:[a,b]\times[c,d]\to\mathbb{R}$ *be a continuous mapping such that the following partial derivatives* $\frac{\partial^{l+k}f(\cdot,\cdot)}{\partial t^k \partial s^l}$, $k=0,1,...,n-1$, $l=0,1,...,m-1$ *exist and are continuous on* $[a,b]\times[c,d]$. *Further, consider* $K_n:[a,b]^2\to\mathbb{R}$, $S_m:[c,d]^2\to\mathbb{R}$ *given by*

$$K_n(x,t) := \begin{cases} \frac{(t-a)^n}{n!}, & t\in[a,x], \\ \frac{(t-b)^n}{n!}, & t\in(x,b], \end{cases} \qquad S_m(y,s) := \begin{cases} \frac{(s-c)^m}{m!}, & s\in[c,y], \\ \frac{(s-d)^m}{m!}, & s\in(y,d], \end{cases} \tag{4.2}$$

then for all $(x,y) \in [a,b] \times [c,d]$*, we have the identity:*

$$\begin{aligned}\int_a^b \int_c^d f(t,s)\,ds\,dt &= \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} X_k(x)Y_l(y)\frac{\partial^{l+k} f(x,y)}{\partial t^k \partial s^l} \\ &+ (-1)^m \sum_{k=0}^{n-1} X_k(x)\int_c^d S_m(y,s)\frac{\partial^{k+m} f(x,s)}{\partial t^k \partial s^m}ds \\ &+ (-1)^n \sum_{l=0}^{m-1} Y_l(y)\int_a^b K_n(x,t)\frac{\partial^{n+l} f(t,y)}{\partial t^n \partial s^l}dt \\ &+ (-1)^{m+n}\int_a^b\int_c^d K_n(x,t)S_m(y,s)\frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m}ds\,dt\,. \end{aligned} \tag{4.3}$$

where

$$X_k(x) = \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}, \qquad Y_l(y) = \frac{(d-y)^{l+1} + (-1)^l (y-c)^{l+1}}{(l+1)!}. \tag{4.4}$$

Proof. Applying the identity (see Cerone *et al.* (1999a))

$$\int_a^b g(t)\,dt = \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] g^{(k)}(x) + (-1)^n \int_a^b P_n(x,t)\, g^{(n)}(t)\,dt, \tag{4.5}$$

where

$$P_n(x,t) = \begin{cases} \frac{(t-a)^n}{n!} & \text{if} \quad t \in [a,x], \\ \frac{(t-b)^n}{n!} & \text{if} \quad t \in (x,b], \end{cases}$$

which has been used essentially in the proof of Theorem 4.1, for the partial mapping $f(\cdot,s)$, $s \in [c,d]$, we can write

$$\int_a^b f(t,s)\,dt = \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right]\frac{\partial^k f(x,s)}{\partial t^k} + (-1)^n \int_a^b K_n(x,t)\,\frac{\partial^n f(t,s)}{\partial t^n}dt, \tag{4.6}$$

for every $x \in [a,b]$ and $s \in [c,d]$. Integrating (4.6) over s on $[c,d]$, we deduce

$$\int_a^b\int_c^d f(t,s)\,ds\,dt = \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right]\int_c^d \frac{\partial^k f}{\partial t^k}(x,s)\,ds + (-1)^n\int_a^b K_n(x,t)\left(\int_c^d \frac{\partial^n f}{\partial t^n}(t,s)\,ds\right)dt \tag{4.7}$$

for all $x \in [a, b]$.

Applying the identity (4.5) again for the partial mapping $\frac{\partial^k f}{\partial t^k}(x, \cdot)$ on $[c, d]$, we obtain

$$\begin{aligned}\int_c^d \frac{\partial^k f}{\partial t^k}(x, s)\, ds &= \sum_{l=0}^{m-1} \left[\frac{(d-y)^{l+1} + (-1)^l (y-c)^{l+1}}{(l+1)!}\right] \frac{\partial^l}{\partial s^l}\left(\frac{\partial^k f}{\partial t^k}\right)(x, y) \\ &\quad + (-1)^m \int_c^d S_m(y, s) \frac{\partial^m}{\partial s^m}\left(\frac{\partial^k f}{\partial t^k}\right)(x, s)\, ds \\ &= \sum_{l=0}^{m-1} \left[\frac{(d-y)^{l+1} + (-1)^l (y-c)^{l+1}}{(l+1)!}\right] \frac{\partial^{l+k}}{\partial t^k \partial s^l} f(x, y) \\ &\quad + (-1)^m \int_c^d S_m(y, s) \frac{\partial^{k+m}}{\partial t^k \partial s^m} f(x, s)\, ds. \quad (4.8)\end{aligned}$$

In addition, the identity (4.5) applied for the partial derivative $\frac{\partial^n f}{\partial t^n}(t, \cdot)$ also gives

$$\begin{aligned}\int_c^d \frac{\partial^n f}{\partial t^n}(t, s)\, ds &= \sum_{l=0}^{m-1} \left[\frac{(d-y)^{l+1} + (-1)^l (y-c)^{l+1}}{(l+1)!}\right] \frac{\partial^{n+l}}{\partial t^n \partial s^l} f(t, y) \\ &\quad + (-1)^m \int_c^d S_m(y, s) \frac{\partial^{n+m}}{\partial t^n \partial s^m} f(t, s)\, ds. \quad (4.9)\end{aligned}$$

Using (4.8) and (4.9) and substituting into (4.7) will produce the result (4.3), and thus the theorem is proved. □

Utilizing the result (4.3) we will produce the mid-point cubature rule for two-dimensional rectangular regions for n-times differentiable mappings as well as the trapezoidal cubature rule as shown in the following two corollaries respectively.

Corollary 4.3. *With the assumptions as in Theorem 4.2, we have the representation*

$$\begin{aligned}\int_a^b \int_c^d f(t, s)\, ds\, dt &= \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} X_k\left(\frac{a+b}{2}\right) Y_l\left(\frac{c+d}{2}\right) \frac{\partial^{l+k} f}{\partial t^k \partial s^l}\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ &\quad + (-1)^m \sum_{k=0}^{n-1} X_k\left(\frac{a+b}{2}\right) \int_c^d \tilde{S}_m(s) \frac{\partial^{k+m} f}{\partial t^k \partial s^m}\left(\frac{a+b}{2}, s\right) ds \\ &\quad + (-1)^n \sum_{l=0}^{m-1} Y_l\left(\frac{c+d}{2}\right) \int_a^b \tilde{K}_n(t) \frac{\partial^{n+l} f}{\partial t^n \partial s^l}\left(t, \frac{c+d}{2}\right) dt \\ &\quad + (-1)^{m+n} \int_a^b \int_c^d \tilde{K}_n(t) \tilde{S}_m(s) \frac{\partial^{n+m} f}{\partial t^n \partial s^m}(t, s)\, ds\, dt, \quad (4.10)\end{aligned}$$

where $X_k(\cdot)$ and $Y_l(\cdot)$ are as given in (4.4) and so

$$X_k\left(\frac{a+b}{2}\right) = \left[\frac{1+(-1)^k}{(k+1)!}\right] \frac{(b-a)^{k+1}}{2^{k+1}}, \quad Y_l\left(\frac{c+d}{2}\right) = \left[\frac{1+(-1)^l}{(l+1)!}\right] \frac{(d-c)^{l+1}}{2^{l+1}},$$

and $\tilde{K}_n : [a,b] \to \mathbb{R}$, $\tilde{S}_m : [c,d] \to \mathbb{R}$ *are given by*

$$\tilde{K}_n(t) = K_n\left(\frac{a+b}{2}, t\right) \quad \text{and} \quad \tilde{S}_m(s) = S_m\left(\frac{c+d}{2}, s\right) \tag{4.11}$$

on using (4.2).

Corollary 4.4. *Let f be as in Theorem 4.2. Then we have the following identity*

$$\begin{aligned}
\int_a^b \int_c^d f(t,s)\,ds\,dt &= \frac{1}{4}\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\left[\frac{(b-a)^{k+1}}{(k+1)!}\right]\times\left[\frac{(d-c)^{l+1}}{(l+1)!}\right] \\
&\times\left[\frac{\partial^{l+k} f}{\partial t^k \partial s^l}(a,c) + (-1)^l \frac{\partial^{l+k} f}{\partial t^k \partial s^l}(a,d) + (-1)^k \frac{\partial^{l+k} f}{\partial t^k \partial s^l}(b,c) + (-1)^{l+k}\frac{\partial^{l+k} f}{\partial t^k \partial s^l}(b,d)\right] \\
&+ \frac{1}{4}(-1)^m \sum_{k=0}^{n-1}\left[\frac{(b-a)^{k+1}}{(k+1)!}\right]\times\left[\int_c^d Y_{m-1}(s)\left[\frac{\partial^{k+m} f}{\partial x^k \partial s^m}(a,s) + (-1)^k \frac{\partial^{k+m} f}{\partial x^k \partial s^m}(b,s)\right]ds\right] \\
&+ \frac{1}{4}(-1)^n \sum_{l=0}^{m-1}\left[\frac{(d-c)^{l+1}}{(l+1)!}\right]\times\left[\int_a^b X_{n-1}(t)\left[\frac{\partial^{n+l} f}{\partial t^n \partial s^l}(t,c) + (-1)^k \frac{\partial^{n+l} f}{\partial t^n \partial s^l}(t,d)\right]dt\right] \\
&+ \frac{1}{4}\int_a^b \int_c^d X_{n-1}(t)\cdot Y_{m-1}(s)\frac{\partial^{n+m} f}{\partial t^n \partial s^m}(t,s)\,ds\,dt.
\end{aligned} \tag{4.12}$$

where $X_{n-1}(t)$ and $Y_{m-1}(s)$ are as given by (4.4).

Proof. By substituting $(x,y) = (a,c), (a,d), (b,c), (b,d)$ respectively and summing the resulting identities and after some simplification, we get the desired identity (4.12). □

In the above section we promote some of two-dimensional integral identities, for n-times differentiable mappings, which are avail in themselves and exploit them in the next section to obtain two-dimensional integral inequalities on the Lebesgue spaces, $L_\infty[[a_1,b_1]\times[a_2,b_2]]$, $L_p[[a_1,b_1]\times[a_2,b_2]]$, $L_1[[a_1,b_1]\times[a_2,b_2]]$,

4.3 Some Integral Inequalities

In this section we tap the equalities of Section 4.2 and develop inequalities for the depiction of the two-dimensional integral of a function with respect to one-dimensional integrals at the boundary, function evaluations at interior or boundary points and/or its derivatives at a multiple number of points over the given region.

We start with the following result.

THEOREM 4.5. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be continuous on $[a,b] \times [c,d]$, and assume that $\frac{\partial^{n+m} f}{\partial t^n \partial s^m}$ exist on $(a,b) \times (c,d)$. Then we have the inequality*

$$\begin{aligned}
&\left| \int_a^b \int_c^d f(t,s)\,ds\,dt - \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} X_k(x)\cdot Y_l(y) \frac{\partial^{l+k} f}{\partial t^k \partial s^l}(x,y) \right. \\
&\left. -(-1)^m \sum_{k=0}^{n-1} X_k(x) \int_c^d S(y,s) \frac{\partial^{k+m} f}{\partial t^k \partial s^m}(x,s)\,ds - (-1)^n \sum_{l=0}^{m-1} Y_l(y) \int_a^b K(x,t) \frac{\partial^{n+l} f}{\partial t^n \partial s^l}(t,y)\,dt \right| \\
&\leq \begin{cases}
\frac{1}{(n+1)!(m+1)!}\left[(x-a)^{n+1}+(b-x)^{n+1}\right] \times \left[(y-c)^{m+1}+(d-y)^{m+1}\right] \times \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_\infty, \\
\qquad \text{if } \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_\infty([a,b]\times[c,d]); \\
\frac{1}{n!m!}\left[\frac{(x-a)^{nq+1}+(b-x)^{nq+1}}{nq+1}\right]^{\frac{1}{q}} \times \left[\frac{(y-c)^{mq+1}+(d-y)^{mq+1}}{mq+1}\right]^{\frac{1}{q}} \times \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_p, \\
\qquad \text{if } \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_p([a,b]\times[c,d]),\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\frac{1}{4n!m!}\left[(x-a)^n+(b-x)^n+\left|(x-a)^n-(b-x)^n\right|\right] \\
\qquad \times \left[(y-c)^m+(d-y)^m+\left|(y-c)^m-(d-y)^m\right|\right] \times \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_1, \\
\qquad \text{if } \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_1([a,b]\times[c,d]).
\end{cases}
\end{aligned} \tag{4.13}$$

for all $(x,y) \in [a,b] \times [c,d]$, where

$$\begin{aligned}
\left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_\infty &= \sup_{(t,s)\in[a,b]\times[c,d]} \left| \frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m} \right| < \infty, \\
\left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_p &= \left(\int_c^d \int_a^b \left| \frac{\partial^{n+m}}{\partial t^n \partial s^m} f(t,s) \right|^p dtds \right)^{\frac{1}{p}} < \infty.
\end{aligned}$$

Proof. Using Theorem 4.2, we get from (4.3)

$$\begin{aligned}
&\left| \int_a^b \int_c^d f(t,s)\,ds\,dt - \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} X_k(x)\cdot Y_l(y) \frac{\partial^{l+k} f(x,y)}{\partial t^k \partial s^l} \right. \\
&\qquad -(-1)^m \sum_{k=0}^{n-1} X_k(x) \int_c^d S(y,s) \frac{\partial^{k+m} f(x,s)}{\partial t^k \partial s^m} ds \\
&\qquad \left. -(-1)^n \sum_{l=0}^{m-1} Y_l(y) \int_a^b K(x,t) \frac{\partial^{n+l} f(t,y)}{\partial t^n \partial s^l} dt \right| \\
&= \left| \int_a^b \int_c^d K_n(x,t)\, S_m(y,s)\, \frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^n} ds\,dt \right| \\
&\leq \int_a^b \int_c^d \left| K_n(x,t)\, S_m(y,s) \right| \left| \frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^n} \right| ds\,dt.
\end{aligned} \tag{4.14}$$

Using Hölder's inequality and properties of the modulus and integral, then we have that

$$\int_a^b \int_c^d |K_n(x,t) S_m(y,s)| \left| \frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m} \right| ds\, dt$$
$$\leq \begin{cases} \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_\infty \int_a^b \int_c^d |K_n(x,t) S_m(y,s)|\, dt\, ds \\ \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_p \left(\int_a^b \int_c^d |K_n(x,t) S_m(y,s)|^q\, dt\, ds \right)^{\frac{1}{q}}, \\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_1 \sup_{(t,s) \in [a,b] \times [c,d]} |K_n(x,t) S_m(y,s)|. \end{cases} \tag{4.15}$$

Now, the result in (4.15) can be further simplified by application of (4.2).

$$\begin{aligned} \int_a^b \int_c^d |K_n(x,t) S_m(y,s)|\, dt\, ds &= \int_a^b |K_n(x,t)|\, dt \int_c^d |S_m(y,s)|\, ds \\ &= \left[\int_a^x \frac{(t-a)^n}{n!} dt + \int_x^b \frac{(b-t)^n}{n!} dt \right] \\ &\quad \times \left[\int_c^y \frac{(s-c)^m}{m!} ds + \int_y^d \frac{(d-s)^m}{m!} ds \right] \\ &= \frac{\left[(x-a)^{n+1} + (b-x)^{n+1} \right] \left[(y-c)^{m+1} + (d-y)^{m+1} \right]}{(n+1)!\,(m+1)!} \end{aligned}$$

giving the first inequality in (4.13). Further,

$$\begin{aligned} &\left(\int_a^b \int_c^d |K_n(x,t)\ S_m(y,s)|^q\, ds\, dt \right)^{\frac{1}{q}} \\ &= \left(\int_a^b |K_n(x,t)|^q\, dt \right)^{\frac{1}{q}} \left(\int_c^d |S_m(y,s)|^q\, ds\, dt \right)^{\frac{1}{q}} \\ &= \frac{1}{n!m!} \left[\int_a^x (t-a)^{nq}\, dt + \int_x^b (b-t)^{nq}\, dt \right]^{\frac{1}{q}} \\ &\quad \times \left[\int_c^y (s-c)^{mq}\, ds + \int_y^d (d-s)^{mq}\, ds \right]^{\frac{1}{q}} \\ &= \frac{1}{n!m!} \left[\frac{(x-a)^{nq+1} + (b-x)^{nq+1}}{nq+1} \right]^{\frac{1}{q}} \times \left[\frac{(y-c)^{mq+1} + (d-y)^{mq+1}}{mq+1} \right]^{\frac{1}{q}} \end{aligned}$$

producing the second inequality in (4.13). Finally,

$$\begin{aligned}
\sup_{(t,s)\in[a,b]\times[c,d]} |K_n(x,t)\, S_m(y,s)| &= \sup_{t\in[a,b]} |K_n(x,t)| \sup_{s\in[c,d]} |S_m(y,s)| \\
&= \max\left\{\frac{(x-a)^n}{n!}, \frac{(b-x)^n}{n!}\right\} \times \max\left\{\frac{(y-c)^m}{m!}, \frac{(d-y)^m}{m!}\right\} \\
&= \frac{1}{n!m!}\left[\frac{(x-a)^n+(b-x)^n}{2} + \left|\frac{(x-a)^n-(b-x)^n}{2}\right|\right] \\
&\quad \times \left[\frac{(y-c)^m+(d-y)^m}{2} + \left|\frac{(y-c)^m+(d-y)^m}{2}\right|\right].
\end{aligned}$$

gives the final inequality in (4.13) where we have used the fact that

$$\max\{X,Y\} = \frac{X+Y}{2} + \left|\frac{Y-X}{2}\right|.$$

Thus the theorem is now completely proved. □

Taking in mind that x and y are free parameters. Thus we can produce "mid-point" and "boundary-point" type results by choosing appropriate values for x and y. In addition choosing values for n and m will re-capture the earlier results of Hanna *et al.* (2000) and Dragomir *et al.* (1998).

Corollary 4.6. *With the assumptions of Theorem 4.5, we have the inequality*

$$\begin{aligned}
&\left| \int_a^b \int_c^d f(t,s)\, ds\, dt - \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} X_k\left(\frac{a+b}{2}\right) Y_l\left(\frac{c+d}{2}\right) \frac{\partial^{l+k}}{\partial t^k \partial s^l} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right. \\
&\quad - (-1)^m \sum_{k=0}^{n-1} X_k\left(\frac{a+b}{2}\right) \int_c^d \tilde{S}_m(s) \frac{\partial^{k+m}}{\partial t^k \partial s^m} f\left(\frac{a+b}{2}, s\right) ds \\
&\quad \left. - (-1)^n \sum_{l=0}^{m-1} Y_l\left(\frac{c+d}{2}\right) \int_a^b \tilde{K}_n(t) \frac{\partial^{n+l}}{\partial t^n \partial s^l} f\left(t, \frac{c+d}{2}\right) dt \right| \\
&\leq \begin{cases}
\frac{1}{2^{n+m}(n+1)!(m+1)!} (b-a)^{n+1} (d-c)^{m+1} \times \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_\infty ; \\[2ex]
\frac{1}{2^{n+m} n! m!} \left[\frac{(b-a)^{nq+1}(d-c)^{mq+1}}{(nq+1)(mq+1)}\right]^{\frac{1}{q}} \times \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_p ; \\[2ex]
\frac{1}{2^{n+m} n! m!} (b-a)^n (d-c)^m \times \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_1 .
\end{cases} \qquad (4.16)
\end{aligned}$$

Proof. Taking $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in (4.13) readily produces the result as stated. □

These are the tightest possible for their respective *Lebesgue* norms, because of the symmetric and convex nature of the bounds in (4.13). This is straightforward because these functions attain their maximums at the ends of the intervals, and their minimum values at the mid-point.

