ANALYSIS OF VIBRATION REDUCTION VIA LOCAL STRUCTURAL MODIFICATION

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ABSTRACT*

As a new technique for structural dynamic analysis, local structural modification is playing a more and more important role in solving practical engineering problems. This thesis develops methods for vibration reduction via local structural modification.

In this thesis, the reduction of unwanted vibration in a structural system is principally analysed in two ways, (i) relocation of resonance, and (ii) relocation of anti-resonance - both of which are realised by using local structural modification techniques. The local structural modification is characterised by locally varying the physical parameters of the structural system, and, an algorithm is developed for the finite element implementation. The algorithm which determines the physical parameters to achieve the desired resonances and anti-resonances can be obtained by solving a non-linear polynomial eigen-problem. Examples illustrating the use of various finite element models along with a practical structure are provided to verify the algorithms developed in this thesis and to illustrate the potential of the methods for solving vibration reduction problems in complicated engineering structural systems.

In addition, a procedure for structural dynamic optimisation is developed to eliminate the unwanted resonance peak from frequency response functions using the 'POLE-ZERO' cancellation principle via local structural modification. This procedure enables the structural system to preserve certain dynamic properties after structural modification. Numerical and experimental simulations are presented to illustrate the general procedure and to demonstrate the capability of the method.

*Some of results included in this thesis have been published in the proceedings of local and international conferences and submitted to International Journal of Analytical and Experimental Modal Analysis.

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NOMENCLATURES

c	viscous damping coefficient (Chapter 2), or real constant (Chapter 2), or direction cosines of local coordinate (Chapter 4)
i	complex notation (Chapter 5), or real constant
j	real constant
k	stiffness coefficient, or real constant
l	real constant, or length of beam element (Chapter 4)
m	mass coefficient, or real constant
n	real constant
s	Laplace operator (Chapter 5), or direction sines of local coordinate (Chapter 4)
c_j	real constants (j = 1, 2, 3 ..., n)
c_r	r^{th} modal damping
k_r	r^{th} modal stiffness
m_r	r^{th} modal mass
t	time variable
A	element cross sectional area
E	element Elastic Modulus
F_0	amplitude of excitation force
F_j	amplitude of excitation force applied on the j^{th} coordinate
F_T	amplitude of force transmitted to the foundation
G	element Rigidity Modulus
$H_{ij}(s)$	transfer function of Laplace operator
$H_{ij}(i\omega)$	frequency response function of complex frequency
I	element area moment of inertia
I_p	element polar moment of inertia
J	element torsional constant
$R_i^{n\times n}$	n×n dimension space R_i
T_a	force (displacement) transmissibility

T_{max}	maximum kinetic energy
V_{max}	maximum potential energy
X_0	amplitude of displacement of vibratory system
X_T	amplitude of displacement transmitted to the foundation
Y^*_r	r^{th} element of vector $\{Y^*\}$
SE_{ir}	total strain energy in the i^{th} material for the r^{th} mode
SE_r	total strain energy in the structure for the r^{th} mode
${}_rA_{ij}$	r^{th} modal constant of FRF $\alpha_{ij}(\omega)$
[..]	matrix [..]
[.. : ..]	partitioning of matrix [....]
$[..]_{ij}$	element in the i^{th} row, j^{th} column of matrix [..]
$[..]_m$	matrix of the modified system
$[..]_o$	matrix of the original system
$[..]^{-1}$	inverse of matrix
$[..]^T$	transpose of matrix
$[..]^+$	pseudo inverse of matrix
det[..]	determinant of matrix [..]
$adj[..]_{ij}$	ij^{th} adjoint matrix of matrix [..]
[0]	null matrix
[c]	real constant matrix
$[k_i^{(e)}]_g$	i^{th} element stiffness matrix in global coordinate
$[m_i^{(e)}]_g$	i^{th} element mass matrix in global coordinate
$[k_i^{(e)}]_l$	i^{th} element stiffness matrix in local coordinate
$[m_i^{(e)}]_l$	i^{th} element mass matrix in local coordinate
[I]	identity matrix
[C]	viscous damping matrix
[H]	structural damping matrix
[K]	stiffness matrix
$[K]_v$	stiffness matrix of 'virtual' system, obtained by deleting the i^{th} row and the j^{th} column of [K]
$[K^{(r)}]$	r^{th} perturbation of stiffness matrix
[M]	mass matrix

$[M]_v$	mass matrix of 'virtual' system, obtained by deleting the i^{th} row and the j^{th} column of [M]
$[M^{(r)}]$	r^{th} perturbation of mass matrix
[T]	transformation matrix
[U]	left singular matrix
[V]	right singular matrix
[Z(s)]	system matrix with respect to Laplace operator
$[Z(\omega)]$	system matrix with respect to frequency
$diag[F_r]$	diagonal matrix with diagonal terms of F_r, r = 1, 2, ..., n.
{.}	column vector
$\{.\}^T$	transpose of a column vector
{0}	null column vector
$\{e_r\}$	column vector with unity in r^{th} element and 0 elsewhere
$\{e_{pq}\}$	column vector with unity in p^{th} and q^{th} elements which are opposite in sign and 0 elsewhere
$\{e_r\}_l$	column vector by deleting the l^{th} element of $\{e_r\}$
$\{e_{pq}\}_l$	column vector by deleting the l^{th} element of $\{e_{pq}\}$
$\{x_m\}$	force response vector of modified system
$\{x_o\}$	force response vector of original system
$\{x_{(i)}\}$	i^{th} element displacement vector, sub-set of {X}
$\{x\}_i$	i^{th} Rayleigh's vector
{z}	eigen vector in iteration procedure
{X}	displacement vector in frequency domain
{Y}	eigen vector of original system
$\{Y^*\}$	eigen vector of modified system
$\{Y^*\}_r$	reduced $\{Y^*\}$, sub-set of $\{Y^*\}$
$\{Y^{(L)}\}$	left side eigen vector
$\{Y^{(R)}\}$	right side eigen vector
{Z}	state vector
{x(t)}	displacement vector in time domain
{f(t)}	force vector in time domain

δm_i	mass variation of the i^{th} coordinate
δk_{ij}	stiffness variation between the i^{th} and the j^{th} coordinates
Δm_i	variation of the i^{th} modal mass
Δk_i	variation of i^{th} modal stiffness
$\Delta\gamma$	constant, scaling factor
$\Delta\gamma_i$	variation of structural physical parameter of the i^{th} element
$\Delta\omega_i$	variation of the i^{th} natural frequency
$\Delta\lambda_i$	variation of the i^{th} Rayleigh's quotient
$\Delta[M]$	mass incremental matrix
$\Delta[K]$	stiffness incremental matrix
$\Delta[M_j]$	element mass incremental matrix scaled by $\Delta\gamma_j$ of the j^{th} element
$\Delta[K_j]$	element stiffness incremental matrix scaled by $\Delta\gamma_j$ of the j^{th} element
$\Delta[\alpha(\omega)]$	receptance FRF incremental matrix
$\Delta\{\psi_i\}$	variation of the i^{th} mode shape vector
$\Delta\{\psi\}_r$	variation of the r^{th} mode shape vector

α	real constant, scaling factor
$\alpha(\omega)_{ij}$	receptance FRF between the i^{th} and the j^{th} coordinate
β	real constant, scaling factor
γ	real constant, scaling factor
γ_i	structural physical parameter of the i^{th} element
γ_i^*	element participant ratio of the i^{th} element
ε	perturbation factor
$\varepsilon_{(i)}$	i^{th} element sensitivity index for natural frequency
$\in_{(i)}$	i^{th} element sensitivity index for anti-resonance
ζ	damping lost factor for viscous damping
ζ_r	damping lost factor for proportional damping of the r^{th} mode
η_i	damping lost factor for structural damping of the i^{th} mode
η_r	mass modification ratio of the r^{th} coordinate
κ_{ij}	stiffness modification ratio between the i^{th} and the j^{th} coordinate
λ_i	i^{th} Rayleigh's quotient

$\lambda_{i(0)}$	i^{th} Rayleigh's quotient of original system
λ_i^*	i^{th} Rayleigh's quotient of modified system
$\lambda_{i,r}$	i^{th} Rayleigh's quotient in the r^{th} iteration
$\lambda_i^{(r)}$	i^{th} Rayleigh's quotient in the r^{th} perturbation
ρ	element density constant
ϕ_{ir}	i^{th} element of the r^{th} mass normalised mode shape vector
ψ_{ir}	i^{th} element of the r^{th} mode shape vector
Ω_r	r^{th} Zero of the system transfer function
ω	frequency of sinusoidal excitation force
ω_a	anti-resonance frequency (eigen frequency of 'virtual' system)
ω^*	natural frequency of modified system
ω_a^*	eigen frequency of modified 'virtual' system
ω_0	natural frequency of original system
ω_r	r^{th} natural frequency of original system
ω_n	natural frequency of a multi-degree of freedom system
$\omega_{n,r}$	r^{th} natural frequency of a multi-degree of freedom system
$[\Phi]$	mass normalised mode shape matrix
$[\Phi^*]$	mass normalised mode shape matrix of modified system
$[\alpha(\omega)]$	receptance FRF matrix
$[\alpha_v(\omega)]$	receptance FRF matrix of 'virtual' system
$[\alpha(\omega)]_o$	receptance FRF matrix of original system
$[\alpha_{ij}(\omega)]$	ij^{th} sub-matrix of $[\alpha(\omega)]$
$[\alpha(\omega)]_r$	reduced $[\alpha(\omega)]$, sub-matrix of $[\alpha(\omega)]$
$[\eta]$	mass incremental matrix scaled by $\Delta\gamma$
$[\eta_i]$	element mass matrix scaled by $\Delta\gamma_i$
$[\eta]_r$	reduced $[\eta]$, sub-matrix of $[\eta]$
$[\kappa]$	stiffness incremental matrix scaled by $\Delta\gamma$
$[\kappa_i]$	element stiffness matrix scaled by $\Delta\gamma_i$
$[\kappa]_r$	reduced $[\kappa]$, sub-matrix of $[\kappa]$
$[\beta_i]$	Boolean mapping matrix for the i^{th} element
$\{\psi\}_0$	mode shape vector of original system
$\{\psi\}_r$	r^{th} mode shape vector

$\{\psi\}_{r,s}$	sub-set of the r^{th} mode shape vector
$\{\psi_i\}$	i^{th} mode shape vector
$\{\psi_i\}_r$	i^{th} mode shape vector in the r^{th} iteration
$\{\psi_i^{(r)}\}$	i^{th} mode shape vector in the r^{th} perturbation
$\{\psi^*\}_r$	r^{th} mode shape vector of modified system
$\{\mu\}$	eigen vector of original 'virtual' system
$\{\mu^*\}$	eigen vector of modified 'virtual' system
$\{\upsilon\}$	displacement vector of 'virtual' system

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CHAPTER 1

INTRODUCTION

1.1 Traditional Vibration Reduction Methods

Vibration reduction in structures and components has been a topic of longstanding interest in the field of vibration engineering. Traditionally, problems relating to this topic were tackled by four principal methods, namely: (i) vibration isolation, (ii) use of vibration absorbers, (iii) introduction of damping, and (iv) active vibration control. Without knowing the innermost characteristics of a structural system, these methods have been rigorously explored in recent decades and effectively used in dealing with some practical vibration reduction problems based on a much simplified mathematical model of the system (mass-spring model). However, the factor that the objective system remains a 'black box', and the reliability of using such model to represent a real engineering system is problem dependent, restricts the further development of the above methods. Therefore, the shortcomings of these traditional methods with the increasing demands in the vibration performance of large, complex structural systems force structural analyst to seek for an alternative.

1.2 A New Structural Analysis Technique - Local Structural Modification

Local Structural modification is a new structural analysis technique that has been gaining more and more attention in recent years. This technique involves determining the changes to the dynamic properties of a linear elastic structural system that arise due to the re-distribution of the mass, stiffness and damping of the system. The re-distribution can be realised by local modification of physical or geometrical parameters of the system, such as

local thickness, width or Elastic modulus. The emergence of this technique has benefited from well developed theories of Finite Element Analysis (FEA) and Experimental Modal Analysis (EMA) and the increasing computational power offered by sophisticated computers. Having been successfully applied in several disciplines of structural mechanics, such as model identification and dynamic prediction, this technique, by its considerable potential, is attracting more and more researchers in the area of vibration reduction. Since the technique is capable of examining the dynamic properties of a structural system from its innermost details, it is of particular importance from the structural design and optimisation view point. A typical structural modification procedure is shown in Figure 1.2.1.

1.3 Scope of Present Work

The research work presented in this thesis is mainly concerned with vibration reduction via structural modification - an alternative to traditional vibration reduction methods. The feasibility of using local structural modification techniques to deal with the vibration reduction problem of a sinusoidally excited linear elastic structural system will be investigated and new vibration reduction methods based on structural modification technique will be developed. As one focus of the work, the theoretical background of existing vibration reduction methods, their recent development and application as well as limitations will be introduced briefly in Chapter 2. Since a good understanding of structural modification technique and acknowledgment of its recent advance is a pre-requisite of the work, the theoretical development of the technique will be reviewed and a detailed literature research base of structural modification technique will be established as the central part of Chapter 2. In addition, the problem definition of this research is given in Chapter 2 to re-highlight the need for the work.

To develop new vibration reduction methods using the structural modification technique, a multi-degree-of-freedom mass-spring system will be investigated by relocating its resonance and anti-resonance frequencies to suppress the vibration level at a certain location in the system. As two most important indicators for evaluating the vibration level of linear vibratory systems, resonance and anti-resonance frequencies will be relocated to

desired locations by locally modifying the mass and stiffness coefficients of the mass-spring system. New methods will be developed to determine the mass and stiffness variations in order to achieve desired resonances and anti-resonances for the purpose of vibration reduction of a system subjected to sinusoidal loading. The limitation of the methods developed on the mass-spring model will be overcome by implementing the methods on a FEA model. An algorithm based on the state space theory is developed to solve a high order eigen problem in which the local structural modification will be characterised by variations of certain physical parameters of the structural system. Optimum structural modification can also be achieved by using a Sensitivity Index which is developed from sensitivity analysis theory. Detailed theoretical development of these methods and algorithm are given in Chapter 3. To validate these methods and algorithm, numerical and experimental results are presented in Chapter 4. To ensure the consistency of analytical results and experimental results, both FEA and EMA will be carried out prior to the local structural modification. Computer codes written in Fortran 77 are developed for analytical modal analysis and impact excitation and single-degree-of-freedom curve fitting methods will be used for EMA.

A dynamic optimisation procedure is developed in Chapter 5 for the purpose of vibration reduction. By applying local structural modification, a resonance peak can be eliminated in a specified frequency response function so that a significant vibration reduction can be achieved in relevant locations of the structure. This method is developed based on the 'Pole-Zero' cancellation principle and can be proved to be an effective new vibration reduction method by numerical and experimental verification as presented in Chapter 5.

The final chapter, Chapter 6, presents conclusions and addresses possible extension and application of the results achieved.

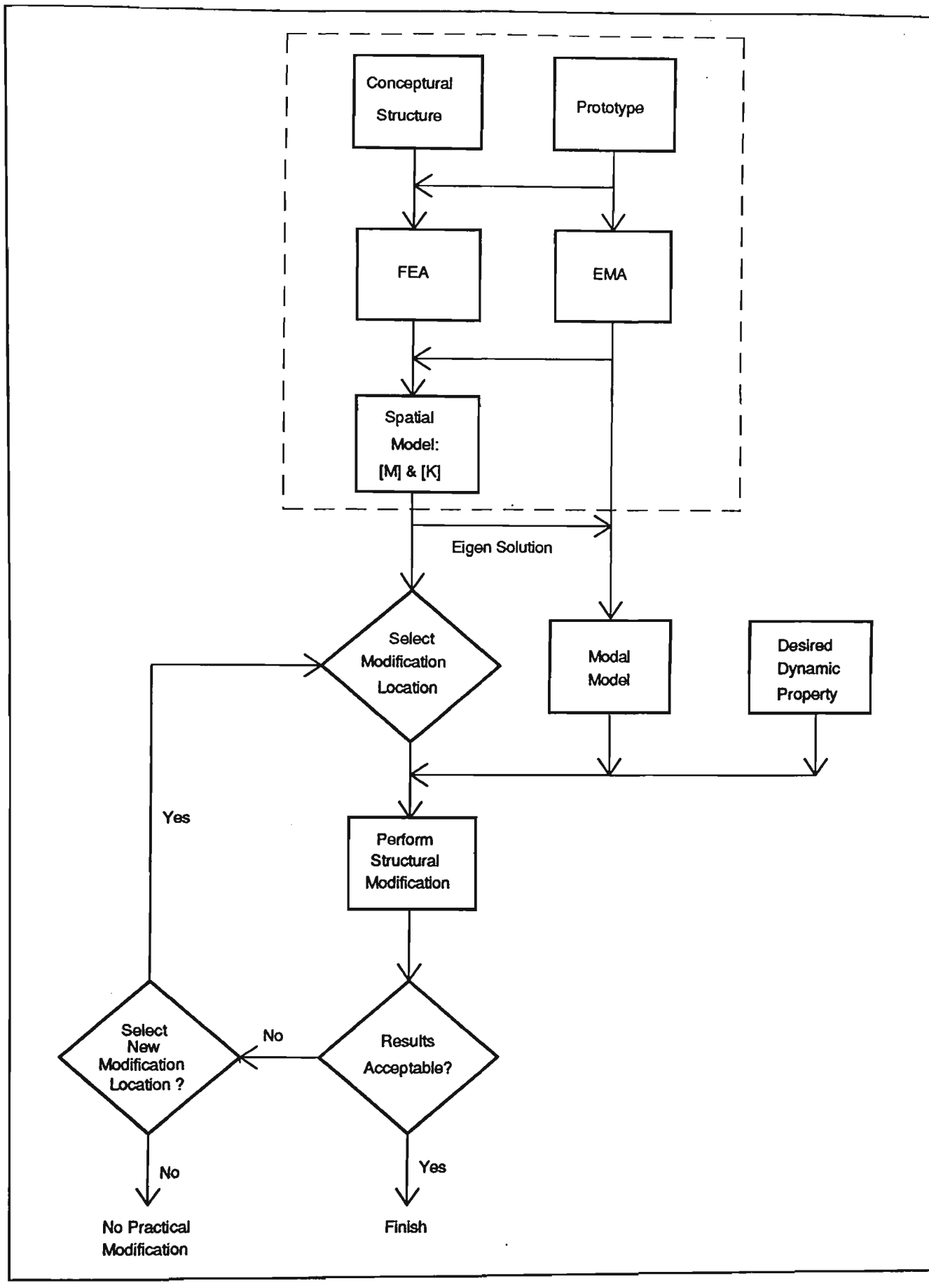


Figure 1.2.1 - General Procedure for Local Structural Modification

CHAPTER 2

THEORETICAL BACKGROUND AND PROBLEM DEFINITION

In this chapter, first, a general introduction will be given for various aspects of vibration reduction. Then, the need for the work which is focused on structural modification will be motivated. As well, a general literature review will be provided.

2.1 Methods and General Procedure for Vibration Reduction

The attempt of reducing unwanted vibration levels is usually made to minimise the magnitude of the vibration sources. However, this attempt is often limited by practical considerations. When the required amount of vibration reduction is impossible to be achieved by improving the performance of the vibration sources, the structure which is excited has to be designed to respond to a minimum extent.

It is known that the response level of a structure due to a given excitation is determined by the threefold characteristics of mass, stiffness, and damping. Different structural response under different conditions of excitation will depend on these characteristics in different ways. Therefore, various vibration reduction methods present their own advantages and disadvantages. The following of this section provide a general introduction to some existing vibration reduction methods, and the method of structural modification will be discussed later in details.

2.1.1 Vibration Isolation

The most elementary form of vibration isolation is the introduction of an additional loadpath between a structure and its mounting surface (vibration source) to reduce the transmitted forces and displacements over a frequency range of concern. Vibration isolation is applied in an attempt either to protect a delicate object from excessive vibration transmitted to it from its surroundings or to prevent vibratory forces generated by a machine from being transmitted to the surroundings. The basic objectives of vibration isolation systems are in fact the same, that is to reduce the transmitted forces as much as possible.

A typical vibration isolation system is commonly idealised as a single-degree-of-freedom (SDOF) system as shown in **Figure 2.1.1**. Its performance can be assessed by the force (displacement) transmissibility T_a which is defined as the ratio of the force (displacement) amplitude transmitted to the foundation to the exciting force (displacement) amplitude.

$$T_a = \left| \frac{X_T}{X_0} \right| = \left| \frac{F_T}{F_0} \right| = \sqrt{\frac{1+(2\zeta\omega/\omega_0)^2}{(1-\omega^2/\omega_0^2)^2+(2\zeta\omega/\omega_0)^2}} \qquad (2.1.1)$$

The system shown in **Figure 2.1.1** is a SDOF, viscously damped system. The isolator for the system is represented by a spring k and a viscous dashpot c. The assumptions for which equation (2.1.1) holds are as follows:

(i) The damping of the isolator is viscous and the dashpot is rigidly connected to the main system (represented by the mass m).

(ii) The main system is connected to the rigid foundation only through the isolator.

(iii) The mass of the isolator is negligible.

In equation (2.1.1), to minimise the transmissibility of the system, which is always desired, the optimum ω_0 and ζ can be determined. It is noted that T_a is also frequency dependent - For $\omega/\omega_0 < \sqrt{2}$, greater ζ is desired and for $\omega/\omega_0 > \sqrt{2}$, zero ζ is desired. This implies that in certain frequency ranges softer spring and less damping are required for

better performance of a vibration isolator. However, the applicability of the above conclusion may vary according to different idealisations of a practical system. Rao[1] discussed the isolation system by considering a flexible foundation and arrived at a similar conclusion. Crede and Ruzicka[2] used two other criteria to assess the performance of vibration isolation systems, namely, the relative transmissibility and the motion response. The relative transmissibility is defined as the ratio of the relative deflection amplitude of the isolator to the displacement amplitude imposed at the foundation whereas the motion response is defined as the ratio of the displacement amplitude of the equipment to the quotient obtained by dividing the excitation force amplitude with the static stiffness of the isolator. On considering the influence of different types of damper encountered in an isolation system and the forms in which they are connected to the main structures which needs to be isolated, a more detailed discussion was also given by Crede and Ruzicka[2].

Schiff[3] discussed the vibration isolation problem of a multi-degree-of-freedom system. Using the example of a two-degree-of-freedom undamped mass-spring system, it was indicated that for systems with parallel spring and dashpot combinations in series, the reverse octave rule should be adopted to keep the transmissibility less than 1 and the octave rule should be adopted to avoid large transmissibility values in the resonance area. The octave rule states that the resonant frequency of a substructure should be at least twice that of its support; and the reverse octave rules states that the resonant frequency of a substructure should be no more than one-half that of its support. However, no further results were given in this study for more complicated systems.

Most of vibration isolation theories were developed on the basis of mass-spring-damper systems and this limits their application. In spite this, using these mass-spring-damper models to investigate the simple isolation system is convenient, and to some extent, is capable of giving sound guidance for practical problems, but it becomes less meaningful when dealing with large complex structures, particularly in the high frequency ranges. Instead, the mobility (or, impedance) approach is commonly used for solving these problems. The mobility of a system is defined as the ratio of the velocity to the excitation force acting on the system and its inverse is called the mechanical impedance. Using

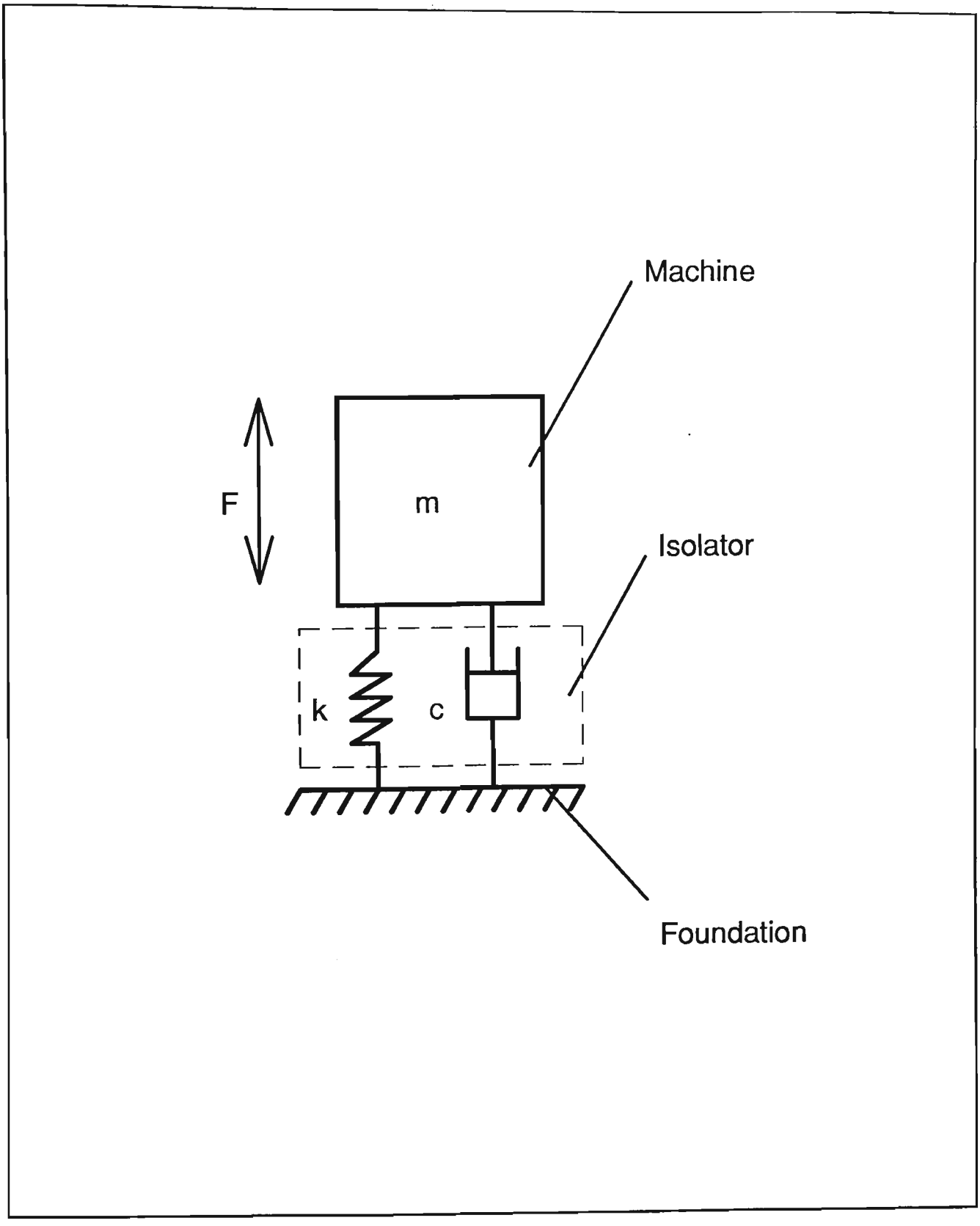


Figure 2.1.1 - Vibration Isolation

this approach, the motion transmissibility of the vibration source, isolator and receiver assembly may be evaluated by considering of the dynamic characteristics of the system in term of frequency response properties. White[4] summarised detailed applications of this technique. Based on this mobility approach, White also introduced a method using the power flow concept which can be seen as unifying vibration control methods by seeking to minimise the power input to the system and then to minimise the power transmission through the system.

The general procedure of using the vibration isolation techniques has been summarised by Hain, *et al*[5], along with the factors which should be considered in isolator selection, and isolator installation.

2.1.2 Introduction of Damping

Damping is a dynamic characteristic which reflects energy dissipation during structural vibration. As one of the effective means of vibration reduction in a wide frequency band, introducing damping by the addition of extra damping materials and mechanisms has long been used.

The damping of a mechanical system may commonly be created from (i) the hysteresis of the structural materials (structural damping), (ii) the friction at the structural joints and contacting dry surfaces (coulomb damping), and (iii) the viscous damping at the lubricated sliding surfaces (viscous damping).

Damping is more difficult to predict compared to mass and stiffness properties. Since damping of a structural system usually can not be inferred from simple modelling or static measurement, there is no mathematical model which will provide an exact representation of the damping properties. This restrains further study of damping characteristics of structures and hence limits the application of damping for vibration reduction. However, it is noted that from a system response viewpoint, the influence of damping is only significant in the vicinity of resonances. In theoretical studies, a viscous damping model,

in which the energy dissipation per radian is often assumed proportional to the velocity, is normally employed and is usually assumed to be proportional to the mass and stiffness of the structural system (so called proportional damping) for convenience.

The primary consequences of damping to the response of structures are twofold: (i) Suppression of resonance responses, and (ii) more rapid decay of free vibration. The suppression of vibration response at resonances by introducing more damping into a structure can be easily visualised by investigating the frequency response properties of the system against the damping changes. To enable a mathematical solution with physical significance, the proportional damping model which is first recommended by Lord Rayleigh [6], and then expressed in matrix form by Wilson [7] is widely adopted,

$$[C] = \alpha[M] \tag{2.1.2a}$$

or

$$[C] = \beta[K] \tag{2.1.2b}$$

Using the damping model given in equation(2.1.2a) and equation (2.1.2b), Ewins[8] gave the receptance frequency response functions of a multi-degree-of-freedom system, which is defined as the ratio of the displacement of certain response coordinates, X_i, to the only external sinusoidal force acting on the certain excitation coordinate, F_j,

$$\alpha(\omega)_{ij} = \frac{X_i}{F_j} = \sum_{r=1}^{n} \frac{\psi_{ir}\psi_{jr}}{(k_r - \omega^2 m_r) + i2\zeta_r\sqrt{k_r m_r}} \tag{2.1.3}$$

In equation (2.1.3),

$$m_r = \{\psi\}_r^T [M]\{\psi\}_r \tag{2.1.4a}$$

$$k_r = \{\psi\}_r^T [K]\{\psi\}_r \tag{2.1.4b}$$

$$c_r = \{\psi\}_r^T [C] \{\psi\}_r \tag{2.1.4c}$$

$$\zeta_r = \frac{c_r}{2\sqrt{k_r m_r}} = \frac{\beta \omega_r}{2} \tag{2.1.4d}$$

where ω_r is the r-th natural frequency of the system. From the above equations it is noted that constant β can be assigned from a known value of modal damping ratio ζ_r, which can be identified experimentally as the ratio of actual damping to the critical damping for a particular natural mode of vibration. **Figure 2.1.2a** shows a series of typical frequency response function curves of a MDOF system as the β varies. It is observed that the resonance peak will be more suppressed as β increases, but the anti-resonance responses will be increased. This observation is also applicable to other damping models. However, since the parameter β is a theoretical approximation, it depends to a large extent on the validity of the proportional viscous damping model. In practice, a higher β is expected from materials with higher internal damping, such as cast iron, or some composite materials.

Using the same damping model, Newland[9] gave an impulse response function which was defined as the displacement of response coordinate i due to a unit impulse excitation applied on input coordinate j as a function of time t.

$$h_{ij}(t) = \sum_{r=1}^{n} \frac{\phi_{ir}\phi_{jr}}{\omega_r\sqrt{1-\zeta_r^2}} e^{-\zeta_r \omega_r t} \sin\omega_r\sqrt{1-\zeta_r^2}\, t \tag{2.1.5}$$

Figure 2.1.2b shows a series of impulse response function curves as β varies. It is observed that as β increases, the attenuation of the impulse response becomes rapid.The effect of improving the damping characteristics of a structure in regard to its vibration response is significant, as stated above. However, due to the difficulty of the identification of real damping properties, vibration reduction by the introduction of damping is mainly applied to practical cases relying mainly on trial-and-error techniques.