Other values of x and y may produce the boundary-point cubature rules.

Remark 4.7. *For $n = m = 1$ in (4.16) and $\frac{\partial^2 f}{\partial t \partial s}$ belonging to the appropriate Lebesgue spaces on $[a,b] \times [c,d]$, we have*

$$\left| \int_a^b \int_c^d f(t,s)\, ds\, dt - (b-a)(d-c) f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + (b-a) \int_c^d \tilde{S}_1(s) \frac{\partial}{\partial s} f\left(\frac{a+b}{2}, s\right) ds + (d-c) \int_a^b \tilde{K}_1(t) \frac{\partial}{\partial t} f\left(t, \frac{c+d}{2}\right) dt \right|$$
$$\leq \begin{cases} \frac{1}{16}(b-a)^2 (d-c)^2 \times \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty ; \\ \\ \frac{1}{4}\left[\frac{(b-a)^{q+1}(d-c)^{q+1}}{(q+1)^2}\right]^{\frac{1}{q}} \times \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_p ; \\ \\ \frac{1}{4}(b-a)(d-c) \times \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_1 , \end{cases} \tag{4.17}$$

and thus some of the results of Hanna et al. *(2000), Pachpatte (2001) and Dragomir* et al. *(1998) are re-captured.*

Corollary 4.8. *With the assumptions on f as outlined in Theorem 4.5, we can obtain another result which is a generalization of the trapezoid inequality*

$$\left| \int_a^b \int_c^d f(t,s)\, ds\, dt - \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \frac{(b-a)^{k+1}}{(k+1)!} \cdot \frac{(d-c)^{l+1}}{(l+1)!} \times \left[\frac{f(a,c) + (-1)^l f(a,d) + (-1)^k f(b,c) + (-1)^{k+l} f(b,d)}{4}\right] - (-1)^m \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} \int_c^d Y_l(s) \frac{\partial^{k+m}}{\partial t^k \partial s^m} \left[\frac{f(a,s) + (-1)^k f(b,s)}{4}\right] ds - (-1)^n \sum_{l=0}^{m-1} \left[\frac{(d-c)^{l+1}}{(l+1)!}\right] \int_a^b X_k(t) \frac{\partial^{l+n}}{\partial t^n \partial s^l} \left[\frac{f(t,c) + (-1)^l f(t,d)}{4}\right] dt \right|$$

$$\leq \begin{cases} \kappa_{n,m} \frac{(b-a)^{n+1}(d-c)^{m+1}}{(n+1)!(m+1)!} \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_\infty ; \\ \qquad if \ \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_\infty ([a,b] \times [c,d]) ; \\ \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_p \left(\int_a^b |T_n(a,b;t)|^q \, dt \right)^{\frac{1}{q}} \left(\int_c^d |T_m(c,d;s)|^q \, ds \right)^{\frac{1}{q}} \\ \qquad if \ \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_p ([a,b] \times [c,d]) , \ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(b-a)^n (d-c)^m}{4n!m!} \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_1 , \\ \qquad if \ \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_1 ([a,b] \times [c,d]). \end{cases} \tag{4.18}$$

where
$$\kappa_{n,m} := \begin{cases} 1 & \text{if } n = 2r_1 \text{ and } m = 2r_2 , \\ \frac{2^n - 1}{2^n} & \text{if } n = 2r_1 + 1 \text{ and } m = 2r_2 , \\ \frac{2^m - 1}{2^m} & \text{if } n = 2r_1 \text{ and } m = 2r_2 + 1 , \\ \frac{(2^n-1)}{2^n} \cdot \frac{(2^m-1)}{2^m} & \text{if } n = 2r_1 + 1 \text{ and } m = 2r_2 + 1 \end{cases}$$

Proof. Using the identity (4.12), we find that

$$\left| \int_a^b \int_c^d f(t,s) \, ds \, dt - \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \frac{(b-a)^{k+1}}{(k+1)!} \cdot \frac{(d-c)^{l+1}}{(l+1)!} \right.$$

$$\times \frac{1}{4} \left[\frac{\partial^{l+k} f}{\partial t^k \partial s^l} (a,c) + (-1)^l \frac{\partial^{l+k} f}{\partial t^k \partial s^l} (a,d) + (-1)^k \frac{\partial^{l+k} f}{\partial t^k \partial s^l} (b,c) + (-1)^{k+l} \frac{\partial^{l+k} f}{\partial t^k \partial s^l} (b,d) \right]$$

$$-(-1)^m \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} \int_c^d \frac{(s-c)^m + (s-d)^m}{4m!} \times \left[\frac{\partial^{k+m} f}{\partial t^k \partial s^m} (a,s) + (-1)^k \frac{\partial^{k+m} f}{\partial t^k \partial s^m} (b,s) \right] ds$$

$$\left. -(-1)^n \sum_{l=0}^{m-1} \frac{(d-c)^{l+1}}{(l+1)!} \int_a^b \frac{(t-a)^n + (t-b)^n}{4n!} \times \left[\frac{\partial^{n+l} f}{\partial t^n \partial s^l} (t,c) + (-1)^l \frac{\partial^{n+l}}{\partial t^n \partial s^l f} (t,d) \right] dt \right|$$

$$= \left| \int_a^b \int_c^d T_n(a,b;t) \, T_m(c,d;s) \frac{\partial^{n+m} f}{\partial t^n \partial s^m} ds dt \right|$$

$$\leq \begin{cases} \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_\infty \int_a^b \int_c^d |T_n(a,b;t) \, T_m(c,d;s)| \, dt \, ds \\ \qquad \text{if } \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_\infty [[a,b] \times [c,d]] ; \\ \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_p \left(\int_a^b |T_n(a,b;t)|^q \, dt \right)^{\frac{1}{q}} \left(\int_c^d |T_m(c,d;s)|^q \, ds \right)^{\frac{1}{q}} \\ \qquad \text{if } \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_p [[a,b] \times [c,d]] , \ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_1 \sup_{(t,s) \in [a,b] \times [c,d]} |T_n(a,b;t) \, T_m(c,d;s)| \\ \qquad \text{if } \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_1 [[a,b] \times [c,d]] . \end{cases}$$

where

$$T_n(a,b;t) = \left[\frac{(b-t)^n + (-1)^n (t-a)^n}{2n!}\right], \qquad T_m(c,d;s) = \left[\frac{(d-s)^m + (-1)^m (s-c)^m}{2m!}\right].$$

Now consider $\int_a^b |T_n(a,b;t)|\,dt$. As may be seen, explicit evaluation of the integral depends on whether n is even or odd.

(*i*) If n is even, put $n = 2r_1$. Therefore,

$$\begin{aligned}
\int_a^b |T_n(a,b;t)|\,dt &= \frac{1}{(2r_1)!}\int_a^b \frac{(b-t)^{2r_1} + (t-a)^{2r_1}}{2}dt \\
&= \frac{1}{(2r_1)!}\cdot\frac{1}{2}\left[\frac{(b-a)^{2r_1+1}}{2r_1+1} + \frac{(b-a)^{2r_1+1}}{2r_1+1}\right] \\
&= \frac{(b-a)^{2r_1+1}}{(2r_1+1)!} = \frac{(b-a)^{n+1}}{(n+1)!}.
\end{aligned}$$

Similarly,

$$\int_c^d |T_m(c,d;s)|\,ds = \frac{1}{(2r_2)!}\int_c^d \frac{(d-s)^{2r_2} + (s-c)^{2r_2}}{2}ds = \frac{(d-c)^{m+1}}{(m+1)!}, \qquad n \text{ even}.$$

(*ii*) Now, if n is odd, that is, $n = 2r_1 + 1$, then

$$T_n(a,b;t) = \frac{(b-t)^{2r_1+1} - (t-a)^{2r_1+1}}{2(2r_1+1)!}.$$

Let $g(t) = (b-t)^{2r_1+1} - (t-a)^{2r_1+1}$.

We can observe that

$$\begin{cases} g(t) < 0 \text{ for all } t \in \left(\frac{a+b}{2}, b\right] \\ g(t) = 0 \ \text{ at } t = \frac{a+b}{2} \\ g(t) > 0 \text{ for all } t \in \left[a, \frac{a+b}{2}\right). \end{cases}$$

Thus

$$\begin{aligned}
2(2r_1+1)!\int_a^b |T_n(a,b;t)|\,dt &= \left[\int_a^{\frac{a+b}{2}} \left[(b-t)^{2r_1+1} - (t-a)^{2r_1+1}\right]dt\right. \\
&\quad \left. + \int_{\frac{a+b}{2}}^b \left[(t-a)^{2r_1+1} - (b-t)^{2r_1+1}\right]dt\right] \\
&= \left[2\cdot\frac{(b-a)^{2r_1+2}}{2r_1+2} - 4\frac{\left(\frac{b-a}{2}\right)^{2r_1+2}}{2r_1+2}\right]
\end{aligned}$$

and so

$$\begin{aligned}\int_a^b |T_n(a,b;t)|\,dt &= \frac{(b-a)^{2r_1+2}}{(2r_1+2)(2r_1+1)!}\left[1-\frac{1}{2^{2r_1+1}}\right]\\ &= \frac{(b-a)^{2r_1+2}}{(2r_1+2)!}\left[\frac{2^{2r_1+1}-1}{2^{2r_1+1}}\right]=\frac{(b-a)^{n+1}}{(n+1)!}\left[\frac{2^n-1}{2^n}\right].\end{aligned}$$

Similarly,

$$\int_c^d |T_m(c,d;s)|\,ds=\frac{(d-c)^{m+1}}{(m+1)!}\left[\frac{2^m-1}{2^m}\right],\qquad m \text{ even.}$$

and this gives the first inequality in (4.18). Now, for the third inequality we have,

$$\sup_{t\in[a,b]}|T_n(a,b;t)|=\frac{1}{2n!}\times\begin{cases}\sup\limits_{t\in[a,b]}\left((b-t)^n+(t-a)^n\right)=\frac{(b-a)^n}{2n!}, & \text{for all } n \text{ even}\\[2ex] \sup\limits_{t\in[a,b]}\left|(b-t)^n-(t-a)^n\right| = \frac{(b-a)^n}{2n!}, & \text{for all } n \text{ odd}\end{cases}$$

and this gives last part of the inequality in (4.18). The corollary is thus completely proved.

□

Remark 4.9. *For $n=m=1$, we have that*

$$\left|\int_a^b\int_c^d f(t,s)\,ds\,dt+\frac{(b-a)(d-c)}{4}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right]\right.$$

$$\left.-\frac{b-a}{2}\left[\int_c^d (f(a,s)+f(b,s))\,ds\right]-\frac{d-c}{2}\left[\int_a^b (f(t,c)+f(t,d))\,dt\right]\right|$$

$$\leq\begin{cases}\frac{(b-a)^2(d-c)^2}{4}\left[(x-a)^2+(b-x)^2\right]\left[(y-c)^2+(d-y)^2\right]\times\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty\\[2ex] \frac{1}{4}\left[\frac{((b-a)(d-c))^{q+1}}{(q+1)^2}\right]^{\frac{1}{q}}\times\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_p,\ p>1,\frac{1}{p}+\frac{1}{q}=1;\\[2ex] \frac{(b-a)(d-c)}{4}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_1.\end{cases}$$

Again, the same result was obtained by Hanna *et al.* (2000) and Dragomir *et al.* (1998).

In the following Section we will utilize the inequalities obtained in this section and demonstrate their capabilities to numerical integrations.

4.4 Applications to Numerical Integration.

Consider f to be a two-dimensional n-times differentiable mapping in and that its all partial derivatives in both direction exist and they are integrable. We apply the inequalities obtained before using a uniform mesh for numerical implementation. Thus the following application in Numerical Integration is natural to be considered.

THEOREM 4.10. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be as in Theorem 4.5. In addition, let I_v and J_μ be arbitrary divisions of $[a,b]$ and $[c,d]$ respectively, that is,*

$$I_v : a = \xi_0 < \xi_1 < \dots < \xi_\nu = b,$$

where $x_i \in (\xi_i, \xi_{i+1})$ for $i = 0, 1, \dots, \nu - 1$, and

$$J_\mu : c = \tau_0 < \tau_1 < \dots < \tau_\mu = d,$$

with $y_j \in (\tau_j, \tau_{j+1})$ for $j = 0, 1, \dots, \mu - 1$, then we have the cubature formula

$$\int_a^b \int_c^d f(t,s)\, ds\, dt = \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \sum_{i=0}^{\nu-1} \sum_{j=0}^{\mu-1} X_k^{(i)}(x_i) Y_l^{(j)}(y_j) \frac{\partial^{i+j} f}{\partial t^i \partial s^j}(x_i, y_j) \tag{4.19}$$

$$+ (-1)^m \sum_{k=0}^{n-1} \sum_{i=0}^{\nu-1} \sum_{j=0}^{\mu-1} X_k^{(i)}(x_i) \int_{\tau_j}^{\tau_{j+1}} S_m^{(j)}(y_j, s) \frac{\partial^{k+m} f}{\partial t^k \partial s^m}(x_i, s)\, ds$$

$$+ (-1)^n \sum_{l=0}^{m-1} \sum_{i=0}^{\nu-1} \sum_{j=0}^{\mu-1} Y_l^{(j)}(y_j) \int_{\xi_i}^{\xi_{i+1}} K_n^{(i)}(x_i, t) \frac{\partial^{n+l} f}{\partial t^n \partial s^l}(t, y_j)\, dt$$

$$+ R(f, I_v, J_\mu, x, y),$$

where the remainder term satisfies the condition

$$
|R(f,I_n,J_m,x,y)| \leq \begin{cases}
\frac{\left\|\frac{\partial^{n+m}f}{\partial t^n \partial s^m}\right\|_\infty}{(n+1)!(m+1)!} \times \sum_{i=0}^{\nu-1}\left[(x_i-\xi_i)^{n+1}+(\xi_{i+1}-x_i)^{n+1}\right] \\
\qquad \times \sum_{j=0}^{\mu-1}\left[(y_j-\tau_j)^{m+1}+(\tau_{j+1}-y_j)^{m+1}\right] \\
\text{if } \frac{\partial^{n+m}f}{\partial t^n \partial s^m} \in L_\infty([a,b]\times[c,d]); \\
\frac{\left\|\frac{\partial^{n+m}f}{\partial t^n \partial s^m}\right\|_p}{n!m!(nq+1)^{\frac{2}{q}}} \times \sum_{i=0}^{\nu-1}\left[(x_i-\xi_i)^{nq+1}+(\xi_{i+1}-x_i)^{nq+1}\right]^{\frac{1}{q}} \\
\qquad \times \sum_{j=0}^{\mu-1}\left[(y_j-\tau_j)^{mq+1}+(\tau_{j+1}-y_j)^{mq+1}\right]^{\frac{1}{q}} \\
\text{if } \frac{\partial^{n+m}f}{\partial t^n \partial s^m} \in L_p([a,b]\times[c,d]),\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\frac{\left\|\frac{\partial^{n+m}f}{\partial t^n \partial s^m}\right\|_1}{4n!m!} \sum_{i=0}^{\nu-1}\left[(x_i-\xi_i)^{n}+(\xi_{i+1}-x_i)^{n}+\left|(x_i-\xi_i)^{n}-(\xi_{i+1}-x_i)^{n}\right|\right] \\
\qquad \times \sum_{j=0}^{\mu-1}\left[(y_j-\tau_j)^{m}+(\tau_{j+1}-y_j)^{m}+\left|(y_j-\tau_j)^{m}-(\tau_{j+1}-y_j)^{m}\right|\right] \\
\text{if } \frac{\partial^{n+m}f}{\partial t^n \partial s^m} \in L_1([a,b]\times[c,d]);
\end{cases}
$$

where

$$X_k^{(i)}(k=0,1,...n-1;i=0,1,...\nu-1)\ ,\ Y_l^{(j)}(l=0,1,...m-1;j=0,1,...\mu-1)$$

and

$$K_n^{(i)}(i=0,1,...\nu-1),\ S_m^{(j)}(j=0,1,...\mu-1)\ \text{ are defined by}$$

$$X_k^{(i)}(x_i) := \frac{(\xi_{i+1}-x_i)^{k+1}+(-1)^k(x_i-\xi_i)^{k+1}}{(k+1)!}, \qquad Y_l^{(j)}(y_j) := \frac{(\tau_{j+1}-y_j)^{l+1}+(-1)^l(y_j-\tau_j)^{l+1}}{(l+1)!},$$

$$K_n^{(i)}(x_i,t) := \begin{cases} \frac{(t-\xi_i)^n}{n!}, & t\in[\xi_i,x_i] \\ \frac{(t-\xi_{i+1})^n}{n!}, & t\in(x_i,\xi_{i+1}] \end{cases} \quad \text{and} \quad S_m^{(j)}(y_j,s) := \begin{cases} \frac{(s-\tau_j)^m}{m!}, & s\in[\tau_i,y_i] \\ \frac{(s-\tau_{j+1})^m}{m!}, & s\in(y_i,\tau_{j+1}] \end{cases}$$

The proof is obvious by Theorem 4.5 applied on the interval $[\xi_i,\xi_{i+1}]\times[\tau_j,\tau_{j+1}]$, $(i=0,1,...\nu-1;j=0,1,...\mu-1)$, and we omit the details.

Remark 4.11. *A similar process can be undertaken in producing composite rules if we use the other results obtained in Section 4.3, but we omit the details.*

In the following chapter, we develope a method of obtaining "weighted" integral inequalities, where the derivatives of the integrand may be unbounded unlike the last two chapters.

Chapter 5

Weighted Quadrature Rules

In the previous two chapters two-dimensional integral inequalities were developed where the upper bounds were expressed in measures (or norms) of derivatives of the integrand. Often in practical applications the integrand may possess some singularity structure which precludes its consideration from this analysis. That is, the function is integrable, but not analytic hence one or more derivatives may be unbounded and non-integrable. In other cases the integrand may be perfectly analytic, but the region of integration is infinite or semi-infinite. Here again, this cannot be considered using the techniques of Chapters 2-4. These cases are most often managed using product or weighted integrands. In this chapter, we will consider extending the Ostrowski result to one-dimensional weighted integrals. This analysis will then be taken up in the subsequent chapter where we again focus on two-dimensional integral inequalities.

5.1 Product and Weighted Interior Point Integral Inequalities

In chapter 7 of Dragomir and Rassias (2001), J. Roumeliotis developed some weighted (or product) integral inequalities using the Ostrowski approach. These inequalities furnish an error estimate for weighted integrals where both the quadrature rule and error bound are given in terms of (at most) the first three moments of the weight. Also, the upper bound is a function of the first few derivatives of the mapping.

A weighted one dimensional counterpart of the Ostrowski inequality involving the first derivative is obtained. But, before going any further, let us consider the following definition.

Definition 5.1. *Let $w : (a,b) \to [0,\infty)$ be an integrable function, i.e. $\int_a^b w(t)dt < \infty$, and non-negative, then define*

$$m_i(a,b) = \int_a^b t^i w(t)dt, \qquad i = 0,1,\ldots \tag{5.1}$$

as the i^{th} moment of w.

Thus, we have the following theorem (see also, Dragomir *et al.* (1999))

THEOREM 5.1. *Let w be as defined in Definition 5.1 and let $f : [a,b] \to \mathbb{R}$ be absolutely continuous and have bounded first derivative, then*

$$\begin{aligned}\left|\int_a^b w(t)f(t)dt - m_0(a,b)f(x)\right| &\le \|f'\|_\infty \int_a^b |x-t|w(t)dt \\ &= \|f'\|_\infty\{x(m_0(a,x) - m_0(x,b)) + m_1(x,b) - m_1(a,x)\}\end{aligned} \tag{5.2}$$

Proof. Define the mapping $K(.,.) : [a,b] \to \mathbb{R}$ by

$$K(x,t) = \begin{cases} m_0(a,t), & t \in [a,x], \\ m_0(b,t), & t \in (x,b]. \end{cases} \tag{5.3}$$

Integrating by parts gives

$$\begin{aligned}\int_a^b K(x,t)f'(t)dt &= \int_a^x m_0(a,t)f'(t)dt + \int_x^b m_0(b,t)f'(t)dt \\ &= m_0(a,t)f(t)]_{t=a}^x + m_0(b,t)f(t)]_{t=x}^b - \int_a^b w(t)f(t)dt.\end{aligned}$$

Yielding the *product Montgomery* identity

$$\int_a^b K(x,t)f'(t)dt = m_0(a,b) - \int_a^b w(t)f(t)dt. \tag{5.4}$$

Taking the modulus and using Hölder inequality gives

$$\begin{aligned}\left|\int_a^b K(x,t)f'(t)dt\right| &\le \|f'\|_\infty \int_a^b |K(x,t)|dt \\ &= \|f'\|_\infty\{\int_a^x m_0(a,t)dt + \int_x^b m_0(t,b)dt\}.\end{aligned} \tag{5.5}$$

The last result being obtained by using the fact that for fixed x, K is positive in $t \in (a,x)$ and negative in $t \in (x,b)$. Utilizing (5.4), recalling Leibniz identity and using integrating by parts gives the desired result (5.2). $\square$

Remark 5.2. *substituting $w(t) = 1$ into (5.2) gives the Ostrowski inequality (2.1).*

Corollary 5.3. *Let the condition in Theorem 5.1 hold and let $x \in [a,b]$, where $[a,b]$ is a finite interval. The following product integral inequality holds.*

$$\left|\int_a^b w(t)f(t)dt - m_0(a,b)f(x)\right| \leq \|f'\|_\infty m_0(a,b)\left(\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right). \tag{5.6}$$

Roumeliotis *et al.* (1999) used Hölder's inequality to produce another estimation in term of the $\|.\|_p$ norm of $w(t)$ as shown in the following corollary.

Corollary 5.4. *Let the conditions in Theorem 5.1 hold and let $w(t) \in L_p[a,b]$, we have the inequality*

$$\left|\int_a^b w(t)f(t)dt - m_0(a,b)f(x)\right| \leq \|f'\|_\infty \|w\|_p \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1}\right]^{1/q}, \tag{5.7}$$

for all $x \in [a,b], p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. It is easily shown that the bound is minimized at the mid-point $x = \frac{(a+b)}{2}$

The same author proved the following theorem.

THEOREM 5.5. *Let w as be given in Definition 5.1 and let $f : [a,b] \to \mathbb{R}$ be such that $f' \in L_1(a,b)$. The following inequality holds*

$$\left|\int_a^b w(t)f(t)dt - m_0(a,b)f(x)\right| \leq \frac{1}{2}\|f'\|_1\{m_0(a,b) + |m_0(a,x) - m_0(x,b)|\}. \tag{5.8}$$

Dragomir *et al.* (1999) developed a weighted Ostrowski type inequality for Hölder mappings as a generalization for (5.2).

THEOREM 5.6. *Let w be as given in Definition 5.1 and let f be of $r - H$-Hölder type. That is*

$$|f(x) - f(y)| \leq H|x - y|^r \tag{5.9}$$

for all $x, y \in (a,b)$, $H > 0$ and $r \in (0,1]$. If $w(t)f(t)$ is integrable, then the following inequality of the weighted integral holds

$$\left|\int_a^b w(t)f(t)dt - m_0(a,b)f(x)\right| \leq H\int_a^b |x-t|^r w(t)dt, \tag{5.10}$$

for all $a < x < b$.

Roumeliotis *et al.* (1999) presented analogous Ostrowski type inequalities for weighted integrals in one dimension where f'' is assumed to exist.

From the point of view that the following definitions will prove useful.

Definition 5.2. *Define the mean of the interval [a,b] with respect to the density w as*

$$\mu(a,b) = \frac{m_1(a,b)}{m_0(a,b)} \tag{5.11}$$

and the variance by

$$\sigma^2(a,b) = \frac{m_2(a,b)}{m_0(a,b)} - \mu^2(a,b). \tag{5.12}$$

Then the following inequality holds

THEOREM 5.7. *Let $f, w : (a,b) \in \mathbb{R}$ be two mappings on (a,b) with the following properties:*

(1) $\sup_{t\in(a,b)} |f''(t)| < \infty$,

(2) $w(t) \geq 0 \ \forall t \in (a,b)$,

(3) $\int_a^b w(t)dt < \infty$,

then the following inequalities hold

$$\left|\frac{1}{m_0(a,b)} \int_a^b w(t)f(t)dt - f(x) + (x - \mu(a,b))f'(x)\right|$$

$$\leq \frac{||f''||_\infty}{2}[(x - \mu(a,b)^2 + \sigma^2(a,b)] \tag{5.13}$$

$$\leq \frac{||f''||_\infty}{2}\left(\left|x - \frac{a+b}{2}\right| + \frac{b-a}{2}\right)^2 \tag{5.14}$$

for all $x \in [a,b]$.

Proof. . Define the mapping $K(.,.) : [a,b]^2 \in \mathbb{R}$ by

$$K(x,t) := \begin{cases} \int_a^t (t-u)w(u)du, & a \leq t \leq x, \\ \\ \int_b^t (t-u)w(u)du, & x < t \leq b. \end{cases} \tag{5.15}$$

Integrating by parts gives the identity

$$\int_a^b K(x,t)f''(t)dt = \int_a^b w(t)f(t)dt - m_0(a,b)f(x) + m_0(a,b)(x - \mu(a,b))f'(x). \tag{5.16}$$

□

Taking the modulus of (5.16), after some computations and simplifying we obtain the desired results.

It should be emphasized that, the optimal point $x = \mu(a, b)$ can be interpreted in many ways. In a physical context, $\mu(a, b)$ represents the center of mass of a one dimensional rod with mass density w. Equivalently, this point can be viewed as that which minimizes the error variance for the probability density w. Finally (5.11) is also the Gauss node point for a one-point rule see (Stroud and Secrest 1966).

5.2 Weighted Three Point Rules

Weighted three point quadrature rules are investigated in this section in which sampling occurs at the boundary points and an interior point. Explicit *a priori* bounds are obtained, thus enabling the determination of the partition required for a prescribed error bound to be fulfilled. This approach contrasts to that commonly used of mesh refinement followed by a successive *a posteriori* comparison of the results (see for example Atkinson (1988)). Other quadrature rules have been developed that differ from those given here. Three point quadrature rules of Newton-Cotes type have been examined in Cerone and Dragomir (1999) in which the error involved the behaviour of, at most, a first derivative. Riemann and Riemann-Stieltjes integrals were examined. In the current section, weighted three point rules are investigated in which the error relies on the behaviour of the first derivative. Also, composite quadrature rules for a log weight function are given and compared with a product-trapezoidal rule of Atkinson (1988)

5.2.1 Weighted Three Point Inequalities

Weighted (or product) inequalities are developed involving function evaluation at three points (see also, Cerone *et al.* (2000)).

THEOREM 5.8. *Utilizing Definition 5.1 and under the conditions of Theorem 2.11 where*

$n = 1$. *Then for* $x \in [a, b]$, $\alpha \in [a, x]$, $\beta \in (x, b]$, *the following inequality holds*

$$\left| \int_a^b w(t) f(t)\, dt - [m_0(\alpha, \beta) f(x) + m_0(a, \alpha) f(a) + m_0(\beta, b) f(b)] \right| \leq I(\alpha, x, \beta) \|f'\|_\infty, \quad (5.17)$$

where

$$I(\alpha, x, \beta) = \int_a^b k(x, t)\, w(t)\, dt \quad \textit{and} \quad k(x, t) = \begin{cases} t - a, & t \in [a, \alpha] \\ |x - t|, & t \in (\alpha, \beta] \\ b - t, & t \in (\beta, b] \end{cases} \quad (5.18)$$

Proof. Define the mapping $K(\cdot, \cdot) : [a, b]^2 \to \mathbb{R}$ by

$$K(x, t) = \begin{cases} m_0(\alpha, t), & t \in [a, x] \\ m_0(\beta, t), & t \in (x, b] \end{cases}, \quad (5.19)$$

where $m_0(a, b)$ is the zeroth moment of $w(\cdot)$ over the interval $[a, b]$. It should be noted that $m_0(c, d)$ will be non-negative for $d \geq c$. Integration by parts gives, on using (5.19),

$$\begin{aligned} \int_a^b K(x, t) f'(t)\, dt &= \int_a^x m_0(\alpha, t) f'(t)\, dt + \int_x^b m_0(\beta, t) f'(t)\, dt \\ &= m_0(\alpha, t) f(t) \Big]_{t=a}^{x} + m_0(\beta, t) f(t) \Big]_{t=x}^{b} - \int_a^b w(t) f(t)\, dt, \end{aligned}$$

producing the identity

$$\int_a^b K(x, t) f'(t)\, dt = m_0(\alpha, \beta) f(x) + m_0(a, \alpha) f(a) + m_0(\beta, b) f(b) - \int_a^b w(t) f(t)\, dt, \quad (5.20)$$

valid for all $x \in [a, b]$. Taking the modulus of (5.20) gives

$$\left| \int_a^b w(t) f(t)\, dt - [m_0(\alpha, \beta) f(x) + m_0(a, \alpha) f(a) + m_0(\beta, b) f(b)] \right| = \left| \int_a^b K(x, t) f'(t)\, dt \right| \leq \|f'\|_\infty \int_a^b |K(x, t)|\, dt. \quad (5.21)$$

Now, we wish to determine $\int_a^b |K(x, t)|\, dt$ explicity. To this end notice that, from (5.19), $K(x, t)$ is a monotonically non-decreasing function of t over each of its branches. Thus, there are points $\alpha \in [a, x]$ and $\beta \in [x, b]$ such that $K(x, \alpha) = K(x, \beta) = 0$. Thus,

$$\int_a^b |K(x, t)|\, dt = -\int_a^\alpha m_0(\alpha, t)\, dt + \int_\alpha^x m_0(\alpha, t)\, dt - \int_x^\beta m_0(\beta, t)\, dt + \int_\beta^b m_0(\beta, t)\, dt. \quad (5.22)$$

Integration by parts gives, for example,

$$-\int_a^\alpha m_0(\alpha,t)\,dt = -(t-a)\,m_0(\alpha,t)\Big]_{t=a}^{\alpha} + \int_a^\alpha (t-a)\,w(t)\,dt = \int_a^\alpha (t-a)\,w(t)\,dt.$$

A similar development for the remainder of the three integrals on the right hand side of (5.22) produces the result

$$\int_a^b |K(x,t)|\,dt = I(\alpha,x,\beta), \tag{5.23}$$

where $I(\alpha,x,\beta)$ is as given by (5.18). Combining (5.21) and (5.23) produces the result (5.17) and hence the theorem is proved. □

It should be noted at this stage that taking $w(\cdot)\equiv 1$ reproduces the results of Cerone and Dragomir (1999). If $\alpha=a$ and $\beta=b$ then a weighted interior point rule is obtained. If $\alpha=\beta=x$, then a weighted rule results where the function is evaluated at the boundary points. For $\alpha=a$ or $\beta=b$ then Radau type rules are obtained while the current work will focus on rules allowing sampling at both ends of the boundary.

Corollary 5.9. *Inequality (5.17) is minimized at $x=x^*$ where x^* satisfies*

$$m_0(\alpha^*,x^*) = m_0(x^*,\beta^*), \quad \alpha^* = \frac{a+x^*}{2} \quad \textit{and} \quad \beta^* = \frac{x^*+b}{2}. \tag{5.24}$$

Proof. From (5.17) - (5.18), $I(\alpha,x,\beta)$ may be written as

$$\begin{aligned} I(\alpha,x,\beta) = \int_a^\alpha (t-a)\,w(t)\,dt + \int_\alpha^x (x-t)\,w(t)\,dt \\ + \int_x^\beta (t-x)\,w(t)\,dt + \int_\beta^b (b-t)\,w(t)\,dt, \end{aligned} \tag{5.25}$$

where $\alpha\in[a,x]$ and $\beta\in(x,b]$. Equation (5.25) could equivalently be written in terms of its zeroth and first moments. Differentiating (5.25) with respect to α,β and x gives

$$\frac{\partial I}{\partial\alpha} = A(\alpha,x)\,w(x), \quad \frac{\partial I}{\partial\beta} = B(\beta,x)\,w(x) \text{ and } \frac{\partial I}{\partial x} = m_0(\alpha,x) - m_0(x,\beta), \tag{5.26}$$

where

$$A(\alpha,x) = 2\alpha-(a+x), \quad B(\beta,x) = 2\beta-(x+b) \tag{5.27}$$

An inspection of the second derivatives demonstrates that (5.25) is convex on using the fact that $w(t)$ is non-negative for $t\in(a,b)$. Thus, I is minimal at the zeros of (5.26) and so the corollary is proven. □

Corollary 5.9 investigates the problem of determining the optimal choice of α, x and β that produce the tightest bound. The following corollary gives coarser bounds although the bound may be easier to implement.

Corollary 5.10. *Let the conditions be as in Theorem 5.8. Then the following inequalities hold*

$$\left|\int_a^b w(t) f(t)\, dt - \left[m_0(\alpha,\beta) f(x) + m_0(a,\alpha) f(a) + m_0(\beta,b) f(b)\right]\right|$$
$$\leq \|f'\|_\infty \times \begin{cases} \|w\|_\infty \cdot K_1(x) \\ \\ \|w\|_1 \cdot K_\infty(x) \end{cases}, \quad (5.28)$$

where

$$K_1(x) = \frac{1}{2}\left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2\right] + \left(\alpha - \frac{a+x}{2}\right)^2 + \left(\beta - \frac{x+b}{2}\right)^2 \quad (5.29)$$

and

$$K_\infty(x) = \frac{1}{2}\left[\frac{b-a}{2} + \left|\alpha - \frac{a+x}{2}\right| + \left|\beta - \frac{x+b}{2}\right| + \left|x - \frac{a+b}{2} + \left|\alpha - \frac{a+x}{2}\right| - \left|\beta - \frac{x+b}{2}\right|\right|\right] \quad (5.30)$$

with $\|g\|_1 := \int_a^b |g(s)|\, ds$ *meaning* $g \in L_1[a,b]$, *the linear space of absolutely integrable functions and* $\|g\|_\infty := \sup_{t\in[a,b]} |g(t)| < \infty$.

Proof. From Theorem 5.8 and equations (5.17) - (5.18), (5.19) and (5.23) we have

$$I(\alpha,x,\beta) = \int_a^b |K(x,t)|\, w(t)\, dt = \int_a^b k(x,t)\, w(t)\, dt.$$

Now,

$$\int_a^b k(x,t)\, w(t)\, dt \leq \begin{cases} \|w\|_\infty \int_a^b |k(x,t)|\, dt \\ \\ \|w\|_1 \sup_{t\in[a,b]} |k(x,t)| \end{cases},$$

where $k(x,t)$ is as defined in (5.18). Some straight forward evaluation gives

$$\int_a^b |k(x,t)|\, dt = \frac{1}{2}\left[(\alpha-a)^2 + (x-\alpha)^2 + (\beta-x)^2 + (b-\beta)^2\right],$$

which may readily be shown to equal $K_1(x)$ as given by (5.29) through using the identity

$$\frac{X^2+Y^2}{2}=\left(\frac{X+Y}{2}\right)^2+\left(\frac{X-Y}{2}\right)^2$$

three times. Further,

$$\sup_{t\in[a,b]}|k(x,t)|=\max\{\alpha-a,x-\alpha,\beta-x,b-\beta\}\ ,$$

which can be shown to equal $K_\infty(x)$ as given by (5.30) from using the result

$$\max\{X,Y\}=\frac{X+Y}{2}+\frac{|X-Y|}{2}, \tag{5.31}$$

three times. □

It should be noted that the tightest bounds are obtained at $x=\frac{a+b}{2}$ and $\alpha=\frac{a+x}{2}$, $\beta=\frac{x+b}{2}$. That is, at their respective mid-points. The optimal sampling scheme is independent of the weight.

THEOREM 5.11. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on $\mathring{I}$ (the interior of I) and $a,b\in\mathring{I}$ are such that $b>a$. If $f'\in L_1[a,b]$, then $\|f'\|_1=\int_a^b|f'(t)|\,dt<\infty$. In addition, let a non-negative weight function $w(\cdot)$ have the properties as outlined in Definition 5.1. Then for $x\in[a,b]$, $\alpha\in[a,x]$ and $\beta\in(x,b]$ the following inequality holds.*

$$\left|\int_a^b w(t)\,f(t)\,dt-[m_0(\alpha,\beta)\,f(x)+m_0(a,\alpha)\,f(a)+m_0(\beta,b)\,f(b)]\right| \\ \leq\theta(\alpha,x,\beta)\,\|f'\|_1, \tag{5.32}$$

where

$$\theta(\alpha,x,\beta)=\frac{1}{4}\{m_0(a,b)+|m_0(\alpha,x)-m_0(a,\alpha)|+|m_0(\beta,b)-m_0(x,\beta)| \\ +|m_0(a,x)-m_0(x,b)+|m_0(\alpha,x)-m_0(a,\alpha)|-|m_0(\beta,b)-m_0(x,\beta)||\} \tag{5.33}$$

and $m_0(a,b)$ is the zeroth moment of $w(\cdot)$ over $[a,b]$

Proof. From identity (5.20) we obtain, from taking the modulus

$$\theta(\alpha,x,\beta)=\sup_{t\in[a,b]}|K(x,t)|\ ,$$

where $K(x,t)$ is as given by (5.19). As discussed in the proof of Theorem 5.8, $K(x,t)$ is a monotonic non-decreasing function of t in each of its two branches so that

$$\theta(\alpha, x, \beta) = \max\{m_0(a,\alpha), m_0(\alpha,x), m_0(x,\beta), m_0(\beta,b)\}.$$

Now, using equation (5.31) we have

$$\begin{aligned} M_1 &= \max\{m_0(a,\alpha), m_0(\alpha,x)\} = \frac{1}{2}\left[m_0(a,x) + |m_0(\alpha,x) - m_0(a,\alpha)|\right] \\ \text{and } M_2 &= \max\{m_0(x,\beta), m_0(\beta,b)\} = \frac{1}{2}\left[m_0(x,b) + |m_0(\beta,b) - m_0(x,\beta)|\right], \end{aligned}$$

to give

$$\theta(\alpha, x, \beta) = \max\{M_1, M_2\} = \frac{M_1 + M_2}{2} + \left|\frac{M_1 - M_2}{2}\right|$$

and hence the result (5.33) is obtained after some simplification and the theorem is proved.

□

Remark 5.12. *It should be noted that the tightest bound in (5.33) is obtained when α, x and β are taken as their respective medians. Thus, the best quadrature rule in the above sense is given by*

$$\left|\int_a^b w(t) f(t)\,dt - \left[m_0(a,\tilde{\alpha}) f(a) + m_0(\tilde{\alpha},\tilde{\beta}) f(\tilde{x}) + m_0(\tilde{\beta},b) f(b)\right]\right| \le \frac{m_0(a,b)}{4} \|f'\|_1, \tag{5.34}$$

where

$$m_0(a,\tilde{x}) = m_0(\tilde{x},b),\ m_0(a,\tilde{\alpha}) = m_0(\tilde{\alpha},\tilde{x}) \text{ and } m_0(\tilde{\beta},b) = m_0(\tilde{x},\tilde{\beta}).$$

5.2.2 Development of a Quadrature Rule

The following theorem will be useful in determining the partition for composite quadrature rules. The optimal partition in terms of the partition that provides the tightest bounds will be determined. The optimal quadrature rules will result for $f' \in L_\infty[a,b]$. If $f' \in L_1[a,b]$ a similiar development may be followed but will not be pursued further here.