A: $\beta = 0.15$
B: $\beta = 0.20$
C: $\beta = 0.25$
D: $\beta = 0.30$

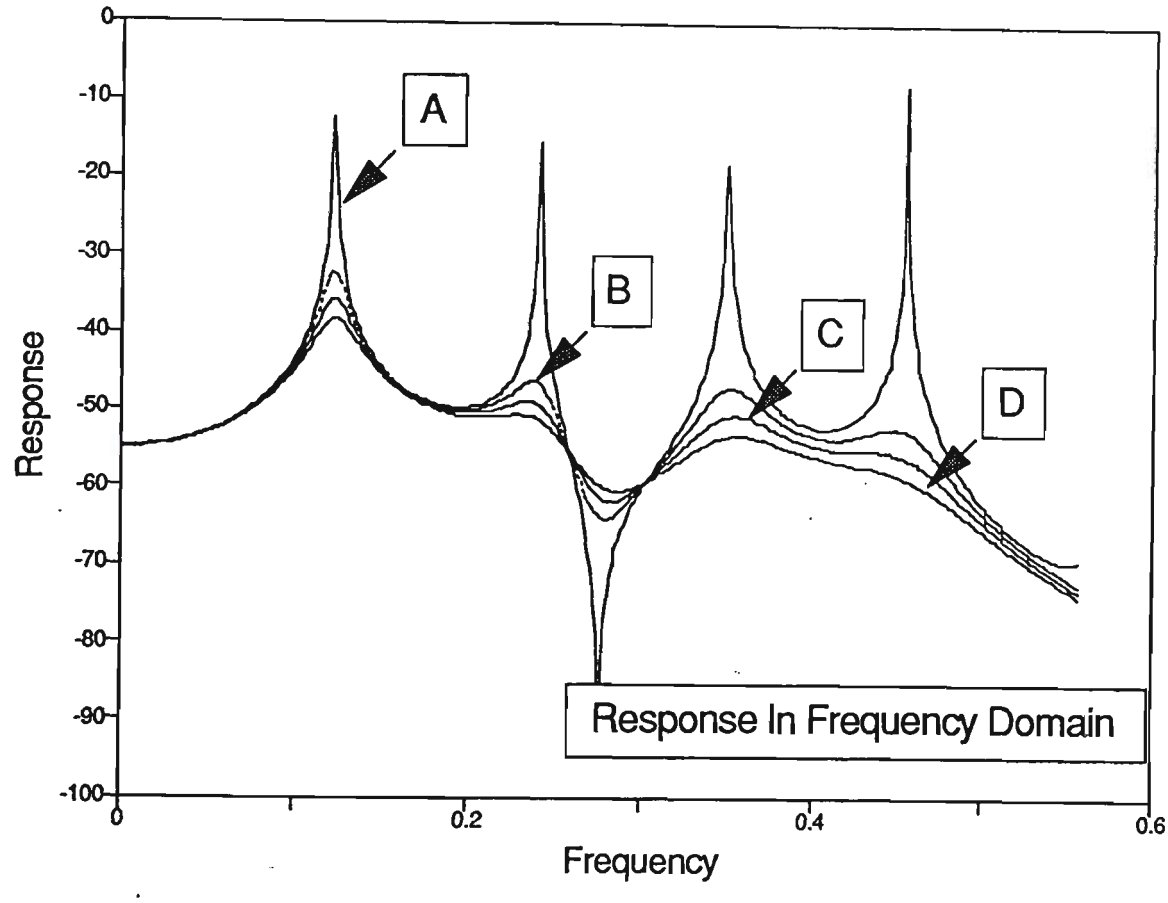


Figure 2.1.2a - Frequency Response Comparison

A: $\beta = 0.15$
B: $\beta = 0.20$
C: $\beta = 0.25$
D: $\beta = 0.30$

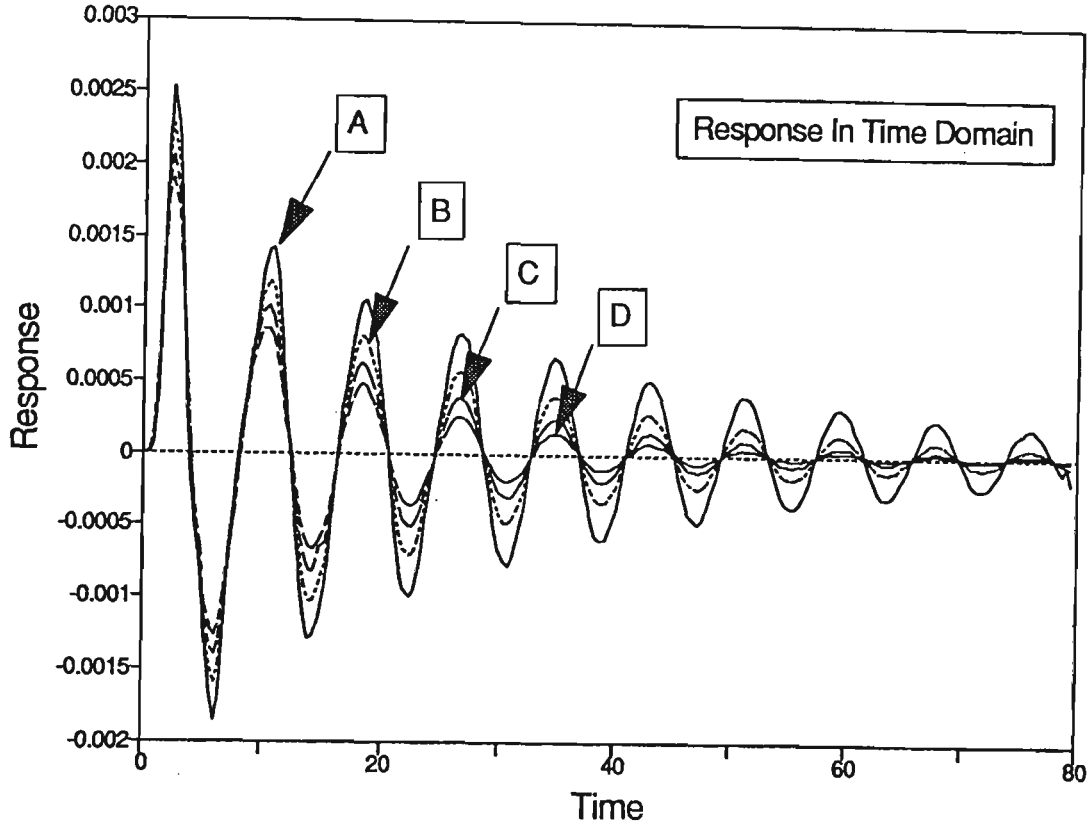


Figure 2.1.2b - Comparison of Time Domain Response

On an experimental basis, Ungar[10] measured systematically the influence of damping on the vibration behaviour of panels. As well, friction damping and viscoelastic damping inherent in the panel structure and imposed by an external mechanism were discussed separately. Ungar also discussed the cases which include the configuration design and material selection to achieve maximum damping from the viewpoint of a damping loss factor η, a parameter defined by Lazan[11], as the ratio of the energy dissipated per radian. As another factor related to energy dissipation, the effect of edges and other discontinuities which can be seen as a form of damping was also discussed by Ungar.

Ikegami *et al*[12] presented a method which applied viscoelastic passive damping to the suppression of vibration level in a satellite structure. Using a commercial finite element analysis package, the viscoelastic components of the structure were modelled with elastic solid elements and the damping loss factor was given using the modal strain energy method:

$$\eta_r = \frac{\sum_{i=1}^{n} (\eta_i SE_{ir})}{SE_r} \tag{2.1.6}$$

The effects of introducing high damping materials into the structure may be evaluated quantitatively by this formula. However, it was also indicated that the accompanying decrease in the strength and stiffness by the introduction of the damping materials limited the application of this method. Using viscoelastic material, Dutt and Nakra[13] also presented a vibration response reduction method for a rotor shaft system.

As another existing damping method for vibration reduction, the application of friction damping has also well studied, particularly for practical applications. Relevant research can be found in Han[14] and Gameron *et al*[15]. However, this method is only applicable when there exist contacting surfaces, which implies that in most situations extra mechanisms have to be introduced.

Panossian[16] summarised the existing methods of using damping for vibration reduction,

namely, (i) viscoelastic materials application; (ii) friction devices; (iii) impact dampers; and (iv) fluid dampers. A new method using non-obstructive particle damping (NOPD) technique for enhancing the structural damping was also presented. The NOPD technique involves the potential of energy absorption/dissipation through friction, momentum exchange between moving particles and vibrating walls, heat, and viscous and shear deformations. Compared with other damping methods, it has been found that the NOPD technique has many advantages, particularly from the economic, environmental and design viewpoints.

2.1.3 Dynamic Absorbers and Mass Dampers

Dynamic absorbers and mass dampers are also called vibration absorbers. The technique of using vibration absorbers to reduce the vibration level of a mechanical system involves attaching auxiliary masses to a vibrating system through spring and damping devices. If a vibration absorber does not involve damping, then it is called a "dynamic absorber". Otherwise, it is called a "mass damper". Compared with other methods such as improving the damping properties, the main advantage of the vibration absorber is that it can be added to machinery in order to reduce vibration after the machinery has been constructed. However, the extra dimension and weight due to the introduction of a vibration absorber to an original system thus becomes the main limitation of the application of this method.

The theoretical basis of a vibration absorber can simply be explained using an undamped single-degree-of-freedom vibrating system which represents the main system and an auxiliary mass connected to the main system by a spring and a dashpot which represents the mass damper. The system thus constructed is a damped two-degree-of-freedom system, as shown in **Figure 2.1.3**. This idealisation is usually that based on the work of Ormondroyd and Den Hartog[17]. The purposes of introduction of a mass damper will be twofold, (i) to eliminate the resonance response of the original system due to a given sinusoidal excitation force by selecting appropriate mass and stiffness coefficients for the mass damper, (ii) to suppress the resonance peaks, which result from introducing the additional degree-of-freedom (mass damper) into the system, by selecting an appropriate

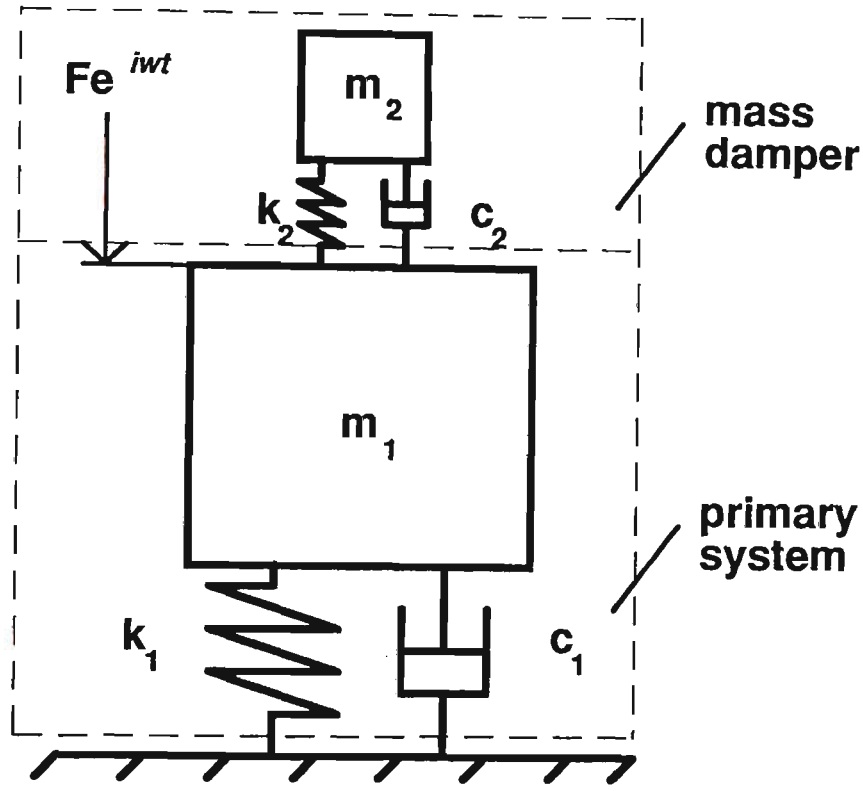


Figure 2.1.3 - A Dynamic Absorber

damping coefficient for the mass damper. The response of the main system and the auxiliary mass due to a given sinusoidal excitation of magnitude F acting on the main system may be expressed, in terms of mass, stiffness and damping of the system, as,

$$X_1 = \frac{F(k_2 - m_2\omega^2 + ic_2\omega)}{[(k_1 - m_1\omega^2)(k_2 - m_2\omega^2) - m_2k_2\omega^2] + i\omega c_2(k_1 - m_1\omega^2 - m_2\omega^2)} \qquad (2.1.7a)$$

$$X_2 = \frac{X_1(k_2 + ic_2\omega)}{k_2 - m_2\omega^2 + ic_2\omega^2} \qquad (2.1.7b)$$

If the sinusoidal excitation acts on the original main system with frequency $\omega = \sqrt{(k_1/m_1)}$, resonance will occur. However, from equation (2.1.7) and equation (2.1.8), it can be seen that this resonance peak is eliminated by adding a dynamic absorber (c_2=0) with $\sqrt{(k_2/m_2)} = \sqrt{(k_1/m_1)}$, resulting in $X_1(\omega=\sqrt{(k_1/m_1)} = 0$. Because of the additional degree of freedom, it is found that two resonances will be created on each side of the eliminated resonance. This makes the dynamic absorber effective only for a system subjected to a narrow-banded frequency excitation. For this reason, in spite of the fact that the introduction of the damping will increase the response at the tuned frequency (for $X_1(\omega) = 0$), mass dampers are more commonly used to reduce the over-all vibration level.

Although lower vibration response is always expected for higher damping, as stated in Section 2.1.2, it becomes changes when dealing with the mass damper problems. If the damping of the mass damper is too high, the two systems will be locked together and vibrate as an undamped single-degree-of-freedom system. Thus it becomes necessary to choose the damping of the mass damper in such a way that the main system has the maximum possible damping at the lower frequency range. This is because the first resonance is usually of major concern when dealing with a vibration reduction problem. Jones[18] has shown that an optimum coefficient can be given as,

$$c_{2,optimum} = 1.63m_2\sqrt{\frac{3m_2k_1}{2(1+\frac{m_2}{m_1})^3}} \qquad (2.1.8)$$

Further work related to the optimum design of a mass damper can also be found in Puksand[19] and Soom and Lee[20].

Large machinery and structures can not normally be treated as lumped parameter (mass-spring) systems. Considerations for which the vibration absorbers need to be designed in order to give optimum conditions at each resonance frequency of a large structure have to be accounted for when applying the absorbers to continuous systems. Hunt[21] gave a detailed discussion about this topic. Hunt also discussed the application of continuous and miscellaneous absorbers for the vibration reduction of typical continuous systems, such as plates and beams. An approach using anti-resonance theory to study the effectiveness of applying vibration absorbers to a continuous system was presented by Wang *et al*[22]. According to their work, anti-resonances exist at the point when any change in the frequency of a sinusoidal excitation causes an increase in the response at this point. Using this approach, an anti-resonance point can be created at a specified location by attaching an appropriate absorber at the specified location of the system. As well, the sensitivity of the response with respect to parameters (mass, stiffness coefficients) of the absorber was also examined in search of the optimum design of the absorber.

For excitations within a wide frequency band a normal vibration absorber is obviously inadequate. Usually, this problem can be overcome by using active vibration absorbers but inevitably the complexity is increased. For this reason, Igusa and Xu[23] developed a method of using distributed tune mass dampers and demonstrated that such devices, which are constructed by mass dampers with natural frequencies distributed over a frequency range, are more effective than a single mass damper in reducing vibration response to wide band excitation. An alternative method has also been presented by Semercigil *et al*[24], where a simple vibration absorber was used in conjunction with an impact damper. An impact damper is a small rigid mass placed in a container which is firmly attached to a resonant main system. Using a normal vibration absorber in conjunction with this impact damper, significant attenuation was achieved in the response of the main system over approximately the entire frequency range studied.

2.1.4 Active Vibration Control

Since passive vibration control devices become less effective in responding to higher demand vibration situations, efforts have been focused in recent years on active vibration control methods which were developed on the basis of feedback control theory. The passive vibration isolator, vibration absorber introduced in the previous sections may be upgraded using servomechanism which are characterised by sensors and actuators, to become active isolators and active absorbers, where the variation of vibration is continuously monitored and the physical parameters (mass, stiffness, damping) comprising these devices can be adjusted according to these variations, so that the desired response can be obtained. Relevant research outcomes can be found in literature[25], [26], [27]. However, the application of active vibration control devices still experiences two shortcomings, namely, (i) it requires external energy, thus increasing the risk of generating unstable states, and (ii) complex servomechanisms increase the weight, dimension and cost.

2.1.5 Local Structural Modification and Redesign

Dynamic properties of a structural system may be improved by changing its mass and stiffness magnitudes and distributions at certain locations. This technique is referred to as 'local structural modification' or 'structural redesign'. The structural modification technique is normally applied at the design stage or/and carried out once a prototype is available. Compared with other vibration reduction methods, the local structural modification method will not involve auxiliary mechanism and normally does not consider the damping effect. Since the modification can be carried out locally, the configuration of the conceptual structural system or an existing design will not be changed significantly. The approach of using local structural modification for vibration reduction is similar to some of other methods, namely, (i) relocating the resonance frequency out of the excitation frequency range, (ii) relocating the anti-resonance frequency into the excitation frequency range. A particular structural modification may be realised by means of locally varying design variables, such as local thickness, width, or of local addition of mass and

stiffness, or of selecting different materials. Recent developments in structural dynamic analysis, such as the application of Finite Element Analysis (FEA) and/or Experimental Modal Analysis (EMA) have facilitated vibration researcher in the application of structural modification technique in structural vibration reduction.

2.2 Local Structural Modification

The study of local structural modification in this work addresses the problem of predicting how the structural modifications affect the dynamic properties and assumes that the mathematical representation of a structural system is available. The fundamentals of local structural modification and a general survey of various methods in this area were presented by Wang *et al*[28], Starkey[29], Brandon[30], and are generally grouped into two categories: methods of prediction and methods of specification.

2.2.1 The Mathematical Representations for Structural Dynamics Analysis

As the prerequisite for theoretical structural dynamic analysis, the development of an accurate and economic mathematical representation is essential. One of the most broadly used mathematical representation of a structural system is the matrix model, which represents a structural system in terms of its mass, stiffness and damping matrices in the spatial (physical) domain as a "spatial model" or in terms of mode shape and natural frequency matrices in the modal domain as a "modal model". The spatial model may be obtained by mass-spring-dashpot idealisation or FEA, whereas the modal model may be obtained by eigensolution to the spatial model or by EMA. Normally, the connectivity information - the magnitude and distribution of mass, stiffness and damping can be visualised from their respective matrices. Thus, any modification of the structural system can be reflected in the change of these matrices. Modelling a continuous structural system into a matrix form implies the dicretisation of the system. For a simple mechanical system, the mass-spring-dashpot system is usually adopted, whereas for a complex structural system, the matrix model should be established from FEA or EMA.

The FEA method has been the focus of many researchers in structural analysis over the past two decades. The construction of mass and stiffness matrices using the FEA method can be found in standard finite element text books[31], [32]. The development of many commercial FEA packages based on the state-of-the-art manner further facilitates the application of this method. However, FEA still suffers certain shortcomings. Firstly, the inclusion of damping is not feasible in the original description of a problem. Therefore, damping must be incorporated separately and estimated based on previous experimental work. Work in this regard can be found in He[33] and Wei *et al*[34]. Secondly, it is difficult to model joints, complicated stamping, spot welds and so on, which are often found in complex structural assemblies.

Using experimental data to correct the analytical model (FEA model) of a structural system has been considered as one of the effective methods to obtain a mathematical model which is more accurate in representing a real structure. EMA plays a vital role in this respect. The mode shapes and natural frequencies of a prototype of a structure obtained from experimental modal analysis are usually considered to be accurate and thus they constitute the experimental modal model of the structure. The fundamentals of this technique can be found in Ewins[8]. If all the modal information is available from the EMA, then the discrepancy between the analytical modal model and the experimental modal model can be projected from the modal space to the spatial space in order to adjust the analytical spatial model. This method is called model updating. However, only a small number of modes can be determined within a limited frequency band, which implies that only a truncated modal model can be obtained experimentally. This decreases the reliability of this method. Numerous methods have been developed in the research of recent years. Up-to-date advances in this topic can be found, for instance, in the publications by Kabe[35], Lim[36], Ibrahim[37] and many others.

Detailed discussion of developing an accurate model for structural modification is beyond the scope of this work. In the following sections accurate models to which local structural modification is applied are assumed to be available based mainly on the FEA and/or on lumped mass-spring systems. Therefore, the following assumptions have been made in this

study:

(i) The system is linear and conservative.

(ii) The mass matrix is symmetric and positive definite, and the stiffness matrix is symmetric and at least positive semi-definite.

(iii) the damping effect will be neglected.

(iv) The eigenvalues are distinct.

2.2.2 Prediction Methods

Prediction methods are characterised by using the modal model or response model (analytical or experimental) of the original structural system to estimate the new modal model or response properties of the modified structural system without re-undertaking a complete procedure of eigensolution or EMA. Therefore, these methods are usually only considered effective if the solution thus obtained is believed to be more accurate, and the computational cost is significantly less than that of using the original procedure to obtain the new modal model of the modified structural system by incorporating the modification information into original spatial model. Considerable effort has been focused on this area and the existing methods are normally classified into two categories, namely: (i) localised modification methods, (ii) small modification methods.

Perhaps the most often quoted work in the literature of localised modification methods is that of Weissenburger[38], although similar work can be traced back to that by Young[39]. Based on an undamped structural system, Wissenburger[38] developed a method where the localised structural modification was characterised by using a single lumped mass or a linear spring connecting two specified coordinates to predict the natural frequencies and mode shapes of the modified system. The equation of motion of the modified system was given by,

$$\left[[K]+\Delta[K]-\omega^{*2}([M]+\Delta[M])\right]\{X\} = \{0\} \tag{2.2.1}$$

Since the mass or the stiffness modification can be expressed by a matrix whose rank is unity,

$$\Delta[M] = \delta m\{e_r\}\{e_r\}^T \tag{2.2.1a}$$

or

$$\Delta[K] = \delta k\{e\}_{ij}\{e\}_{ij}^T \tag{2.2.1b}$$

the method is also called the 'unit rank modification' method. The computational efficiency achieved by the unit rank mass modification may be exploited by transferring equation (2.2.1a) into the modal domain, (assuming that Δ[K] = [0])

$$\left[(diag[\omega_p^2]-\omega^{*2}[\ I\ ])diag[m_p]\right]\{q\} = -\delta m\{r\}\{r\}^T\{q\} \tag{2.2.2}$$

where

$$diag[m_p] = [\Psi]^T[M][\Psi] \tag{2.2.2a}$$

$$diag[k_p] = [\Psi]^T[K][\Psi] \tag{2.2.2b}$$

$$diag[\omega_p^2] = diag[k_p]diag[m_p]^{-1} \tag{2.2.2c}$$

$$\{r\} = [\Psi]^T\{e_r\} \tag{2.2.2d}$$

$$\{X\} = [\Psi]\{q\} \tag{2.2.2e}$$

Considering a general row of equation (2.2.2), noting that

$$-\delta m\omega^{*2}\sum_{p=1}^{n} r_p q_p = (\omega_1^2-\omega^{*2})\frac{q_1}{r_1} = (\omega_2^2-\omega^{*2})\frac{q_2}{r_2} = \ldots\ \ldots = (\omega_n^2-\omega^{*2})\frac{q_n}{r_n} \tag{2.2.2f}$$

the eigenvalue of the modified system can be solved from the characteristic equation (2.2.3) by using the Newton-Raphson algorithm and the solution will be given in terms of natural frequencies of the original system.

$$\sum_{p=1}^{n} \frac{r_p}{m_p(\omega_k^2-\omega^{*2})} = -\frac{1}{\delta m \omega^2} \tag{2.2.3}$$

Once the natural frequencies of the modified system are found, the eigenvectors(mode shapes) will be given by,

$$\{\psi\} = \beta\left\{\frac{r_1}{m_1(\omega^{*2}-\omega_1^2)} \quad \frac{r_2}{m_2(\omega^{*2}-\omega_2^2)} \quad \cdots \quad \cdots \quad \frac{r_n}{m_n(\omega^{*2}-\omega_n^2)}\right\}^T \tag{2.2.4}$$

where β is a scaling factor.

The fact that the method may be extended to the combination of unit rank mass and stiffness modifications was also discussed by Weissenburger[38]. More complex modifications (with the rank greater than unity) can be carried out by the repeated applications of the unit rank modification. Because of the high computational efficiency, this method has gained wide acceptance and has been generalised to the non-conservative system by Pomazal and Snyder[40]. Further studies on utilising the computational advantages of low rank modification matrices can be found in Skingle and Ewins[41] which is based largely on an experimental interpretation of the structural system, ie. experimental modal model. The main shortcoming of those methods derived from Weissenburger's method is that they restrict the modification into single rank, despite the fact that, a mass modification would affect terms of the mass matrix corresponding to the coordinates of three perpendicular directions, (as stated by Ewins[8]). More details of the development of these methods can also be found in Hallquist[42] and Brandon *et al*[43].

Another type of well developed approach to predict the dynamic properties of the locally modified system is the response predicting method (also called receptance method). The theoretical basis of this method is the Shermann-Morrison Identity,

$$[[A]+[B][D]]^{-1} = [A]^{-1}-[A]^{-1}[B][[\ I\ ]+[B][A]^{-1}[D]]^{-1}[D][A]^{-1} \qquad (2.2.5)$$

The numerical advantage of equation (2.2.5) was firstly exploited by Kron[44] for analysing the dynamic characteristics of electrical networks and then extended into structural dynamic analysis, which the receptance matrix (details of receptance matrix can be found in Section 3.1.1 or Ewins[8]) of the modified structural system $[\alpha(\omega)]_m = [\alpha(\omega)]_o+\Delta[\alpha(\omega)]$ can be derived in terms of the local modifications and recptance matrix $[\alpha(\omega)]_o$ of the original system. Given that the localised modification may be expressed in equations (2.2.2a,b), and assuming the receptance matrix of the original system is available, then, the receptance matrix of the modified structural system due to a unit rank mass modification can be expressed as,

$$\begin{aligned}[\alpha(\omega)]_o+\Delta[\alpha(\omega)] &= \left[-\omega^2[M]+[K]-\omega^2\Delta[M]\right]^{-1} \\ &= [\alpha(\omega)]_o - \frac{\omega^2[\alpha(\omega)]_o\{e\}\delta m\{e\}^T[\alpha(\omega)]_o}{1+\omega^2\delta m\{e\}^T[\alpha(\omega)]_o\{e\}}, \quad \left(\delta m\neq -\frac{1}{\{e\}^T[\alpha(\omega)]_o\{e\}}\right)\end{aligned} \qquad (2.2.6)$$

The computational efficiency of equation (2.2.6) is high, since the computation of a full matrix inverse is replaced by a series of matrix multiplications. This method is of particular significance in dealing with low rank modification (characterised by low rank of Δ[M] and Δ[K]). Based on this method, Moraux *et al*[45] gave the forced response $\{x_m\}$ of the modified system in terms of that of the original system $\{x_o\}$ as,

$$\{x_m\} = \{x_o\} - \frac{\omega^2\delta m[\alpha(\omega)]_o\{e\}\{e\}^T}{1+\omega^2\delta m\{e\}^T[\alpha(\omega)]_o\{e\}}\{x_o\} \qquad (2.2.7)$$

They also further reduced the order of the problem by considering that only one coordinate, say $\{x_{(2)}\}_{(1\times1)}$, was involved in modification. Rearranging and partitioning $[\alpha(\omega)]_o$ and Δ[M] yields,

$$[\alpha(\omega)]_o = \begin{bmatrix}[\alpha_{11}(\omega)]_o & [\alpha_{12}(\omega)]_o \\ [\alpha_{21}(\omega)]_o & [\alpha_{22}(\omega)]_o\end{bmatrix} \qquad (2.2.7a)$$

$$\Delta[M] = \begin{bmatrix} [0]_{11} & [0]_{12} \\ [0]_{21} & [\Delta M]_{22} \end{bmatrix} \tag{2.2.7b}$$

where

$$[\Delta M]_{22} = [\delta m] \tag{2.2.7c}$$

thus the equation (2.2.6) is reduced to,

$$\begin{bmatrix} \alpha_{12}(\omega) \\ \alpha_{22}(\omega) \end{bmatrix}_m = \begin{bmatrix} \alpha_{12}(\omega) \\ \alpha_{22}(\omega) \end{bmatrix}_o - \frac{\omega^2 \begin{bmatrix} \alpha_{12}(\omega) \\ \alpha_{22}(\omega) \end{bmatrix}_o \delta m[\alpha_{22}(\omega)]_o}{1 + \omega^2 \delta m[\alpha_{22}(\omega)]_o} \tag{2.2.8}$$

Moraux's work was carried out on a general structural system. Similar results have also been given by Hirai *et al*[46] based on an undamped system. The limitation of this method is that it requires evaluation of new response properties at each frequency of interest, these make it uneconomical if the response of the modified system across a wide range of frequency is required. Nevertheless, the method is still regarded as the most direct method for response prediction.

Details of other methods which utilise low rank properties of localised modification can be found in the publications by Palazzoro[47] and Sadeghipour *et al*[48].

To predict the dynamic properties of a modified structural system due to small modification, three methods have been discussed in literature, namely: Rayleigh's method, first order perturbation method and modal sensitivity analysis method.

The basic assumption of Rayleigh's method for predicting the effect of small structural modification is that the mode shape vectors of a modified structural system may be approximated by those of the original system. Thus the Rayleigh's Quotient of the modified system may be expressed as,

$$\lambda_i^* = (\omega_i+\Delta\omega_i)^2 = \frac{\{\phi_i\}[[K]+\Delta[K]]\{\phi_i\}^T}{\{\phi_i\}[[M]+\Delta[M]]\{\phi_i\}^T} = \frac{\lambda+\{\phi_i\}\Delta[K]\{\phi_i\}^T}{1+\{\phi_i\}\Delta[M]\{\phi_i\}^T} \tag{2.2.9}$$

Using the generalised Rayleigh's method, which involves an iterative procedure, the methods for eigenvalue reanalysis by Rayleigh's quotient, Timoshenko's quotient and inverse iteration was presented by Wang and Pilkey[49]. The upper and lower bounds of the predicted eigenvalues of the locally modified structural system using Rayleigh's method were investigated as well. Similar work is also found in Ram *et al*[50]. Once the eigenvalue of the modified structural system is identified, a procedure which was proposed by To and Ewins[51] may be employed to predict the eigenvector of the modified system,

(1) $\{\psi\}_0$ is given, $\|\{\psi\}_0\|_2=1$ ($\|\{\psi\}_0\|_2$ is the Euclidean norm of vector $\{\psi\}_0$)

(2) For r=0, 1, ...

$$\lambda_{i,r} = \frac{\{\psi_i\}_r^T[[K]+\Delta[K]]\{\psi_i\}_r}{\{\psi_i\}_r^T[[M]+\Delta[M]]\{\psi_i\}_r} \tag{2.2.10}$$

(3) Solve

$$\left[[K]+\Delta[K]-\lambda_{i,r}([M]+\Delta[M])\right]\{z\}_{r+1} = [[M]+\Delta[M]]\{\psi_i\}_r \tag{2.2.11}$$

for $\{z\}_{r+1}$

(4)

$$\{\psi_i\}_{r+1} = \frac{\{z\}_{r+1}}{\|\{z\}_{r+1}\|_2} \tag{2.2.12}$$

The accuracy of this method depends to large extent on the validity of the assumption that the mode shape vector of the modified structural system for estimating its Rayleigh's Quotient can be approximated by the corresponding mode shape vector of the original system. Higher accuracy can be achieved if the above procedure is made iterative, which in turn implies more computational effort.

Similar to the Rayleigh's Quotient method, the first order perturbation method and sensitivity analysis method are also based on the small modification approximations to simplify the analysis and computations along with the assumed mode assumption (see details in Section 2.2.3). Both methods are achieved by neglecting the higher order terms encountered in the analysis procedure.