THEOREM 5.13. *Let the conditions of Theorem 5.8 hold and let ξ partition the interval $[a,b]$ into two. Then the following inequality holds*

$$\begin{aligned} \Bigg|\int_a^b w(t) f(t)\,dt - \big[&m_0(a,\alpha_1) f(a) + m_0(\alpha_1,\beta_1) f(x_1) + m_0(\beta_1,\alpha_2) f(\xi) \\ &+ m_0(\alpha_2,\beta_2) f(x_2) + m_0(\beta_2,b) f(b)\big]\Bigg| \le J(\mathbf{z},\xi)\, \|f'\|_\infty, \end{aligned} \tag{5.35}$$

where

$$J(\mathbf{z}, \xi) = J_1(\mathbf{z}_1, \xi) + J_2(\mathbf{z}_2, \xi) \tag{5.36}$$

with

$$\begin{aligned} \mathbf{z}_i^T &= (\alpha_i, x_i, \beta_i), \ i = 1, 2, \ \mathbf{z} = \mathbf{z}_1 \cup \mathbf{z}_2, \\ J_1(\mathbf{z}_1, \xi) &= \int_a^\xi k_1(x_1, t) w(t) \, dt, \ J_2(\mathbf{z}_2, \xi) = \int_\xi^b k_2(x_2, t) w(t) \, dt \end{aligned} \tag{5.37}$$

and

$$k_1(x_1, t) = \begin{cases} t - a, & t \in [a, \alpha_1] \\ |x_1 - t|, & t \in (\alpha_1, \beta_1] \\ \xi - t, & t \in (\beta_1, \xi] \end{cases}, \ k_2(x_2, t) = \begin{cases} t - \xi, & t \in [\xi, \alpha_2] \\ |x_2 - t|, & t \in (\alpha_2, \beta_2] \\ b - t, & t \in (\beta_2, b] \end{cases}. \tag{5.38}$$

Further, $a \le \alpha_1 \le x_1 \le \beta_1 \le \xi$ *and* $\xi \le \alpha_2 \le x_2 \le \beta_2 \le b$.

Proof. The proof follows that of Theorem 5.8. A subscript of 1 is used to denote parameters in the interval $[a, \xi]$ and 2 for parameters in $(\xi, b]$. Integration by parts of $\int_a^\xi K(x_1, t) f'(t) \, dt$ produces an identity similar to (5.20) with b replaced by ξ and x by x_1. Similarly for $\int_\xi^b K(x_2, t) f'(t) \, dt$ produces an identity like (5.20) with a replaced by ξ and x by x_2. Summing the two results produces an identity over $[a, b]$. Taking the modulus and using the triangle inequality, relying heavily on (5.18) gives the stated result after collecting the terms in order. Here on $[a, \xi]$, (α, x, β, b) are replaced by $(\alpha_1, x_1, \beta_1, \xi)$ and on $[\xi, b]$, (a, α, x, β) are replaced by $(\xi, \alpha_2, x_2, \beta_2)$. Hence the theorem is proved. □

Corollary 5.14. *The optimal location of the parameters in Theorem 5.13 are* $\alpha_1 = \alpha_1^* = \frac{a + x_1^*}{2}$, $\beta_1 = \beta_1^* = \frac{x_1^* + \xi^*}{2}$, $\alpha_2 = \alpha_2^* = \frac{\xi^* + x_2^*}{2}$, $\beta_2 = \beta_2^* = \frac{x_2^* + b}{2}$ *and* x_1^*, x_2^* *and* ξ^* *satisfy the following respective equations*

$$m_0(\alpha_1^*, x_1^*) = m_0(x_1^*, \beta_1^*), \ m_0(\alpha_2^*, x_2^*) = m_0(x_2^*, \beta_2^*) \ \text{ and } m_0(\beta_1^*, \xi^*) = m_0(\xi^*, \alpha_2^*).$$

Proof. The proof of this corollary closely follows that of Corollary 5.9. From (5.36) - (5.38), differentiation of J with respect to $(\alpha_1, x_1, \beta_1, \xi, \alpha_2, x_2, \beta_2)$ produces, on equating to zero, seven simultaneous equations. Using the fact that the weight function is assumed to be positive, then the solution of the seven simultaneous equations give the point at which an optimal bound is produced, since an inspection of the second derivatives readily demonstrates the convexity of the function J. □

The results in Theorem 5.13 may be used to develop a composite quadrature rule. To this end, define a grid $I_n : a = \xi_0 < \xi_1 < \cdots < \xi_{n-1} < \xi_n = b$ on the interval [a,b], with $x_i \in [\xi_i, \xi_{i+1}]$ for $i = 0, 1, \ldots, n-1$. The following quadrature formula for weighted integrals is obtained which relies only on the first two moments of the weight function.

THEOREM 5.15. *Let the conditions in Theorem 5.13 hold, then following weighted quadrature rule holds*

$$\int_a^b w(t)f(t)\,dt = A(f, \boldsymbol{\xi}, \boldsymbol{x}) + R(f, \boldsymbol{\xi}, \boldsymbol{x}) \tag{5.39}$$

where

$$A(f, \boldsymbol{\xi}, \boldsymbol{x}) = \sum_{i=1}^{n} m_0(\alpha_i, \beta_i) f(x_i) + m_0(\xi_0, \alpha_1) f(\xi_0) + 2\sum_{i=1}^{n-1} m_0(\beta_i, \xi_i) f(\xi_i) + m_0(\beta_n, \xi_n) f(\xi_n) \tag{5.40}$$

and

$$\begin{aligned} |R(f, \boldsymbol{\xi}, \boldsymbol{x})| \le \|f'\|_\infty \bigg(& M(\xi_0, \xi_n) - 2\sum_{i=}^{n} [M(\alpha_i, \beta_i) + M(\beta_i, \xi_i)] \\ & + \xi_n m_0(\beta_n, \xi_n) - \xi_0 m_0(\xi_0, \alpha_1) \bigg). \end{aligned} \tag{5.41}$$

The parameters x_i, α_i, β_i and ξ_i satisfy

$$m_0(\alpha_i, x_i) = m_0(x_i, \beta_i), \qquad \alpha_i = \frac{\xi_{i-1} + x_i}{2}, \qquad \beta_i = \frac{x_i + \xi_i}{2} \tag{5.42}$$

for $i = 1, 2, \ldots, n$, and

$$m_0(\beta_i, \xi_i) = m_0(\xi_i, \alpha_{i+1}), \tag{5.43}$$

for $i = 1, 2, \ldots, n-1$.

Proof. Using the results of Theorems 5.8 and 5.13 over $[\xi_i, \xi_{i+1}]$ for $i = 0, 1, \ldots, n-1$ and summing readily produces the result after using Corollaries 5.9 and 5.14 to simplify. □

5.2.3 Numerical Results

In this section we illustrate the application of the composite quadrature rule developed in the previous section to approximate the integrals

$$\int_0^1 \frac{\ln(1/t)}{t+2}\,dt = 0.4484142069 \quad \text{and} \quad \int_0^1 e^{-1/t} \ln(1/t)\,dt = 0.05065230956 \tag{5.44}$$

The integrals are evaluated using the composite rule (5.39) and the product-trapezoidal as described in Atkinson (1988, p. 310). The first integral, $(5.44)_1$, has been used to demonstrate the product-trapezoidal and as a result we can compare the performance with the rule developed here. Note that (5.39) is a first-order rule in that it was derived for the class of once-differentiable functions. This contrasts with the product-trapezoidal rule which is of second order. Thus, to investigate the effects of rule order, we also apply these rules to $(5.44)_2$. In contrast with $(5.44)_1$, the integrand of $(5.44)_2$ increases with the order of its derivative. Table 5.1 shows the actual error in evaluating (5.44) using (5.39) for an increasing number of intervals. We note that the nodes and weights of the quadrature rule are obtained by solving the $4n-1$ simultaneous equations (5.42) and (5.43). It is a simple matter to implement a numerical procedure to solve these equations iteratively with an initial uniform mesh. For example on a Pentium-90 personal computer, with $n = 32$, calculating (5.42) and (5.43) to 14 digit accuracy took close to 42 seconds. Inspection of Table 5.1 reveals that a more accurate result is obtained for $(5.44)_1$ than for $(5.44)_2$. This is probably due to the nature of the integrands, since the integrand in the second integral $(5.44)_2$ has an essential singularity at the origin (lower bound of the integral). On the other side, the plot t=-2 is far-away from the support of the measure in the first example $(5.44)_1$. The estimated error ratio is consistently close to 2. This value confirms that, due to its development, the quadrature rule is at least of first order. The actual error ratios are somewhat larger, these values suggest an asymptotic form of the error bound

$$|R(f, \boldsymbol{\xi}, \boldsymbol{x})| \sim O\left(\frac{1}{n^{\gamma}}\right), \qquad \text{where} \quad \gamma \leq 2. \tag{5.45}$$

In Table 5.2 the errors in employing the product-trapezoidal rule are presented. The error ratios are consistently close to 4 which simply reflects the fact that the rule is of second order. This rule was developed by employing a linear approximation for the weighted integrand - a higher order approximation than that used here. This rule performs better than (5.39) for $(5.44)_1$ since the integrand is well behaved and its magnitude decreases as its derivatives increase. In contrast, the product-trapezoidal rule is inferior to (5.39) for $(5.44)_2$. This integrand is not well behaved and its integral is better suited to (5.39) which was developed for a more general class of function.

We note that the product-trapezoidal rule employs a uniform mesh and the behaviour of the weight function, $w(t)$, is accounted for in the quadrature rule weight. Rules of this type

n	Equation $(5.44)_1$		Equation $(5.44)_2$		Theoretical
	Relative Error	Error Ratio	Relative Error	Error Ratio	Error Ratio
2	1.64(-2)		7.27(-2)		
4	4.53(-3)	3.64	2.62(-2)	2.78	1.70
8	1.23(-3)	3.69	8.47(-3)	3.09	2.81
16	3.29(-4)	3.73	2.57(-3)	3.30	2.08
32	8.77(-5)	3.75	7.52(-4)	3.41	2.05
64	2.33(-5)	3.77	2.15(-4)	3.50	2.03

Table 5.1: The relative error in evaluating (5.44) using (5.39), where n is the number of intervals.

n	Equation $(5.44)_1$		Equation $(5.44)_2$	
	Relative Error	Error Ratio	Relative Error	Error Ratio
2	7.12(-3)		4.29(-1)	
4	1.98(-3)	3.60	8.08(-2)	5.30
8	5.17(-4)	3.83	1.90(-2)	4.25
16	1.32(-4)	3.92	4.74(-3)	4.01
32	3.33(-5)	3.96	1.18(-3)	4.00
64	8.35(-6)	3.98	2.96(-4)	4.00

Table 5.2: The relative error in evaluating (5.44) using the product trapezoidal rule , where n is the number of intervals.

were explored in Roumeliotis *et al.* (1999), where a one-point, second order product rule was developed. In this article, Roumeliotis *et al.* (1999), showed that, for the log weight, employing a non-uniform mesh, similiar to (5.43) increases accuracy by a factor of more than two for $f'' \in L_\infty[a,b]$. Finally, we note that the rule developed here is composite in nature and identifies an "optimal" partition for an arbitrary weight. This contrasts with Gauss quadrature Stroud and Secrest (1966) which is not composite and hence provides no information as to how one should partition.

5.2.4 Concluding Remarks

The approach described enables the user to predetermine the partition required to *assure* the result to be within a certain error tolerance. This approach is somewhat different from that commonly used of systematic mesh refinement followed by a comparison of successive approximations which forms the basis of a stopping rule. See (Atkinson (1988), Engels (1980) and Krommer and Ueberhuber (1994)) for a comprehensive treatment of traditional methods. Although the bounds were obtained in terms of the behaviour of the first derivative the methodology may be extended to involve higher derivatives. It may be advantageous to rely on the behaviour of lower derivatives as demonstrated in the evaluation of $(5.44)_2$ in which the higher derivatives are badly behaved.

The analysis discussed in this chapter is then taken up in next chapter, where we again focus on two-dimensional integral inequalities. We derive a second order weighted double integral inequality. Weighted second order cubature rules are developed and we devise a method for calculating cubature grids that rely only on the first two moments of the weight.

Chapter 6

Weighted Integral Inequalities in Two Dimensions

In Chapter 3, two dimensional integration was considered and error bounds were expressed in terms of the first mixed partial derivative of the integrand. In Chapter 5, weighted one dimensional Ostrowski type inequalities were reviewed and weighted quadrature rules developed. A quadrature grid influenced by the weight function was evaluated via minimization of the error bound. In this chapter we combine and extend the work of these previous chapters and develop weighted first and second order double integral inequalities. We pay particular attention to the influence of the two dimensional weight function on the error bound and explore this influence for different weights and weight null-spaces. Furthermore, weighted second order cubature rules are developed and we devise a method for calculating cubature grids that rely only on the first two moments of the weight.

The material in this chapter is presented in the following order. In Section 6.1, we use a two variable Taylor expansion to develop weighted two dimensional integral inequalities. Milovanović (1975) used this method to extend Ostrowski's inequality to multiple dimensions. Here we will content ourselves with two dimensions, but extend the order of the rule to two. We undertake an examination of the error bound and identify parameters that will minimize the bound. In Section 6.2, we present a Peano kernel method, based on analagous results in Chapter 3, to derive a second order weighted double integral inequality. Error bounds are expressed in terms of the L_1 and L_∞ norms of the first mixed partial derivative of the integrand. Particular attention is paied to minimizing this integrand for different weights and weight null-spaces. Finally, the results of this section are extended in Section 6.4 to develop a weighted cubature formula. Minimizing the error bound furnishes a set of non-linear coupled equations in the first two moments of the weight whose solution produces a cubature grid influenced by the weight function. Plots of the grid for various weights are given.

6.1 Taylor's Formula

Milovanović (1975) generalised the Ostrowski inequality to multiple dimensions using the multiple variable Taylor formula. As per the Ostrowksi result, the inequality was expressed in terms of the first partial derivatives of the integrand. We state the two dimensional formula below.

Following Milovanović (1975), let $D = \{(x_1, x_2)|a_i < x_i < b_i (i = 1, 2)\}$ and let $\bar{D}$ be the closure of D.

THEOREM 6.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function defined on $\bar{D}$ and let $\left|\frac{\partial f}{\partial t_i}\right| \leq M_i$ $(M_i > 0;\ i = 1, 2)$ in D. Then, for every $X = (x_1, x_2) \in \bar{D}$,*

$$\left|\frac{1}{(b_1 - a_1)(b_2 - a_2)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(t_1, t_2)\, dt_2 dt_1 - f(x_1, x_2)\right|$$
$$\leq M_1(b_1 - a_1) \left(\frac{\left(x_1 - \frac{a_1+b_1}{2}\right)^2}{(b_1 - a_1)^2} + \frac{1}{4}\right) + M_2(b_2 - a_2) \left(\frac{\left(x_2 - \frac{a_2+b_2}{2}\right)^2}{(b_2 - a_2)^2} + \frac{1}{4}\right). \quad (6.1)$$

The weighted version of Theorem 6.1 appears below.

THEOREM 6.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function defined on $\bar{D}$ and let $\left|\frac{\partial f}{\partial t_i}\right| \leq M_i$ $(M_i > 0;\ i = 1, 2)$ in D. Furthermore, let the function $X \mapsto w(X)$ be defined, integrable and $w(X) > 0$ for every $X \in \bar{D}$. Then for every $X \in \bar{D}$,*

$$\left|\frac{\int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1, t_2) f(t_1, t_2)\, dt_2 dt_1}{\int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1, t_2)\, dt_2 dt_1} - f(x_1, x_2)\right|$$
$$\leq \frac{1}{\int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1, t_2)\, dt_2 dt_1} \left(M_1 \int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1, t_2) |x_1 - t_1|\, dt_2 dt_1\right.$$
$$\left. + M_2 \int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1, t_2) |x_2 - t_2|\, dt_2 dt_1\right). \quad (6.2)$$

Theorem 6.2 can be extended to higher orders and below we provide such an extension to second order.

THEOREM 6.3. *Let $f : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ be such that all its partial derivatives up to order 2 exist and be continuous, i.e. $\frac{\partial^i f}{\partial t_1^j \partial t_2^k} < \infty, i = 1, 2; j = 0, \ldots, i; k = i - j$. Furthermore, let $w : (a_1, b_1) \times (a_2, b_2) \to (0, \infty)$ be integrable (i.e. $\iint w\, dA < \infty$). Then for all $(x_1, x_2) \in [a_1, b_1] \times [a_2, b_2]$ the following second order product double integral inequality holds*

$$\begin{aligned}
&\Bigg| \int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1, t_2) f(t_1, t_2)\, dt_2 dt_1 - f(x_1, x_2) \int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1, t_2)\, dt_2 dt_1 \\
&+ \frac{\partial f}{\partial t_1}(x_1, x_2) \int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1, t_2)(x_1 - t_1)\, dt_2 dt_1 + \frac{\partial f}{\partial t_2}(x_1, x_2) \int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1, t_2)(x_2 - t_2)\, dt_2 dt_1 \Bigg| \\
&\leq \frac{\|\frac{\partial^2 f}{\partial t_1^2}\|_\infty}{2} \int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1, t_2)(x_1 - t_1)^2\, dt_2 dt_1 + \frac{\|\frac{\partial^2 f}{\partial t_2^2}\|_\infty}{2} \int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1, t_2)(x_2 - t_2)^2\, dt_2 dt_1 \\
&+ \left\| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right\|_\infty \int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1, t_2) |x_1 - t_1| |x_2 - t_2|\, dt_2 dt_1. \qquad (6.3)
\end{aligned}$$

Proof. The two-variable Taylor formula states that

$$\begin{aligned}
f(t_1, t_2) &= f(x_1, x_2) + (t_1 - x_1) \frac{\partial f}{\partial t_1}(x_1, x_2) + (t_2 - x_2) \frac{\partial f}{\partial t_2}(x_1, x_2) \\
&+ \frac{(t_1 - x_1)^2}{2} \frac{\partial^2 f}{\partial t_1^2}(\xi_1, \xi_2) + (t_1 - x_1)(t_2 - x_2) \frac{\partial^2 f}{\partial t_1 \partial t_2}(\xi_1, \xi_2) + \frac{(t_2 - x_2)^2}{2} \frac{\partial^2 f}{\partial t_2^2}(\xi_1, \xi_2), \quad (6.4)
\end{aligned}$$

where $\xi_i = t_i + \theta(x_i - t_i)$, $i = 1, 2$, $0 < \theta < 1$. Multiplyling (6.4) by w and integrating produces the identity

$$\begin{aligned}
&\int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1, t_2) f(t_1, t_2)\, dt_2 dt_1 - f(x_1, x_2) \int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1, t_2)\, dt_2 dt_1 \\
&\qquad + \frac{\partial f}{\partial t_1}(x_1, x_2) \int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1, t_2)(x_1 - t_1)\, dt_2 dt_1 \\
&\qquad + \frac{\partial f}{\partial t_2}(x_1, x_2) \int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1, t_2)(x_2 - t_2)\, dt_2 dt_1 \\
&\qquad = \int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1, t_2) \frac{(t_1 - x_1)^2}{2} \frac{\partial^2 f}{\partial t_1^2}(\xi_1, \xi_2)\, dt_2 dt_1 \\
&\qquad + \int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1, t_2)(t_1 - x_1)(t_2 - x_2) \frac{\partial^2 f}{\partial t_1 \partial t_2}(\xi_1, \xi_2)\, dt_2 dt_1 \\
&\qquad + \int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1, t_2) \frac{(t_2 - x_2)^2}{2} \frac{\partial^2 f}{\partial t_2^2}(\xi_1, \xi_2),\, dt_2 dt_1. \qquad (6.5)
\end{aligned}$$

Taking the modulus of both sides of (6.5), applying the triangle inequality and then Hölder's inequality on the right hand side gives (6.3). □

Corollary 6.4. *Let the conditions for f be as in Theorem 6.3. Then the following double integral inequality holds*

$$\left|\frac{1}{(b_1-a_1)(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(t_1,t_2)\,dt_2dt_1 - f(x_1,x_2) + \frac{\partial f}{\partial t_1}(x_1,x_2)\left(x_1-\frac{a_1+b_1}{2}\right)\right.$$
$$\left.+\frac{\partial f}{\partial t_2}(x_1,x_2)\left(x_1-\frac{a_1+b_1}{2}\right)\right| \leq \left\|\frac{\partial^2 f}{\partial t_1^2}\right\|_\infty \frac{(b_1-a_1)^2}{2}\left(\frac{\left(x_1-\frac{a_1+b_1}{2}\right)^2}{(b_1-a_1)^2}+\frac{1}{12}\right)$$
$$+\left\|\frac{\partial^2 f}{\partial t_1\partial t_2}\right\|_\infty (b_1-a_1)(b_2-a_2)\left(\frac{\left(x_1-\frac{a_1+b_1}{2}\right)^2}{(b_1-a_1)^2}+\frac{1}{4}\right)\left(\frac{\left(x_2-\frac{a_2+b_2}{2}\right)^2}{(b_2-a_2)^2}+\frac{1}{4}\right)$$
$$+\left\|\frac{\partial^2 f}{\partial t_2^2}\right\|_\infty \frac{(b_2-a_2)^2}{2}\left(\frac{\left(x_2-\frac{a_2+b_2}{2}\right)^2}{(b_2-a_2)^2}+\frac{1}{12}\right) \tag{6.6}$$

Proof. Substituting $w(t_1,t_2)=1$ into (6.3) and simplifying produces the desired result. □

The point (x_1,x_2), the sample point of the integration rule, is free to be chosen. Often, such points are chosen to simplify the rule. For example, in (6.3) if we choose the weight mean

$$x_i = \frac{\int_{a_1}^{b_1}\int_{a_2}^{b_2} t_i w(t_1,t_2)\,dt_2dt_1}{\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)\,dt_2dt_1}, \qquad i=1,2$$

then the partial derivative terms vanish. Fortuitously, in this case, this point also minimizes the bound. In the following sub-section, and indeed this chapter, we will not be concerned with simplifying the integration rule, but instead attempt to determine such parameters in order for the error bound to be minimized.

6.1.1 Minimizing the upper bound

Corollary 6.5. *The bound in equation (6.2) is minimized at the median point (x_1,x_2) satisfying*

$$\int_{a_1}^{x_1}\int_{a_2}^{b_2} w(t_1,t_2)\,dt_2dt_1 = \int_{x_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)\,dt_2dt_1 \tag{6.7}$$

and

$$\int_{a_2}^{x_2}\int_{a_1}^{b_1} w(t_1,t_2)\,dt_1dt_2 = \int_{x_2}^{b_2}\int_{a_1}^{b_1} w(t_1,t_2)\,dt_1dt_2. \tag{6.8}$$

Proof. It is a simple matter to show that

$$I(x_1,x_2) = M_1 \int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)|x_1-t_1|\,dt_2dt_1 + M_2 \int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)|x_2-t_2|\,dt_2dt_1$$

is a convex function. Hence the upper bound in (6.2) is minimized at the stationary point of I. Evaluating the first partial derivatives of I produces equations (6.7) and (6.8). □

That is, the minimum point is the "median" of the weight in each direction. This is consistent with first order rules reported in Roumeliotis (2001).

Minimization of the second order bound in Theorem 6.3 is not as simple. It is quite difficult to identify a minimum point for the upper bound of (6.3). This bound is comprised of three components; the first and last are minimized at the mean (in each direction)

$$x_1 = \frac{\int_{a_1}^{b_1}\int_{a_2}^{b_2} t_1 w(t_1,t_2)\,dt_2dt_1}{\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)\,dt_2dt_1}, \qquad x_2 = \frac{\int_{a_1}^{b_1}\int_{a_2}^{b_2} t_2 w(t_1,t_2)\,dt_2dt_1}{\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)\,dt_2dt_1}, \tag{6.9}$$

while the second is minimized at the root of a median-type expression

$$\int_{a_1}^{x_1}\int_{a_2}^{b_2} |x_2-t_2|w(t_1,t_2)\,dt_2dt_1 = \int_{x_1}^{b_1}\int_{a_2}^{b_2} |x_2-t_2|w(t_1,t_2)\,dt_2dt_1 \tag{6.10}$$

and

$$\int_{a_2}^{x_2}\int_{a_1}^{b_1} |x_1-t_1|w(t_1,t_2)\,dt_1dt_2 = \int_{x_2}^{b_2}\int_{a_1}^{b_1} |x_1-t_1|w(t_1,t_2)\,dt_1dt_2. \tag{6.11}$$

Of course, for weights in which the solutions of (6.9) are identical to those of (6.10) and (6.11) then identification of the minimum point presents little challenge. For example if w is a product weight and symmetric about the midpoint $(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2})$ then the minimum point is the midpoint. That is, if $w(t_1,t_2) = w_1(t_1)w_2(t_2)$ and $w_i((a+b)/2-t) = w_i((a+b)/2+t)$ $(i = 1,2)$, then it can be shown that the solution of (6.9)–(6.11) is the mid-point. This is the case when $w = 1$ and Corollary 6.4 shows that the upper bound is minimized at $x_i = (a_i+b_i)/2$, $i = 1,2$.

The major difficulty with (6.3) is that the upper bound is comprised of a linear combination of three terms involving norms of the partial derivative of the integrand. Hence it would be near impossible to find a global minimum that depends only on the weight and not f. To obtain a global minimum for a general second order rule will require either simplification of (6.3) or the derivation of another expression for the bound. The first point is dealt with in the corollary below, while the second is taken up in the next section.

Corollary 6.6. *Let f and w be as given in Theorem 6.3. Then for all $(x_1, x_2) \in [a_1, b_1] \times [a_2, b_2]$ the following second order product double integral inequality holds*

$$\begin{aligned}
&\bigg| \int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1,t_2) f(t_1,t_2)\, dt_2 dt_1 - f(x_1,x_2) \int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1,t_2)\, dt_2 dt_1 \\
&+ \frac{\partial f}{\partial t_1}(x_1,x_2) \int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1,t_2)(x_1 - t_1)\, dt_2 dt_1 + \frac{\partial f}{\partial t_2}(x_1,x_2) \int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1,t_2)(x_2 - t_2)\, dt_2 dt_1 \bigg| \\
&\le \left\| \frac{\partial^2 f}{\partial t_1^2} \right\|_\infty \frac{\|w\|_1}{2} \left[\left| x_1 - \frac{a_1+b_1}{2} \right| + \frac{b_1 - a_1}{2} \right]^2 + \left\| \frac{\partial^2 f}{\partial t_2^2} \right\|_\infty \frac{\|w\|_1}{2} \left[\left| x_2 - \frac{a_2+b_2}{2} \right| + \frac{b_2 - a_2}{2} \right]^2 \\
&\quad + \left\| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right\|_\infty \|w\|_1 \left[\left| x_1 - \frac{a_1+b_1}{2} \right| + \frac{b_1 - a_1}{2} \right] \left[\left| x_2 - \frac{a_2+b_2}{2} \right| + \frac{b_2 - a_2}{2} \right], \qquad (6.12)
\end{aligned}$$

where $\|w\|_1 = \int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1,t_2)\, dt_2 dt_1$ is the zero-th moment of the weight.

Proof. The proof involves taking an upper bound of (6.3) using Hölder's inequality. Thus, consider

$$\begin{aligned}
\int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1,t_2)(x_1 - t_1)^2\, dt_2 dt_1 &\le \sup_{t_1 \in [a_1,b_1]} (x_1 - t_1)^2 \int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1,t_2)\, dt_2 dt_1 \\
&= \max\{(x_1 - a_1)^2, (x_1 - b_1)^2\} \|w\|_1 \\
&= \left[\left| x_1 - \frac{a_1+b_1}{2} \right| + \frac{b_1 - a_1}{2} \right]^2 \|w\|_1. \qquad (6.13)
\end{aligned}$$

Similarly

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1,t_2)(x_2 - t_2)^2\, dt_2 dt_1 \le \left[\left| x_2 - \frac{a_2+b_2}{2} \right| + \frac{b_2 - a_2}{2} \right]^2 \|w\|_1. \qquad (6.14)$$

Finally,

$$\begin{aligned}
&\int_{a_1}^{b_1} \int_{a_2}^{b_2} w(t_1,t_2) |x_1 - t_1| |x_2 - t_2|\, dt_2 dt_1 \\
&\qquad \le \sup_{(t_1,t_2) \in [a_1,b_1] \times [a_2,b_2]} |x_1 - t_1||x_2 - t_2| \|w\|_1 \\
&\qquad = \max\{x_1 - a_1, b_1 - x_1\} \max\{x_2 - a_2, b_2 - x_2\} \|w\|_1 \\
&\qquad = \left[\left| x_1 - \frac{a_1+b_1}{2} \right| + \frac{b_1 - a_1}{2} \right] \left[\left| x_2 - \frac{a_2+b_2}{2} \right| + \frac{b_2 - a_2}{2} \right] \|w\|_1 \qquad (6.15)
\end{aligned}$$

Making use of (6.13), (6.14) and (6.15) gives (6.3). $\square$

It is clear that the bound in (6.12) is minimized at the mid-point of the rectangular region. Unfortunately, the weight does not influence this minimum point.

Taylor's theorem is a popular vehicle for developing cubature and higher dimension rules. Stroud (1971) uses Taylor's expansion to develop cubature rules and recently Qi (2001), used this technique to derive weighted Iyengar-type multiple integrals. The drawback is in the size of the error bound. For two dimensions, an n-th order rule has a Taylor remainder of $n+1$ terms. Minimizing any rule with order greater than one would be extremely difficult. Thus, in the next section, we turn to the Peano kernel and use the results of Chapter 3 to derive a second order weighted double integral inequality that contains only one term in the upper bound.

6.2 Main Results

Lemma 6.1. *Let $f : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ be bounded and integrable and whose first partial derivatives exist and are also bounded and integrable. Furthermore, let $w : (a_1, b_1) \times (a_2, b_2) \to (0, \infty)$ be integrable. Then following identity holds*

$$I = \int_{a_1}^{b_1} \int_{a_2}^{b_2} [f(x_1, x_2) - f(x_1, t_2) - f(t_1, x_2) + f(t_1, t_2)]\, w(t_1, t_2)\, dt_2 dt_1$$
$$= \int_{a_1}^{b_1} \int_{a_2}^{b_2} P(t_1, t_2) \frac{\partial^2 f}{\partial t_1 \partial t_2}\, dt_2 dt_1 \quad (6.16)$$

where $x_1 \in [a_1, b_1]$, $x_2 \in [a_2, b_2]$ and

$$P(t_1, t_2) = \begin{cases} \displaystyle\int_{a_2}^{t_2} p(t_1, u_2)\, du_2, & a_2 \le t_2 \le x_2, \\ \displaystyle\int_{b_2}^{t_2} p(t_1, u_2)\, du_2, & x_2 < t_2 \le b_2, \end{cases} \quad (6.17)$$

$$p(t_1, t_2) = \begin{cases} \displaystyle\int_{a_1}^{t_1} w(u_1, t_2)\, du_1, & a_1 \le t_1 \le x_1, \\ \displaystyle\int_{b_1}^{t_1} w(u_1, t_2)\, du_1, & x_1 < t_1 \le b_1. \end{cases} \quad (6.18)$$

Proof. To begin, let $I = \int_{a_1}^{b_1} I_2\, dt_1$ and consider I_2 where

$$I_2 = \int_{a_2}^{b_2} P(t_1, t_2) \frac{\partial^2 f(t_1, t_2)}{\partial t_1 \partial t_2}\, dt_2$$
$$= \int_{a_2}^{x_2} P(t_1, t_2) \frac{\partial^2 f(t_1, t_2)}{\partial t_1 \partial t_2}\, dt_2 + \int_{x_2}^{b_2} P(t_1, t_2) \frac{\partial^2 f(t_1, t_2)}{\partial t_1 \partial t_2}\, dt_2$$

$$= \int_{a_2}^{x_2} \left(\int_{a_2}^{t_2} p(t_1, u_2) du_2 \right) \frac{\partial^2 f(t_1, t_2)}{\partial t_1 \partial t_2} dt_2 + \int_{x_2}^{b_2} \left(\int_{b_2}^{t_2} p(t_1, u_2) du_2 \right) \frac{\partial^2 f(t_1, t_2)}{\partial t_1 \partial t_2} dt_2$$
$$= I_{21} + I_{22}.$$

Using integration by parts, we find that

$$\begin{aligned} I_{21} &= \int_{a_2}^{t_2} p(t_1, u_2) du_2 \left. \frac{\partial f(t_1, t_2)}{\partial t_1} \right|_{a_2}^{x_2} - \int_{a_2}^{x_2} \frac{\partial f(t_1, t_2)}{\partial t_1} p(t_1, t_2)\, dt_2 \\ &= \int_{a_2}^{x_2} p(t_1, u_2) du_2 \frac{\partial f(t_1, x_2)}{\partial t_1} - \int_{a_2}^{x_2} \frac{\partial f(t_1, t_2)}{\partial t_1} p(t_1, t_2)\, dt_2 \\ &= \int_{a_2}^{x_2} p(t_1, t_2) \left(\frac{\partial f(t_1, x_2)}{\partial t_1} - \frac{\partial f(t_1, t_2)}{\partial t_1} \right) dt_2. \end{aligned}$$

Similarly

$$I_{22} = \int_{x_2}^{b_2} p(t_1, t_2) \left(\frac{\partial f(t_1, x_2)}{\partial t_1} - \frac{\partial f(t_1, t_2)}{\partial t_1} \right) dt_2.$$

Thus I_2 becomes

$$I_2 = \int_{a_2}^{b_2} p(t_1, t_2) \left(\frac{\partial f(t_1, x_2)}{\partial t_1} - \frac{\partial f(t_1, t_2)}{\partial t_1} \right) dt_2$$

and substituting into I gives

$$\begin{aligned} I = \int_{a_1}^{b_1} \int_{a_2}^{b_2} P(t_1, t_2) \frac{\partial^2 f}{\partial t_1 \partial t_2} dt_2 dt_1 &= \int_{a_1}^{b_1} \int_{a_2}^{b_2} p(t_1, t_2) \left(\frac{\partial f(t_1, x_2)}{\partial t_1} - \frac{\partial f(t_1, t_2)}{\partial t_1} \right) dt_2 dt_1 \\ &= \int_{a_2}^{b_2} \int_{a_1}^{b_1} p(t_1, t_2) \left(\frac{\partial f(t_1, x_2)}{\partial t_1} - \frac{\partial f(t_1, t_2)}{\partial t_1} \right) dt_1 dt_2 \\ &= \int_{a_2}^{b_2} I_3\, dt_2, \end{aligned} \tag{6.19}$$

where

$$I_3 = \int_{a_1}^{b_1} p(t_1, t_2) \left(\frac{\partial f(t_1, x_2)}{\partial t_1} - \frac{\partial f(t_1, t_2)}{\partial t_1} \right) dt_1.$$

Applying the same treatment to I_3 as for I_2 gives

$$I_3 = \int_{a_1}^{b_1} w(t_1, t_2)[f(x_1, x_2) - f(t_1, x_2) - f(x_1, t_2) + f(t_1, t_2)]\, dt_1.$$

Substituting I_3 into (6.19) we find that the identity (6.16) is thus proved. $\square$

The upper bound of the integration rule will depend on P. Below, we detail some properties of P that will be subsquently used in analysis of the bound.

Lemma 6.2. *The kernel $P : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ as defined in Lemma 6.1 has the following properties:*

1. *P vanishes on the boundary of the rectangle $[a_1, b_1] \times [a_2, b_2]$,*

2. *$P(t_1, \cdot) : (a_2, b_2) \to \mathbb{R}$ is monotonic increasing for all $t_1 \in (a_1, x_1)$,*

3. *$P(t_1, \cdot) : (a_2, b_2) \to \mathbb{R}$ is monotonic decreasing for all $t_1 \in (x_1, b_1)$,*

4. *P is positive on $(a_1, x_1) \times (a_2, x_2)$ and $(x_1, b_1) \times (x_2, b_2)$,*

5. *P is negative on $(a_1, x_1) \times (x_2, b_2)$ and $(x_1, b_1) \times (a_2, x_2)$.*

for all $(x_1, x_2) \in (a_1, b_1) \times (a_2, b_2)$.

Proof. These properties are quite simple to prove via inspection of the first partial derivatives of P. $\square$

In Figure 6.1, we plot the surface and contours of (6.17) for two different weights. The plots exhibit the properties discussed in Lemma 6.2. It is obvious that the kernel achieves its maximum deviation on of its branches at the discontinuos point (x_1, x_2).

In the following theorem we state the main result by employing the identity in Lemma 6.1 to produce second order weighted double integral inequalities. In contrast with the inequalities of the previous section, the upper bound here is comprised of just one term.

THEOREM 6.7. *Let the conditions of Lemma 6.1 hold. The following double integral inequalities involving the usual Lebesgue norms of the first mixed partial derivative of f hold,*

$$|I| \leq \left\| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right\|_\infty \int_{a_1}^{b_1} \int_{a_2}^{b_2} |x_1 - t_1| \, |x_2 - t_2| \, w(t_1, t_2) \, dt_1 dt_2, \tag{6.20}$$

if $\frac{\partial^2 f}{\partial t_1 \partial t_2} \in L_\infty[a_1, b_1] \times [a_2, b_2]$ and

$$\begin{aligned} |I| \leq \left\| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right\|_1 \max\Bigg\{ & \int_{a_1}^{x_1} \int_{a_2}^{x_2} w(t_1, t_2) \, dt_2 dt_1, \int_{a_1}^{x_1} \int_{x_2}^{b_2} w(t_1, t_2) \, dt_2 dt_1, \\ & \int_{x_1}^{b_1} \int_{a_2}^{x_2} w(t_1, t_2) \, dt_2 dt_1, \int_{x_1}^{b_1} \int_{x_2}^{b_2} w(t_1, t_2) \, dt_2 dt_1 \Bigg\} \end{aligned} \tag{6.21}$$

if $\frac{\partial^2 f}{\partial t_1 \partial t_2} \in L_1[a_1, b_1] \times [a_2, b_2]$, where I is defined in equation (6.16).