The difference between the Rayleigh's Quotient method and the first order perturbation method is that the latter provides a clear understanding of the involvement of the mode shape. The basic formulation of the first order perturbation method can be found from Rudisill[52] and is described as below:

According to the perturbation theory, the mass and stiffness matrices of a modified structural system due to small modification may be expressed as,

$$[M]+\Delta[M] = [M]+\varepsilon[M^{(1)}]+\varepsilon^2[M^{(2)}]+ \ldots \ldots \tag{2.2.13a}$$

$$[K]+\Delta[K] = [K]+\varepsilon[K^{(1)}]+\varepsilon^2[K^{(2)}]+ \ldots \ldots \tag{2.2.13b}$$

respectively, the eigenvalue and corresponding eigenvector may be written as,

$$\lambda_i+\Delta\lambda_i = \lambda_i+\varepsilon\lambda_i^{(1)}+\varepsilon^2\lambda_i^{(2)}+ \ldots \ldots \tag{2.2.14a}$$

$$\{\psi_i\}+\Delta\{\psi_i\} = \{\psi_i\}+\varepsilon\{\psi_i^{(1)}\}+\varepsilon^2\{\psi_i^{(2)}\}+ \ldots \ldots \tag{2.2.14b}$$

Substituting equations (2.2.13a,b) and (2.2.14a,b) into equation (2.2.1), and neglecting the terms which contain ε with higher order than one, yields,

$$\lambda_i[M]\{\psi_i^{(1)}\}+\lambda_i^{(1)}[M]\{\psi_i\}+\lambda_i[M^{(1)}]\{\psi_i\}+[K]\{\psi_i^{(1)}\}+[K^{(1)}]\{\psi_i\} = \{0\} \tag{2.2.15}$$

Pre-multiplying equation (2.2.15) by $\{\psi_i\}^T$ and considering the orthorgonality properties given by Lancaster[53] which is:

$$\{\psi_i\}^T[M]\{\psi_i^{(1)}\} = \{\psi_i\}^T[M]\{\psi_i^{(2)}\} = 0 \tag{2.2.16a}$$

the first order perturbation of the eigenvalue can then be obtained as,

$$\{\psi_i\}^T[K]\{\psi_i^{(1)}\} = \{\psi_i\}^T[K]\{\psi_i^{(2)}\} = 0 \tag{2.2.16b}$$

$$\lambda_i^{(1)} = \frac{\{\psi_i\}^T[K^{(1)}]\{\psi_i\} - \lambda_i\{\psi_i\}^T[M^{(1)}]\{\psi_i\}}{\{\psi_i\}^T[M]\{\psi_i\}} \tag{2.2.17}$$

The first order perturbation of the eigenvector can then be obtained as,

$$\begin{aligned}\{\psi_i^{(1)}\} &= -\frac{\left[\{\psi_i\}^T[M^{(1)}]\{\psi_i\}\right]\{\psi_i\}}{2\{\psi_i\}^T[M]\{\psi_i\}} \\ &+ \sum_{s=1,\ s\neq i}^{n} \frac{\left[\{\psi_s\}^T[K^{(1)}]\{\psi_i\} - \lambda_i\{\psi_i\}^T[M^{(1)}]\{\psi_i^{(1)}\}\right]\{\psi_s\}}{(\lambda_i - \lambda_s)\{\psi_S\}^T[M]\{\psi_s\}}\end{aligned} \tag{2.2.18}$$

If more accurate prediction is required, higher order perturbation in equations (2.2.14a,b) should be taken into account. Flax[54] discussed this case and presented the higher order perturbation of the eigenvalue and eigenvector in terms of results of the first order perturbation. For mass modification, To and Ewins[51] showed that the first order perturbation can also be obtained from the power series expansion with respect to the Rayleigh Quotient. Similar results were also found from Romstad *et al*[55]. Since the first order perturbation method and Rayleigh's Quotient method are both based on small modification, the study on the numerical stability of using these methods becomes vitally important. Discussion of this topic can be found in Stewart[56]. A detailed comparison between the first order perturbation method and the Rayleigh's Quotient method was given by Kaminski[57].

As perhaps the most accepted method in dealing with small modification problems, the modal sensitivity method has been studied with great enthusiasm by numerous researchers in both theoretical and experimental arenas. The modal sensitivities are the derivatives of the modal properties (natural frequencies and mode shapes) of a structural system with respect to selected structural variables. Using this method to predict the effects of proposed structural modification relies on the matrix Taylor series expansion of the modal

properties of the original system. Thus, the truncation error stemmed from neglecting the higher order terms in the series limits the application of this method for small modification of the structural system. The first order sensitivities of an undamped system with respect to structural parameter γ are given below.

The derivative of the equation of motion of a structural system with respect to γ is given as,

$$\left[\frac{\partial[K]}{\partial\gamma}-\frac{\partial\omega_i^2}{\partial\gamma}[M]-\omega_i^2\frac{\partial[M]}{\partial\gamma}\right]\{\psi_i\}+\left[[K]-\omega_i^2[M]\right]\frac{\partial\{\psi\}}{\partial\gamma} = \{0\} \tag{2.2.19}$$

Pre-multiplying equation (2.2.19) by $\{\psi\}^T$ and solving for $\partial\omega_i^2 / \partial\gamma$ gives:

$$\frac{\partial\omega_i^2}{\partial\gamma} = \frac{\{\psi_i\}^T\left[\frac{\partial[K]}{\partial\gamma}-\omega_i^2\frac{\partial[M]}{\partial\gamma}\right]\{\psi_i\}}{\{\psi_i\}^T[M]\{\psi_i\}} \tag{2.2.20}$$

The mode shape derivative can be obtained by assuming that it can be expressed as a linear combination of the eigenvectors of the original system (so called assumed mode assumption),

$$\frac{\partial\{\psi\}}{\partial\gamma} = \sum_{j=1}^{n} c_j\{\psi_j\} \tag{2.2.21}$$

The constants c_j are given as,

$$c_{j,\ (j=1,2,\ldots,n\ \ j\neq i)} = \frac{\{\psi_j\}^T\left[\frac{\partial[K]}{\partial\gamma}-\omega_i^2\frac{\partial[M]}{\partial\gamma}\right]\{\psi_i\}}{(\omega_j^2-\omega_i^2)\{\psi_j\}^T[M]\{\psi_j\}} \tag{2.2.22a}$$

$$c_{j,\ (j=i)} = \{\psi_j\}^T\frac{\partial[M]}{2\partial\gamma}\{\psi_i\} \tag{2.2.22b}$$

Although the mathematical foundation of sensitivity analysis had been laid by

Lancaster[58] and others, extending this method into structural vibration analysis is attributed to Fox and Kapoor[59]. Since the calculation of the mode shape derivative from equations (2.2.21) and (2.2.22a,b) requires the knowledge of all mode shapes of the original system, which is not always possible in practice, Nelson[60] developed a simplified method by which only the modal data associated with particular mode is required.

Provided that the modal sensitivity information is available, the effect of changes of the structural parameter γ can then be predicted by using a Taylor's series expansion about the modal properties of the original system,

$$\omega_i^2+\Delta\omega_i^2 = \omega_i^2+\frac{\partial\omega_i^2}{\partial\gamma}\Delta\gamma+\frac{\partial^2\omega_i^2}{\partial\gamma^2}\Delta\gamma^2+ \dots \dots \tag{2.2.23a}$$

$$\{\psi\}_i+\Delta\{\psi_i\} = \{\psi_i\}+\frac{\partial\{\psi_i\}}{\partial\gamma}\Delta\gamma+\frac{\partial^2\{\psi_i\}}{\partial\gamma^2}\Delta\gamma^2+ \dots \dots \tag{2.2.23b}$$

Normally, for small structural modification, the higher order terms in equations (2.2.23a,b) are assumed negligible and therefore these equations can be readily applied to structural modification problems. However, higher order derivatives are required in some cases for more accurately predicted results. Brandon[61] discussed these cases in detail and formulated higher order modal derivatives in his work. As well, Brandon[62] and He[63] discussed the response sensitivities in terms of receptance derivatives. For large amount of structural modifications, the computation of higher order modal derivatives are rather complicated and lower order approximations is inaccurate. Hence a repeated application of modal sensitivities based on the piece-wise linear principle is usually employed. Although this approach has defects which are difficult to overcome, it is still considered as an effective approach in structural analysis due to the following advantages (i) suitable for distributed modification, (ii) straightforward to compute using information from experimental modal analysis, (iii) provides the information as to the most sensitive location for a particular structural modification.

2.2.3 Specification Methods

The specification methods for local structural modification comprise those methods used to predict the variations in the structural parameters of a structural system which will lead to specified modal properties (natural frequency and/or mode shape). These methods can also be described as 'inverse methods' - the opposite of the prediction methods discussed in Section 2.2.2. Therefore, the problem characterised in equation (2.2.1) becomes to one of searching for $\Delta[M]$ and $\Delta[K]$ of [M] and [K] respectively in order to produce ω^*, and/or $\{\psi^*\}$.

The localised modification approach in prediction methods can readily be applied in specification methods. Returning to equations (2.2.3) and (2.2.8), it is noted that the problems defined by these equations will be significantly simpler if the unknown becomes the unit rank mass modification. However, since these prediction methods are only valid for lumped mass systems with unit rank modification, they are only suitable for parametric studies. A similar method was also presented by Tsuei and Yee[64], where only the information (physical and modal) of the original system related to the location of modification are required to obtain a proposed unit rank mass or stiffness modification for a specified natural frequency. This method was then extended by Li *et al*[65] to solve multiple mass and stiffness modifications for both a specified natural frequency and a specified anti-resonance frequency. As well, an approach for finite element implementation of this method which is characterised by a set of non-linear eigenpolynomials has been developed by Li *et al*[66] based on the state space theory.

A similar problem was discussed earlier by Lancaster[58] to search for a mass and stiffness incremental matrices which, when applied to the structural system, will mathematically satisfy a specified eigenproblem. He stated that for a fully determined undamped system with distinct engen values, unique solutions for mass and stiffness matrices exist. Nowerdays, theories for such a problem have been well developed in the area of analytical model updating. Most of them attempt to use experimental modal data of a structure from EMA to update or correct its analytical model obtained by FEA. The

differences between the analytical and the 'correct' models may be described as the incremental matrices. Some well known model updating methods are: Objective Function Method where solutions are obtained in a least square sense (Baruch[67]); Error Matrix Method based on an inverse perturbation approach (Sidbu and Ewins[68]); Pole Placement Method originating from control theory (Starkey[69]); Error Location Method using simple dynamic equation (He and Ewins[70]) and Inverse Sensitivity Method (Lin and He[71]). Most of these methods may readily be, or have been extended to be, applied as the specification methods in the structural modification problem. Although the general procedure for solving inverse structural modification problems and model updating problems may be the same, the problems themselves are essentially different. Model updating methods aim at identifying the location and quantity of the errors between the analytical and the 'exact' models using would-be errorous mass and stiffness matrices and correct modal properties, whereas the inverse structural modification methods aim at finding the location and quantity of the modifications which lead to specified modal properties using 'exact' mass and stiffness matrices. The consequence of this difference implies that the 'error' matrices in model updating problems will always exist and with explicit physical significance. However, the local structural modification represented by the incremental matrices in inverse structural modification may not be always physically realisable, meaning that the desired modal properties may not always be achievable through a practical structural modification. In the remainder of this Section, a general examination will be undertaken into some of the widely adopted methods which focus on the inverse problems of structural modification. Most of these methods are based on two assumptions which have already been stated in the prediction methods, namely: the small modification assumption and assumed mode assumption.

As one of the most popular specification approaches based on the small modification assumption, the inverse perturbation approach has been well studied by many researchers (Stetson and Palma[72], Stetson *et al*[73]). The processes to obtain the mass and stiffness incremental matrices by using these approaches is rather complicated. Thus realisation processes of these incremental matrices are often featured by expressing the mass and stiffness incremental matrices as the functions of variation of a particular structural

parameter. Hoff *et al*[74] introduced an algorithm which involved a typical procedure of using inverse perturbation approach in specification methods. In their work, the mass and stiffness incremental matrices were expressed as,

$$\Delta[M] = \sum_{j=1}^{p} \Delta\gamma_j \Delta[M_j] \tag{2.2.24a}$$

and

$$\Delta[K] = \sum_{j=1}^{p} \Delta\gamma_j \Delta[K_j] \tag{2.2.24b}$$

However, it is noted that the expression of such linear functions may not always be possible in practice (see detailed discussion in Section 3.2.2).

According to Hoff *et al*, the algorithm was developed in two stages:

(i) Predictor stage: $\Delta\gamma$ is obtained in terms of the modal properties of the original system as,

$$\Delta\lambda_i = \frac{\sum_{j=1}^{p} \Delta\gamma_j \{\psi_i\}^T \Delta[K_j]\{\psi_i\} - \lambda_i \{\psi_i\}^T \Delta[M_j]\{\psi_i\}}{\{\psi_i\}^T [M]\{\psi_i\}} \tag{2.2.25}$$

Equation (2.2.25) becomes a set of linear equations from which the parameter changes $\Delta\gamma$ can be solved subject to the desired changes in multiple natural frequencies, and the first order perturbation of mode shapes to the $\Delta\gamma$ can then be written in term of the parameter's change as,

$$\Delta\{\psi_i\} = \sum_{j=1}^{p} \Delta\gamma_j \left[-\frac{\left[\{\psi_i\}^T \Delta[M_j]\{\psi_i\}\right]\{\psi_i\}}{2\{\psi_i\}^T [M]\{\psi_i\}} + \sum_{s=1,\ s\neq i}^{n} \frac{\left[\{\psi_s\}^T \Delta[K_j]\{\psi_i\} - \lambda_i \{\psi_i\}^T \Delta[M_j]\{\psi_i\}\right]\{\psi_s\}}{(\lambda_i - \lambda_s)\{\psi_s\}^T [M]\{\psi_s\}} \right] \tag{2.2.26}$$

(ii) Corrector stage: the pre-defined natural frequencies and the mode

shapes obtained from equation (2.2.26) are used to solve the corrected structural parameter's change,

$$\sum_{j=1}^{p} (\{\psi^*\}_s^T \Delta[K_j]\{\psi^*\}_i - \omega_i^{*2}\{\psi^*\}_s^T \Delta[M_j]\{\psi^*\}_i)\Delta\gamma_j = \omega_i^{*2}\{\psi^*\}_s^T \Delta[M_j]\{\psi^*\} - \{\psi^*\}_s^T \Delta[K]\{\psi^*\} \tag{2.2.27}$$

where

$$\{\psi^*\}_s = \{\psi\}_s + \Delta\{\psi\}_s \tag{2.2.27a}$$

$$\omega_i^{*2} = \omega_i^2 + \Delta\omega_i^2 \tag{2.2.27b}$$

Equation (2.2.27) is a set of linear equations whose the number depends on the number of the specified natural frequencies.

In case of large modal property changes, the algorithm can be applied in several small steps. This will then require more computational effort. A similar method has also been presented by Zhang and Wang[75] using a viscously damped mass-spring system. Due to the involvement of complex mode shapes, the computation becomes more complex and an iteration procedure must be employed. The application of the higher order perturbation was also discussed in [75]. However, the accuracy of specification methods based on the inverse perturbation approach still suffer from shortcomings stemming from: (i) computational cost because they always involve iterative processes, (ii) the limitation on the amount of modification because they always neglect the higher order terms in the perturbation equation, and (iii) the fact that the modification predictions are not always realisable because the modification must be idealised as concentrated mass modification or the incremental matrices are expressed as linear function of structural parameter. The fundamentals of methods based on inverse perturbation are clearly depicted by Stetson and Palma[72] and Auburn[76].

The assumed mode assumption has become one of the most important theoretical

foundation of many structural dynamic analysis methods - the mode shape vector of the modified system belongs to the space which is spanned by a truncated set of mode shape vectors of the original system. Widely accepted not only in the prediction methods, but also in the specification methods, this assumption is of great significance when using an incomplete set of modal properties to deal with structural modification problems. Firstly, the assumption implies that the mode shape vectors and/or their variations due to local structural modification can be expressed as a linear combination of an incomplete set of mode shape vectors of the original system. Secondly, it implies that the order of the problem can be greatly reduced by transferring the problem into the modal space spanned by this incomplete set of original mode shape vectors. Nevertheless, this may sacrifice the physical connectivity information. The methods involving this assumption are also referred to as assumed mode methods by some researchers (see also White and Maytum[77] for the application of this method). A series of investigations about this assumption have been carried out by Braun *et al*[78],[79],[80] and some valuable conclusions have been drawn. Based on this assumption, Ram and Braun[81] proposed a new method to calculate mathematically the mass and stiffness incremental matrices subjected to given natural frequencies and corresponding mode shapes without any knowledge of the mass and stiffness matrices of the original system. Assuming the mode shapes of the modified structural system can be expressed as,

$$[\Phi] = [\Phi_1 \vdots \Phi_2]_{[\Phi]\in R^{n\times n},\ [\Phi_1]\in R_1^{n\times p},\ [\Phi_2]\in R_2^{n\times q},\ n=p+q} \tag{2.2.28}$$

$$[\Phi_1^*] = [\Phi_1][c]_{[\Phi^*]\in R_1^{n\times p},\ [c]\in R_2^{p\times p},\ n=p+q} \tag{2.2.29}$$

The equations of motion of the modified system can be written as,

$$[[K]+\Delta[K]][\Phi_1^*] = [[M]+\Delta[M]][\Phi_1^*]diag[\omega_r^{*2}] \tag{3.3.30a}$$

Substituting equation (2.2.29) into equation (2.2.30a) and pre-multiplying by $[c]^T$, yield,

$$\begin{aligned}&[c]^T\left[diag[\omega_r^2]+[\Phi_1]^T\Delta[K][\Phi_1]\right][c] = \\ &[c]^T\left[[\ I\ ]+[\Phi_1]^T\Delta[M][\Phi_1]\right][c]diag[\omega_r^{*2}]\end{aligned} \tag{3.3.30b}$$

where

$$[\Phi_1]^T[M][\Phi_1] = [\ I\ ] \tag{2.2.30c}$$

$$[\Phi_1]^T[K][\Phi_1] = diag[\omega_r^2] \tag{2.2.30d}$$

considering

$$[c]^T\Big[[\ I\ ]+[\Phi_1]^T\Delta[M][\Phi_1]\Big][c] = [\ I\ ] \tag{2.2.31}$$

$$[c]^T\Big[diag[\omega_r^2]+[\Phi_1]^T\Delta[K][\Phi_1]\Big][c] = diag[\omega_r^{*2}] \tag{2.2.32}$$

The non-unique symmetric matrices Δ[M] and Δ[K] can be obtained according to Ben-Isreal and Grevillle[82] as,

$$\begin{aligned}\Delta[M] = {} & [\Phi_1]^{T+}\Big[[c]^{-T}[c]^{-1}-[\ I\ ]\Big][\Phi_1]^{+} \\ & +[Y]-[\Phi_1]^{T+}[\Phi_1]^T[Y][\Phi_1][\Phi_1]^{+}\end{aligned} \tag{2.2.33}$$

$$\begin{aligned}\Delta[K] = {} & [\Phi_1]^{T+}\Big[[c]^{-T}diag[\omega_r^{*2}][c]^{-1}-diag[\omega_r^2]\Big][\Phi_1]^{+} \\ & +[Z]-[\Phi_1]^{T+}[\Phi_1]^T[Z][\Phi_1][\Phi_1]^{+}\end{aligned} \tag{2.2.34}$$

the matrices Δ[M] and Δ[K] may then be realised using finite element method as,

$$\Delta[M] = \sum_{i=1}^{\sigma}\Delta\gamma_i[\beta_i]^T[M_e][\beta_i] \tag{2.2.35}$$

and

$$\Delta[K] = \sum_{i=1}^{\sigma}\Delta\gamma_i[\beta_i]^T[K_e][\beta_i] \tag{2.2.36}$$

The realisation problem given in equations(2.2.35) and (2.2.36) may not always have exact solutions. In many cases, it will exist as a over-determined problem, which is similar to that discussed by Lim[36]. The mathematical conditions on which this assumed mode

assumption based were also discussed and given by Bucher and Braun[83].

2.3 Problem Definition

It was shown in the previous Sections that theories in areas of establishing mathematical models and structural modification analysis have been well developed from both theoretical and practical application viewpoints. As one of the most important applications, using these theories in vibration reduction is attracting more and more attention. For this application, the resonance and anti-resonance phenomena are of major concern. The goal of this work is to investigate and develop an effective method for vibration reduction by local structural modification. In this work, two algorithms will be developed along with the theoretical background based on analytical (mass-spring or FEA) models.

The first algorithm performs the relocation of resonance and anti-resonance by local structural modification to serve for the vibration reduction purposes. As well, some important properties of anti-resonance relevant to the vibration reduction will be explored. The element sensitivity index will be defined in order to provide guidance for local structural modifications. The algorithm developed will readily be applicable to an existing finite element model. Both numerical and experimental examples will be examined to validate the algorithm.

The second algorithm performs the structural dynamic optimisation by local structural modification. It aims at the elimination of a resonance peak from certain frequency response functions. The theoretical basis of the algorithm will be investigated in detail. Numerical examples will be provided to verify the effectiveness of the algorithm.

The problems addressed in this work will also be focused on the applicability of using the local structural modification approach in vibration reduction. The frequency response in the vicinity of resonance and anti-resonance will become the main criterion to evaluate the effectiveness of the outcome.

CHAPTER 3

RELOCATION OF RESONANCE AND ANTI-RESONANCE BY LOCAL STRUCTURAL MODIFICATION

This Chapter aims to present a complete theoretical development of a procedure to perform local structural modification which is focused on vibration reduction. As a manifestation of vibration level, the frequency response characteristics of a structural system are of major concern. These characteristics are commonly interpreted in terms of receptance, mobility, inertance, resonances and anti-resonances. In this Chapter, mass and stiffness modifications will be determined which will result in specified relocation of resonance and/or anti-resonance in an FRF.

3.1 Receptance FRF

For an NDOF structural system undergoing forced vibration, the equations of motion in matrix form are,

$$[M]\{\ddot{x}(t)\}+[C]\{\dot{x}(t)\}+i[H]\{x(t)\}+[K]\{x(t)\} = \{f(t)\} \tag{3.1.1}$$

If the system is lightly damped, the inherent damping of either viscous type or hysteresis type is assumed to be negligible. Thus, damping is not considered in local structural modification. The symmetrical mass and stiffness matrices, [M] and [K], are normally derived from the finite element analysis (FEA) of an existing or conceptual structure. It is generally assumed that matrix [M] be positive definite and matrix [K] be at least positive semi-definite. The [M] and [K] matrices thus derived comprise the mathematical model of the structural system which is used to express the magnitudes and distributions of mass and stiffness of the system.

The transformation relations, which are used to convert equation (3.1.1) from time domain to frequency domain are,

$$\{x(t)\} = \{X\}e^{i\omega t} \tag{3.1.2a}$$

$$\{f(t)\} = \{F\}e^{i\omega t} \tag{3.1.2b}$$

which imply that the structural system is subjected to harmonic excitation. Substituting equations (3.1.2a,b) into equation (3.1.1), and assuming damping is negligible, will lead to,

$$-\omega^2[M]\{X\}+[K]\{X\} = \{F\} \tag{3.1.3}$$

The equation of motion of the structural system subject to free vibration can be written as,

$$-\omega_n^2[M]\{Y\}+[K]\{Y\} = \{0\} \tag{3.1.4}$$

Rearranging equation (3.1.3) yields,

$$\{X\} = \left[-\omega^2[M]+[K]\right]^{-1}\{F\} = [Z(\omega)]^{-1}\{F\} \tag{3.1.5}$$

The matrix $[Z(\omega)] = [-\omega^2[M]+[K]]$ is referred to as the 'dynamic stiffness matrix' or the 'system matrix'. Provided $[Z(\omega)]$ is non-singular, matrix $[\alpha(\omega)]$ is given as,

$$[\alpha(\omega)] = [Z(\omega)]^{-1} = \left[-\omega^2[M]+[K]\right]^{-1} \tag{3.1.6}$$

It is customary to define the matrix $|\alpha(\omega)|$ as receptance frequency response functions (FRF) matrix, for the structural system, the condition of its existing is that $\omega \neq \omega_n$, since equation (3.1.5) can not be solved if the structural system is excited at one of its natural frequencies. The general elements in the receptance FRF matrix are defined as follows,

$$\alpha_{ij}(\omega) = \frac{X_i}{F_j}\Big|_{r = 0\ (r=1,2,\ldots n, r \neq j)} \tag{3.1.7}$$

The physical significance of $\alpha_{ij}(\omega)$ is explicit. It represents the displacement response of coordinate i to a unit sinusoidal excitation with frequency ω applied on coordinate j, as no other excitation is assumed. However, to solve for each element of $[\alpha(\omega)]$ using equation (3.1.6) involves the matrix inverse of $[Z(\omega)]$ at each frequency ω. Ewins[8] has discussed the disadvantages of this direct matrix inverse method as follows:

(i) costly for large-order systems,

(ii) inefficient if only a few of the individual FRF expressions are required,

(iii) provides no insight into the form of various FRF properties.

Using the orthogonality properties of the structural system which consists of [M] and [K], an alternative form of expression of $\alpha_{ij}(\omega)$ can be derived as,

$$[\Phi]^T[M][\Phi] = [\ I\ ] \tag{3.1.8a}$$

$$[\Phi]^T[K][\Phi] = diag[\omega_{n,r}^2] \tag{3.1.8b}$$

Rearranging equation(3.1.6) yields,

$$\left[-\omega^2[M]+[K]\right] = [\alpha(\omega)]^{-1} \tag{3.1.9}$$

pre-multiplying both sides by $[\Phi]^T$ and post-multiplying both sides by $[\Phi]$ yields,

$$[\Phi]^T\left[-\omega^2[M]+[K]\right][\Phi] = [\Phi]^T[\alpha(\omega)]^{-1}[\Phi] \tag{3.1.10}$$

which results in the spectral decomposition form of $[\alpha(\omega)]$,

$$[\alpha(\omega)] = [\Phi]\left[diag[\omega_{n,r}^2-\omega^2]\right]^{-1}[\Phi]^T \tag{3.1.11}$$

It is clear from this equation that the receptance matrix is symmetric. This demonstrates the principle of reciprocity, in other words,

$$\alpha_{ij} = \frac{X_i}{F_j} = \alpha_{ji} = \frac{X_j}{F_i} \tag{3.1.12}$$

Response of coordinate i to the excitation applied at coordinate j will be the same as that of coordinate j to the excitation applied at coordinate i. This property is important for vibration reduction which involves certain coordinates of the structure system, as will be discussed later.

To compute an individual FRF at any frequency of interest, $\alpha_{ij}(\omega)$ may be expressed in the form of partial fractions as,

$$\alpha_{ij}(\omega) = \sum_{r=1}^{n} \frac{\phi_{ir}\phi_{jr}}{\omega_{n,r}^2 - \omega^2} \qquad (3.1.13)$$

equation (3.1.13) is normally written in the form of equation (3.1.14) by introducing modal constant - ${}_rA_{ij}$, as,

$$\alpha_{ij}(\omega) = \sum_{r=1}^{n} \frac{{}_rA_{ij}}{\omega_{n,r}^2 - \omega^2} \qquad (3.1.14)$$

So far, we have defined the receptance FRF $\alpha_{ij}(\omega)$. It will be used as the main parameter to assess the system vibration behaviour. From the vibration reduction point of view, the objective of this research is to decrease the value of the receptance frequency response within the frequency range of interests. Three methods will developed in this thesis, they being,

(i) shifting a natural frequency away from the frequency range of concern,

(ii) relocating an anti-resonance (the concept of anti-resonance will be explained later) into the frequency range of interest,

(iii) eliminating a resonance peak of a required FRF.

3.2 RELOCATION OF RESONANCE IN FRF

3.2.1 Relocation of Resonance for a Lumped Mass System

It is well known that when encouraged to vibrate at a natural frequency, systems joyfully respond. This frequency is also called a resonance frequency. It is so called because of the fact that if the excitation frequency is equal to a natural frequency, then violent motion may be expected. This extreme magnification of motion is known as 'resonance'. This phenomenon is described mathematically in equation (3.1.14). Rewriting equation (3.1.14) yields,

$$\alpha_{ij}(\omega) = \frac{{}_pA_{ij}}{\omega_{n,p}^2 - \omega^2} + \sum_{r=1, r\neq p}^{n} \frac{{}_rA_{ij}}{\omega_{n,r}^2 - \omega^2}, \quad (1<p<n) \qquad (3.2.1)$$

From equation (3.2.1), it can be seen that when the exciting frequency is equal, or close,

to the natural frequency of the p^{th} mode of the structural system, the response of the system to the excitation will tend to be infinite in the case of no damping. The total response will be dominated by the first term on the right side of equation (3.2.1). In the other words, it is dominated by the contribution of the p^{th} natural mode of the system. However, when the p^{th} modal constant is zero or is approximately zero, the p^{th} natural mode will not show a resonance peak in FRF curves relating certain coordinates. This will be discussed in Chapter 5.

A forced vibration usually becomes significant only if resonance occurs. In practice, it has been recognised as an efficient means to 'detune' a system so that the natural frequency is shifted away from the exciting frequency and unwanted resonance vibration is hence averted. The practical determination of changes in mass and stiffness distribution to relocate a natural frequency is therefore vital.

If equation (3.1.4) is used as the equation of motion of a structural system, then it is recognised that the system be fully identified and the mass and stiffness matrices obtained from FEA or other methods be capable of representing the mass and stiffness distribution of the real or the conceptual structure. The global sum of mass and stiffness modification applied on the system can be expressed as mass and stiffness incremental matrices, $\Delta[M]$ and $\Delta[K]$ respectively,

Recasting equation (3.1.4) with mass and stiffness incremental matrices yields,

$$-\omega^{*2}\left[[M]+\Delta[M]\right]\{Y^*\}+\left[[K]+\Delta[K]\right]\{Y^*\} = \{0\} \qquad (3.2.2)$$

Equation (3.2.2) is the equation of motion of the modified structural system subjected to free vibration, where the $\Delta|M|$ and $\Delta|K|$ matrices can be constructed as follows:

For a lumped mass system, where an individual mass modification represented by a mass increment in the r^{th} coordinate, typical of the attachment of a 'lumped' mass, $\Delta[M]$ can be constructed by using $\{e_r\}$ mapping vector,

$$\Delta[M] = \{e_r\}\delta m_r\{e_r\}^T \qquad (3.2.3a)$$

Therefore, multiple mass modifications, say, on coordinates i, j, k, can be represented as a summation of individuals,

$$\Delta[M] = \sum_{r=i,j,k} \{e_r\}\delta m_r\{e_r\}^T \qquad (3.2.3b)$$

Similarly, the global sum of stiffness modification between coordinates p, q(p=i,j,k, q=i,j,k), can be written as,

$$\Delta[K] = \sum_{p,q=i,j,k} \{e_{pq}\}\delta k_{pq}\{e_{pq}\}^T \qquad (p\neq q) \qquad (3.2.3c)$$

Re-arranging equation (3.2.2) yields,

$$-\omega^{*2}[M]\{Y^*\}+[K]\{Y^*\} = \omega^{*2}\Delta[M]\{Y^*\}-\Delta[K]\{Y^*\} \qquad (3.2.4)$$

which leads to,

$$\{Y^*\} = \left[-\omega^{*2}[M]+[K]\right]^{-1}\left[\omega^{*2}\Delta[M]-\Delta[K]\right]\{Y^*\} \qquad (3.2.5)$$

Considering equation (3.1.6), and assuming $\omega=\omega^*$, equation (3.2.5) becomes,

$$\{Y^*\} = [\alpha(\omega^*)]\left[\omega^{*2}\Delta[M]-\Delta[K]\right]\{Y^*\} \qquad (3.2.6)$$

If we only intend to change one natural frequency which may be close to the excitation frequency, then the problem of determining $\Delta|M|$ and $\Delta[K]$ becomes simple, since any modification of mass and stiffness to any part of a vibrating system is likely attended by the variation of the system natural frequencies of interest. However, the goal is to relocate a natural frequency so that the exciting frequency can be out of the resonance region (normally it is banded by approximate $\omega/\omega_n=1\pm(1-\sqrt{2})$, where ω and ω_n are respectively the excitation frequency and a natural frequency of the original structural system). To achieve it, the optimum amount of mass and stiffness modification have to be determined.

For a lumped mass system, without loss of generality, assume the mass modification is applied on coordinates i, j, k which are δm_i, δm_j, δm_k (more mass modifications can be studied likewise). Assume also that such mass modification do not affect stiffness

distribution of the original system (in practice this situation occurs when lumped masses are added to or subtracted from the system). Then, the mass modification can be expressed in matrix form to,

$$\Delta[M] = \begin{bmatrix} [0] & & \\ & [\Delta M] & \\ & & [0] \end{bmatrix} \tag{3.2.7a}$$

where

$$[\Delta M] = \begin{bmatrix} \delta m_i & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \delta m_j & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \delta m_k \end{bmatrix} \tag{3.2.7b}$$

By introducing mass modification ratios $\eta_{i,\ j,\ k}$, which are pre-defined ratios of the modified masses (positive ratios mean the addition of masses, and negative ratios mean the subtraction of masses), the relationship among δm_i, δm_j, and δm_k will be as follows,

$$\{\delta m_i \quad \delta m_j \quad \delta m_k\}^T = \gamma\{\eta_i \quad \eta_j \quad \eta_k\}^T \tag{3.2.8}$$

where γ is a scaling factor of a unit of mass.