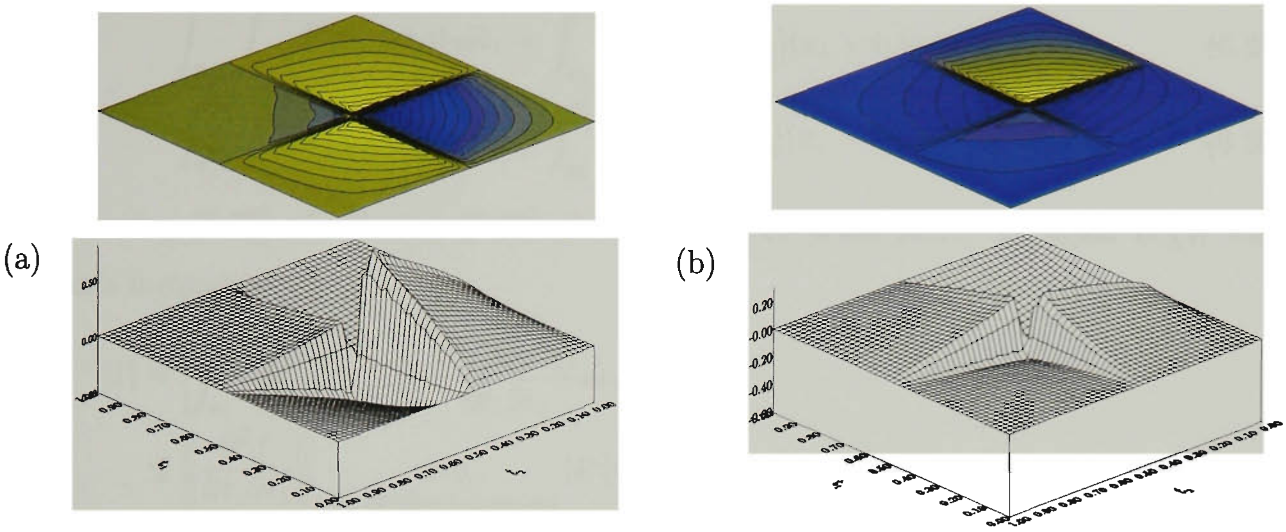


Figure 6.1: Surface and contour plots of the Peano type kernels P defined in (6.17) for different weights. (a) $w(t_1,t_2) = -\ln(t_1t_2)$ over the unit square and $x_1 = x_2 = 0.5$, (b) $w(t_1,t_2) = \sqrt{t_1/t_2}$ over the unit square and $x_1 = x_2 = 0.5$.

Proof. To prove (6.20) we begin with Hölder's inequality and then simplify using Lemma 6.2

$$\begin{aligned}
|I| &= \left|\int_{a_1}^{b_1}\int_{a_2}^{b_2} P(t_1,t_2)\frac{\partial^2 f\,(t_1,t_2)}{\partial t_1\partial t_2}dt_1dt_2\right| \\
&\le \left\|\frac{\partial^2 f}{\partial t_1\partial t_2}\right\|_\infty \int_{a_1}^{b_1}\int_{a_2}^{b_2}|P(t_1,t_2)|\,dt_2dt_1 \\
&= \left\|\frac{\partial^2 f}{\partial t_1\partial t_2}\right\|_\infty \left(\int_{a_1}^{x_1}\int_{a_2}^{x_2} P(t_1,t_2)\,dt_2dt_1 - \int_{a_1}^{x_1}\int_{x_2}^{b_2} P(t_1,t_2)\,dt_2dt_1\right. \\
&\qquad \left. - \int_{x_1}^{b_1}\int_{a_2}^{x_2} P(t_1,t_2)\,dt_2dt_1 + \int_{x_1}^{b_1}\int_{x_2}^{b_2} P(t_1,t_2)\,dt_2dt_1\right). \qquad (6.22)
\end{aligned}$$

Now each of the terms in (6.22) can be evaluated via partial integration and simplified using Lemma 6.1 and equations (6.17) and (6.18). For the first term

$$\begin{aligned}
\int_{a_1}^{x_1}\int_{a_2}^{x_2} P(t_1,t_2)\,dt_2dt_1 &= \int_{a_1}^{x_1}\left\{(t_2-x_2)P\Big|_{a_2}^{x_2} - \int_{a_2}^{x_2}(t_2-x_2)p\,dt_2\right\}dt_1 \\
&= -\int_{a_2}^{x_2}\int_{a_1}^{x_1}(t_2-x_2)p\,dt_1dt_2 \\
&= -\int_{a_2}^{x_2}(t_2-x_2)\left\{(t_1-x_1)p\Big|_{a_1}^{x_1} - \int_{a_1}^{x_1}(t_1-x_1)w\,dt_1\right\}dt_2 \\
&= \int_{a_2}^{x_2}\int_{a_1}^{x_1}(x_2-t_2)(x_1-t_1)w(t_1,t_2)\,dt_1dt_2. \qquad (6.23)
\end{aligned}$$

Employing the same procedure for the other terms we find

$$\int_{a_1}^{x_1}\int_{x_2}^{b_2} P(t_1,t_2)\,dt_2dt_1 = \int_{a_1}^{x_1}\int_{x_2}^{b_2}(x_2-t_2)(x_1-t_1)w(t_1,t_2)\,dt_1dt_2, \qquad (6.24)$$

$$\int_{x_1}^{b_1}\int_{a_2}^{x_2} P(t_1,t_2)\,dt_2dt_1 = \int_{x_1}^{b_1}\int_{a_2}^{x_2}(x_2-t_2)(x_1-t_1)w(t_1,t_2)\,dt_1dt_2, \tag{6.25}$$

$$\int_{x_1}^{b_1}\int_{x_2}^{b_2} P(t_1,t_2)\,dt_2dt_1 = \int_{x_1}^{b_1}\int_{x_2}^{b_2}(x_2-t_2)(x_1-t_1)w(t_1,t_2)\,dt_1dt_2. \tag{6.26}$$

Substituting (6.23)–(6.26) into (6.22) gives (6.20). To prove (6.21) we again begin with Hölder's inequality

$$\begin{aligned}
|I| &= \left|\int_{a_1}^{b_1}\int_{a_2}^{b_2} P(t_1,t_2)\frac{\partial^2 f(t_1,t_2)}{\partial t_1\partial t_2}dt_1dt_2\right| \\
&\leq \left\|\frac{\partial^2 f}{\partial t_1\partial t_2}\right\|_1 \sup_{(t_1,t_2)\in[a_1,b_1]\times[a_2,b_2]}|P(t_1,t_2)| \\
&= \left\|\frac{\partial^2 f}{\partial t_1\partial t_2}\right\|_1 \max\left\{\int_{a_1}^{x_1}\int_{a_2}^{x_2} w(t_1,t_2)\,dt_2dt_1, \int_{a_1}^{x_1}\int_{x_2}^{b_2} w(t_1,t_2)\,dt_2dt_1,\right. \\
&\qquad \left.\int_{x_1}^{b_1}\int_{a_2}^{x_2} w(t_1,t_2)\,dt_2dt_1, \int_{x_1}^{b_1}\int_{x_2}^{b_2} w(t_1,t_2)\,dt_2dt_1\right\}.
\end{aligned} \tag{6.27}$$

The last line being computed by appealing to the properties of P as listed in Lemma 6.2. Thus the theorem is proved. □

If the first moments of the weight w are known, as well as the "one dimensional" integrals

$$\int_{a_1}^{b_1} f(t_1,x_2)\left(\int_{a_2}^{b_2} w(t_1,t_2)\,dt_2\right)dt_1 \quad \text{and} \quad \int_{a_2}^{b_2} f(x_1,t_2)\left(\int_{a_1}^{b_1} w(t_1,t_2)\,dt_1\right)dt_2 \tag{6.28}$$

then (6.20) can form the basis of a cubature formula for the evaluation of the weighted double integral $\iint_D f(t_1,t_2)w(t_1,t_2)\,dA$ over a rectangular region D. A major drawback is that is most cases the integrals (6.28) are unknown. These can be eliminated using the one-dimensional weighted results in Roumeliotis *et al.* (1999). It was shown that for mappings f with bounded second derivative that

$$\left|\int_a^b w(t)f(t)\,dt - f(x)\int_a^b w(t)\,dt + f'(x)\int_a^b (x-t)w(t)\,dt\right| \leq \frac{\|f''\|_\infty}{2}\int_a^b (x-t)^2 w(t)\,dt, \tag{6.29}$$

where $x\in(a,b)$ and w is a weight function. Thus making use of (6.29), the following inequalities hold

$$\begin{aligned}
&\left|\int_{a_1}^{b_1} f(t_1,x_2)\left(\int_{a_2}^{b_2} w(t_1,t_2)\,dt_2\right)dt_1 - f(x_1,x_2)\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)\,dt_2\,dt_1\right. \\
&\left.+\frac{\partial f}{\partial t_1}(x_1,x_2)\int_{a_1}^{b_1}\int_{a_2}^{b_2}(x_1-t_1)w(t_1,t_2)\,dt_2\,dt_1\right| \leq \left\|\frac{\partial^2 f}{\partial t_1^2}\right\|_\infty \int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)\frac{(x_1-t_1)^2}{2}\,dt_2dt_1
\end{aligned} \tag{6.30}$$

and

$$\left| \int_{a_2}^{b_2} f(x_1,t_2)\left(\int_{a_1}^{b_1} w(t_1,t_2)\,dt_1\right)dt_2 - f(x_1,x_2)\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)\,dt_2\,dt_1 \right.$$
$$\left. + \frac{\partial f}{\partial t_2}(x_1,x_2)\int_{a_1}^{b_1}\int_{a_2}^{b_2}(x_2-t_2)w(t_1,t_2)\,dt_2\,dt_1\right| \le \left\|\frac{\partial^2 f}{\partial t_2^2}\right\|_\infty \int_{a_1}^{b_1}\int_{a_2}^{b_2} w(t_1,t_2)\frac{(x_2-t_2)^2}{2}\,dt_2dt_1. \tag{6.31}$$

It is of interest to note that combining (6.20), (6.30) and (6.31) will produce (6.3). Thus, in one sense, (6.20) is more general than (6.3) since it is not obvious how one may derive (6.20) from (6.3).

One advantange of (6.20) over (6.3) is that the upper bound involves one term instead of three. Thus, with (6.20) we can find points (x_1, x_2) that will minimize the upper bound in terms of the weight and independent of the integrand. In the following corollary we will identify points (x_1, x_2) to minimize the bound

$$\mathcal{J}(x_1,x_2) = \int_{a_1}^{b_1}\int_{a_2}^{b_2} |x_1-t_1|\,|x_2-t_2|\,w\,(t_1,t_2)\,dt_2dt_1. \tag{6.32}$$

Corollary 6.8. *$\mathcal{J}(x_1,x_2)$ as defined in (6.32) is minimized at (x_1^*, x_2^*) where x_1^* and x_2^* satisfy the equations*

$$\int_{a_1}^{x_1^*}\int_{a_2}^{b_2} |x_2^*-t_2|w(t_1,t_2)\,dt_2dt_1 = \int_{x_1^*}^{b_1}\int_{a_2}^{b_2} |x_2^*-t_2|w(t_1,t_2)\,dt_2dt_1 \tag{6.33}$$

and

$$\int_{a_2}^{x_2^*}\int_{a_1}^{b_1} |x_1^*-t_1|w(t_1,t_2)\,dt_2dt_1 = \int_{x_2^*}^{b_2}\int_{a_1}^{b_1} |x_1^*-t_1|w(t_1,t_2)\,dt_2dt_1. \tag{6.34}$$

Proof. Evaluating the partial derivatives of $\mathcal{J}$ gives

$$\mathcal{J}^{(1)} = \frac{\partial \mathcal{J}}{\partial x_1}(x_1,x_2) = \int_{a_1}^{x_1}\int_{a_2}^{b_2} |x_2-t_2|w(t_1,t_2)\,dt_2dt_1 - \int_{x_1}^{b_1}\int_{a_2}^{b_2} |x_2-t_2|w(t_1,t_2)\,dt_2dt_1, \tag{6.35}$$

$$\mathcal{J}^{(2)} = \frac{\partial \mathcal{J}}{\partial x_2}(x_1,x_2) = \int_{a_2}^{x_2}\int_{a_1}^{b_1} |x_1-t_1|w(t_1,t_2)\,dt_1dt_2 - \int_{x_2}^{b_2}\int_{a_1}^{b_1} |x_1-t_1|w(t_1,t_2)\,dt_1dt_2. \tag{6.36}$$

Inspection of (6.35) reveals that, for fixed x_2, $\mathcal{J}^{(1)}$ is monotonic increasing and $\mathcal{J}^{(1)}(a_1,x_2) = -\mathcal{J}^{(2)}(b_1,x_2) \le 0$. $\mathcal{J}^{(2)}$ also exhibits similar properties and hence there exists a unique point (x_1^*, x_2^*) that is the zero of (6.35) and (6.36) and minimizes $\mathcal{J}$. □

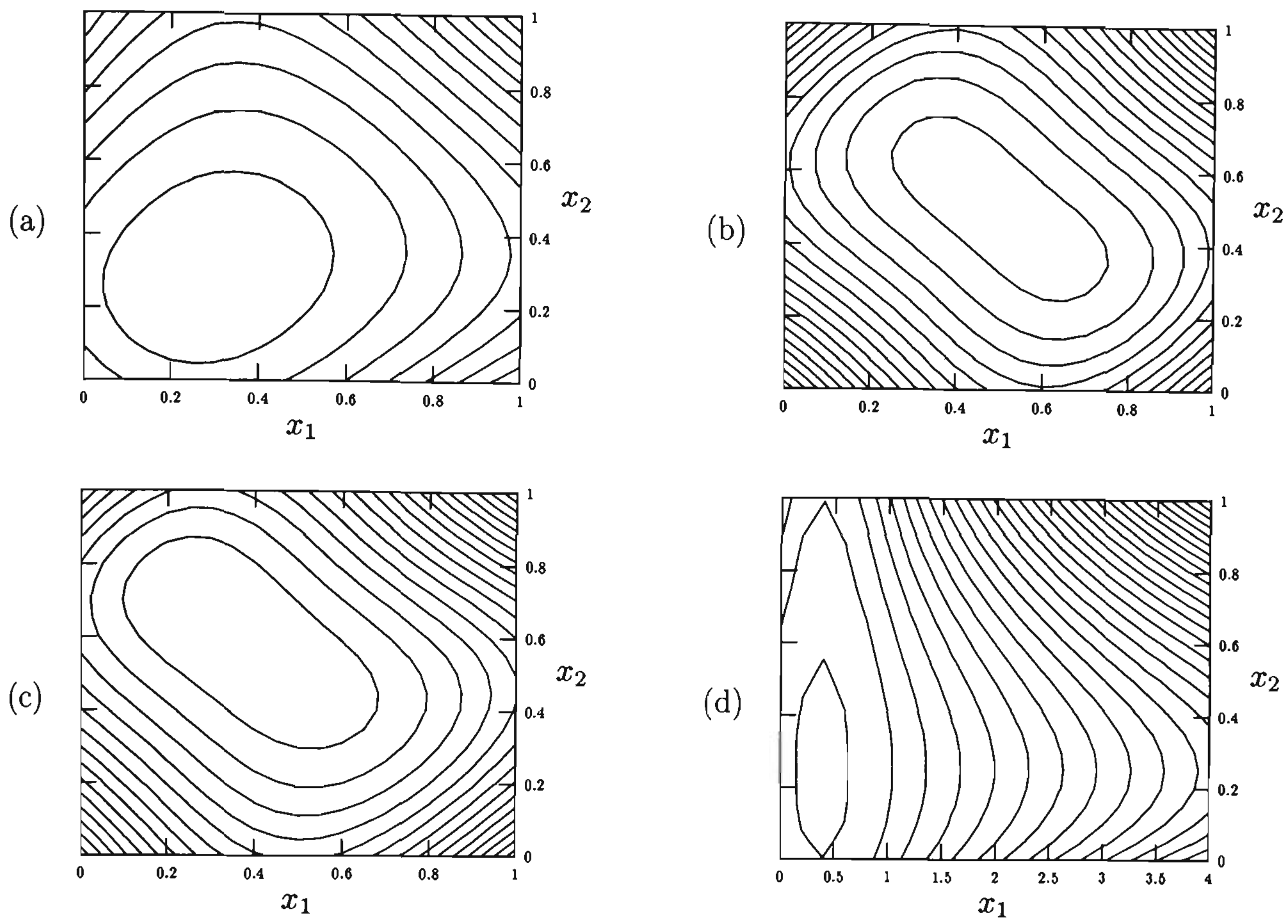


Figure 6.2: Contour plots of the $\mathcal{J}(x_1, x_2)$ given by (6.32) for various weight functions. (a) $w(t_1, t_2) = -\ln(t_1 t_2)$, $(t_1, t_2) \in (0,1) \times (0,1)$, (b) $w(t_1, t_2) = -\ln|t_1 - t_2|$, $(t_1, t_2) \in (0,1) \times (0,1)$, (c) $w(t_1, t_2) = -\ln|t_1 - t_2^2|$, $(t_1, t_2) \in (0,1) \times (0,1)$ and (d) $w(t_1, t_2) = e^{-t_1}/\sqrt{t_2}$, $(t_1, t_2) \in (0,4) \times (0,1)$.

The behaviour of (6.32) is very dependant on the behaviour of the weight. In Figure 6.2 contours of $\mathcal{J}$ are plotted for different weight functions. In each case, the minimum point is readily observed and its location depends on the weight and weight null-space.

In the following section, properties of the minimum point of $\mathcal{J}$ are identified for various conditions on w.

6.3 Minimizing the bound

Solution of equations (6.33) and (6.34) provide the point that minimizes the bound (6.32). The equations are non-linear and two dimensional, thus, in most cases, require numerical treatment. In this section we identify solutions or simplifications to (6.33) and (6.34) for specific weight types. Some of these weights are of importance since they appear in the important areas of integral transforms and integral equations.

With functions of two or more variables it is common that an identifiable relationship between the variables is observed. That is, $w(t_1, t_2) = w(\phi(t_1, t_2))$ for some ϕ. For singular weights, the null-space of ϕ, $\{(t_1, t_2) : \phi(t_1, t_2) = 0\}$, is of interest since this gives rise to a mapping which furnishes the singularity structure. Below, we explore the properties of $\mathcal{J}$ for ϕ being the difference mapping on a square and generalise to more general null-spaces in other corollaries.

Corollary 6.9 (Difference weight). *Let $w : (a, b) \to (0, \infty)$ be integrable and let $a < x_1, x_2 < b$. Then the bound*

$$\mathcal{J}(x_1, x_2) = \int_a^b \int_a^b |x_1 - t_1| \, |x_2 - t_2| \, w \, |t_1 - t_2| \, dt_2 dt_1.$$

is minimized at the midpoint $x_1 = x_2 = \frac{a+b}{2}$.

Proof. As stated in Corollary 6.8, $\mathcal{J}$ is minimized at the root of equations (6.33) and (6.34). Substituting the midpoint in (6.33) gives

$$\begin{aligned}
&\int_a^{(a+b)/2} \int_a^b \left| \frac{a+b}{2} - t_2 \right| w|t_1 - t_2| \, dt_2 dt_1 - \int_{(a+b)/2}^b \int_a^b \left| \frac{a+b}{2} - t_2 \right| w|t_1 - t_2| \, dt_2 dt_1 \\
&= \int_{(a+b)/2}^b \int_a^b \left| \frac{a+b}{2} - v \right| w|u - v| \, dv du - \int_{(a+b)/2}^b \int_a^b \left| \frac{a+b}{2} - t_2 \right| w|t_1 - t_2| \, dt_2 dt_1 \\
&= 0,
\end{aligned}$$

where $u = a + b - t_1$ and $v = a + b - t_2$ are integral substitutions. The same treatment on (6.34) shows that the midpoint minimizes the bound ☐

The following two corollaries show that the simultaneous equations (6.33) and (6.34) may be decoupled under certain conditions for the weight.