Substituting equation (3.2.8) into equation (3.2.6) when $\Delta[K]=0$, yields,

$$\{Y^*\} = [\alpha(\omega^*)](\omega^{*2}\gamma[\eta])\{Y^*\} \tag{3.2.9}$$

where

$$[\eta] = \begin{bmatrix} [0] & & \\ & [\eta]_o & \\ & & [0] \end{bmatrix} \tag{3.2.9a}$$

and

$$[\eta]_o = \begin{bmatrix} \eta_i & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \eta_j & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \eta_k \end{bmatrix} \qquad (3.2.9b)$$

By considering the equations corresponding to coordinates i, j, k only, equation (3.2.9) can then be reduced to,

$$\frac{1}{\gamma}\{Y^*\}_r = [\alpha(\omega^*)]_r(\omega^{*2}[\eta]_r)\{Y^*\}_r \qquad (3.2.10)$$

where

$$[\alpha(\omega^*)]_r = \begin{bmatrix} \alpha(\omega^*)_{ii} & \alpha(\omega^*)_{ij} & \alpha(\omega^*)_{ik} \\ \alpha(\omega^*)_{ji} & \alpha(\omega^*)_{jj} & \alpha(\omega^*)_{jk} \\ \alpha(\omega^*)_{ki} & \alpha(\omega^*)_{kj} & \alpha(\omega^*)_{kk} \end{bmatrix} \qquad (3.2.11a)$$

$$[\eta]_r = \begin{bmatrix} \eta_i & 0 & 0 \\ 0 & \eta_j & 0 \\ 0 & 0 & \eta_k \end{bmatrix} \qquad (3.2.11b)$$

$$\{Y^*\}_r^T = \{Y_i^* \; Y_j^* \; Y_k^*\}^T \qquad (3.2.11c)$$

Let $1/\gamma = \lambda$ and $[\alpha(\omega)]_r[\eta]_r = [C]$, equation (3.2.10) can be re-written as,

$$\lambda\{Y^*\}_r = \omega^{*2}[C]\{Y^*\}_r \qquad (3.2.12)$$

Equation (3.2.12) is a reduced ('reduced' implies that the order of the original problem has been reduced to the number of coordinates involved in the modification) standard eigen problem. By solving this eigen problem, the eigenvalues, which are λ, will be obtained. The multiple eigenvalues may indicate there exist several acceptable modifications. The mass modifications which are applied on coordinates i, j, k, to obtain desired new natural

frequency ω^* can then be determined as follows,

$$\delta m_i = \gamma \eta_i \tag{3.2.13a}$$

$$\delta m_j = \gamma \eta_j \tag{3.2.13b}$$

$$\delta m_k = \gamma \eta_k \tag{3.2.13c}$$

Similarly, stiffness modification between the p^{th} and the q^{th} coordinates (p=i, j, k, q=i, j, k, $p \neq q$), which can be expressed as δk_{ij}, δk_{jk}, δk_{ik}, with stiffness modification ratio κ_{ij}, κ_{jk}, κ_{ik}, can be solved as follows,

The matrix form of stiffness modification may be expressed by equations (3.2.14a,b)

$$\Delta[K] = \begin{bmatrix} [0] & & \\ & [\Delta K] & \\ & & [0] \end{bmatrix} \tag{3.2.14a}$$

where

$$[\Delta K] = \begin{bmatrix} \delta k_{ij}+\delta k_{ik} & \cdots & -\delta k_{ij} & \cdots & -\delta k_{ik} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\delta k_{ij} & \cdots & \delta k_{ij}+\delta k_{jk} & \cdots & -\delta k_{jk} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\delta k_{ik} & \cdots & -\delta k_{jk} & \cdots & \delta k_{jk}+\delta k_{ij} \end{bmatrix} \tag{3.2.14b}$$

Assume that the stiffness modification is carried out without changing the mass distribution of the original system. By substituting equations (3.2.14a,b) into equation (3.2.6), and considering $\Delta[M] = [0]$, the stiffness modifications between the p^{th} coordinate and the q^{th} coordinate can then be obtained in the same manner as mass modification to obtain the desired natural frequency of the system by solving equation (3.2.15).

$$\lambda\{Y^*\}_r = [C]\{Y^*\}_r \tag{3.2.15}$$

where

$$\lambda = 1/\gamma \tag{3.2.15a}$$

$$[C] = [\alpha(\omega)]_r[\kappa]_r \tag{3.2.15b}$$

$$[\kappa]_r = \begin{bmatrix} \kappa_{ij}+\kappa_{ik} & -\kappa_{ij} & -\kappa_{ik} \\ -\kappa_{ij} & \kappa_{ij}+\kappa_{jk} & -\kappa_{jk} \\ -\kappa_{ik} & -\kappa_{jk} & \kappa_{jk}+\kappa_{ik} \end{bmatrix} \tag{3.2.15c}$$

$$\{\delta k_{ij} \quad \delta k_{jk} \ \ldots \ \ldots \quad \delta k_{kk}\}^T = \gamma\{\kappa_{ij} \quad \kappa_{jk} \ \ldots \ \ldots \quad \kappa_{kk}\}^T \tag{3.2.15d}$$

3.2.2 Finite Element Implementation of the Local Structural Modification

The methods described above to determine the mass or stiffness modification of a lumped mass system to achieve a desired natural frequency have significant importance and can find immediate practical application. However, its effectiveness becomes limited because the method assumes that either mass or stiffness modification be made, but not both. To utilise the full potential of the method, a structural modification procedure with both mass and stiffness being involved has to be explored. This is particularly important in the case where a consistent mass matrix rather than a lumped mass matrix is used to deal with a structural system. Traditionally, by using FEA, the global mass and stiffness matrices are assembled from element mass and stiffness matrices which are usually governed by the element types and their physical properties. Once the elements used to construct the system have been defined, they can be described in terms of physical structural parameters (elasticity modulus, poisson ratio, density, cross section area, thickness, etc). Details of the derivation of mass and stiffness matrices can be found in standard FEA text books.

The finite element implementation of the local structural modification on a vibratory structural system can be realised by varying the physical parameters of local elements to achieve the desired dynamic properties. For a single element, assume γ is one of its physical structural parameters which is to be modified, and mass and stiffness incremental matrices with respect to γ are assumed to be linear (for instance, the mass and stiffness matrices of a truss bar element is linear with respect to its cross sectional area) and are

denoted as $\Delta[M]$ and $\Delta|K|$ respectively, then they can be expressed as,

$$\Delta[M_i] = \Delta\gamma_i[\eta_i] \tag{3.2.16a}$$

$$\Delta[K_i] = \Delta\gamma_i[\kappa_i] \tag{3.2.16b}$$

In case more than one element is involved in modification, $\Delta[M]$ and $\Delta[K]$ can be expressed as the summation of the individual mass and stiffness incremental matrices,

$$\Delta[M] = \sum_{i=1}^{n} \Delta\gamma_i[\beta_i]^T[\eta_i][\beta_i] \tag{3.2.17a}$$

and

$$\Delta[K] = \sum_{i=1}^{n} \Delta\gamma_i[\beta_i]^T[\kappa_i][\beta_i] \tag{3.2.17b}$$

Using a similar concept of mass and stiffness modification ratios, element participant ratio γ_i^* (i=1,2...n) is introduced which can be predetermined. Then, equation (3.2.17a) and (3.2.17b) can be re-written as,

$$\Delta|M| = \Delta\gamma\sum_{i=1}^{n} \gamma_i^*|\beta_i|^T[\eta_i]|\beta_i| \tag{3.2.18a}$$

and

$$\Delta[K] = \Delta\gamma\sum_{i=1}^{n} \gamma_i^*[\beta_i]^T[\kappa_i][\beta_i] \tag{3.2.18b}$$

where

$$\{\Delta\gamma_1 \quad \Delta\gamma_2 \quad \dots \quad \Delta\gamma_n\}^T = \Delta\gamma\{\gamma_1^* \quad \gamma_2^* \quad \dots \quad \gamma_n^*\}^T \tag{3.2.19}$$

Substituting equations (3.2.18a) and (3.2.18b) into equation (3.2.6) yields,

$$\{Y^*\} = [\alpha(\omega^*)]\left[\Delta\gamma\sum_{i=1}^{n} \gamma_i^*[\beta_i]^T(\omega^{*2}[\eta_i]-[\kappa_i])[\beta_i]\right]\{Y^*\} \tag{3.2.20}$$

Rearranging equation (3.2.20) yields,

$$\frac{1}{\Delta\gamma}\{Y^*\} = [\alpha(\omega^*)]\left[\sum_{i=1}^{n}\gamma_i^*[\beta_i]^T(\omega^{*2}[\eta_i]-[\kappa_i])[\beta_i]\right]\{Y^*\} \qquad (3.2.21)$$

Equation(3.2.21) is a standard eigen problem from which $\Delta\gamma$ can be solved. Once $\Delta\gamma$ is determined, the modifications of parameters γ_i can be computed from equation (3.2.19).

For some finite element types, the relationship between the element matrices and the structural parameters to be modified may not be expressed ideally as equations (3.2.16a) and (3.2.16b), for instance, the incremental matrices may not be linear functions of the structural parameter γ_i. For the sake of simplicity, assume $\Delta[M_i]$ is a first order function of γ_i and $\Delta[K_i]$ is the p^{th} order function of γ_i, respectively, the mass and stiffness incremental matrices may be derived as follows:

$$\Delta[M_i] = (\gamma_i+\Delta\gamma_i)[\eta]-[M_i] = \Delta\gamma_i[\eta_i] \qquad (3.2.22a)$$

$$\Delta[K_i] = (\gamma_i+\Delta\gamma_i)^q[\kappa_i]-[K_i] = \sum_{p=1}^{q}\binom{q}{p}\gamma_i^{q-p}\Delta\gamma_i^{p}[\kappa_i] \qquad (3.2.22b)$$

where

$$\binom{q}{p} = \frac{q!}{p!(q-p)!} \qquad (p = 1,2,3,\cdots\cdots\cdots,q) \qquad (3.2.22c)$$

Substituting equations (3.2.22a) and (3.2.22b) into (3.2.6) yields,

$$\{Y^*\} = [\alpha(\omega^*)][\Delta\gamma\sum_{i=1}^{n}\gamma_i^*[\beta_i]^T\omega^{*2}[\eta_i][\beta_i]- \sum_{i=1}^{n}[\beta_i]^T(\sum_{p=1}^{q}\binom{q}{p}\Delta\gamma^p\gamma_i^{q-p}\Delta\gamma_i^{*p}[\kappa_i])[\beta_i]]\{Y^*\} \qquad (3.2.23)$$

Re-arranging yields,

$$\{Y^*\} = [\alpha(\omega^*)][\Delta\gamma\sum_{i=1}^{n}\gamma_i^*[\beta_i]^T\omega^{*2}[\eta_i][\beta_i]- \sum_{p=1}^{q}\Delta\gamma^p\sum_{i=1}^{n}[\beta_i]^T\binom{q}{p}\gamma_i^{q-p}\Delta\gamma_i^{*p}[\kappa_i][\beta_i]]\{Y^*\} \qquad (3.2.24)$$

Equation (3.2.24) is a polynomial eigenequation, which can be re-written as,

$$\{Y^*\} = (\Delta\gamma[A] - \sum_{p=1}^{q} \Delta\gamma^p [B]_p)\{Y^*\} \tag{3.2.25}$$

where

$$[A] = [\alpha(\omega^*)][\sum_{i=1}^{n} \gamma_i^* [\beta_i]^T \omega^{*2} [\eta_i][\beta_i]] \tag{3.2.26a}$$

and

$$[B]_p = [\alpha(\omega^*)][\sum_{i=1}^{n} [\beta_i]^T \binom{q}{p} \gamma_i^{q-p} \Delta\gamma_i^{*p} [\kappa_i][\beta_i] \tag{3.2.26b}$$

re-arrange equation (3.2.25), yields,

$$(\sum_{p=1}^{q} \Delta\gamma^p [B]_p)\{Y^*\} - \Delta\gamma[A]\{Y^*\} + [\ I\ ]\{Y^*\} = \{0\} \tag{3.2.27}$$

By combining the terms including $\Delta\gamma$, equation (3.2.27) can be written as,

$$(\sum_{p=2}^{q} \Delta\gamma^p [B]_p)\{Y^*\} - \Delta\gamma([A] + [B]_1)\{Y^*\} + [\ I\ ]\{Y^*\} = \{0\} \tag{3.2.28}$$

Equation (3.2.28) is a non-linear polynomial eigenequation. Here a method using the 'state space' theory is introduced to solve this equation. By means of the state space concept the non-linear eigenequation can be linearised so that it can be solved using a general eigen solution routine.

Assuming a system is characterised by the following differential equation of order p,

$$\sum_{p=2}^{q} [B]_p \{Y^{*(p)}\} - ([A] + [B]_1)\{Y^{*(1)}\} + [\ I\ ]\{Y^*\} = \{0\} \tag{3.2.29}$$

where $\{Y^{*(p)}\}$; the p^{th} order derivative of $\{Y^*\}$ with respect to τ.

Let $\{Y^*\} = \{\mu\}e^{\Delta\gamma\tau}$, then,

$$\{Y^{*(p)}\} = \Delta\gamma^p\{\mu\}e^{\Delta\gamma\tau} = \Delta\gamma^p\{Y^*\}$$

$$\vdots \qquad\qquad \vdots$$

$$\vdots \qquad\qquad \vdots$$

$$\{Y^{*(2)}\} = \Delta\gamma^2\{\mu\}e^{\Delta\gamma\tau} = \Delta\gamma^2\{Y^*\}$$

$$\{Y^{*(1)}\} = \Delta\gamma\{\mu\}e^{\Delta\gamma\tau} = \Delta\gamma\{Y^*\}$$

Thus, it may be noted that equation (3.2.29) is actually equivalent to equation (3.2.28). By defining the state vector $\{Z\}$ of the assumed system as,

$$\{Z\}^T = \{\, Y^{*(p-1)}\ Y^{*(p-2)} \ \ldots\ \ldots\ Y^{*(0)}\,\}^T \tag{3.2.30a}$$

$$\{\dot{Z}\}^T = \{\, Y^{*(p)}\ Y^{*(p-1)} \ \ldots\ \ldots\ Y^{*(1)}\,\}^T \tag{3.2.30b}$$

equation (3.2.29) can be recast into,

$$\left[[B]_p : [B]_{p-1} : [B]_{p-2} \cdots\ \cdots\ -[A]+[B]_1\right]\{\dot{Z}\}+\left[[0] : [0] : [0] \cdots\ \cdots\ [\ I\ ]\right]\{Z\} = \{0\} \tag{3.2.31}$$

This equation can be adjoined by following (p-1) sets of identity which are in the form of,

$$\left[[0] : [\ I\ ] : [0] \ \cdots\ \cdots\ [0]\right]\{\dot{Z}\}+\left[-[\ I\ ] : [0] : [0] \ \cdots\ \cdots\ [0]\right]\{Z\} = \{0\} \tag{3.2.31,1}$$

$$\left[[0] : [0] : [\ I\ ] \ \cdots\ \cdots\ [0]\right]\{\dot{Z}\}+\left[[0] : -[\ I\ ] : [0] \ \cdots\ \cdots\ [0]\right]\{Z\} = \{0\} \tag{3.2.31,2}$$

$$\vdots \qquad\qquad \vdots$$

$$\vdots \qquad\qquad \vdots$$

$$\left[[0] : [0] \ \cdots\ \cdots\ [0] : [\ I\ ]\right]\{\dot{Z}\}+\left[[0] : [0] \ \cdots\ \cdots\ -[\ I\ ] : [0]\right]\{Z\} = \{0\} \tag{3.2.31,p-1}$$

Thus, equation (3.2.29) becomes,

$$[H]\{\dot{Z}\}+[P]\{Z\} = \{0\} \tag{3.2.32}$$

where,

$$[H] = \begin{bmatrix} [B]_p & [B]_{p-1} & [B]_{p-2} & \cdots & \cdots & -[A]+[B]_1 \\ [\,0\,] & [\,I\,] & [\,0\,] & \cdots & \cdots & [\,0\,] \\ [\,0\,] & [\,0\,] & [\,I\,] & \cdots & \cdots & [\,0\,] \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ [\,0\,] & [\,0\,] & [\,0\,] & \cdots & \cdots & [\,I\,] \end{bmatrix} \qquad (3.2.33a)$$

and

$$[P] = \begin{bmatrix} [\,0\,] & [\,0\,] & [\,0\,] & \cdots & \cdots & [\,I\,] \\ -[\,I\,] & [\,0\,] & [\,0\,] & \cdots & \cdots & [\,0\,] \\ [\,0\,] & -[\,I\,] & [\,0\,] & \cdots & \cdots & [\,0\,] \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ [\,0\,] & [\,0\,] & [\,0\,] & \cdots & -[\,I\,] & [\,0\,] \end{bmatrix} \qquad (3.2.33b)$$

which equation (3.2.31) can be transformed to be a generalised eigen problem as,

$$\Delta\gamma[H]\{Z\}+[P]\{Z\} = \{0\} \qquad (3.2.34)$$

If matrix [H] is singular, and this will occur in most situations due to the singularity in matrix $[B]_p$ which is the global sum of element stiffness incremental matrices, an alternative procedure can be employed for solving equation (3.2.34) as follows.

Let $\Delta\gamma = 1/\zeta$, then equation (3.2.34) becomes,

$$-[P]^{-1}[H]\{Z\} = \zeta\{Z\} \qquad (3.2.35)$$

This is a standard eigen equation. By solving this eigen problem, $\Delta\gamma$ can be determined to give the desired structural parameter variations to achieve the required natural frequency ω^*. Since [H] and [P] are real and non-symmetrical, the eigenvelues of equation (3.2.35) will occur either in real or in complex conjugate pairs. However, only those real eigenvalues will be considered in structural modification. The flexibility of this method is characterised by the multiple eigenvalues and the modification ratio can be assigned differently. This allows the user to choose various alternatives to achieve specific

modification requirements. In addition, it should be noticed that even after $\Delta\gamma$ has been found, it may not be realisable in practice since the final decision may be subject to practical limitations.

3.3 'Virtual' System and Relocation of Anti-Resonance Frequency

3.3.1 Anti-Resonance of an F.R.F

Anti-resonance theoretically occurs when a given excitation induces no response at certain coordinates, such as $\alpha_{ij}(\omega)=0$. An FRF curve will show anti-resonances as 'troughs' if the curve is plotted using dB scales. As an important dynamic property, anti-resonance seems to be have been overlooked. The principle of dynamic vibration absorbers is to create an anti-resonance at the frequency where a resonance previously existed. However, a dynamic vibration absorber is usually applied to a single degree-of-freedom system or a lumped mass system. Unlike a resonance, the appearance of an anti-resonance is the combined effect of many modes of a system. The nature of the anti-resonance may limit its application because:

(i) it is always associated with particular excitation and response locations, thus, for different FRFs, the anti-resonances may be different. However, the resonances are the same.

(ii) since it is always accompanied with low response level, an anti-resonance is much more sensitive to the constraint conditions and noise, compared with the resonance. Therefore, it becomes difficult to predict the anti-resonance accurately by using analytical (FE) models, particularly for complex structures, and to identify anti-resonance experimentally.

In recent years, a number methods for dynamic system identification have been developed, most of them are based on the correlation of experimental data and analytical (FEA) models. They have provided efficient tools for obtaining an acceptable model for the structural engineer to study anti-resonances effectively.

In the vicinity of an anti-resonance, there will always be a region which has a significantly

low response level, and the anti-resonance 'troughs' should have the same sharpness as those neighbouring resonance 'peaks'. This property was discussed by Ewins[8] as a criterion for assessing the quality of the experimental FRF data. Thus, it can be expected that once a resonance peak is replaced by an anti-resonance 'trough', a low response level will appear. This has special significance for vibration reduction.

Whether or not anti-resonances will appear in a FRF between two consecutive resonances depends on the modal constants of two neighbouring vibration modes[8]. If the modal constants are of the same sign, an anti-resonance will occur. Otherwise, a minimum occurs. From Section 3.1, we know that a modal constant is defined as the product of two eigenvector elements, which represent the mode shape cofactors of response and excitation coordinates respectively. For a receptance FRF of a point measurement, the modal constants of all modes will be positive, which implies that there must be an anti-resonance between every two neighbouring resonances. For an FRF of a transfer measurement, since the signs of the modal constants vary, the FRF will usually be in the form of mixture of resonances, anti-resonances and minima. For a lightly damped structural system, the signs of cofactors in a mode shape vector represent the phase difference between excitation and response coordinates. If the mode shape cofactors of two coordinates have different signs, they are 180 degree out of phase. In general, for a structural system, the further apart the two coordinates in question are, the more likely the two corresponding mode shape cofactors are 180 degrees out of phase, or are opposite in sign, as one progresses through the modes, and therefore, the less anti-resonance exhibited in the FRF. From the vibration reduction point of view, more anti-resonances are often preferred.

Assume that the response coordinate and excitation coordinate are rigidly connected, the FRF of transfer measurement between these two coordinates will be the same as those FRFs of point measurement of these coordinates. From this we may draw a useful conclusion that if the stiffness between two coordinates of a structural system is increased, it is likely that anti-resonances will be created in the relevant FRF. However, this should be achieved by introducing new connectivity, which means the increase of 'relative' stiffness other than the 'global' stiffness. It can be further explained as follows: The stiffer the connection between the two coordinates, the more stable the phase relationships of two

coordinates is, namely, the more likely that the signs of the modal constants in the FRF is kept unaltered, particularly in the low frequency range. It can be illustrated with a mass-spring system as shown in **Figure 3.3.1.** The receptance FRF between points 2 and 5 is examined. **Figure 3.3.1** gives the FRF curves comparison and it indicates that anti-resonances can be obtained by locally increasing the stiffness between coordinates 2 and 5, particularly at the lower frequency range.

It is worthy of mentioning that, although the response level at any frequencies is mainly determined by the contribution of the nearby modes, the 'exact' location of an anti-resonance is also affected by the remote modes. In some situations, more than one anti-resonance exist between two consecutive resonances, which indicates that remote modes have a significant effect on the location of anti-resonances. In the following Section, a method for relocating anti-resonance will be developed on a lumped mass system. The method can be applied on the FE model by following the same procedure stated in Section 3.2.2.

3.3.2 'Virtual' System and Relocation of Anti-Resonance

In this section, a method is developed for relocating an anti-resonance of a FRF of a given NDOF vibrating structural system by using the structural modification approach. The pre-requisite of the method is that the system has been fully identified in terms of mass and stiffness matrices.

Considering equation (3.1.6), the individual terms in the matrix $[\alpha(\omega)]$ can be expressed as,

$$\alpha(\omega)_{ij} = \frac{\det(-\omega^2[M]_v+[K]_v)}{\det(-\omega^2[M]+[K])} = \frac{\det(-\omega^2[M]_v+[K]_v)}{\det[Z(\omega)]} \qquad (3.3.1)$$

When an anti-resonance occurs, $\alpha(\omega)_{ij}=0$, where ω can be determined by the positive roots of the following equations:

$$\det(-\omega^2[M]_v+[K]_v) = 0 \qquad (3.3.2)$$

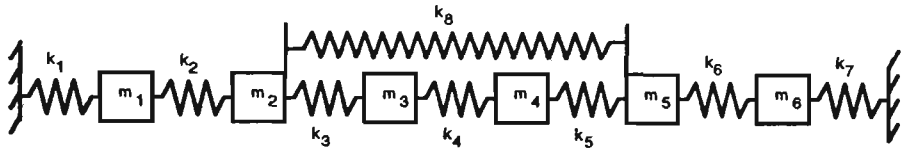
Original System

$m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = 100$ KG

$k_1 = 100$ N/M $\quad k_2 = 200$ N/M $\quad k_3 = 300$ N/M $\quad k_4 = 400$ N/M

$k_5 = 500$ N/M $\quad k_6 = 600$ N/M $\quad k_7 = 700$ N/M

Modified System



$m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = 100$ KG

$k_1 = 100$ N/M $\quad k_2 = 200$ N/M $\quad k_3 = 300$ N/M $\quad k_4 = 400$ N/M

$k_5 = 500$ N/M $\quad k_6 = 600$ N/M $\quad k_7 = 700$ N/M $\quad k_8 = 1000$ N/M

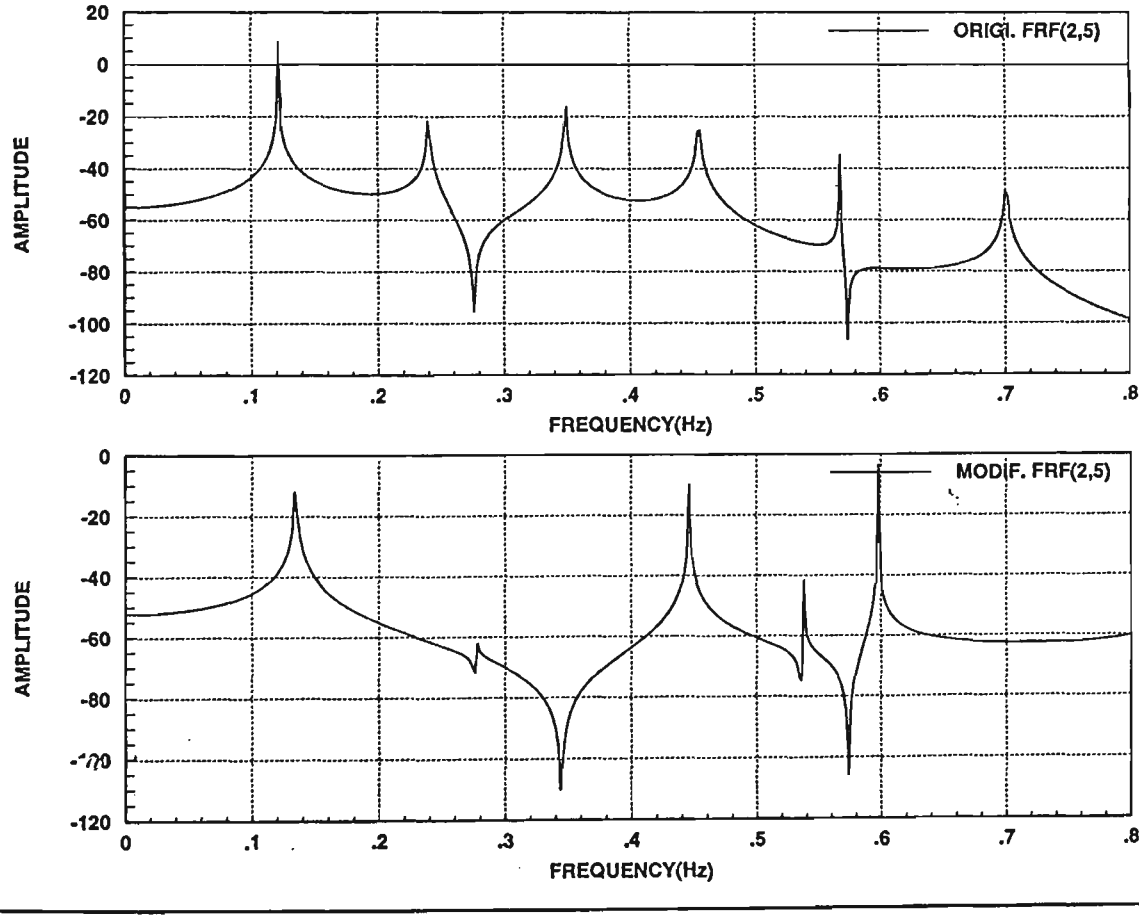


Figure 3.4.1 - Anti-Resonance Creation of a Mass-Spring System

However, the roots of the above equation can also be interpreted as the square roots of positive eigenvalues of the follows equation,

$$(-\lambda_a[M]_v+[K]_v)\{\mu\} = 0 \tag{3.3.3}$$

Assume there exists a multi-degree-of-freedom dynamic system whose equation of motion can be expressed in the same way as equation (3.3.3),

$$(-\omega^2{}_a[M]_v+[K]_v)\{\mu\} = 0 \tag{3.3.4}$$

where $\omega_a = \sqrt{\lambda_a}$ can be interpreted as the natural frequency of this system. Since this system, which is derived from the 'actual' system comprising [K] and [M] matrices does not exist actually, we call it a 'virtual' system whose mass and stiffness matrices are $[M]_v$ and $[K]_v$ respectively. Therefore, the equation of motion of the 'virtual' system subjected to a harmonic excitation force vector {G} with frequency of ω_a^* is,

$$(-\omega_a{}^{*2}[M]_v+[K]_v)\{\upsilon\} = \{G\} \tag{3.3.5}$$

Using a similar definition for $\alpha(\omega)$, equation (3.3.5) can be re-written as,

$$\{\upsilon\} = [\alpha_v(\omega_a^*)]\{G\} \tag{3.3.6}$$

The mass and stiffness modifications applied to the 'virtual' system corresponding to $\alpha_{lm}(\omega)$ can be expressed as incremental matrices $\Delta[M]_v$ and $\Delta[K]_v$ which are sparse and their counterparts $\Delta[M]$ and $\Delta[K]$ for [M] and [K] respectively can be identified from the formation of the 'virtual' system. Considering the above notations, the equation of motion of the modified 'virtual' system becomes,

$$\left[-\omega_a{}^{*2}([M]_v+\Delta[M]_v)+([K]_v+\Delta[K]_v)\right]\{\mu^*\} = 0 \tag{3.3.7}$$

here we have defined ω_a^* as the natural frequency of the modified 'virtual' system, which is an anti-resonance frequency of the modified 'actual' system.

Following the same procedure from equation (3.2.1) to equation (3.2.6), equation (3.3.7) can be rewritten as,

$$\{\mu^*\} = -[\alpha_v(\omega_a^*)]\left[-\omega_a^{*2}\Delta[M]_v+\Delta[K]_v\right]\{\mu^*\} \tag{3.3.8}$$

To demonstrated the procedure of implementing the method for relocating anti-resonance, for a simplicity, an n-degree of freedom lumped mass system is discussed. Assume the local modification of mass and stiffness are expressed in the form of equations (3.2.3b) and (3.2.3c):

$$\Delta[M] = \sum_{r=i,j,k} \{e_r\}\delta m_r\{e_r\}^T \tag{3.2.3b}$$

$$\Delta[K] = \sum_{p,q=i,j,k} \{e_{pq}\}\delta k_{pq}\{e_{pq}\}^T \tag{3.2.3c}$$

Here, δm_r (r=i,j,k) in equation (3.2.3b) and δk_{pq} (p=i,j,k; q=i,j,k; p≠q) in equation (3.2.3c) represent the mass variation of coordinate 'r' and stiffness variation between coordinates 'p' and 'q' respectively. By introducing the user-defined modification ratios η and κ, the following equations can be derived,

$$\Delta[M] = \gamma \sum_{r=i,j,k} \{e_r\}\eta_r\{e_r\}^T \tag{3.3.9a}$$

$$\Delta[K] = \gamma \sum_{p,q=i,j,k} \{e_{pq}\}\kappa_{pq}\{e_{pq}\}^T \tag{3.3.9b}$$

Hence, the mass and stiffness incremental matrices for the 'virtual' system corresponding to receptance FRF $\alpha_{lm}(\omega)$ can be expressed as,

$$\Delta[M]_v = \gamma \sum_{r=i,j,k} \{e_r\}_l\eta_r\{e_r\}_m^T \tag{3.3.10a}$$

$$\Delta[K]_v = \gamma \sum_{p,q=i,j,k} \{e_{pq}\}_l\kappa_{pq}\{e_{pq}\}_m^T \tag{3.3.10b}$$

For the 'virtual' system corresponding to receptance $\alpha(\omega)_{lm}$, after the mass and stiffness modifications applied to the original system have been defined, $\Delta[M]_v$ and $\Delta[K]_v$ can be identified by using the same operations on $\Delta[M]$ and $\Delta[K]$. Therefore, the matrix forms of

$\Delta[M]_v$ and $\Delta[K]_v$ are,

$$\Delta[M]_v = \begin{bmatrix} 0 & & & & & & \\ & \cdots & & & & & \\ & & 0 & \cdots & \delta m_{uv} & & \\ & & & 0 & \cdots & \delta m_{wx} & \\ & & & & \cdots & & \\ & & & & & 0 & \delta m_{yz} \\ & & & & & & \cdots \\ & & & & & & 0 \end{bmatrix}_{(n-1)\times(n-1)} \tag{3.3.11a}$$

$$\delta m_{uv} = \delta m_i, \quad \delta m_{wx} = \delta m_j, \quad \delta m_{uv} = \delta m_k$$

and

$$\Delta[K]_v = \begin{bmatrix} 0 & & & & & & \\ & \cdots & \delta k_{uv} & \cdots & \delta k_{ux} & \cdots & \delta k_{uz} & \cdots \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & \cdots & \delta k_{wv} & \cdots & \delta k_{wx} & \cdots & \delta k_{wz} & \cdots \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & \cdots & \delta k_{yv} & \cdots & \delta k_{yx} & \cdots & \delta k_{yz} & \cdots \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & & & & 0 \end{bmatrix}_{(n-1)\times(n-1)} \tag{3.3.11b}$$

$$\delta k_{uv} = \delta k_{ij} + k_{ik}, \quad \delta k_{ux} = -\delta k_{ij}, \quad \delta k_{uz} = -\delta k_{ik}$$

$$\delta k_{wx} = \delta k_{ij} + k_{jk}, \quad \delta k_{wv} = -\delta k_{ij}, \quad \delta k_{wz} = -\delta k_{jk}$$

$$\delta k_{yz} = \delta k_{ik} + k_{jk}, \quad \delta k_{yx} = -\delta k_{jk}, \quad \delta k_{yv} = -\delta k_{ik}$$

However, there is a slight difference in problem reduction procedures when dealing with a 'virtual' system. As discussed above, $\Delta[M]$ and $\Delta[K]$ for the 'actual' system should always be symmetric and sparse matrices, due to the assumption of system linearity and principle of reciprocity. Since the mass and stiffness matrices of the 'virtual' system are only derived mathematically from those of the actual system, no obvious physical significance

is apparent. In fact, $\Delta[M]_v$ and $\Delta[K]_v$ are obtained from $\Delta[M]$ and $\Delta[K]$ by deleting one row and one column of them so they are no longer symmetric unless the FRF of interest is a drive point FRF.