Corollary 6.10 (Separable weight). *Let the conditions in Corollary 6.8 hold. Furthermore, let w be separable, that is $w(t_1, t_2) = w_1(t_1) w_2(t_2)$, where w_i are themselves weight functions defined on $[a_i, b_i]$, i=1,2. Then $\mathcal{J}$ is minimized at the median of each weight*

$$\int_{a_i}^{x_i} w_i(t_i) \, dt_i = \int_{x_i}^{b_i} w_i(t_i) \, dt_i, \qquad i = 1, 2.$$

Proof. Substituting $w(t_1,t_2) = w_1(t_1)w_2(t_2)$ into (6.33) and (6.34) and simplifying produces the result. $\square$

Corollary 6.11 (Symmetric weight). *Let the conditions in Corollary 6.8 hold and let $w : (a,b) \times (a,b) \to \mathbb{R}$ be symmetric, that is, $w(t_1,t_2) = w(t_2,t_1)$. Then the minimum point is at $x_1 = x_2$.*

Proof. With the above conditions, the two equations in Corollary 6.8 are

$$\frac{\partial \mathcal{J}}{\partial x_1}(x_1,x_2) = \int_a^{x_1}\int_a^b |x_2 - t_2| w(t_1,t_2)\,dt_2dt_1 - \int_{x_1}^b\int_a^b |x_2 - t_2| w(t_1,t_2)\,dt_2dt_1, \tag{6.37}$$

$$\frac{\partial \mathcal{J}}{\partial x_2}(x_1,x_2) = \int_a^{x_2}\int_a^b |x_1 - t_1| w(t_1,t_2)\,dt_1dt_2 - \int_{x_2}^b\int_a^b |x_1 - t_1| w(t_1,t_2)\,dt_1dt_2. \tag{6.38}$$

Beginning with (6.37) we have

$$\begin{aligned}
\frac{\partial \mathcal{J}}{\partial x_1}(x_1,x_2) &= \int_a^{x_1}\int_a^b |x_2 - t_2| w(t_1,t_2)\,dt_2dt_1 - \int_{x_1}^b\int_a^b |x_2 - t_2| w(t_1,t_2)\,dt_2dt_1 \\
&= \int_a^{x_1}\int_a^b |x_2 - t_1| w(t_2,t_1)\,dt_1dt_2 - \int_{x_1}^b\int_a^b |x_2 - t_1| w(t_2,t_1)\,dt_1dt_2 \\
&= \int_a^{x_1}\int_a^b |x_2 - t_1| w(t_1,t_2)\,dt_1dt_2 - \int_{x_1}^b\int_a^b |x_2 - t_1| w(t_1,t_2)\,dt_1dt_2 \\
&= \frac{\partial \mathcal{J}}{\partial x_2}(x_2,x_1).
\end{aligned}$$

Thus the solution of

$$\frac{\partial \mathcal{J}}{\partial x_1}(x_1,x_2) = 0 \quad \text{and} \quad \frac{\partial \mathcal{J}}{\partial x_2}(x_1,x_2) = 0,$$

is identical to

$$\frac{\partial \mathcal{J}}{\partial x_2}(x_2,x_1) = 0 \quad \text{and} \quad \frac{\partial \mathcal{J}}{\partial x_2}(x_1,x_2) = 0,$$

and hence the solution occurs at $x_1 = x_2$. $\square$

In Corollary 6.9 we showed that if a weight has a "difference" null-space on a square then the bound (6.32) is minimized at the centre of the square. The following corollary will generalise this result and we will consider a null space of the form $t_1 = \phi(t_2)$ where ϕ is anti-symmetric on a rectangle.

Corollary 6.12. *Let $w : (-a,a) \times (-A,A) \to (0,\infty)$ be a weight function of the form $w(t_1,t_2) = w|t_1 - \phi(t_2)|$, where $\phi : (-A,A) \to (-a,a)$ is surjective and odd, for some $a, A > 0$, that is $\phi(-t) = -\phi(t)$. Then $\mathcal{J}$ as defined in (6.32) is minimized at the origin.*

Proof. We need to show that

$$\int_{-a}^{0}\int_{-A}^{A}|t_2|w|t_1-\phi(t_2)|\,dt_2dt_1=\int_{0}^{a}\int_{-A}^{A}|t_2|w|t_1-\phi(t_2)|\,dt_2dt_1 \tag{6.39}$$

and

$$\int_{-A}^{0}\int_{-a}^{a}|t_1|w|t_1-\phi(t_2)|\,dt_1dt_2=\int_{0}^{A}\int_{-a}^{a}|t_1|w|t_1-\phi(t_2)|\,dt_1dt_2. \tag{6.40}$$

Making the substitution $t_1=-u$ and $t_2=-v$ in the first integral of (6.39) we have

$$\int_{-a}^{0}\int_{-A}^{A}|t_2|w|t_1-\phi(t_2)|\,dt_2dt_1=\int_{0}^{a}\int_{-A}^{A}|v|w|u-\phi(v)|\,dvdu.$$

Similarly

$$\int_{-A}^{0}\int_{-a}^{a}|t_1|w|t_1-\phi(t_2)|\,dt_1dt_2=\int_{0}^{A}\int_{-a}^{a}|u|w|u-\phi(v)|\,dudv.$$

Hence, the corollary is proved. □

6.4 Cubature and grid generation

Theorem 6.7 can form the basis of a cubature formula for weighted double integrals. That is, we can form a mesh and apply equation 6.20 to each grid rectangle. The minimum point of each rectangle would be given by (6.33) and (6.34). The question that would remain is how would such a grid be "optimally" constructed? For example, for four grid rectangles, as shown in Figure 6.3, how would ξ_1 and ξ_2 be chosen?

Let us consider a partition $a_i\le\xi_i\le b_i$ of the interval $[a_i,b_i]$, with $x_{i,1}\in[a_i,\xi_i]$ and $x_{i,2}\in[\xi_i,b_i]$, for $i=1,2$. In addition, define D to be the rectangular region $[a_1,b_1]\times[a_2,b_2]$ and define the sub-regions $D_{1,1}=[a_1,\xi_1]\times[a_2,\xi_2]$, $D_{1,2}=[\xi_1,b_1]\times[a_2,\xi_2]$, $D_{2,1}=[a_1,\xi_1]\times[\xi_2,b_2]$ and $D_{2,2}=[\xi_1,b_1]\times[\xi_2,b_2]$. A sketch of this partition is shown is Figure 6.3.

THEOREM 6.13. *Let the conditions in Theorem 6.7 hold. Given the partition defined above, the following double integral inequality holds*

$$\Bigg|\iint_D f(t_1,t_2)w(t_1,t_2)\,dt_1dt_2-\sum_{i=1}^{2}\Bigg(\iint_{D_{1,i}+D_{2,i}} f(x_{1,i},t_2)w(t_1,t_2)\,dt_1dt_2$$
$$-\iint_{D_{i,1}+D_{i,2}} f(t_1,x_{2,i})w(t_1,t_2)\,dt_1dt_2\Bigg)+\sum_{i=1}^{2}\sum_{j=1}^{2}f(x_{1,j},x_{2,i})\iint_{D_{i,j}} w(t_1,t_2)\,dt_1dt_2\Bigg|$$
$$\le\left\|\frac{\partial^2 f}{\partial t_1\partial t_2}\right\|_\infty\sum_{i=1}^{2}\sum_{j=1}^{2}\iint_{D_{i,j}}|x_{1,i}-t_1||x_{2,j}-t_2|w(t_1,t_2)\,dt_1dt_2. \tag{6.41}$$

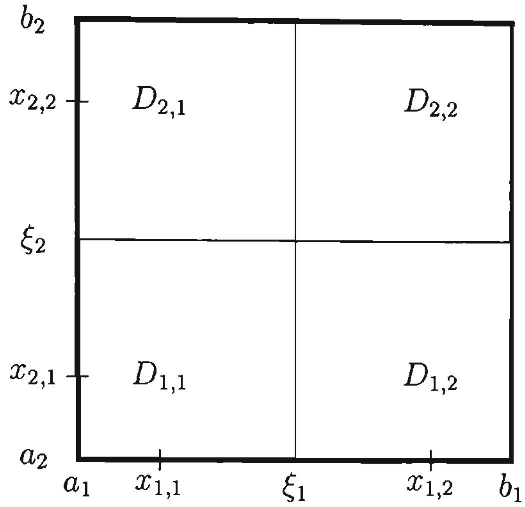


Figure 6.3: A partition of the rectangular region $D = [a_1, b_1] \times [a_2, b_2]$ showing the sub-regions $D_{i,j}$, $i, j = 1, 2$.

The bound is minimized at the points $x_{i,j}, \xi_i$, $(i, j = 1, 2)$ statisfying

$$\begin{aligned}\int_{a_1}^{x_{1,1}}\int_{a_2}^{\xi_2} |x_{2,1} - t_2| w(t_1,t_2)\, dt_2 dt_1 + \int_{a_1}^{x_{1,1}}\int_{\xi_2}^{b_2} |x_{2,2} - t_2| w(t_1,t_2)\, dt_2 dt_1| \\ = \int_{x_{1,1}}^{\xi_1}\int_{a_2}^{\xi_2} |x_{2,1} - t_2| w(t_1,t_2)\, dt_2 dt_1 + \int_{x_{1,1}}^{\xi_1}\int_{\xi_2}^{b_2} |x_{2,2} - t_2| w(t_1,t_2)\, dt_2 dt_1,\end{aligned} \tag{6.42}$$

$$\begin{aligned}\int_{\xi_1}^{x_{1,2}}\int_{a_2}^{\xi_2} |x_{2,1} - t_2| w(t_1,t_2)\, dt_2 dt_1 + \int_{\xi_1}^{x_{1,2}}\int_{\xi_2}^{b_2} |x_{2,2} - t_2| w(t_1,t_2)\, dt_2 dt_1| \\ = \int_{x_{1,2}}^{b_1}\int_{a_2}^{\xi_2} |x_{2,1} - t_2| w(t_1,t_2)\, dt_2 dt_1 + \int_{x_{1,2}}^{b_1}\int_{\xi_2}^{b_2} |x_{2,2} - t_2| w(t_1,t_2)\, dt_2 dt_1,\end{aligned} \tag{6.43}$$

$$\begin{aligned}\int_{a_2}^{x_{2,1}}\int_{a_1}^{\xi_1} |x_{1,1} - t_1| w(t_1,t_2)\, dt_1 dt_2 + \int_{a_2}^{x_{2,1}}\int_{\xi_1}^{b_1} |x_{1,2} - t_1| w(t_1,t_2)\, dt_1 dt_2| \\ = \int_{x_{2,1}}^{\xi_2}\int_{a_1}^{\xi_1} |x_{1,1} - t_1| w(t_1,t_2)\, dt_1 dt_2 + \int_{x_{2,1}}^{\xi_2}\int_{\xi_1}^{b_1} |x_{1,2} - t_1| w(t_1,t_2)\, dt_1 dt_2,\end{aligned} \tag{6.44}$$

$$\begin{aligned}\int_{\xi_2}^{x_{2,2}}\int_{a_1}^{\xi_1} |x_{1,1} - t_1| w(t_1,t_2)\, dt_1 dt_2 + \int_{\xi_2}^{x_{2,2}}\int_{\xi_1}^{b_1} |x_{1,2} - t_1| w(t_1,t_2)\, dt_1 dt_2| \\ = \int_{x_{2,2}}^{b_2}\int_{a_1}^{\xi_1} |x_{1,1} - t_1| w(t_1,t_2)\, dt_1 dt_2 + \int_{x_{2,2}}^{b_2}\int_{\xi_1}^{b_1} |x_{1,2} - t_1| w(t_1,t_2)\, dt_1 dt_2,\end{aligned} \tag{6.45}$$

$$\xi_1 = \frac{x_{1,1} + x_{1,2}}{2} \qquad \textit{and} \qquad \xi_2 = \frac{x_{2,1} + x_{2,2}}{2}. \tag{6.46}$$

Proof. To obtain (6.41), it is a simple matter of applying equation (6.20) of Theorem 6.7 to each region $D_{i,j}$ $(i, j = 1, 2)$, summing and finally employing the triangle inequality.

To show equations (6.42)–(6.46), we calculate the stationary point of the bound

$$\mathcal{J} = \sum_{i=1}^{2}\sum_{j=1}^{2}\iint_{D_{i,j}} |x_{1,i} - t_1||x_{2,j} - t_2|w(t_1,t_2)\,dt_1dt_2. \tag{6.47}$$

For $x_{1,1}$,

$$\begin{aligned}
\frac{\partial \mathcal{J}}{\partial x_{1,1}} &= \frac{\partial}{\partial x_{1,1}}\left\{\sum_{j=1}^{2}\iint_{D_{1,j}} |x_{1,1} - t_1||x_{2,j} - t_2|w(t_1,t_2)\,dt_1dt_2\right\} \\
&= \frac{\partial}{\partial x_{1,1}}\left\{\int_{a_1}^{x_{1,1}}\int_{a_2}^{\xi_2}(x_{1,1} - t_1)|x_{2,1} - t_2|w(t_1,t_2)\,dt_2dt_1\right. \\
&\quad + \int_{x_{1,1}}^{\xi_1}\int_{a_2}^{\xi_2}(t_1 - x_{1,1})|x_{2,1} - t_2|w(t_1,t_2)\,dt_2dt_1 \\
&\quad + \int_{a_1}^{x_{1,1}}\int_{\xi_2}^{b_2}(x_{1,1} - t_1)|x_{2,2} - t_2|w(t_1,t_2)\,dt_2dt_1 \\
&\quad \left. + \int_{x_{1,1}}^{\xi_1}\int_{\xi_2}^{b_2}(t_1 - x_{1,1})|x_{2,2} - t_2|w(t_1,t_2)\,dt_2dt_1\right\} \\
&= \int_{a_1}^{x_{1,1}}\int_{a_2}^{\xi_2}|x_{2,1} - t_2|w(t_1,t_2)\,dt_2dt_1 - \int_{x_{1,1}}^{\xi_1}\int_{a_2}^{\xi_2}|x_{2,1} - t_2|w(t_1,t_2)\,dt_2dt_1 \\
&\quad + \int_{a_1}^{x_{1,1}}\int_{\xi_2}^{b_2}|x_{2,2} - t_2|w(t_1,t_2)\,dt_2dt_1 - \int_{x_{1,1}}^{\xi_1}\int_{\xi_2}^{b_2}|x_{2,2} - t_2|w(t_1,t_2)\,dt_2dt_1.
\end{aligned}$$

Setting the last expression to zero gives (6.42) and the same process can be used to show equations (6.43)–(6.45).

To show (6.46), observe that

$$\begin{aligned}
\frac{\partial \mathcal{J}}{\partial \xi_1} &= \int_{a_2}^{\xi_2}(\xi_1 - x_{1,1})|x_{2,1} - t_2|w(\xi_1,t_2)\,dt_2 - \int_{a_2}^{\xi_2}(x_{1,2} - \xi_1)|x_{2,1} - t_2|w(\xi_1,t_2)\,dt_2 \\
&\quad + \int_{\xi_2}^{b_2}(\xi_1 - x_{1,1})|x_{2,2} - t_2|w(\xi_1,t_2)\,dt_2 - \int_{\xi_2}^{b_2}(x_{1,2} - \xi_1)|x_{2,1} - t_2|w(\xi_1,t_2)\,dt_2 \\
&= 2\int_{a_2}^{\xi_2}\left(\xi_1 - \frac{x_{1,1} + x_{1,2}}{2}\right)|x_{2,1} - t_2|w(\xi_1,t_2)\,dt_2 \\
&\quad + 2\int_{\xi_2}^{b_2}\left(\xi_1 - \frac{x_{1,1} + x_{1,2}}{2}\right)|x_{2,2} - t_2|w(\xi_1,t_2)\,dt_2,
\end{aligned}$$

which obviously has a root at $(6.46)_1$. Similarly, we can show $(6.46)_2$. □

We now proceed to a full weighted cubature formulae.

Define the following partitions of the intervals $[a_i, b_i]$

$$I_i : a_i = \xi_{i,0} \le \xi_{i,1} \le \cdots \le \xi_{i,n} = b_i,$$

and let $x_{i,j} \in [\xi_{i,j-1}, \xi_{i,j}]$ for $i = 1, 2$ and $j = 1, 2, \ldots, n$. Furthermore, let $D_{i,j} = [\xi_{1,i-1}, \xi_{1,i}] \times [\xi_{2,j-1}, \xi_{2,j}]$, $D_i^{(1)} = \bigcup_{k=1}^{n} D_{i,k}$ and $D_i^{(2)} = \bigcup_{k=1}^{n} D_{k,i}$, for $i, j = 1, 2, \ldots, n$.

Consider the weighted cubature formula

$$\begin{aligned} A(f, w, I_1, I_2, \boldsymbol{\xi}, \boldsymbol{x}) &= \sum_{i=1}^{n} \left(\iint_{D_i^{(1)}} f(x_{1,i}, t_2) w(t_1, t_2)\, dt_1 dt_2 + \iint_{D_i^{(2)}} f(t_1, x_{2,i}) w(t_1, t_2)\, dt_1 dt_2 \right) \\ &\quad - \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_{1,i}, x_{2,j}) \iint_{D_{i,j}} w(t_1, t_2)\, dt_1 dt_2. \end{aligned} \tag{6.48}$$

Using the above assumptions, we can write the following theorem.