For the 'virtual system', its eigen equation for modification without change in stiffness can then be written as,

$$\begin{Bmatrix} \mu_1^* \\ \mu_2^* \\ \vdots \\ \vdots \\ \vdots \\ \mu_{n-1}^* \end{Bmatrix} = \omega_a^{*2}[\alpha_v(\omega_a^*)] \begin{bmatrix} \cdots & & & & \\ & 0 \cdots \delta m_{uv} & & & \\ & & 0 \cdots \delta m_{wx} & & \\ & & \cdots & & \\ & & & 0 \quad \delta m_{yz} & \\ & & & \cdots & \end{bmatrix}_{(n-1)\times(n-1)} \begin{Bmatrix} \mu_1^* \\ \mu_2^* \\ \vdots \\ \vdots \\ \vdots \\ \mu_{n-1}^* \end{Bmatrix} \quad (3.3.12)$$

Equation (3.3.12) can then be reduced to:

$$\frac{1}{\gamma}\begin{Bmatrix} \mu_v^* \\ \mu_x^* \\ \mu_z^* \end{Bmatrix} = \omega_a^{*2} \begin{bmatrix} \alpha_v(\omega_a^*)_{vu} & \alpha_v(\omega_a^*)_{vw} & \alpha_v(\omega_a^*)_{vy} \\ \alpha_v(\omega_a^*)_{xu} & \alpha_v(\omega_a^*)_{xw} & \alpha_v(\omega_a^*)_{xy} \\ \alpha_v(\omega_a^*)_{zu} & \alpha_v(\omega_a^*)_{zw} & \alpha_v(\omega_a^*)_{zy} \end{bmatrix} \begin{bmatrix} \eta_i & 0 & 0 \\ 0 & \eta_j & 0 \\ 0 & 0 & \eta_k \end{bmatrix} \begin{Bmatrix} \mu_v^* \\ \mu_x^* \\ \mu_z^* \end{Bmatrix} \quad (3.3.13)$$

Equation (3.3.13) is a standard eigenproblem whose eigenvalue is $1/\gamma$. Therefore, for a specified frequency ω_a^* which is the desired anti-resonance frequency of the original system, and given mass modification ratios η_r (r=i,j,k), the mass modification factor can be determined from one of the eigenvalues of equation (3.3.13). The actual mass variations of the original system can then be determined, which will create an anti-resonance at frequency ω_a^* for the receptance FRF $\alpha(\omega)_{ij}$. Similarly, the eigenequation for stiffness modification can be derived as,

$$\frac{1}{\gamma}\begin{Bmatrix} \mu_v^* \\ \mu_x^* \\ \mu_z^* \end{Bmatrix} = -\begin{bmatrix} \alpha_v(\omega_a^*)_{vu} & \alpha_v(\omega_a^*)_{vw} & \alpha_v(\omega_a^*)_{vy} \\ \alpha_v(\omega_a^*)_{xu} & \alpha_v(\omega_a^*)_{xw} & \alpha_v(\omega_a^*)_{xy} \\ \alpha_v(\omega_a^*)_{zu} & \alpha_v(\omega_a^*)_{zw} & \alpha_v(\omega_a^*)_{zy} \end{bmatrix} \begin{bmatrix} \kappa_{ij}+\kappa_{ik} & -\kappa_{ij} & -\kappa_{ik} \\ -\kappa_{ij} & \kappa_{ij}+\kappa_{jk} & -\kappa_{jk} \\ -\kappa_{ik} & -\kappa_{jk} & \kappa_{jk}+\kappa_{ij} \end{bmatrix} \begin{Bmatrix} \mu_v^* \\ \mu_x^* \\ \mu_z^* \end{Bmatrix} \quad (3.3.14)$$

By following the same procedure, the stiffness modification needed to produce the desired anti-resonance can be obtained.

Similarities and differences of the procedures for relocating resonances and anti-resonances are summarised below:

(i) Both procedures are based on the assumption that the mass and stiffness incremental matrices are obtained analytically, and the FRF data associated with the involved coordinates are available.

(ii) Both procedures can eventually be expressed as reduced eigen-problems whose orders are dependent on the coordinates involved in the modification. No iteration is required.

(iii) For the relocation of a resonance, the FRF data needed can be obtained individually from equation (3.2.4) or by experiment directly. For relocation of an anti-resonance, the 'virtual' mass and stiffness matrices are not longer symetric and posititive (semi-poistive) definite as the actual system, also normally no 'modal properties' will be available before hand and only sigle frequency point are required. Considering above factors, the most economical way of obtaining 'virtual' receptance matrix is by obtained inversion. This needs more computational effort for large, complex structures.

3.4 Optimum Structural Modification

The first question encountered in a structural modification procedure is usually which locations of the structural system are more sensitive to a modification. As mentioned in Chapter 2, the sensitivity analysis which normally involves the calculation of the eigenderivatives is widely employed to answer a question of this kind. By using information obtained from a sensitivity analysis, effort for the determination of optimum modification location can be greatly reduced. Another application of sensitivity analysis is to assess the secondary effect which may occur as a result of structural modification, that is, relocating one natural frequency accompanied by introducing a different one into frequency range of interest. The effectiveness of the sensitivity analysis is problem dependent, which implies that for some structures, the most sensitive locations may vary

as a consequence of the changes of mass and stiffness distribution introduced with the modification. However, it is difficult to predict to what extent a certain modification will invalidate the sensitivity information obtained from the original system. Thus it is advisable that a group of elements with relatively higher sensitivity values be included in the potential modification location by assuming that the global sensitivity distribution of the original structure will vary continuously during modification. An alternative is the introduction of second order sensitivity to indicate the rate of change of the first order sensitivity. Since the second order sensitivity is much more intensive in computation and does not provide as much information as expected without being further processed, it is used only for limited number of cases. The subject on eigenvalue and eigenvector derivatives for a dynamic system has been presented by numerous authors (Brandon[82], Xie *et al*[84]). More details of application of sensitivity analysis have been given in Chapter 2. In this Section, a sensitivity index for a structural system is developed and the derivative of an anti-resonance frequency with respect to structural parameters has been derived based on a 'virtual' system.

3.4.1 Element Sensitivity Index for Natural Frequency

For a linear, conservative, discrete dynamic system, by equating the maximum kinetic energy to the maximum potential energy, we have,

$$T_{max} = V_{max} \tag{3.4.1}$$

where

T_{max}: maximum kinetic energy

V_{max}: maximum potential energy

For a system which is completely identified in term of mass and stiffness matrices [M] and [K], using the natural frequency ω_i and the corresponding assumed or identified mode shape $\{x\}_i$, its T_{max} and V_{max} can be expressed as,

$$T_{\max} = \frac{1}{2}\omega_i^2 \{x\}_i^T [M]\{x\}_i \tag{3.4.2a}$$

$$V_{\max} = \frac{1}{2}\{x\}_i^T [K]\{x\}_i \tag{3.4.2b}$$

The substitution of equations (3.4.2a,b) into equation (3.4.1) leads to the Rayleigh's quotient,

$$\lambda_i = \omega_i^2 = \frac{\{x\}_i^T [K]\{x\}_i}{\{x\}_i^T [M]\{x\}_i} = \frac{k_i}{m_i} \tag{3.4.3}$$

where k_i and m_i are referred to as the modal stiffness and modal mass respectively.

Using the Taylor's series expansion with respect to multi-variables, equation (3.4.3) can be written as,

$$\lambda_i = \lambda_{i(0)} + \left(\frac{\partial}{\partial m_i}\Delta m_i + \frac{\partial}{\partial k_i}\Delta k_i\right)\lambda_i + \cdots \cdots \cdots \tag{3.4.4}$$

neglecting the higher order terms and re-arranging equation(3.4.4), yields,

$$\Delta\lambda_i = -\frac{k_i}{m_i^2}\Delta m_i + \frac{1}{m_i}\Delta k_i = \frac{1}{m_i}(\Delta k_i - \lambda_i \Delta m_i) \tag{3.4.5}$$

The Rayleigh's quotient has a stationary value when the assumed mode shape vector $\{x\}_i$ is in the neighbourhood of an 'exact' mode shape vector. It is reasonable to assume that the mode shape does not change drastically for a small amount of local structural modification. Considering these notations, the Rayleigh's quotient of the modified system λ^* can be expressed as,

$$\lambda_i^* = \frac{\{x\}_i^T ([K]+\Delta[K])\{x\}_i}{\{x\}_i^T ([M]+\Delta[M])\{x\}_i} = \frac{k_i+\Delta k_i}{m_i+\Delta m_i} \tag{3.4.6}$$

where

$$\Delta k_i = \{x\}_i^T \Delta[K]\{x\}_i \tag{3.4.7a}$$

$$\Delta m_i = \{x\}_i^T \Delta[M]\{x\}_i \tag{3.4.7b}$$

The Δ[K] and Δ[M] in equations (3.4.7a,b) can be expressed in terms of element incremental matrices, thus, equation (3.4.5) will lead to,

$$\Delta\lambda_i = \frac{1}{m_i}(\{x_{(i)}\}^T[\kappa_i]\{x_{(i)}\}\sum_{p=1}^{q}\binom{q}{p}\gamma_i^{q-p}\Delta\gamma_{(i)}^p - \lambda_i\{x_{(i)}\}^T[\eta_i]\{x_{(i)}\}\Delta\gamma_i) \tag{3.4.8}$$

The variation of a natural frequency due to the modification of an element physical parameter $\Delta\gamma_i$ can be evaluated roughly form equation (3.4.8). However, its application is limited by the assumption that the mode shape vector dose not change drastically by local modifications. In most cases, equation (3.4.8) can be used as a guide for structural modification by introducing the Element Sensitivity Index $\varepsilon_{(i)}$, which is defined as,

$$\varepsilon_{(i)} = \{x_{(i)}\}^T[\kappa_i]\{x_{(i)}\}\sum_{p=1}^{q}\binom{q}{p}\gamma_{(i)}^{q-p}\Delta\gamma_{(i)}^p - \lambda_i\{x_{(i)}\}^T[\eta_i]\{x_{(i)}\}\Delta\gamma_i \tag{3.4.9}$$

This index $\varepsilon_{(i)}$ can be computed individually for the elements to be modified. The elements with the largest absolute values of $\varepsilon_{(i)}$ will then indicate the most sensitive locations for local structural modification.

3.4.2 Element Sensitivity Index for Anti-resonance

Considering the eigenequation of a 'virtual' system, equation (3.3.4) can be re-written as,

$$(-\lambda_a[M]_v + [K]_v)\{Y^{(R)}\} = \{0\} \tag{3.4.10}$$

and

$$\{Y^{(L)}\}^T(-\lambda_a[M]_v + [K]_v) = \{0\}^T \tag{3.4.11}$$

Since $[M]_v$ and $[K]_v$ are obtained from deleting one row and one column of [M] and [K], λ_a can not be solved using a general eigen routine if $[M]_v$ becomes singular. In fact, for a lumped mass system, this will always happen when dealing with the 'virtual' system for a

transfer FRF of interest. In this case, an alternative form of eigenequation is proposed, this is,

$$(-[M]_v+\zeta_a[K]_v)\{Y^{(R)}\} = \{0\} \tag{3.4.12}$$

and

$$\{Y^{(L)}\}^T(-[M]_v+\zeta_a[K]_v) = \{0\}^T \tag{3.4.13}$$

where $\zeta_a = 1/\lambda_a$.

The Element Sensitivity Index for anti-resonance frequency $\in_{(i)}$ is defined by the eigenvalue derivative with respect to the design variable γ_i of element i. Differentiating equation (3.4.12) and pre-multiplying by $\{Y^{(L)}\}^T$ yields,

$$\{Y^{(L)}\}^T(-[M]_v+\zeta_a[K]_v)\frac{\partial\{Y^{(R)}\}}{\partial\gamma_i}+\{Y^{(L)}\}^T(\frac{-\partial[M]_v}{\partial\gamma_i}+\zeta_a\frac{\partial[K]_v}{\partial\gamma_i})\{Y^{(R)}\} + \frac{\partial\zeta_a}{\partial\gamma_i}\{Y^{(L)}\}^T[K]_v\{Y^{(R)}\} = 0 \tag{3.4.14}$$

Considering equation (3.3.13) and $\partial\lambda/\partial\gamma_i = -\zeta_a^{-2}\partial\zeta_a/\partial\gamma_i$, equation (3.4.14) yields,

$$\in_{(i)} = \frac{\partial\lambda_a}{\partial\gamma} = -\lambda_a^2\frac{\{Y^{(L)}\}^T(\frac{-\partial[M]_v}{\partial\gamma_i}+\zeta_a\frac{\partial[K]_v}{\partial\gamma_i})\{Y^{(R)}\}}{\{Y^{(L)}\}^T[K]_v\{Y^{(R)}\}} \tag{3.4.15}$$

Thus the most sensitive modification location for an anti-resonance of a given FRF can be identified from the biggest Index $\in_{(i)}$.

CHAPTER 4

ANALYTICAL AND EXPERIMENTAL VERIFICATION

This Chapter describes two numerical and one experimental study to illustrate and verify the methods developed in Chapter 3. Several computer codes were developed to facilitate these studies. The first study, which was conducted on an FE planar truss structure of 32-elements, was used to check the validity of methods for relocating resonances and anti-resonances. The second study, which was conducted on a 10-element planar beam structure, was used to illustrate the general procedure of finite element implementation of resonance relocation. The third study, was experimental and was conducted on a cross-stiffened grid (flat grillage) structure which was used in conjunction with FEA results to detail how the methods developed work for a practical engineering problem.

4.1 Example Systems

4.1.1 System #1 - A 32-Element Planar Truss Structure

This system is a finite element plane-truss structure with 30 bar elements and 2 concentrated mass elements located at node 6 and node 8 (ref **Figure 4.1.1a**). Two nodes, node 1 and node 14, are fully constrained. Each element has two nodes with two degree of freedom in each node, namely, the translational displacement along x and y directions in the global coordinate system. No bending displacement was considered. The physical parameters of the system were given in **Table 4.1.1a**. The formulation of element mass and stiffness matrices in the global coordinate is given by [86].For a bar element, the element mass and stiffness matrices in the global coordinate are,

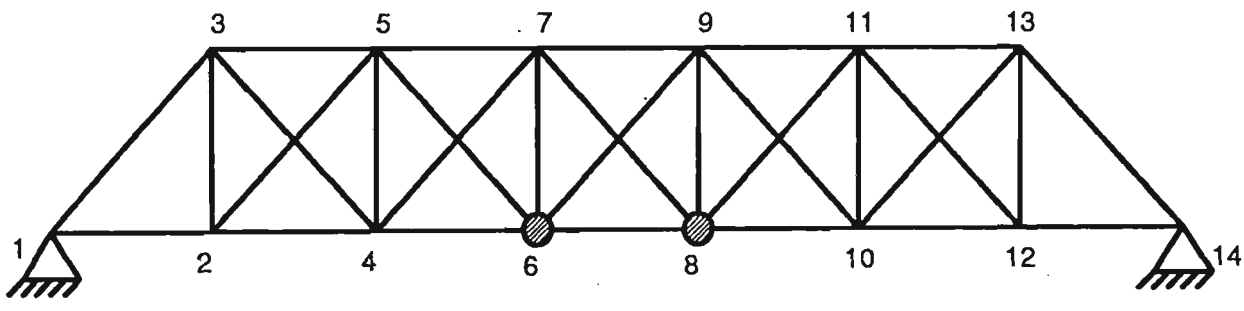


Figure 4.1.1 - 32-Element Planar Truss Structure

Table 4.1.1 - Physical Parameters for Planar Finite Element Truss Structure (System #1)

No. Element	No. Nodes		Cross Sectional Area (M^2)	No. Element	No. Nodes		Cross Sectional Area (M^2)
1	1	2	19.7E-4	17	7	9	19.7E-4
2	1	3	19.7E-4	18	8	9	19.7E-4
3	2	3	19.7E-4	19	8	10	19.7E-4
4	2	4	19.7E-4	20	8	11	19.4E-4
5	2	5	19.7E-4	21	9	10	19.7E-4
6	3	4	19.7E-4	22	9	11	19.7E-4
7	3	5	19.7E-4	23	10	11	19.7E-4
8	4	5	19.7E-4	24	10	12	19.7E-4
9	4	6	19.7E-4	25	10	13	19.7E-4
10	4	7	19.7E-4	26	11	12	19.7E-4
11	5	6	19.7E-4	27	11	13	19.7E-4
12	5	7	19.7E-4	28	12	13	19.7E-4
13	6	7	19.7E-4	29	12	14	19.7E-4
14	6	8	19.7E-4	30	13	14	19.7E-4
15	6	9	19.7E-4	31	6 (Con. Mass)		28 KG
16	7	8	19.7E-4	32	8 (Con. Mass)		28 KG

Elastic Modulus = 68 GPa, Mass Density = 2800 KGM^{-3} for All Bar Elements.

$$[m_i^{(e)}]_g = \frac{\rho A l}{6}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \tag{4.1.1}$$

and

$$[k_i^{(e)}]_g = \frac{EA}{l}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \tag{4.1.2}$$

For the concentrated mass element, the element mass matrices in the global coordinate are,

$$[m_i^{(e)}] = m_i \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.1.3}$$

The system mass and stiffness matrices can be obtained as follows,

$$[M] = \sum_{i=1}^{n} [\beta_i]^T [m_i^{(e)}]_g [\beta_i] \tag{4.1.4}$$

and

$$[K] = \sum_{i=1}^{n} [\beta_i]^T [k_i^{(e)}]_g [\beta_i] \tag{4.1.5}$$

4.1.2 System #2 - A 10-Element Planar Beam Structure

For finite element modelling, an uniform free-free beam is divided into ten two dimensional finite elements (ref **Figure 4.1.2a**). Each element has two nodes. Each node has two degrees of freedom, the translational displacement along the y direction and the rotation displacement about the z direction in the global coordinate. No constraints were given to this structure. The physical parameters are given in **Table 4.1.2a**. The formulation of element mass and stiffness matrices in the global coordinate can be found in [86] as,

$$[m_i^{(e)}]_g = \frac{\rho A l}{420} \begin{bmatrix} 156 & & Sym. & \\ -22l & 4l^2 & & \\ 54 & -13l & 156 & \\ 13l & -3l^2 & 22l & 4l^2 \end{bmatrix} \tag{4.2.1}$$

and

$$[k_i^{(e)}]_g = EI \begin{bmatrix} 12/l^3 & & Sym. & \\ -6/l^2 & 4/l & & \\ -12/l^3 & 6/l^2 & 12/l^3 & \\ -6/l^2 & 2/l & 6/l^2 & 4/l \end{bmatrix} \tag{4.2.2}$$

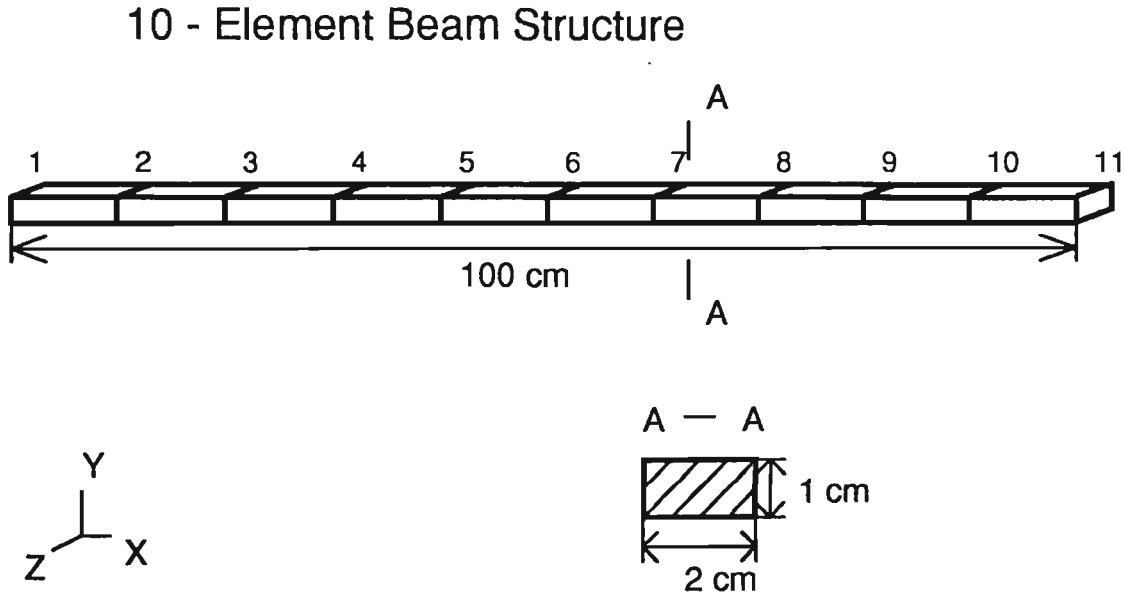


Figure 4.1.2 - 10-Element Planar Beam Structure

Table 4.1.2 - Physical Parameters for Planar Finite Element Beam Structure (System #2)

No. Element	No. Nodes		Element Height (M) (Y Axis Thickness)	Element Width (M) (Z Axis Thickness)
1	1	2	1E-2	2E-2
2	2	3	1E-2	2E-2
3	3	4	1E-2	2E-2
4	4	5	1E-2	2E-2
5	5	6	1E-2	2E-2
6	6	7	1E-2	2E-2
7	7	8	1E-2	2E-2
8	8	9	1E-2	2E-2
9	9	10	1E-2	2E-2
10	10	11	1E-2	2E-2

Elastic Modulus = 209 GPa, Mass Density = 7860 KGM^{-3} for All Beam Elements

The system mass and stiffness matrices can be obtained using equations (4.1.4-5).

4.1.3 System #3 - A Free-Free Cross-Stiffened Grid Structure

A grid (flat grillage) is a structure on which loads are applied perpendicular to the plane of the structure. Therefore, in finite element modelling, only those degrees of freedom which are not in the plane of the structure are considered. In practice, examples of grids include floor and bridge deck systems. The grid structure used in this study is shown in **Fig.4.1.3a**. The physical parameters are given in **Table 4.1.3a**.

The finite element model of this structure was constructed using 22 uniform grid beam elements, and 23 node. Each node had 3 degrees of freedom in the local coordinate, including one torsional rotation DOF about x axis, one bending rotation DOF about z axis

and one translation DOF about y axis. The stiffness and mass matrices of the grid element in global coordinate can be found in [86] as:

$$[k_i^{(e)}]_g = [T]^T[k_i^{(e)}]_l[T] \tag{4.1.8}$$

and

$$[m_i^{(e)}]_g = [T]^T[m_i^{(e)}]_l[T] \tag{4.1.7}$$

where

$$[T] = \begin{bmatrix} 1 & 0 & 0 & & & \\ 0 & c & s & & 0 & \\ 0 & -s & c & & & \\ & & & 1 & 0 & 0 \\ & 0 & & 0 & c & s \\ & & & 0 & -s & c \end{bmatrix} \tag{4.1.10}$$

$$[m_i^{(e)}]_l = \rho Al \begin{bmatrix} \frac{13}{35} & 0 & \frac{-11l}{210} & \frac{9}{70} & 0 & \frac{13l}{420} \\ & \frac{I_p}{3A} & 0 & 0 & \frac{I_p}{6A} & 0 \\ & & \frac{l^2}{105} & \frac{-13l}{420} & 0 & \frac{l^2}{420} \\ & & & \frac{13}{15} & 0 & 0 \\ & sym. & & & \frac{I_p}{3A} & 0 \\ & & & & & \frac{l^2}{105} \end{bmatrix} \tag{4.1.11}$$

Table 4.1.3a - Physical Parameters for Finite Element Grid Structure (System #3)

No. Element	No. Nodes		Element Thickness in Z Axis (M)	Element Thickness in Y Axis (M)	Element Length in X Axis (M)
1	1	4	2E-2	1E-2	2E-1
2	4	8	2E-2	1E-2	1E-1
3	8	10	2E-2	1E-2	1E-1
4	10	12	2E-2	1E-2	1E-1
5	12	16	2E-2	1E-2	1.5E-1
6	16	19	2E-2	1E-2	0.5E-1
7	19	22	2E-2	1E-2	1E-1
8	2	6	2E-2	1E-2	2E-1
9	6	9	2E-2	1E-2	1E-1
10	9	12	2E-2	1E-2	1E-1
11	12	14	2E-2	1E-2	1E-1
12	14	17	2E-2	1E-2	1.5E-1
13	17	21	2E-2	1E-2	0.5E-1
14	21	23	2E-2	1E-2	1E-1
15	3	4	2E-2	1E-2	1E-1
16	4	5	2E-2	1E-2	2E-1
17	5	6	2E-2	1E-2	2E-1
18	6	7	2E-2	1E-2	1E-1
19	10	11	2E-2	1E-2	2E-1
20	11	12	2E-2	1E-2	2E-1
21	15	16	2E-2	1E-2	1E-1
22	17	18	2E-2	1E-2	1E-1
23	19	20	2E-2	1E-2	2E-1
24	20	21	2E-1	1E-2	2E-1

Elastic Modulus = 209 GPa, Mass Density = 7860 KGM^{-3} for All Grid Beam Elements

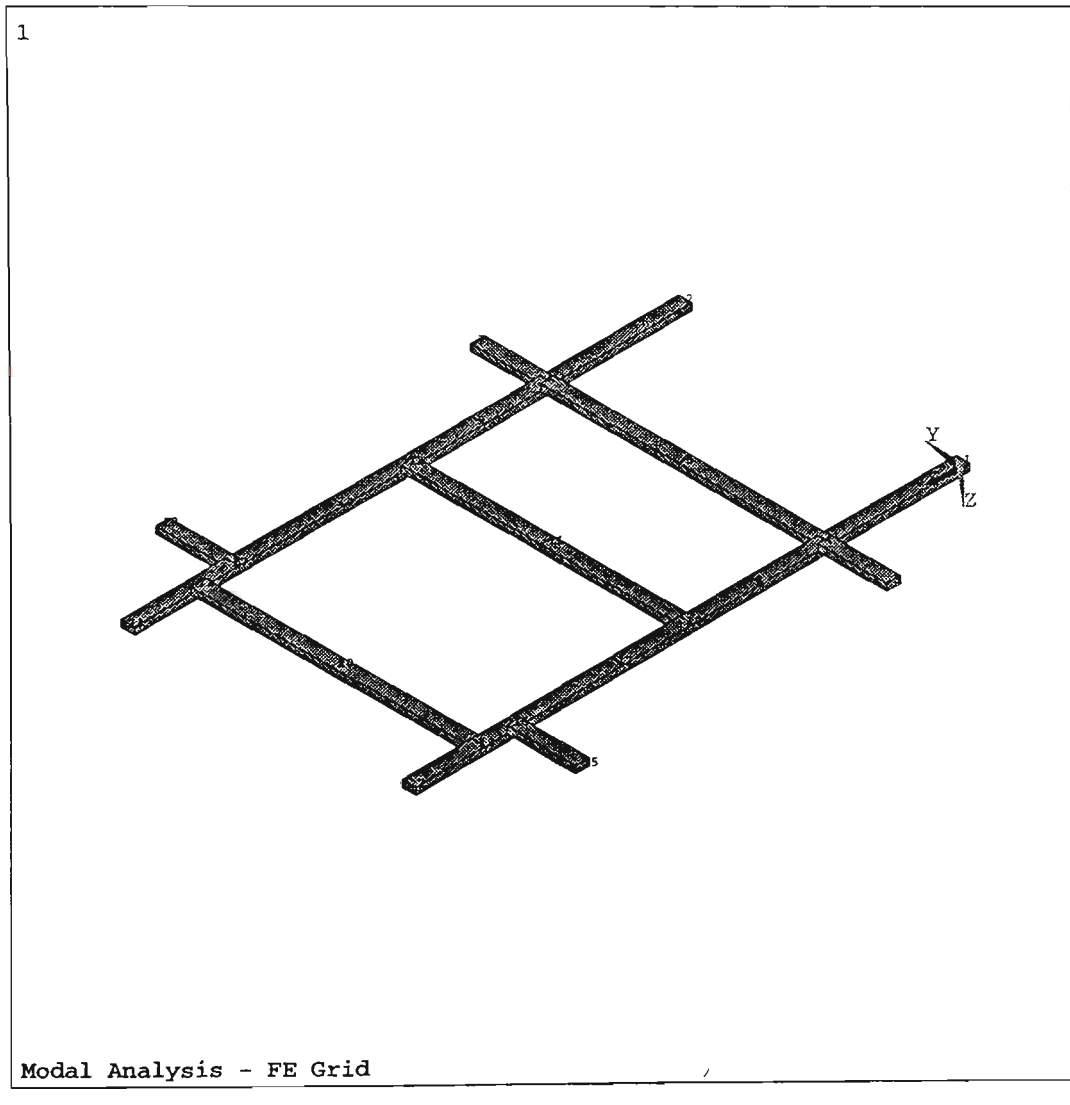


Figure 4.1.3 - Cross-Stiffened Grid Structure

$$[k_i^{(e)}]_l = \begin{bmatrix} \frac{12EI}{l^3} & 0 & \frac{-6EI}{l^2} & \frac{-12EI}{l^3} & 0 & \frac{-6EI}{l^2} \\ & \frac{GJ}{l} & 0 & 0 & -\frac{GJ}{l} & 0 \\ & & \frac{4EI}{l} & \frac{6EI}{l^2} & 0 & \frac{2EI}{l} \\ & & & \frac{12EI}{l^3} & 0 & \frac{6EI}{l^2} \\ & sym. & & & \frac{GJ}{l} & 0 \\ & & & & & \frac{4EI}{l} \end{bmatrix} \qquad (4.1.12)$$

The system mass and stiffness matrices can be obtained using equations (4.1.4) and (4.1.5).

4.2 Numerical Results and Discussions for Example System #1 and System #2

4.2.1 Relocating First Three Resonances of the Finite Element Truss Structure

The 24-DOF planar truss structure given in Section 4.1.1 was used to validate numerically the method developed in Sections 3.2.1 and 3.2.2 for relocating resonances by local structural modification. A computer code *truss.sdm.ftn* was developed to facilitate the studies in this Section and in Section 4.2.2. The first three natural frequencies were given in **Table 4.2.1a** after the eigen solution of the system.

Table 4.2.1a - Natural Frequencies of Original Truss Structure

Mode	Natural Frequency (Hz)
1	44.686
2	110.565
3	182.409

Figure 4.2.1a and **Figure 4.2.1b** illustrated the corresponding mode shapes. For local structural modification, the element cross sectional areas and concentrated masses were utilised as the physical parameters to be modified. Note that in a truss system, the variation of element cross sectional areas will influence both the mass and stiffness distribution of the system linearly. The global mass and stiffness incremental matrices can then be formulated by equation (3.2.17a) and equation (3.2.17b). Therefore, the method developed in Section 3.2.1 can be readily applied on such systems after the desired natural frequency, location of modification and element participant ratios have been specified.

For relocating the first natural frequency of the truss system from 44.486 Hz to 50.000 Hz, the elements selected for modification, the corresponding element participant ratios and modification results were given in **Table 4.2.1b**. **Table 4.2.1c** gives the first three natural frequencies after modification. It can be seen that the first natural frequency of the original truss system has been relocated accurately from 44.686 Hz to 50.000 Hz. **Figure 4.2.1c** graphically shows the comparison of two typical FRF curves belonging to the original and modified systems respectively.

Note from **Table 4.2.1b** that two 'practical' modification results were obtained. In fact, only the first set of modification results is reasonable. The second set of modification was to relocate the third natural frequency down from 110.56 Hz to 50 Hz (Ref **Table 4.2.1c**). However, which set of modification should be used will be subjected to practical considerations. Similarly, the results for relocating the second and third resonances to desired locations were given in **Table 4.2.1d**, **Table 4.2.1e**, **Figure 4.2.1d** and **Table 4.2.1f**, **Table 4.2.1g**, **Figure 4.2.1e**, respectively.

Typical input and output files for *truss.sdm.ftn* are given in **Appendix IV.**

4.2.2 Relocating Anti-Resonances of the Finite Element Truss Structure

To relocate an anti-resonance a receptance FRF of interest has to be specified. Two receptance FRF of the example system #1 were examined: (i) receptance FRF between coordinate 5 (translational DOF along Y axis of node 4) and coordinate 16 (translational

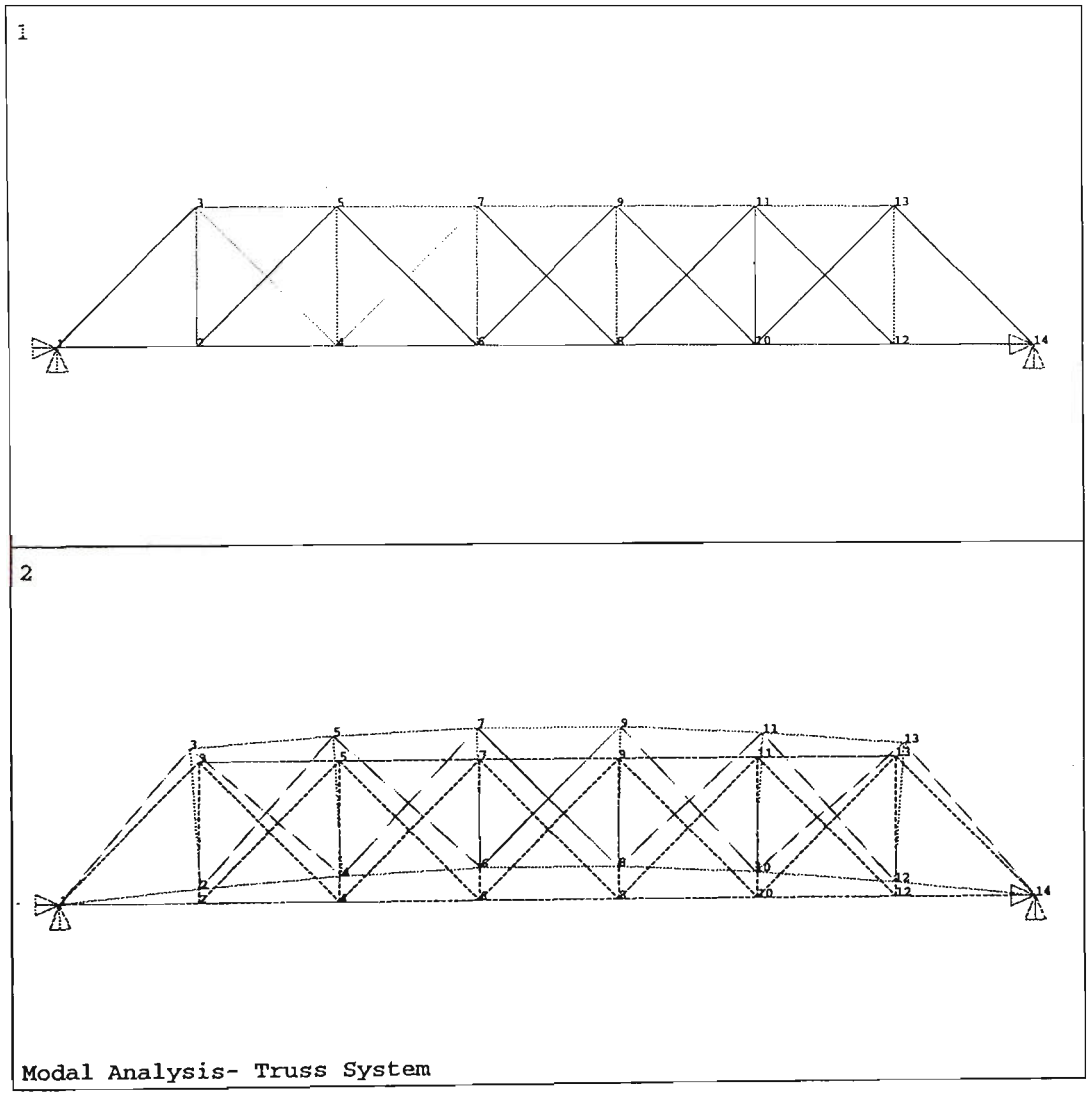


Figure 4.2.1a - Mode Shape of Truss Structure
Upper: Undeformed Structure, Lower: The 1-st Mode Shape

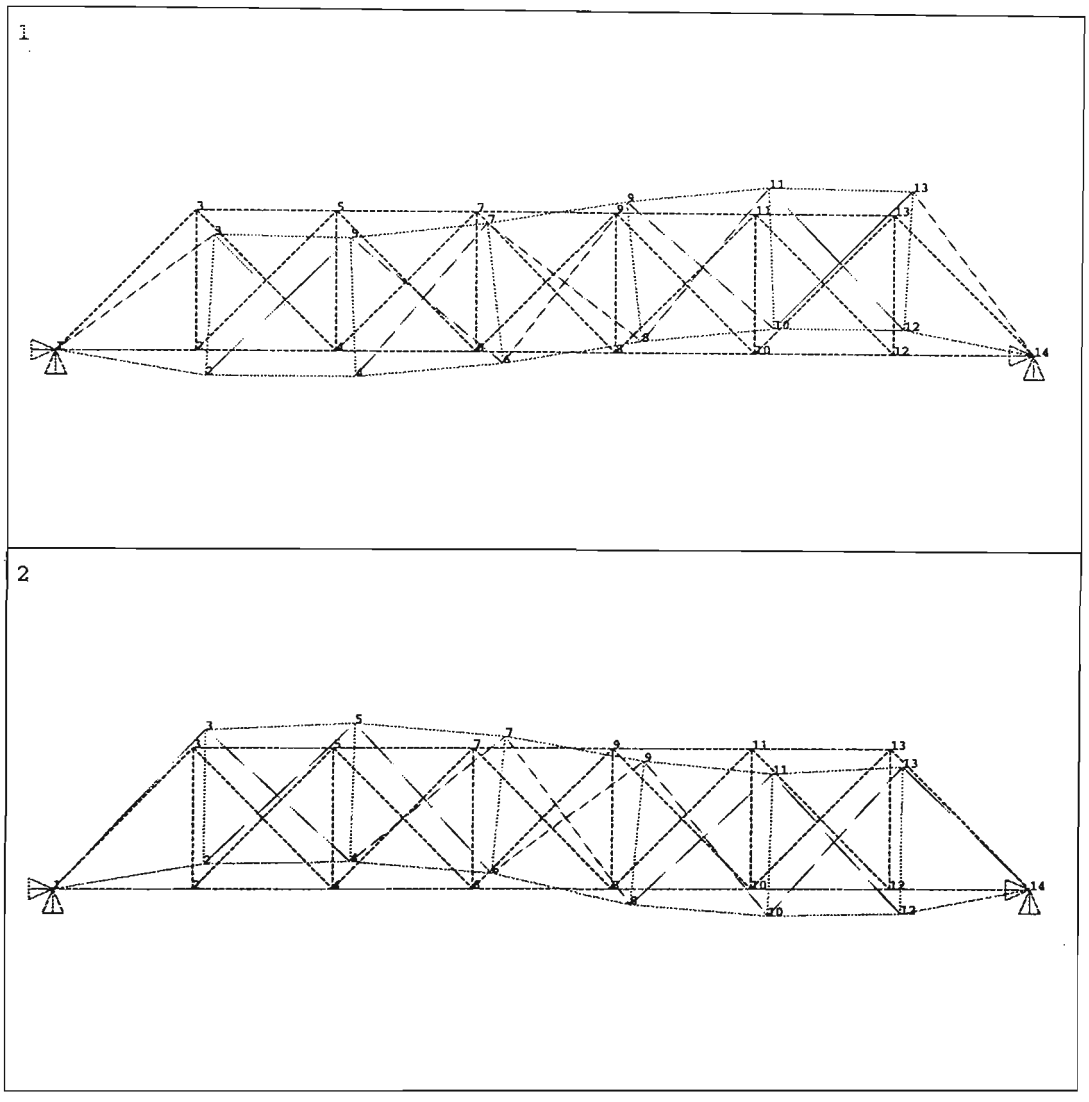


Figure 4.2.1b - Mode Shape of Truss Structure
Upper: The 2-nd Mode Shape, Lower: The 3-rd Mode Shape

Table 4.2.1b - Modification Results For Relocating The 1-st Resonance of Truss Structure From 44.686 Hz To 50.000Hz

No. Element	No. Nodes		1-st Modification Results(M^2)	2-nd Modification Results(M^2)	Element Participant Ratio
9	4	6	2.370E-4	1.42E-1	1
13	6	7	2.370E-4	1.42E-1	1
18	8	9	2.370E-4	1.42E-1	1
23	10	11	2.370E-4	1.42E-1	1
31	6 (Con. Mass)		3.738KG	1988KG	1

Table 4.2.1c - Natural frequencies of Truss Structure Modified by Results in Table 4.2.1b

Mode	Natural Frequency by 1-st Modification (Hz)	Natural Frequency by 2-nd Modification (Hz)
1	50.000	9.966
2	111.453	28.756
3	194.602	50.000

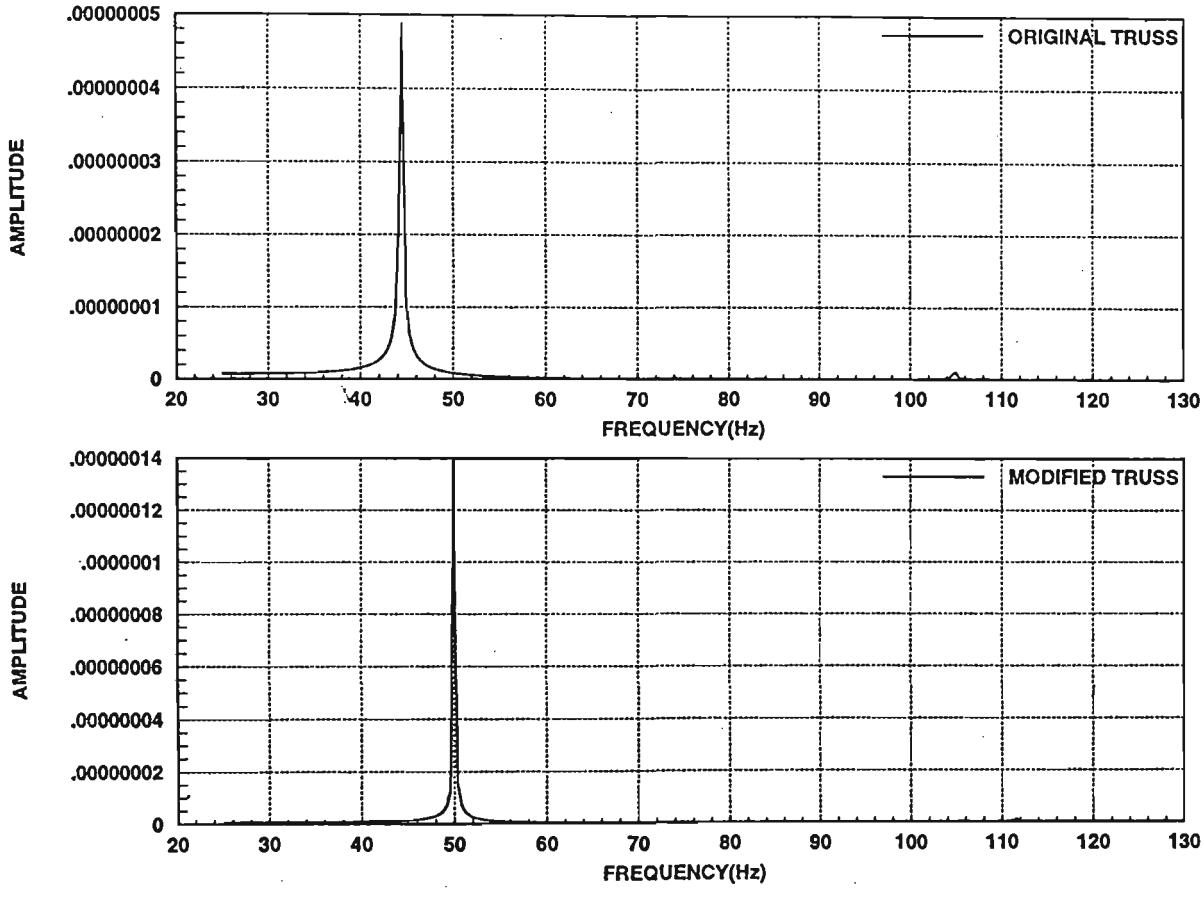


Figure 4.2.1c - FRF Comparison of Truss Structure
Upper: Original, Lower: Modified

Table 4.2.1d - Modification Results For Relocating the 2-nd Resonance of Truss Structure From 110.564 Hz To 80.000Hz

No. Element	No. Nodes		1-st Modification Results(M^2)	2-nd Modification Results(M^2)	Element Participant Ratio
8	4	5	2.798E-2	1.043E-1	2
9	4	6	1.498E-2	5.315E-2	1
10	4	7	2.798E-2	1.043E-1	2
11	5	6	1.498E-2	5.315E-2	1
12	5	7	2.798E-2	1.043E-1	2
13	6	7	1.498E-2	5.315E-2	1

Table 4.2.1e - Natural frequencies of the Truss System Modified by Results in Table 4.2.1d

Mode	Natural Frequency (Hz)
1	29.558
2	80.000
3	114.824

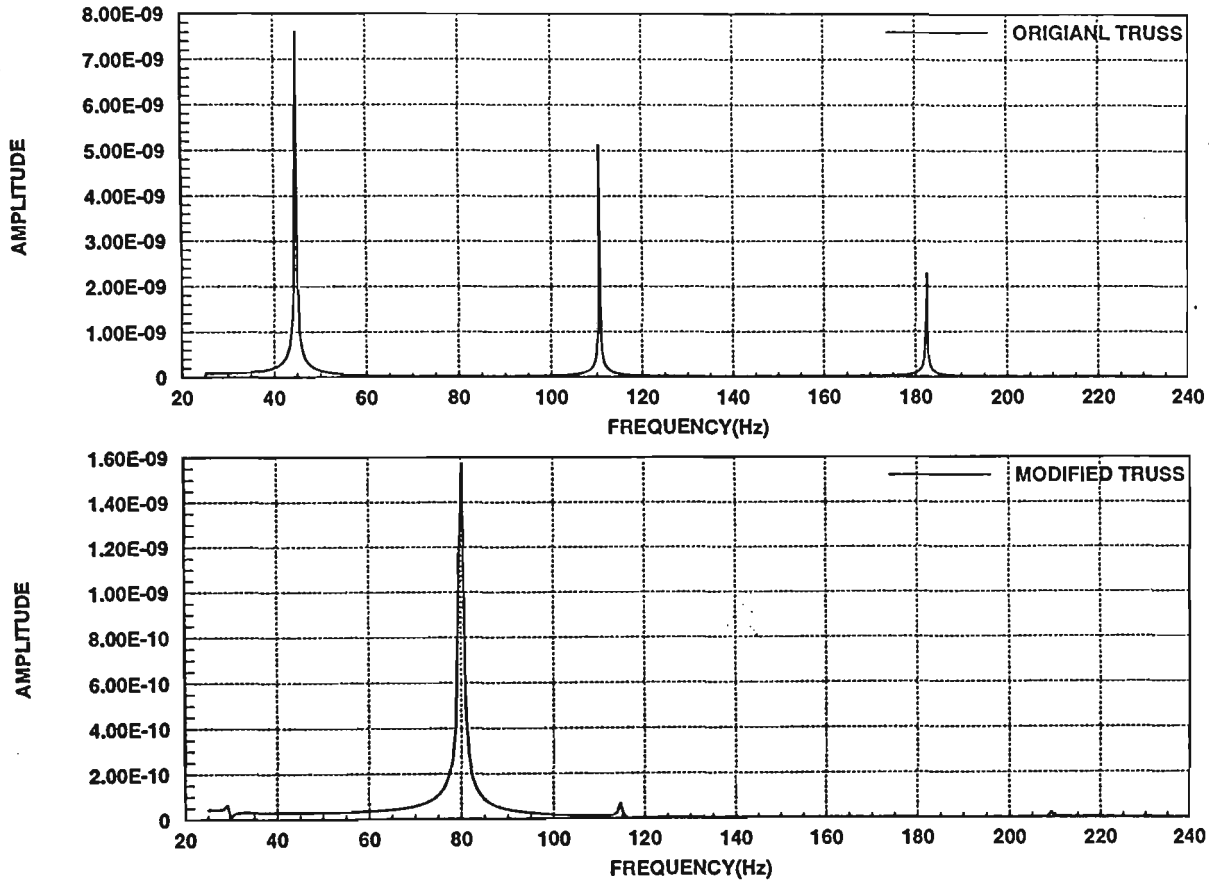


Figure 4.2.1d - FRF Comparison of Truss Structure
Upper: Original, Lower: Modified

Table 4.2.1f - Modification Results For Relocating the 3-rd Resonance of The Truss Structure From 182.409 Hz To 160.000Hz

No. Element	No. Nodes		1-st Modification Results(M^2)	2-nd Modification Results(M^2)	Element Participant Ratio
8	4	5	6.474e-3	1.289E-1	2
9	4	6	4.222E-3	6.545E-2	1
10	4	7	6.474E-3	1.289E-1	2
11	5	6	4.222E-3	6.545E-2	1
12	5	7	6.474E-3	1.289E-1	2
13	6	7	4.222E-3	6.545E-2	1

Table 4.2.1g- Natural Frequencies of Truss Structure Modified by the 1-st Set of Results in Table 4.2.1f

Mode	Natural Frequency (Hz)
1	41.914
2	103.678
3	160.000

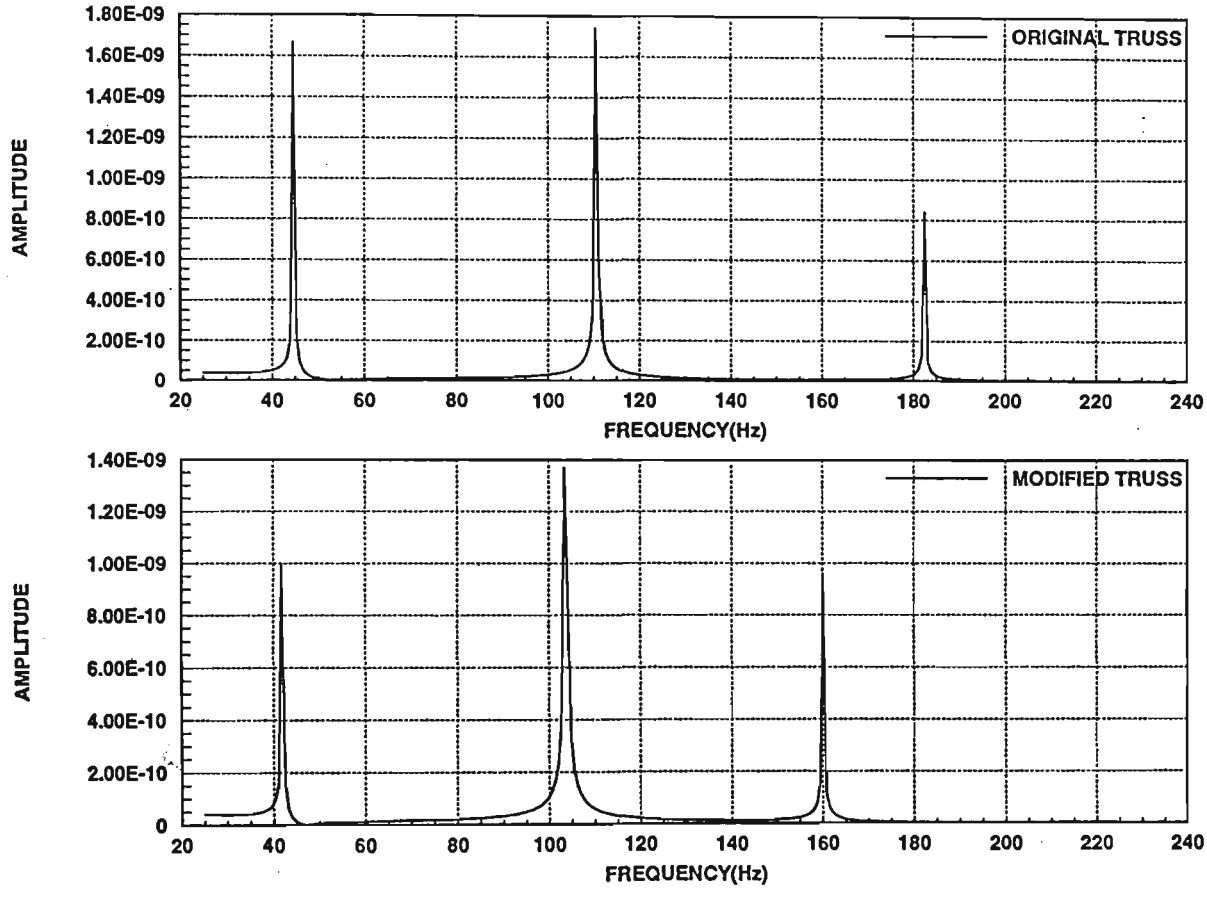


Figure 4.2.1e - FRF Comparison of Truss Structure
Upper: Original, Lower: Modified

DOF along y axis of node 9), ie., $\alpha(5,16)$, and (ii) receptance FRF between coordinate 9 (translational DOF along y axis of node 6) and coordinate 16 (translational DOF along X axis of node 9), ie., $\alpha(9,16)$.

The first three anti-resonances in $\alpha(5,16)$ and $\alpha(9,16)$ of the original truss system can be obtained by solving the eigen problem with respect to the corresponding 'virtual' system and are listed in **Table 4.2.2a**.

Table 4.2.2a - Anti-Resonances in FRFs $\alpha(5,16)$and $\alpha(9,16)$

Anti-Resonance	Receptance Frequency Response Function	
	$\alpha(5,16)$	$\alpha(9,16)$
1	65.427 Hz	52.915 Hz
2	508.500 Hz	300.962 Hz
3	559.397 Hz	353.550 Hz

The problem here is to determine the modification of cross sectional area of selected truss elements which can create the desired anti-resonance at a specified frequency. Thus, the vibration response at this frequency and nearby can be significantly reduced.

Assume that anti-resonances need to be created one at a time at the frequencies of the first and second resonances of the original system which are 44.686 Hz and 110.564 Hz respectively. The target cross sectional area of pre-selected elements listed in **Table 4.2.2b** and **Table 4.2.2c** are required. Using the method developed in Section 3.3, the modification results for $\alpha(5,16)$ and $\alpha(9,16)$ are given in **Table 4.2.2b** and **Table 4.2.2c**. Again, several possible modifications have been found, and the eventual modification is subject to the practical consideration or designer's decision. The anti-resonance frequency of the original truss system had been relocated accurately to a desired location by using the proposed method. **Figure 4.2.2a**, **Figure 4.2.2b** and **Figure 4.2.2c** graphically illustrate comparisons of FRF curves between the original and modified systems, and indicate that within the frequency ranges of interest the response at certain

Table 4.2.2b - Modification Results for Relocating Anti-Resonance of Receptance FRF α(5,16)

Receptance FRF Desired Anti-Resonance			α(5,16) 44.685 Hz		Receptance FRF Desired Anti-Resonance			α(5,16) 110.560 Hz	
No. Elements	No. Nodes		Element Part. Ratio	Modification Results (M^2)	No. Element	No. Nodes		Element Part. Ratio	Modification Results (M^2)
13	6	7	1	1.588E-2	13	6	7	1	1.00E-4
14	6	8	2	2.979E-2	14	6	8	2	2.00E-5
15	6	9	1	1.588E-2	15	6	9	1	1.00E-4
16	7	8	2	2.979E-2	16	7	8	2	2.00E-5
17	7	9	1	1.588E-2	17	7	9	1	1.00E-4
18	8	9	2	2.979E-2	18	8	9	2	2.00E-5

Table 4.2.2c - Modification Results for Relocating Anti-Resonance of Receptance FRF α(9,16)

Receptance FRF Desired Anti-Resonance			α(9,16) 44.685 Hz	
No. Elements	No. Nodes		Element Part. Ratio	Modification Results (M^2)
13	6	7	1	5.683E-3
14	6	8	1	5.683E-3
15	6	9	1	5.683E-3
16	7	8	1	5.683E-3
17	7	9	1	5.683E-3
18	8	9	1	5.683E-3

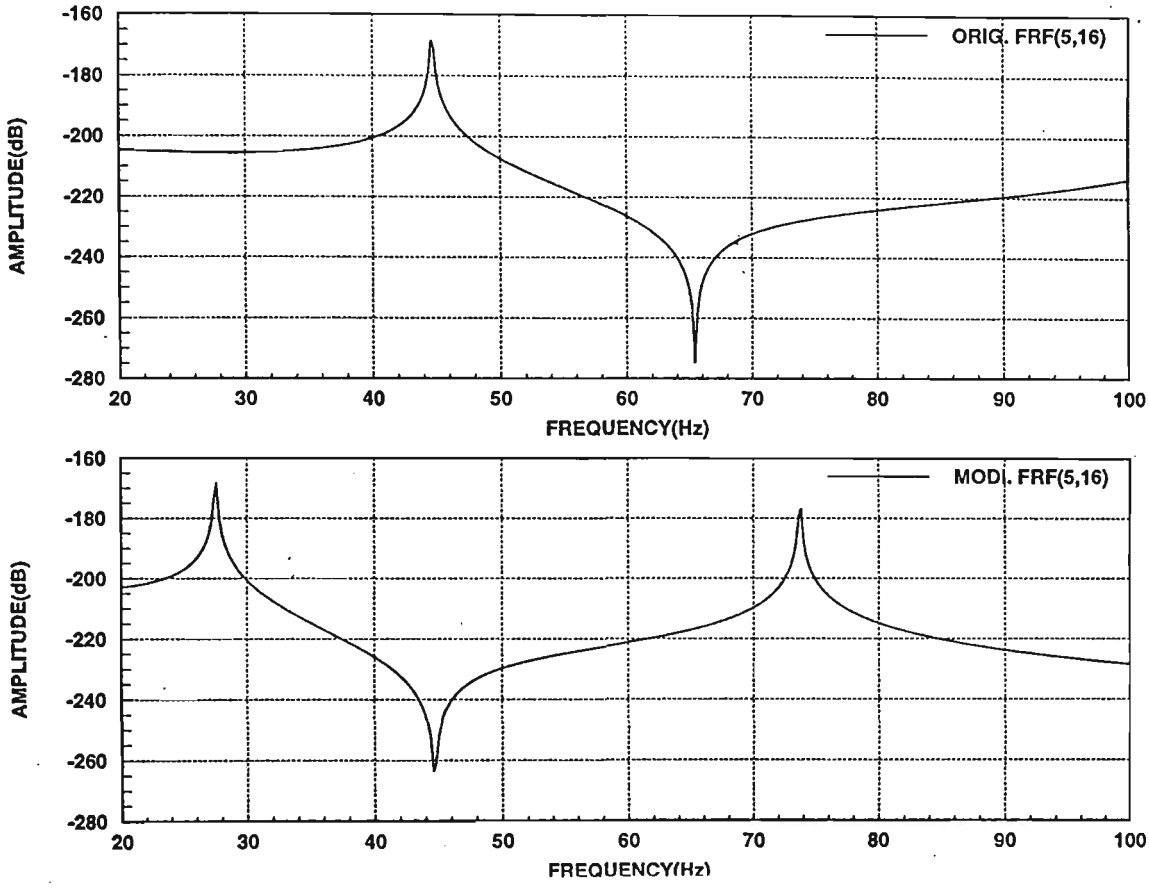


Figure 4.2.2a - FRF Comparison of Truss Structure
Relocation of the 1-st Anti-Resonance of $\alpha(5,16)$
Upper: Original, Lower: Modified

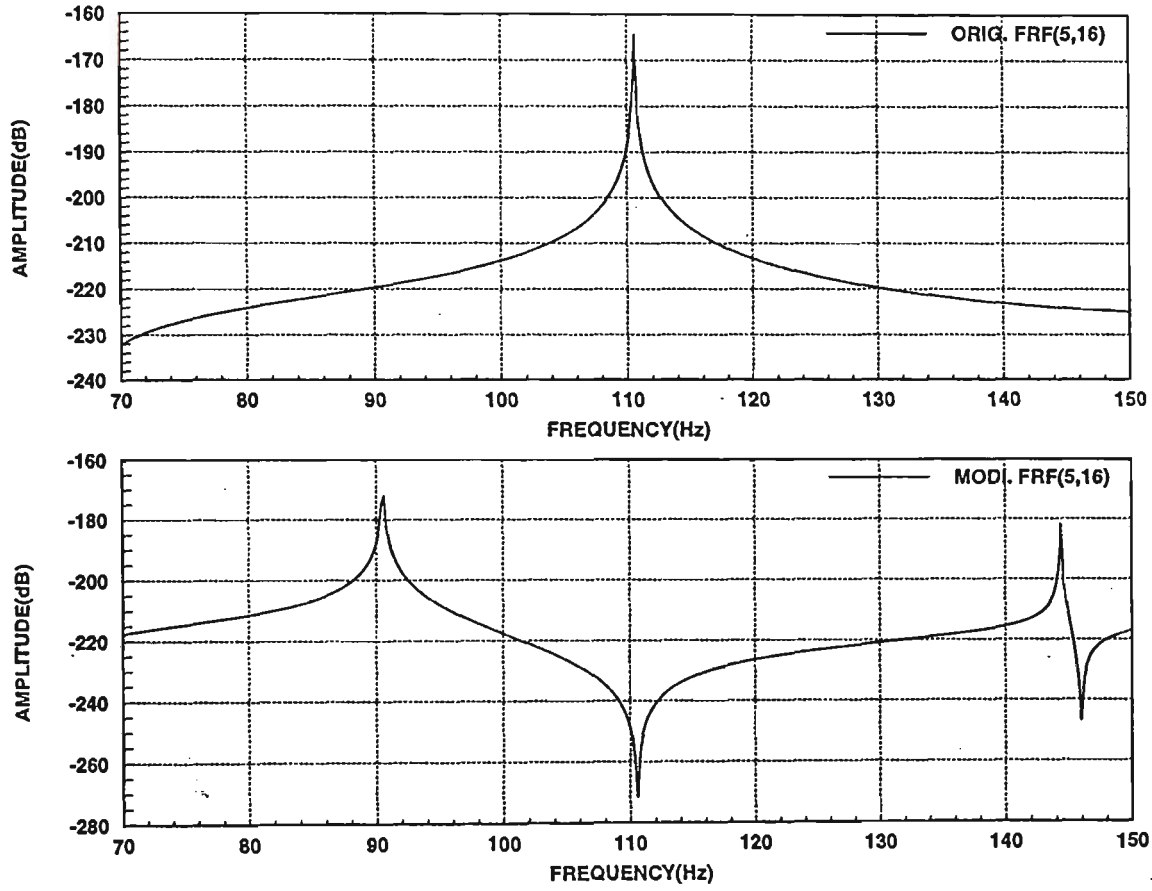


Figure 4.2.2b - FRF Comparison of Truss Structure Relocation of The 2-nd Anti-Resonance of $\alpha(5,16)$ Upper: Original, Lower: Modified

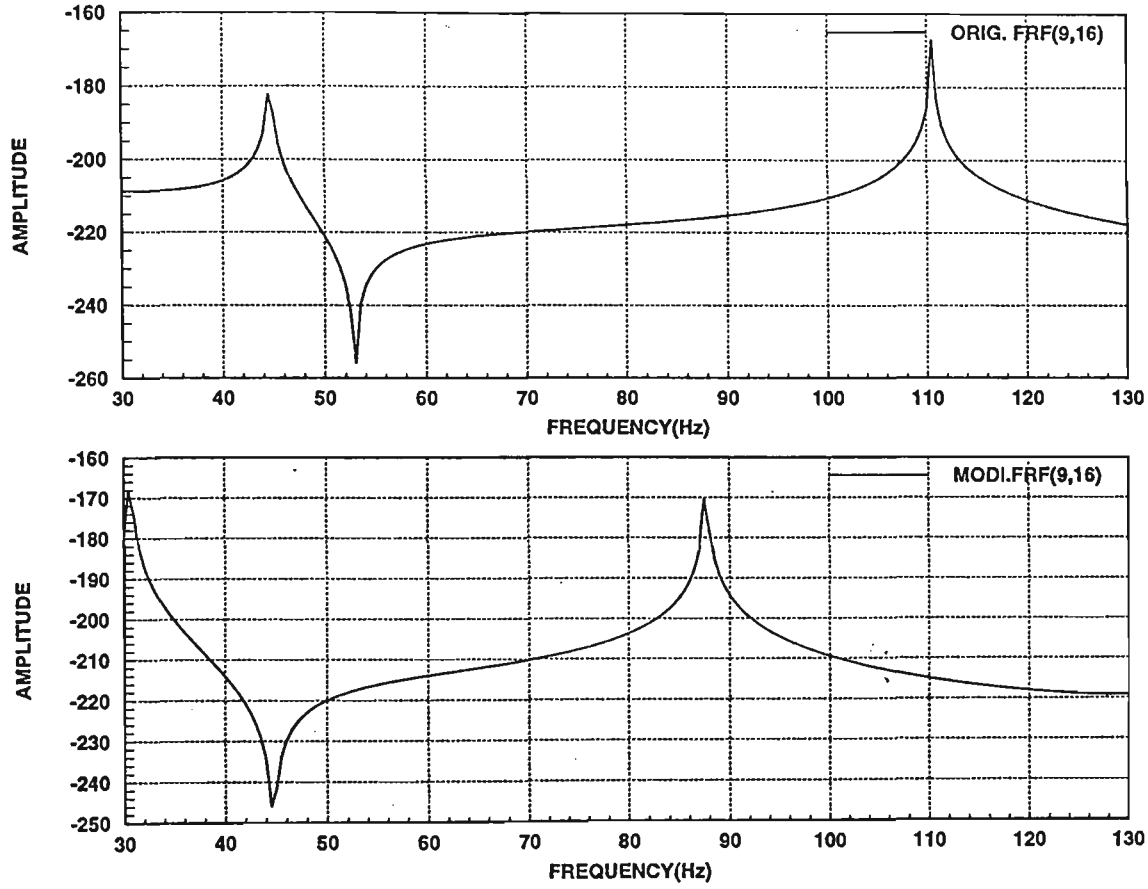


Figure 4.2.2c - FRF Comparison of Truss Structure Relocation of The 1-st Anti-Resonance of $\alpha(9,16)$ Upper: Original, Lower: Modified

locations (coordinates determined by FRFs) have been significantly reduced. Given a sinusoidal excitation with frequency in the vicinity of one of the system's resonances at coordinate 5 (translational DOF along x axis of node 4), a resonance response at coordinate 16 (translational DOF along y axis of node 9) will happen for the original truss system. For the modified truss structure, with the same excitation at the same location, not only the resonance will be attenuated, but an anti-resonance will be created in the desired location (node 5 and node 16). This is of great significance if an extremely low vibration response at a certain location is required for a sinusoidally excited vibratory system.