THEOREM 6.14. *Let $f : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ and $w : (a_1, b_1) \times (a_2, b_2) \to (0, \infty)$ be as in Theorem 6.7 and $I_1, I_2, \boldsymbol{\xi}, \boldsymbol{x}$ be given above. The following weighted cubature formula holds*

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} f(t_1, t_2) w(t_1, t_2)\, dt_2 dt_1 = A(f, w, I_1, I_2, \boldsymbol{\xi}, \boldsymbol{x}) + R(f, w, I_1, I_2, \boldsymbol{\xi}, \boldsymbol{x}), \tag{6.49}$$

where

$$|R(f, w, I_1, I_2, \boldsymbol{\xi}, \boldsymbol{x})| \le \left\| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right\|_\infty \sum_{i=1}^{n} \sum_{j=1}^{n} \iint_{D_{i,j}} |x_{1,i} - t_1||x_{2,j} - t_2| w(t_1, t_2)\, dt_1 dt_2. \tag{6.50}$$

The bound (6.50) is minimized when $\boldsymbol{x}$ and $\boldsymbol{\xi}$satisfy

$$\sum_{j=1}^{n} \int_{\xi_{1,i-1}}^{x_{1,i}} \int_{\xi_{2,j-1}}^{\xi_{2,j}} |x_{2,j} - t_2| w(t_1, t_2)\, dt_2 dt_1 = \sum_{j=1}^{n} \int_{x_{1,i}}^{\xi_{1,i}} \int_{\xi_{2,j-1}}^{\xi_{2,j}} |x_{2,j} - t_2| w(t_1, t_2)\, dt_2 dt_1 \tag{6.51}$$

$$\sum_{j=1}^{n} \int_{\xi_{2,i-1}}^{x_{2,i}} \int_{\xi_{1,j-1}}^{\xi_{1,j}} |x_{1,j} - t_1| w(t_1, t_2)\, dt_1 dt_2 = \sum_{j=1}^{n} \int_{x_{2,i}}^{\xi_{2,i}} \int_{\xi_{1,j-1}}^{\xi_{1,j}} |x_{1,j} - t_1| w(t_1, t_2)\, dt_1 dt_2 \tag{6.52}$$

$$\xi_{k,l} = \frac{x_{k,l} + x_{k,l+1}}{2}, \qquad \textit{for} \;\; i = 1, \ldots, n, \;\; l = 1, \ldots, n-1, \;\; k = 1, 2. \tag{6.53}$$

Proof. The proof follows that of Theorem 6.14. □

To find the $4n - 2$ unknowns

$$x_{i,1} \le \xi_{i,1} \le x_{i,2} \le \cdots \le \xi_{i,n-1} \le x_{i,n},$$

for $i = 1, 2$, we need to solve the $4n-2$ coupled non-linear equations (6.51), (6.52) and (6.53). These equations are easily solved iteratively with a uniform grid as the starting point. With this method of solution all variables are fixed apart from the parameter of interest. Thus for

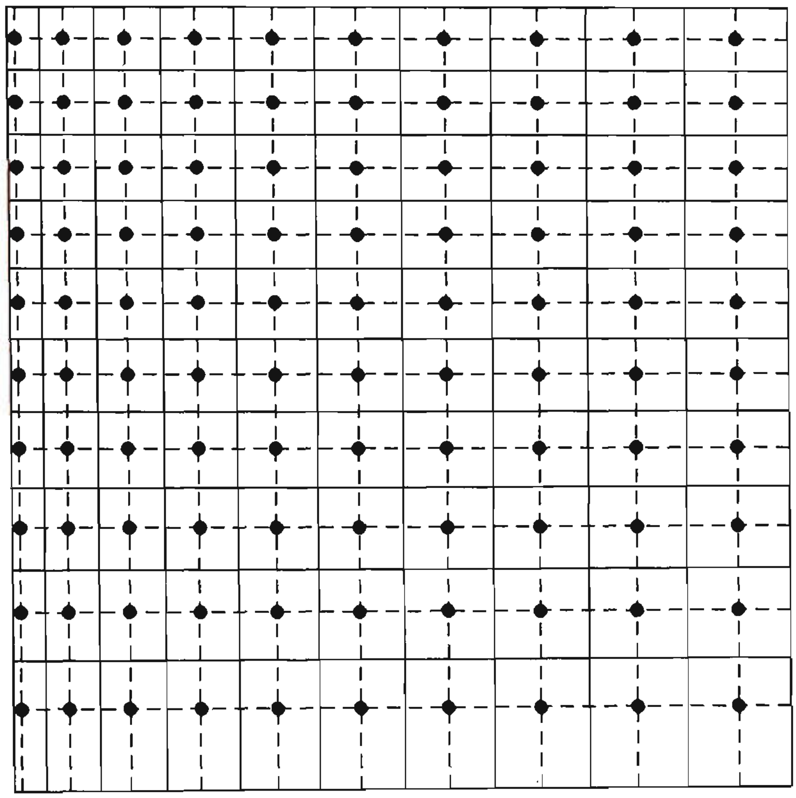


Figure 6.4: Grid generated from the solution of equations (6.51)– (6.53) for the weight $w(t_1,t_2) = \sqrt{t_2/t_1}$ over $[0,1]\times[0,1]$ and $n = 10$. The solid lines indicate the composite grid; in each grid square there is one function evaluation (dot) and two single integral evaluations (dashed lines).

example if $k = 1$ and we fix i, then equation (6.51) may be considered as a function of $x_{1,i}$ only; say $F(x_{1,i})$. It is easy to see that

$$F'(x_{1,i}) = 2\sum_{j=1}^{n}\int_{\xi_{2,j-1}}^{\xi_{2,j}} |x_{2,j} - t_2| w(x_{1,i}, t_2)\, dt_2 \geq 0$$

and $F(\xi_{1,i-1}) \leq 0$, $F(\xi_{1,i}) \geq 0$. Thus F has a unique root and the bisection algorithm would be an appropriate numerical technique to produce the solution.

In Figures 6.4, 6.5, 6.6 and 6.7, the grid obtained via numerical solution of (6.51)–(6.53) is plotted for various weight functions and n. We can see that the grid clustering reflects the weight behaviour.

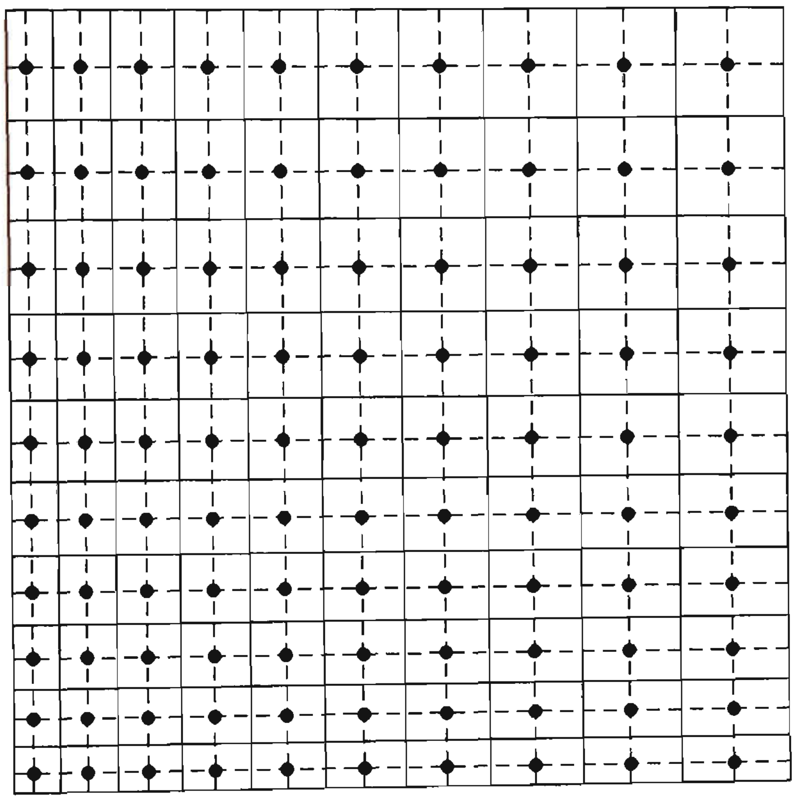


Figure 6.5: Grid generated from the solution of equations (6.51)– (6.53) for the weight $w(t_1, t_2) = -\ln(t_1 t_2)$ over $[0,1] \times [0,1]$ and $n = 10$. The solid lines indicate the composite grid; in each grid square there is one function evaluation (dot) and two single integral evaluations (dashed lines).

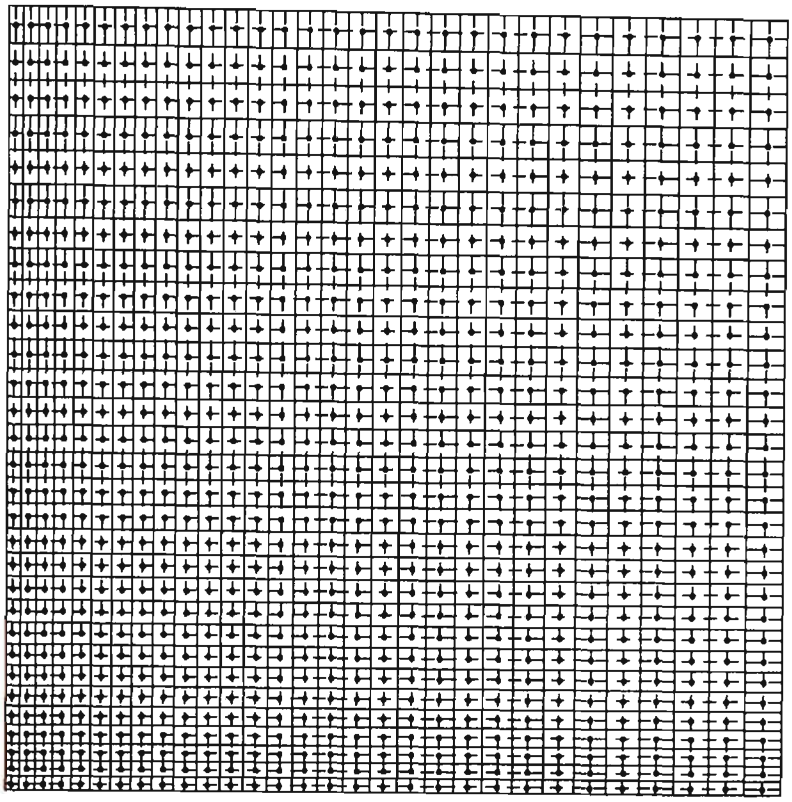


Figure 6.6: Grid generated from the solution of equations (6.51)– (6.53) for the weight $w(t_1, t_2) = -\ln(t_1 t_2)$ over $[0,1] \times [0,1]$ and $n = 30$. The solid lines indicate the composite grid; in each grid square there is one function evaluation (dot) and two single integral evaluations (dashed lines).

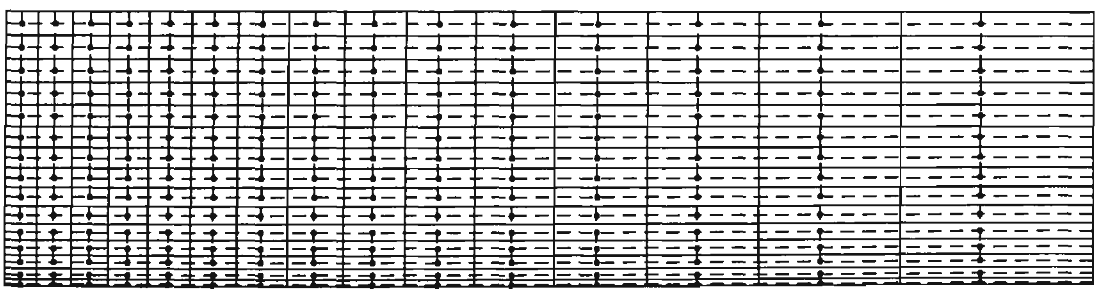


Figure 6.7: Grid generated from the solution of equations (6.51)– (6.53) for the weight $w(t_1, t_2) = e^{-t_1}/\sqrt{t_2}$ over $[0,4] \times [0,1]$ and $n = 15$. The solid lines indicate the composite grid; in each grid square there is one function evaluation (dot) and two single integral evaluations (dashed lines).

Bibliography

Abramowitz, M. and C. A. Stegun (1972). *Handbook of Mathematical Function with Formulas, Graphs and Mathematical Tabels.* New York: Dover.

Anastassiou, G. (1995). Ostrowski type inequalities. *Proc. of the American Math. Soc.* **123**(12), 3775–3781.

Atkinson, K. E. (1988). *An Introduction to Numerical Analysis* 2^{nd} *edition.* Singapor: Wiley and sons.

Barnett, N. S. and S. S. Dragomir (2001). An Ostrowski type inequality for Double Integrals and Application for Cubature Formulae. *Soochow J. Math* **27**(1), 1–10.

Bertsen, J., T. O. Esplied and T. Sorevik (1991). On the Subdivision strategy in Adaptive quadrature Algorithm. *J. Compute. Appl. Math.* **35**, 119–132.

Cerone, P. and S. S. Dragomir (1999). Three Point Quadrature Rules Involving, At Most, a First Derivative. *RGMIA,Research Report Collection* **2**(4).

Cerone, P. and S. S. Dragomir (2000a). Midpoint-type rules from an inequalities point of view. *Handbook of Analytic-Computational Methods in Applied Mathematics* **Editor: G. Anastassiou**, 135–200.

Cerone, P. and S. S. Dragomir (2000b). Trapezoidal-type rules from an inequalities point of view. *Handbook of Analytic-Computational Methods in Ap plied Mathematics* **Editor: G. Anastassiou**, 65–134.

Cerone, P., S. S. Dragomir and J. Roumeliotis (1998). An Ostrowski type inequality for mappings whose second derivatives belong to $L_p(a, b)$ and Applications. Accepted for publication in *The Journal of the Indian Mathematical Society*.

Cerone, P., S. S. Dragomir and J. Roumeliotis (1999b). An Inequality of Ostrowski type for mapping whose second derivatives are bounded and Applications. *East Asian Math.*

J. **15**(1), 1–9.

Cerone, P., S. S. Dragomir and J. Roumeliotis (1999c). On Ostrowski type for mappings whose second derivatives belong to $L_1(a, b)$ and Applications. *Honam. Math. J.* **21**(1), 127–137.

Cerone, P., S. S. Dragomir and J. Roumeliotis (1999a). Some Ostrowski Type inequalities for n- Times Differentiable Mapping and Applications. *Demonstratio Mathematica* **32**(4), 697–712.

Cerone, P., J. Roumeliotis and G. Hanna (2000). On Weighted Three Point Quadrature Rules. *ANZIAM J* **42(E), [ONLINE] Available online from http://anziamj.austms.org.au/V42/CTAC99/Cero**, C340–C361.

Cools, R. (1999). Monomial cubature rules since "Srooud": a compilation - part 2. *http://www.cs.Kuleuven.ac.be/~nines/research/ecf/ecf.html.*

Cools, R., D.Laurie and L. Pluym (1997). Algorithm 764: Cubpack++: A C++ package for automatic two-dimensional cubature. *ACM Trans. Math. Software* **23**, 1–15.

Davis, P. J. and P. Robinowitz (1984). *Methods of Numerical Integration, 2^{nd} edition.* Orland, Florida.

Dragomir, S. S. (1998). Ostrowski's Inequality for monotonous mappings and applications. *J. KSIAM* **3**(1), 127–135.

Dragomir, S. S. (1999). On the Ostrowski's integral inequality for Lipschitzian mapping and applications. *Computers and Mathematics with Applications* **38**, 33–37.

Dragomir, S. S. (2001). A Generalization of Ostrowski Inequality for Mapping whose First Derivatives Belong to $L_1[a, b]$ and Applications in Numerical Integration. *J. Computational Analysis and Applications* **3**(4), 1–14.

Dragomir, S. S., N. S. Barnett and P. Cerone (1998). An Ostrowski type Inequality for double integrals in term of L_p-norms and Applications in numerical integrations. *Anal. Num. Theor. Approx.* **2**(12), 1–10.

Dragomir, S. S., P. Cerone, J. Roumeliotis and S. Wang (1999). A Weighted Version of Ostrowski Inequality for Mapping of Hölder type and application in numerical analysis. *Bull. Math Soc. Sc Math Roumanie* **42(90)**(4), 301–314.

Dragomir, S. S., J. E. Pečarić and S. Wang (2000). The Unified treatment of trapezoid, Simpson and Ostrowski type inequality for monotonic mapping and Applications. *Mathematical and Computer Modelling.* **31**, 61–70.

Dragomir, S. S. and T. M. Rassias (2001). *Ostrowski Type Inequalites and Applications in Numerical Integration, online,* http://rgmia.vu.edu.au. Dordrecht: Kluwer Academic.

Dragomir, S. S. and S. Wang (1997). A New Inequality of Ostrowski's Type in L_1 - norm and Applications to some Special Means and to some Numerical Quadrature Rules. *Tamkang J. of Math.* **28**, 239–244.

Dragomir, S. S. and S. Wang (1998a). A New Inequality of Ostrowski's Type in L_p - norm. *Indian j. Math* **40**(3), 299–304.

Dragomir, S. S. and S. Wang (1998b). Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules. *Appl. Math. Lett* **11**, 105–109.

Dunkl, C. F. and Y. Xu (2001). *Orthogonal Polynomials of Several Variables.* Cambridge: Cambridge University Press.

Engels, H. (1980). *Numerical Quadrature and Cubature.* New York: Academic Press.

Fang, K. T. and Y. Wang (1994). *Number Theoretic Methods in Statistics.* London: Chapman and Hall.

Fang, K. T. and Z. K. Zhang (1999). A two stage algorithm of numerical evaluation of integrals in Number-Theoretical Methods. *J. Comput. Math.* **17**, 285–292.

Fink, A. M. (1992). Bounds on the deviation of a function from its averages. *Czechoslovak Mathematical Journal.* **42**(117), 289–310.

Golomb, M. and H. F. Weinberger (1959). *Optimal Approximation and Error Bounds, in on Numerical Approximations (R.E. Langer, ed.).* Madison: University of Wisconsin Press.

Haber, S. (1967). A modified Monte Carlo quadrature II. *Math. Comput.* **21**, 381–397.

Haber, S. (1970). Numerical evaluation of multiple integrals. *SIAM. Review* **12**, 481–526.

Hanna, G., P. Cerone and J. Roumeliotis (2000). An Ostrowski type inequality in two dimensions using the three point rule. *ANZIAM J* **42(E), [ONLINE] Available online from http://anziamj.austms.org.au/V42/CTAC99/Hann**, C671–C689.

Korobov, N. M. (MR 28# 716 1963). *Number-Theoretical Methods in approximate Analysis.* Moscow: GOZ IZ dat. FIZ. Lit.

Krommer, A. R. and C. W. Ueberhuber (1994). *Numerical Integration on Advanced Computer Systems.* Springer-Verlag, Berlin: Lecture Notes in Computer Science, 848.

Matić, M. and J. E. Pečarić (2001). Two-point Ostrowski inequality. *Math. Inequal. Appl* **4**(2), 215–221.

Milovanović, G. V. (1975). On some integral inequalities. *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* **No. 498-541**, 119–124.

Milovanović, G. V. and J. E. Pečarić (1976). On generalization of the inequality of Ostrowski and some related applications. *Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, 155–158.

Mitrinoviĉ, D. S., J. E. Pečarić and A. Fink (1994). *Inequalities Involving Functions and Their Integrals and Derivatives.* Dordrecht: Kluwer Academic.

Ostrowski, A. (1938). Über die absolutabweichung einer differentiierbaren funktion von ihrem integralmittelwert . *Comment. Math Helv.* **10**, 226–227.

Pachpatte, B. G. (2001). On A New Oserowski Type Inequality In Two Independent Variables. *Tamakang J. of Maths* **32**(1), 45–49.

Piessens, R., E. D. Doncker-Kaperga, C. W. Uberhuber and D. K. Kahauer (1983). *QUADPACK, a subroutine package for automatic integration.* Berlin: Springer-Verlag.

Press, W. H., B. P. Flannery, S. A. Teukolsky and W. T. Vetterlung (1986). *Numerical Recipes: The Art of Scientific Computing.* Cambridge: Cambridge University Press.

Qi, F. (2001). Inequalities for weightrd multiple integral. *J. Math. Anal. Appl.* **253**(2), 381–388.

Rice, J. R. (1973). A Metalgorithm for Adaptive quadrature. *J. ACM,* **22**(1), 61–82.

Romberg, W. (1955). Vereinfachte Numerische Integration. *Det Kong. Norske Vidensk. Selsk. Forhandle* **28**(7).

Roumeliotis, J. (2001). Product inequalities and weighted quadrature *Ostrowski Type Inequalites and Applications in Numerical Integration,* eds. S. S. Dragomir and T. M. Rassias, . Dordrecht: Kluwer Academic. [ONLINE] Preprint available from *http://rgmia.vu.edu.au/monograph.html.*

Roumeliotis, J., P. Cerone and S. S. Dragomir (1999). An Ostrowski Type Inequality for Weighted Mapping with Bounded Second Derivatives. *J. KSIAM* **3**(2), 107–119.

Sloan, I. H. and J. N. Lyness (1989). The Representation of Lattice Quadrature Rules as Multiple Sums. *Math of Comp* **52**, 81–94.

Sloan, I. H. and L. Walsh (1990). A computer search of Rank-2-Lattice Rules for multidimensional Quadrature. *Math of Comp* **54**(189), 281–302.

Sofo, A. and S. S. Dragomir (2001). An Inequality of Ostrowski Type for twice differentiable mappings in terms of L_p-norm and Applications. *Soochow J. Math* **27**(1), 97–111.

Stroud, A. H. (1971). *Approximate Calculation of Multiple Integrals.* New Jersey: Prentice Hall.

Stroud, A. H. and D. Secrest (1966). *Gaussian Quadrature Formulae.* Anglewood Cliffs, N. J.: Prentice Hall.

Traub, J. F. and H. Wozniakowski (1980). *A General Theory of Optimal Algorithms.* Academic Press.

Worlet, R. T. (1991). On integration Lattices. *BIT* **31**, 529–539.