4.2.3 Finite Element Implementation of Structural Modification of the Planar Beam Structure

Studies in Section 4.2.1 and Section 4.2.2 help to establish the validity of the methods developed in Chapter 3. However, these studies are based on the planar truss structure, whose mass and stiffness matrices can be formulated as linear (first order) functions of the modified structural parameters. The problem can then be solved as a first order eigenvalue problem. For most of element types used in FEA, such as beam elements and shell elements, the mass and stiffness matrices can not be idealised as linear functions of structural parameters. Thus, the eigenvalue problem generated in modification will become a high order one. An alternative algorithm has been developed in Section 3.2.2 for these cases.

In this Section, a planar beam structure will be investigated. The elements to be modified whose mass and stiffness matrices can not be expressed as linear functions of a structural parameter are described as 'high order elements'. The eigenvalue problem with 'high order elements' can be solved based on the algorithm developed in Section 3.2.2 (some basic mathematical manipulation may be needed depending on the formation of finite element). Results obtained from example system #3 used in this Section can readily be extended to the structural system with other high order elements.

The structural parameters were given in **Table 4.1.2**. After eigensolution, the first 3 natural frequencies and their corresponding mode shapes are given in **Table 4.2.3a** and

Table 4.2.3a - The 1-st Three Natural Frequencies of Original Free-Free Beam Structure

Mode	Natural Frequency (Hz)
1	53.004
2	146.136
3	286.818

Figure 4.2.3a, **Figure 4.2.3b**, **Figure 4.2.3c** with configurations of the original beam. The element height (thickness in z axis) is considered as the structural parameter to be modified. A computer code called *beam.sdm.ftn* was developed to facilitate this study.

Using the method developed in Section 3.2.2 to relocate the resonance to a desired location, results were given in **Table 4.2.3b**. The corresponding mode shapes and modified configurations were given in **Figure 4.2.3d**, **Figure 4.2.3e** and **Figure 4.2.3f**. It is noted that the original uniform beam has been stepped to achieve the desired natural frequencies.

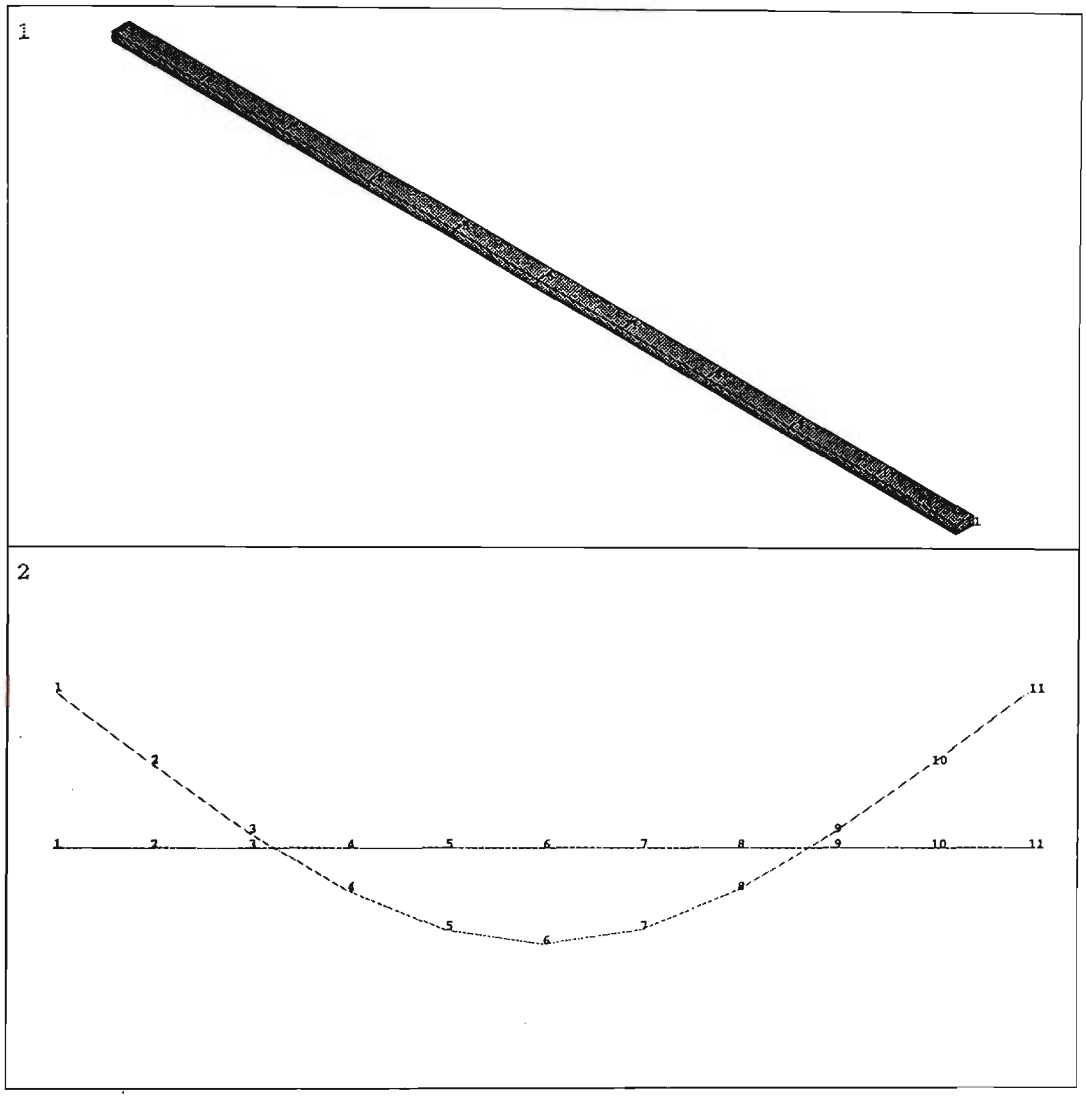


Figure 4.2.3a - The 1-st Mode Shape of Original Beam Structure
Upper: Configuration, Lower: The 1-st Mode Shape

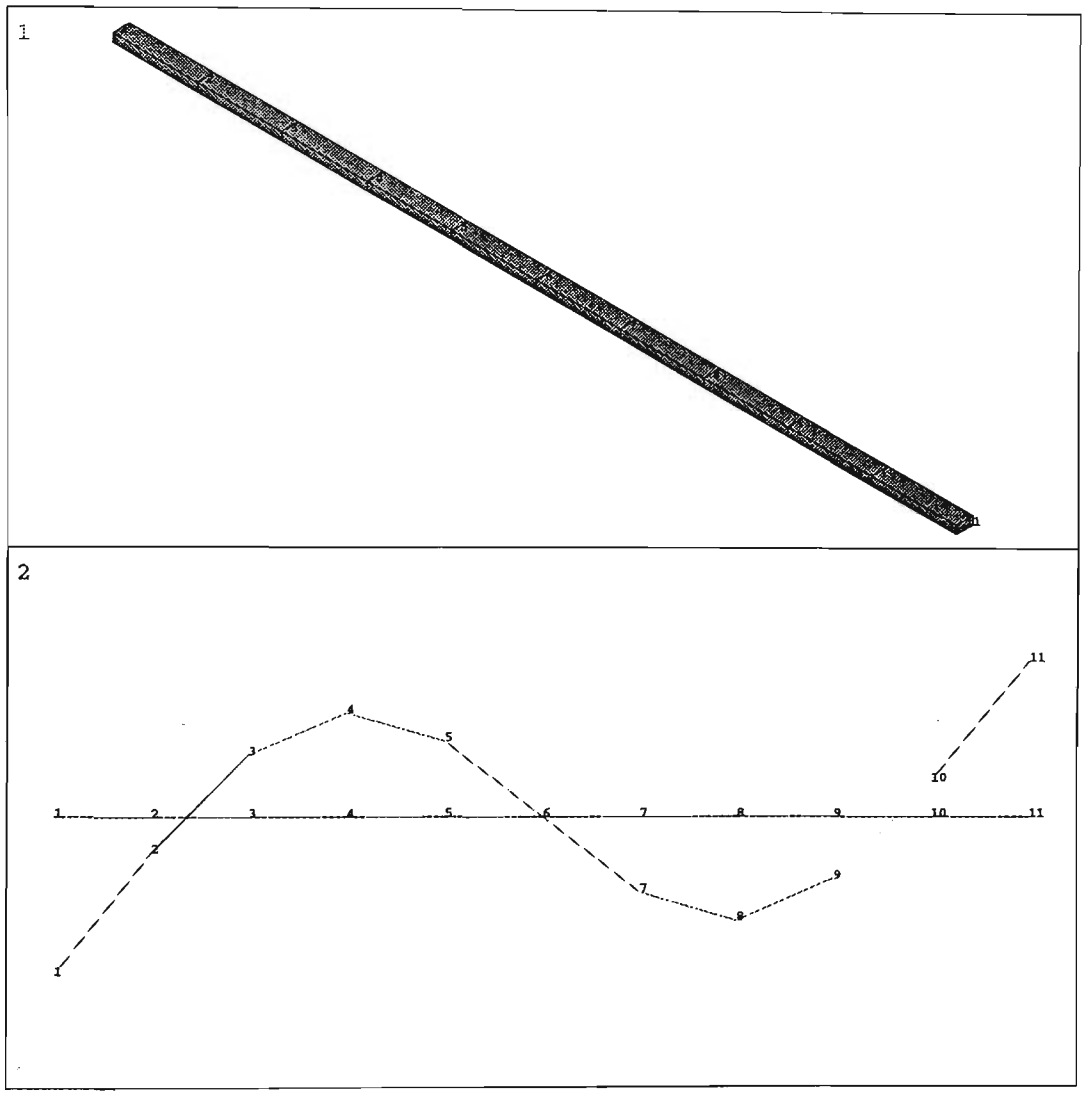


Figure 4.2.3b - The 2-nd Mode Shape of Original Beam Structure
Upper: Configuration, Lower: The 2-nd Mode Shape

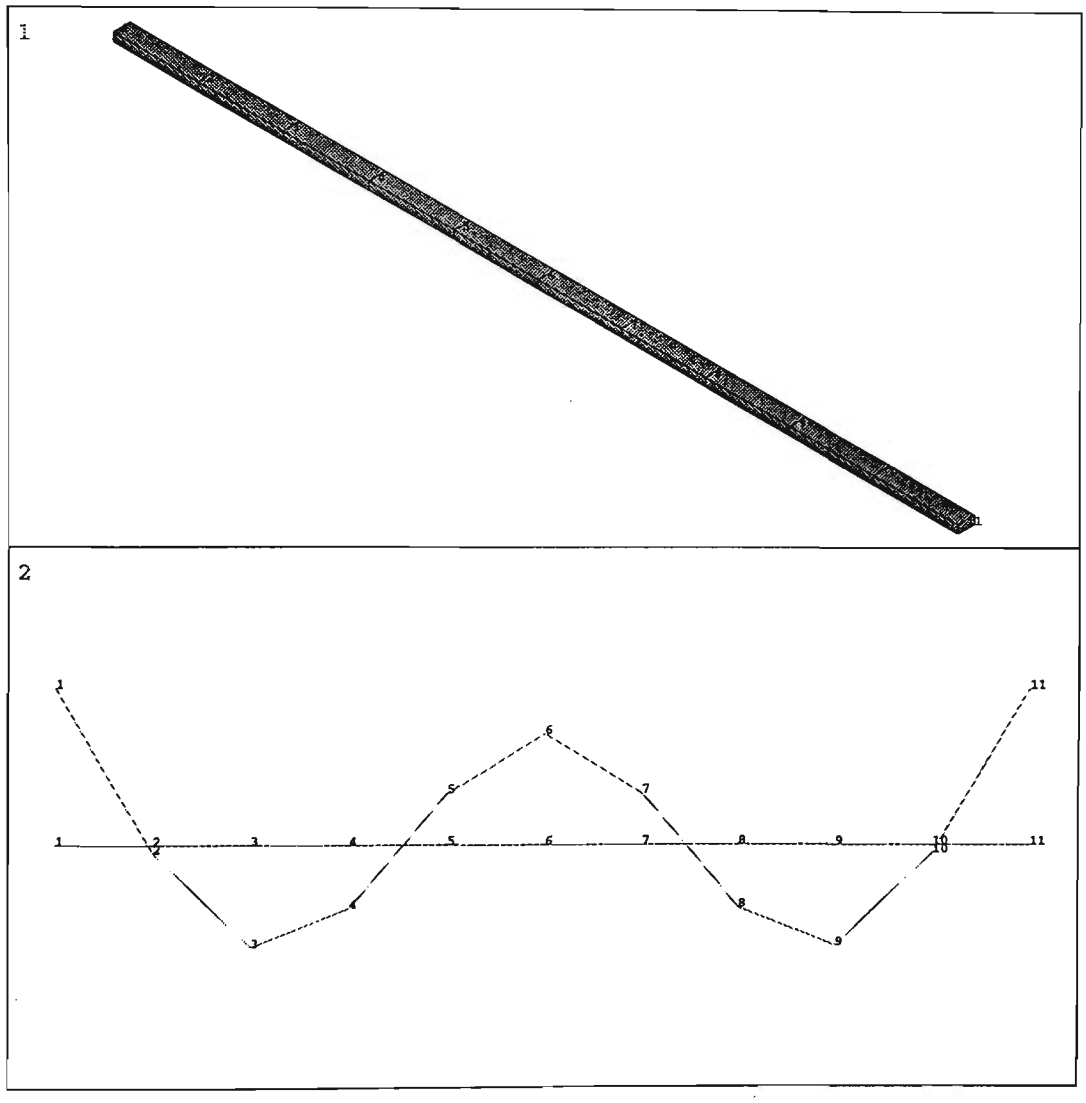


Figure 4.2.3c - The 3-rd Mode Shape of Original Beam Structure
Upper: Configuration, Lower: The 3-rd Mode Shape

Table 4.2.3b - Modification Results for Relocating Resonance of Free-Free Beam Structure

Desired Natural Frequency (Hz)			63.000	
Variation (Hz)			10.000	
No. Element	No. Nodes		E. P. R*	Mod. Res.* (M)
4	4	5	1	1.296E-2
5	5	6	1	1.296E-2
6	6	7	1	1.296E-2
Desired Natural Frequency (Hz)			160.000	
Variation (Hz)			13.468	
No. Element	No. Nodes		E. P. R*	Mod. Res.* (M)
4	4	5	1	1.711E-2
5	5	6	2	2.422E-2
6	6	7	1	1.711E-2
Desired Natural Frequency (Hz)			250.000	
Variation (Hz)			-36.818	
No. Element	No. Nodes		E. P. R*	Mod. Res.* (M)
3	3	4	1	0.735E-2
8	8	9	1	0.735E-2

E.P.R*: Element Participant Ratio. Mod. Res.*(M): Modification results.

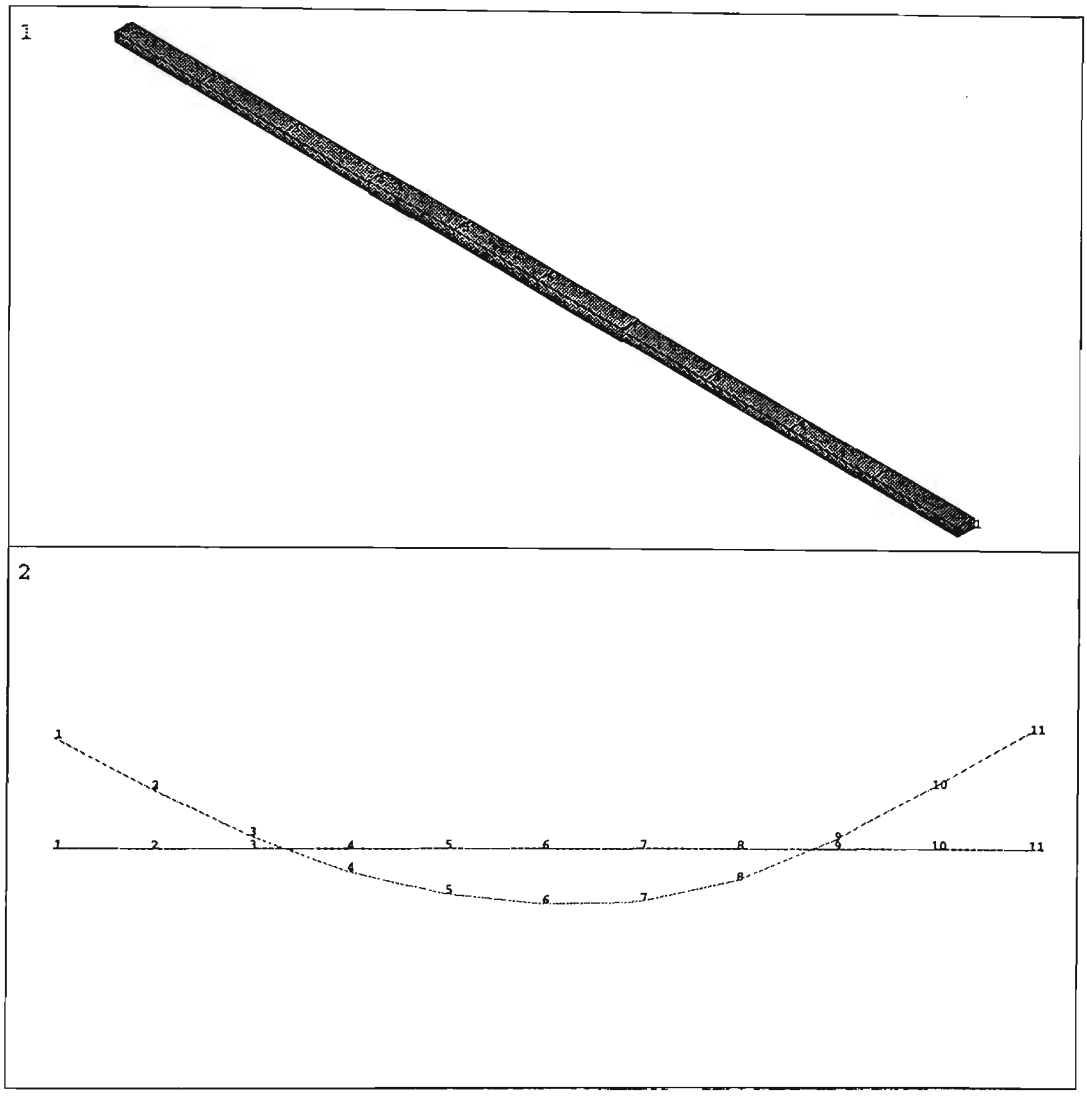


Figure 4.2.3d - The 1-st Mode Shape of Modified Beam Structure
Upper: Configuration, Lower: The 1-st Mode Shape

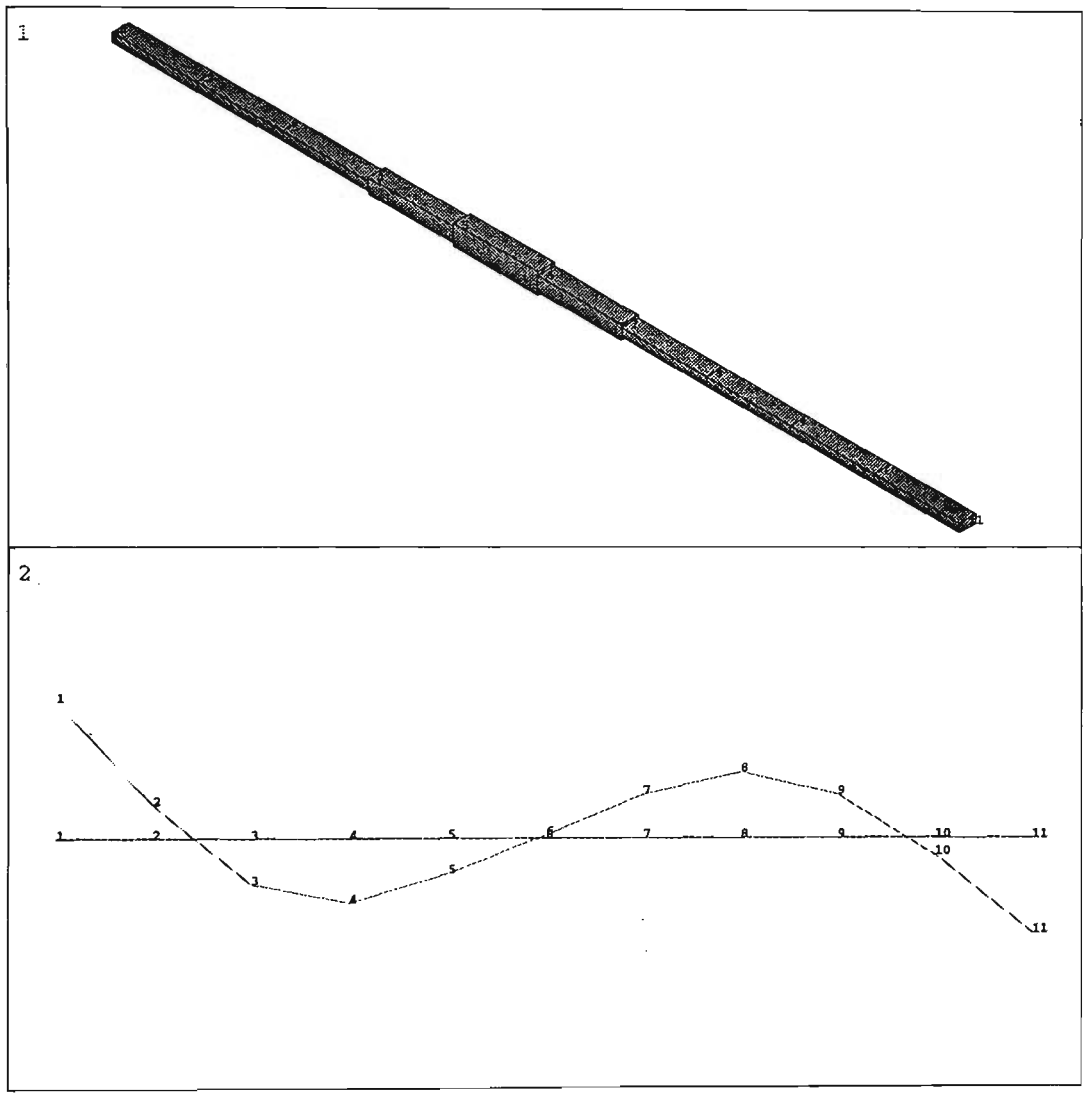


Figure 4.2.3e - The 2-nd Mode Shape of Modified Beam Structure
Upper: Configuration, Lower: The 2-nd Mode Shape

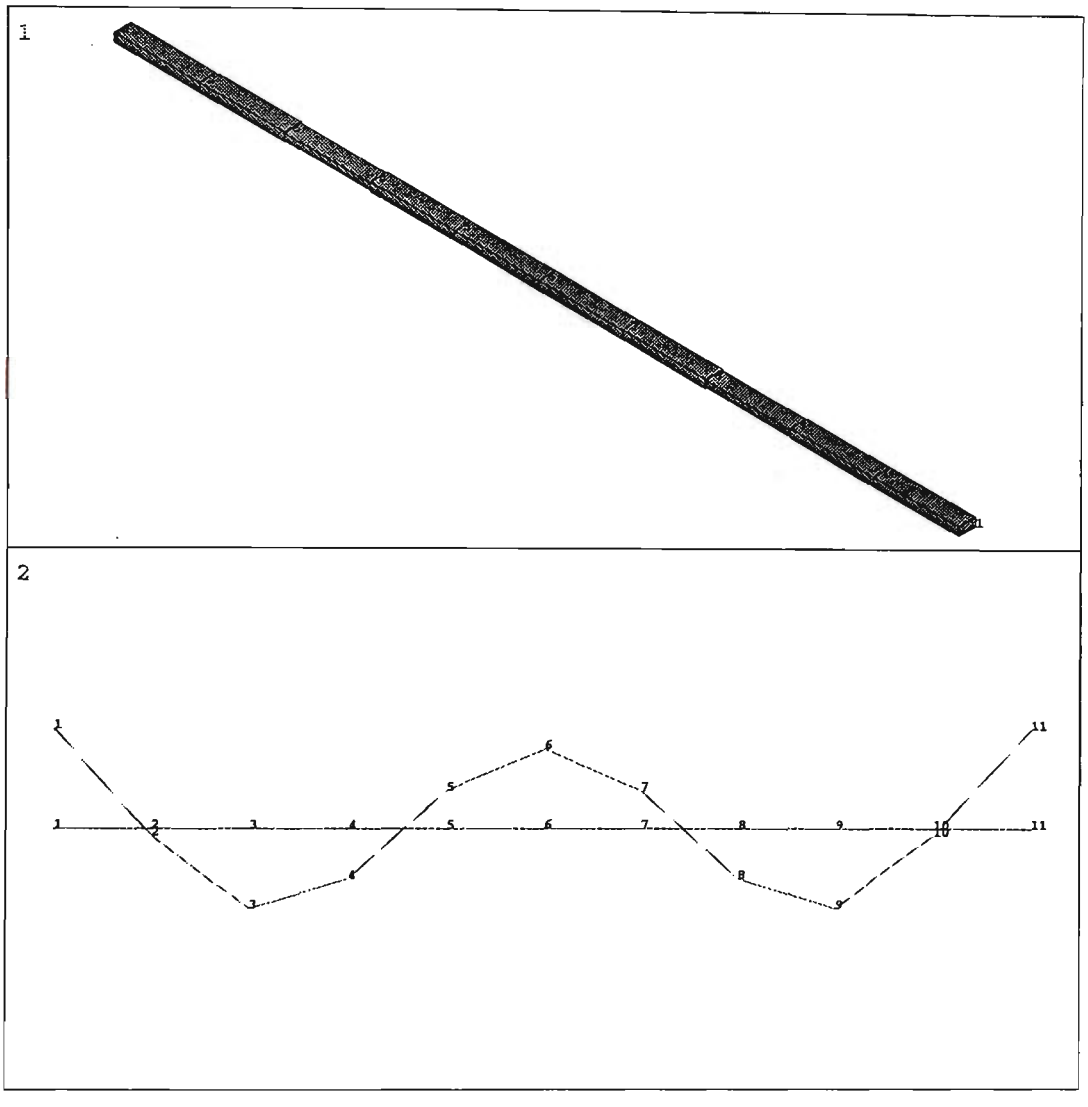


Figure 4.2.3f - The 3-rd Mode Shape of Modified Beam Structure
Upper: Configuration, Lower: The 3-rd Mode Shape

4.3 Application of Methods For Cross-stiffened Grid System (System #3)

This section will demonstrate the application of the methods developed in Chapter 3 in an practical engineering problem. A cross-stiffened grid structure which was given in Section 4.1.3 is investigated for this purpose. The original dynamic model of this structure will be provided from both EMA and FEA.

4.3.1 Experimental Modal Analysis of Cross-stiffened Grid

The modal model (natural frequencies and mode shapes) of a dynamic structure can be experimentally determined by using an experimental modal analysis technique. These techniques involve acquiring the frequency response function data from some points whose mode shape are of interest on the structure, and these data can then be curve-fitted to extract natural frequencies and mode shapes. The theoretical background and general procedure for the EMA have been systematically described by Ewins[8]. A modal analysis software package called MODAL 3.0SE developed by SMS[87] was utilised to facilitate this study.

In this study, a single-degree of freedom polynomial curve fitting algorithm was used. The conditions under which this algorithm can provide sufficient accuracy are:

1. the structure is lightly damped (structures exhibiting resonance conditions and/or modal damping of 10% of critical or less);
2. the structure has low modal density within the frequency range of interest.

For the grid structure under investigation, both conditions were considered satisfied. Detailed discussion of this algorithm can be found in SMS[87].

Figure 4.3.1a and **Figure 4.3.1b** lay out the experimental set-up for this study. The geometry mapping of the grid consisted of 23 coordinate locations. Details are given in **Figure 4.3.1c**. Note that only those translational degrees of freedom perpendicular to the plane of structure are measured. They are named as 1Y, 2Y, ... 23Y in turn. The grid structure is suspended with 4 elastic bands at points 4, 6, 16, 17 to simulate the Free-Free

boundary condition. Impact excitation method was adopted. Specifics for the data acquisition of the modal test are given in **Table 4.3.1a, Table 4.3.1b**.

The first three non-rigid body natural modes which fall into the frequency range from 25Hz to 125Hz are curve-fitted and the results (natural frequencies and mode shapes) are given graphically in **Figure 4.3.1d, Figure 4.3.1e** and **Figure 4.3.1f**. These results will be compared with those from FEA in following Section.

4.3.2 Finite Element Analysis of the Cross-Stiffened Grid

The finite element analysis of the grid structure was carried out on the finite element model described in Section 4.1.3. The location and number of nodes used to discretise the structure are accordance with those in EMA as shown in **Figure 4.1.3c**. Since only those translational DOF perpendicular to the plane of the structure are considered in EMA, The FEA model was reduced to match the translational experimental model by applying the Guyan reduction to eliminate the torsional and bending rotation DOF. A computer code *system.grid.ftn* was developed to facilitate building up the system mass and stiffness matrices for eigen solution and carrying out modification. The first three natural mode shapes of non-rigid body mode and corresponding natural frequency are given in **Figure 4.3.2a, Figure 4.3.2b**. The comparison of natural frequencies from FEA and EMA are given in **Table 4.3.2a.**

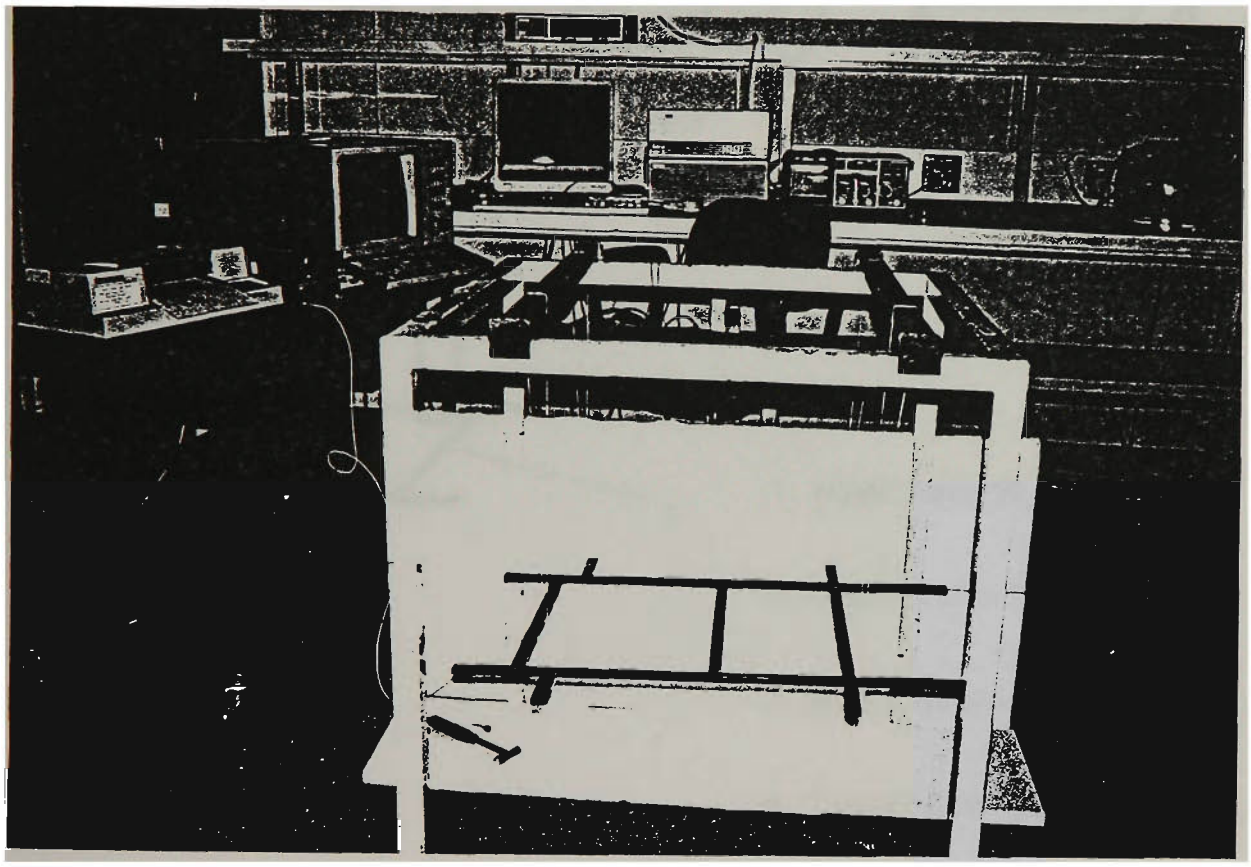


Figure 4.3.1a - EMA Set-Up for Crossed-Stiffened Grid Structure

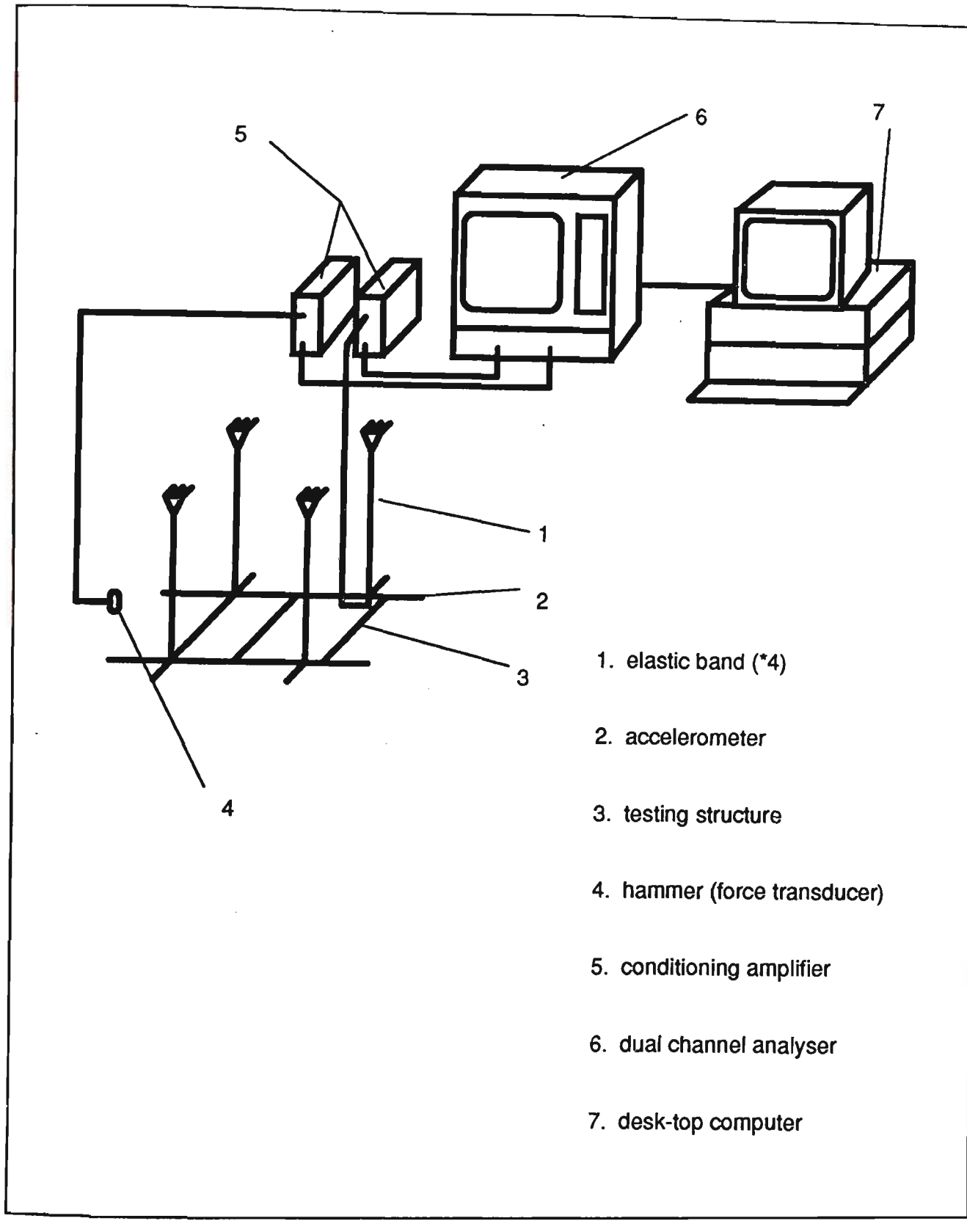


Figure 4.3.1b - EMA Set-Up for Cross-Stiffened Grid Structure

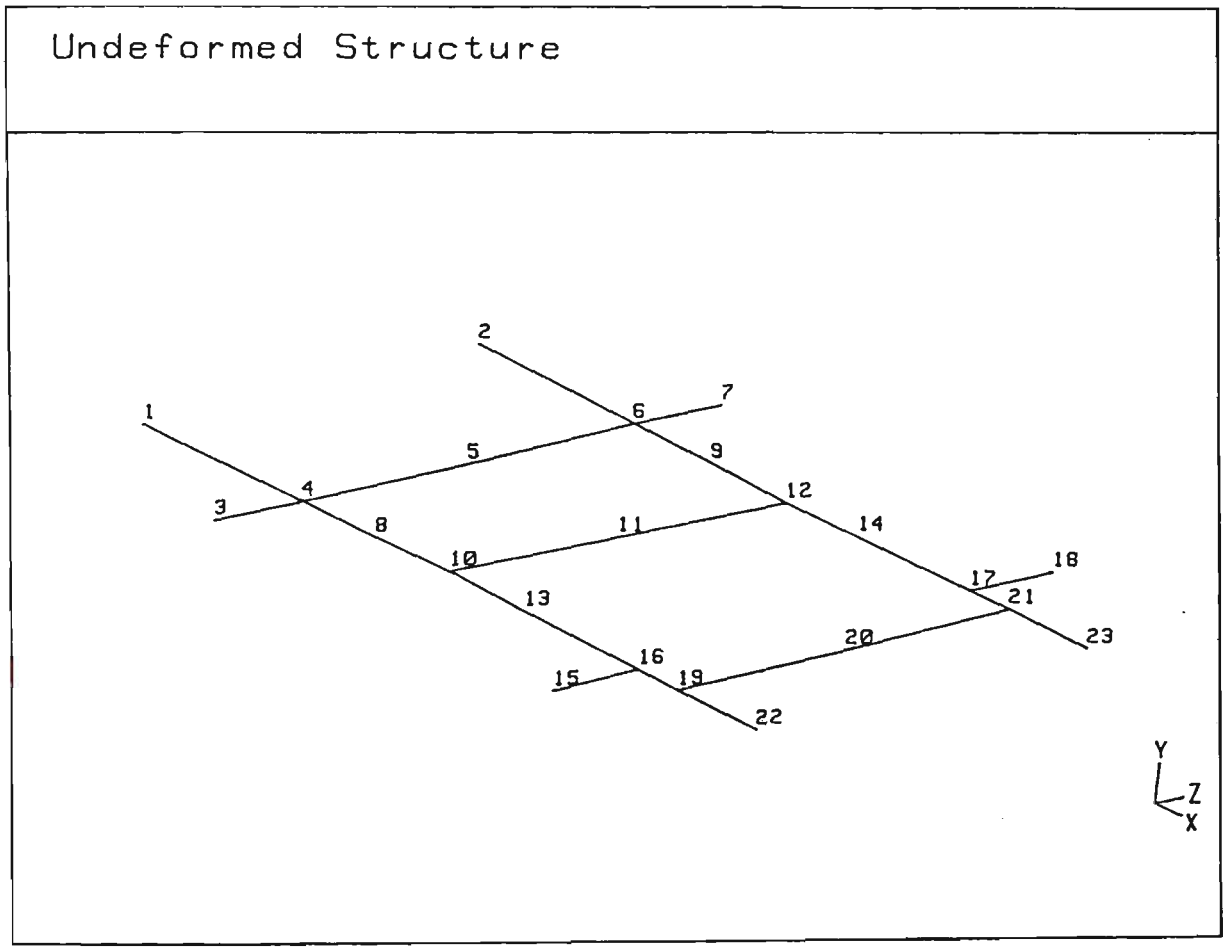


Figure 4.3.1c - EMA Geometry Mapping of Cross-Stiffened Grid Structure

Table 4.3.1.a - Experimental Set-Up for EMA of the Grid Structure

Structure:	Cross Stiffened Grid
Dual Channel Analyser:	Bruel & Kjaer, Type 2032
Exciter:	PCB Hammer, Force Transducer Sensitivity: 2.10 mv/N
Signal Amplifier:	NVMS Conditioning Amplifier
Accelerometers:	PCB 10729, Voltage, Sensitivity: 0.804 mv/g
Data Processor:	HP Desktop Computer, Model 7957A
Plotter:	HP, Model 7440A

Table 4.3.1b - Equipment Setting

Dual Channel Analyser		Signal Amplifier	
Frequency Range:	0 - 800Hz	Gain:	1.0
Resolution:	800 Lines	High Pass:	5.0Hz
Windows:	Excitation: Transient Response: Exponential	Frequency Filter:	All Pass
Sensitivity: (Calibrated)	Excitation: 1mv/N Response: 102μm/ms^{-2}	Sensitivity Unit: (Calibrated)	Excitation:2.10 Response: 0.804
Averages:	22		

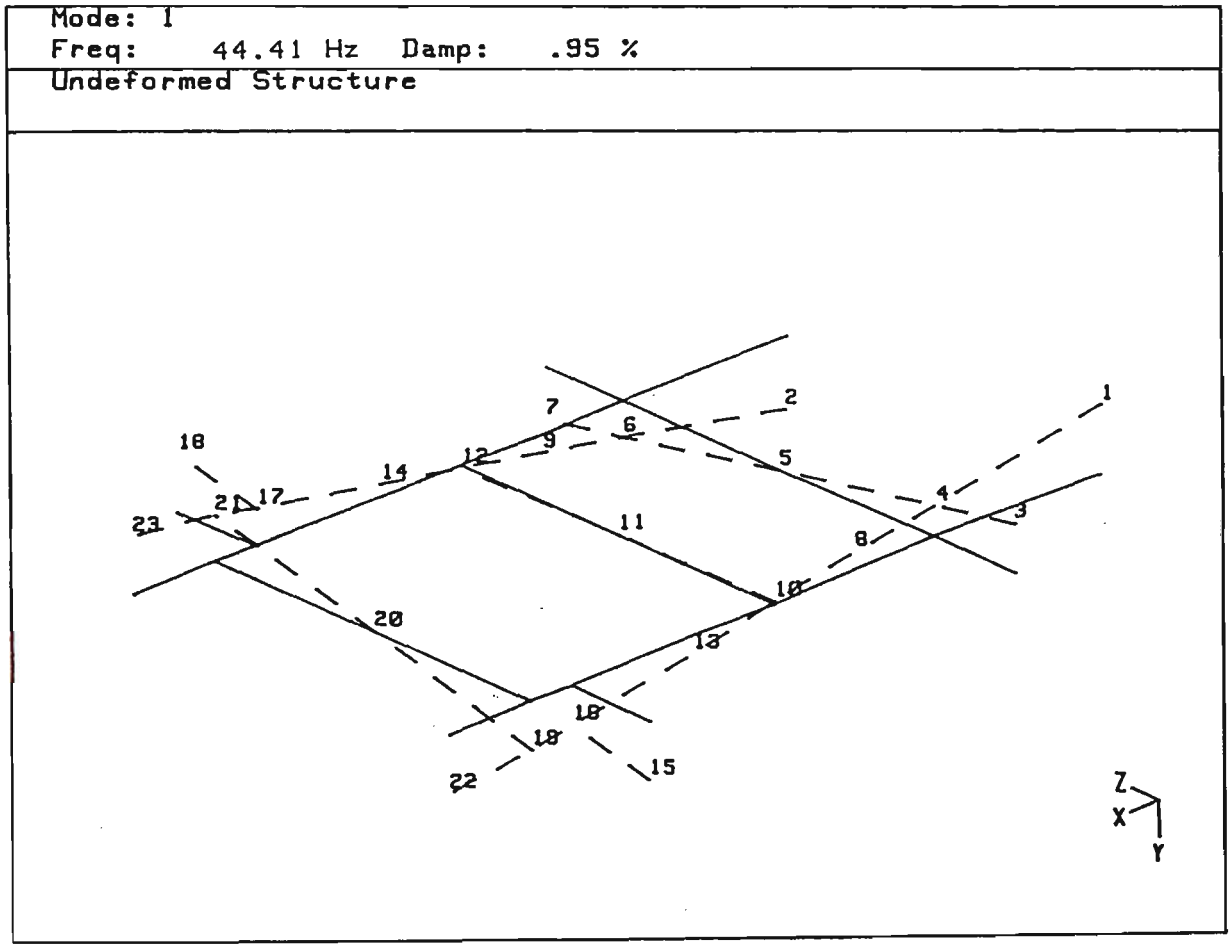


Figure 4.3.1d - The 1-st Mode Shape of Cross-Stiffened Grid Structure (EMA)

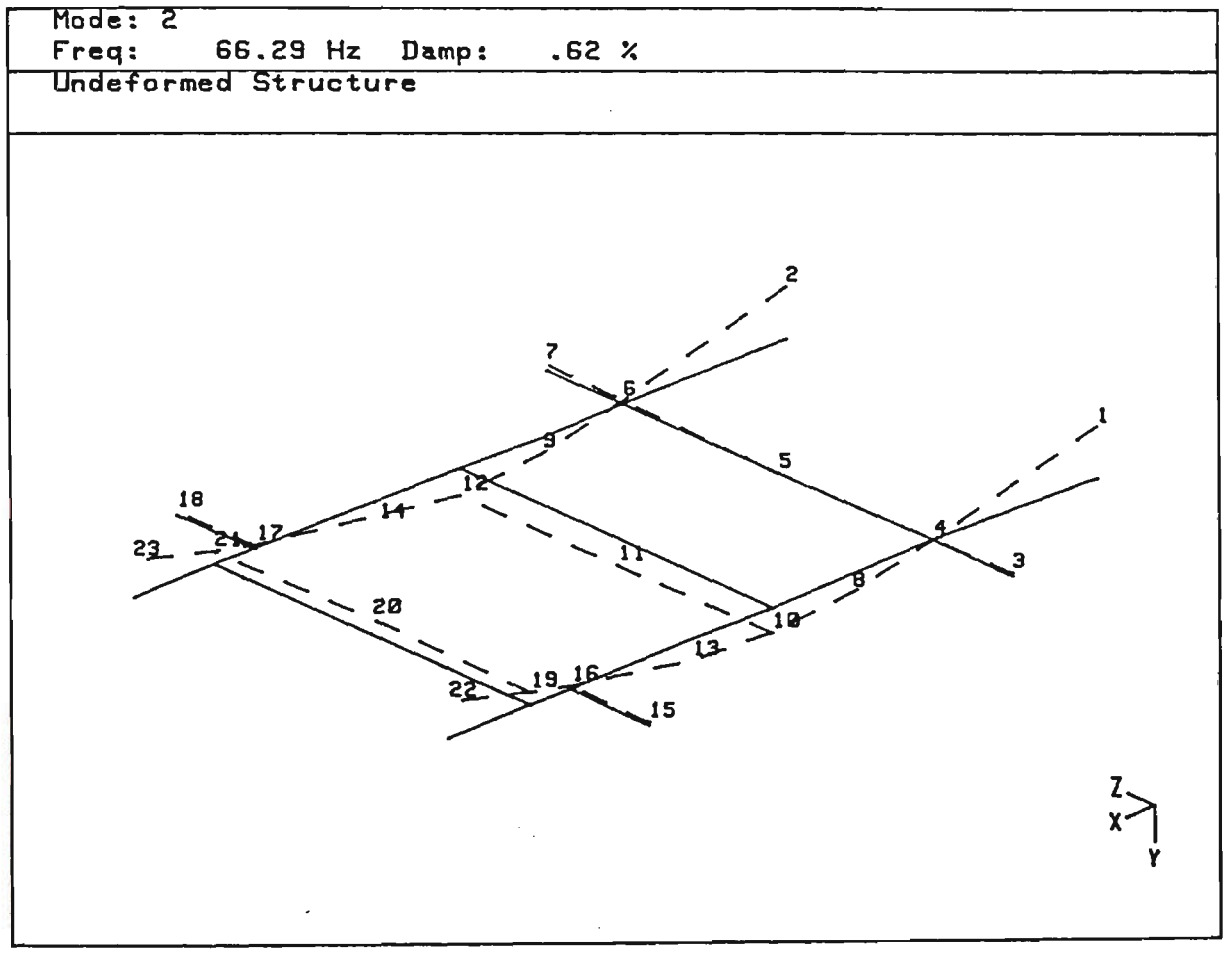


Figure 4.3.1e - The 2-nd Mode Shape of Cross-Stiffened Grid Structure (EMA)

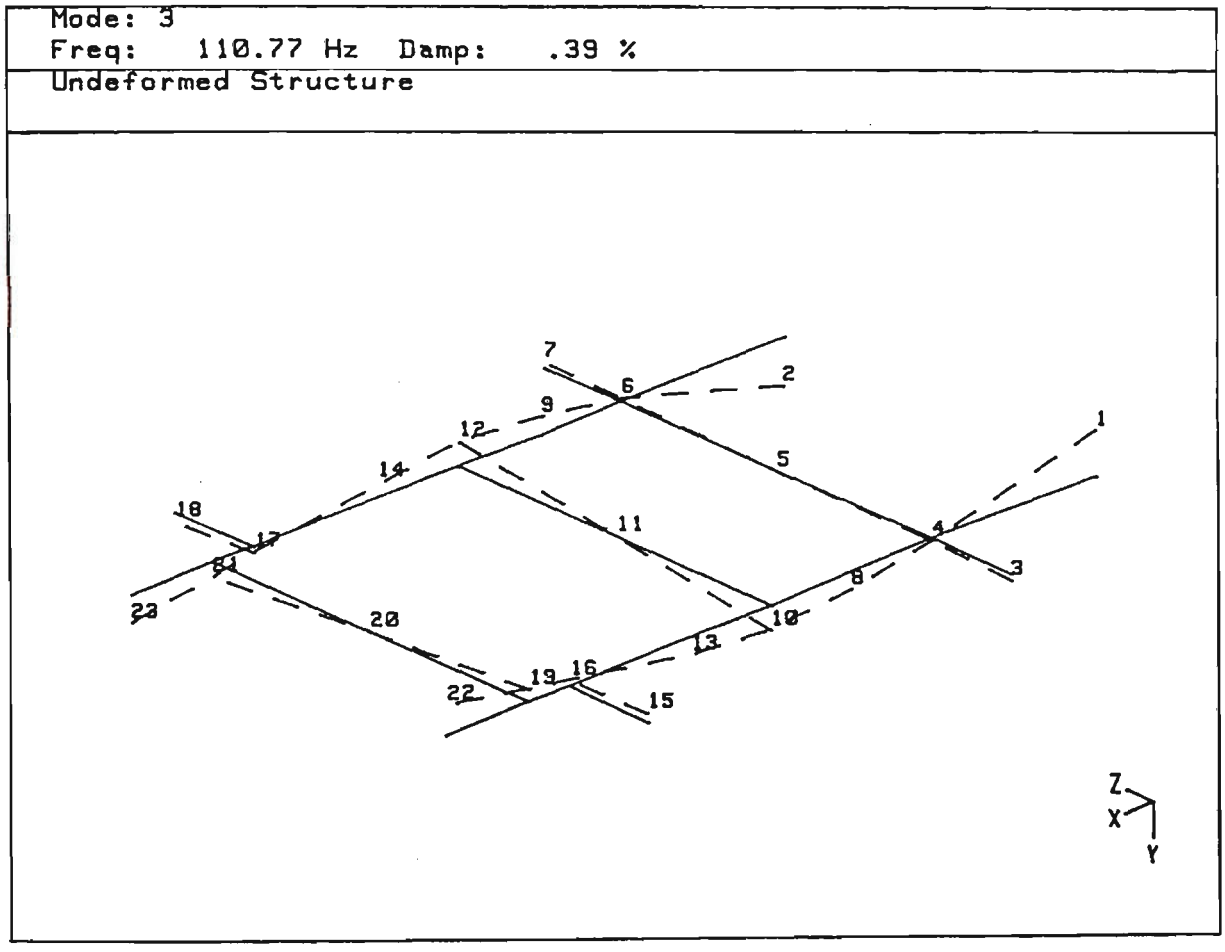


Figure 4.3.1f - The 3-rd Mode Shape of Cross-Stiffened Grid Structure (EMA)

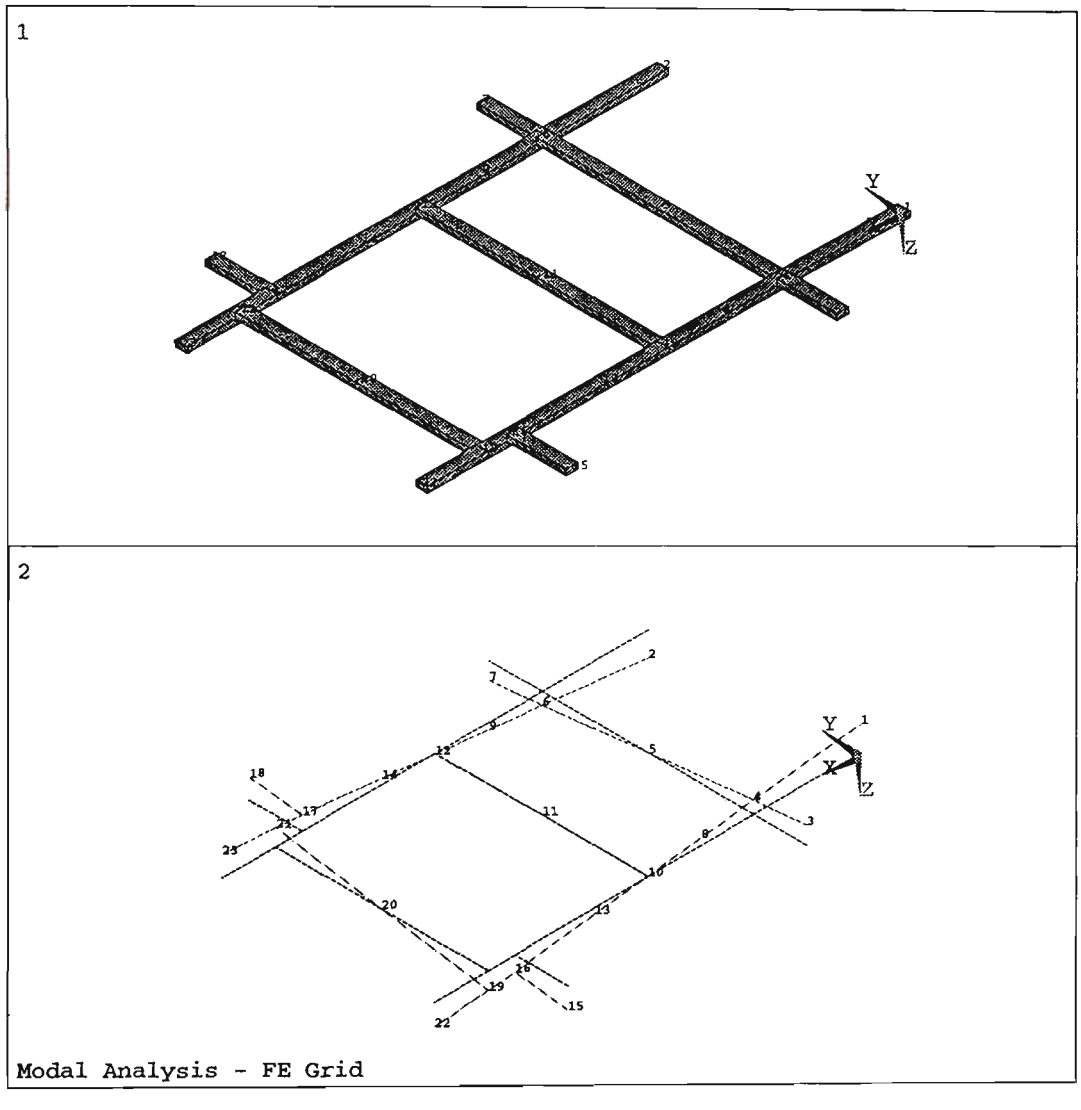


Figure 4.3.2a - The 1-st Mode Shape of Grid Structure (FEA)
Upper: Undeformed, Lower: The 1st Mode Shape

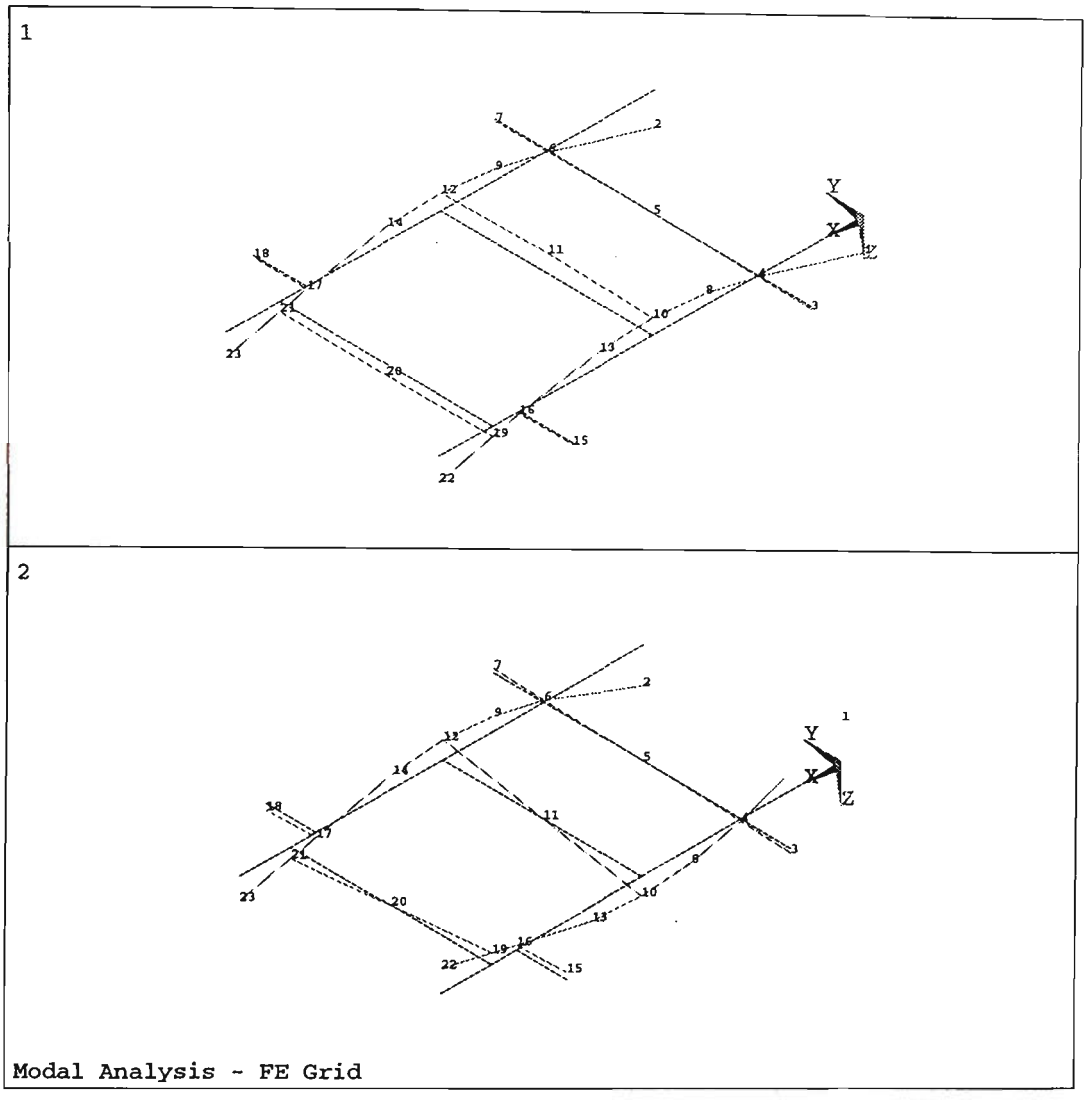


Figure 4.3.2b - The 2-nd and 3-rd Mode Shapes of Grid Structure (FEA)
Upper: The 2-nd Mode Shape, Lower: The 3-rd Mode Shape

Table 4.3.2a - Comparison of Natural Frequencies From EMA and FEA

Non-Rigid Mode	Natural Frequency (Hz) EMA	Natural Frequency (Hz) FEM	Error(%)
1	44.41	44.73	0.72
2	66.29	66.93	0.96
3	110.75	111.72	0.87

4.3.3 Relocating Resonance and Anti-Resonance of the Grid Structure by Concentrated Mass

Results of Sections 4.3.1 and 4.3.2, indicate that the finite element model of the grid structure was reasonably accurate. Comparing the first three natural mode from FEA and EMA, it has shown that the error of natural frequencies between them are within 1% (refer to **Table 4.3.2a**). These errors are propably due to the boundary condition in EMA. In this Section, the numerical results from the finite element model will be applied to the real structure to verify the theoretical prediction.

For easy physical realisation, the concentrated mass will be selected as modification type. To relocate the first natural frequency of the grid structure from 44.41 Hz to 40.00 Hz, points (coordinate locations) 4, 6, 19, 21 were selected to apply point mass modification. This is realised by rigidly attaching 4 brass blocks of calculated weight to the corresponding points. The results are given in **Table 4.3.3a**.

To relocate the anti-resonance from 71.00 Hz to 66.00 Hz, which is the second resonance of the original system, using concentrated mass, the receptance FRF between point 8 and point 16 was examined (note that point 6 and point 8 actually represent the y coordinates of these two points). The resonances and anti-resonances of the FRF within frequency range of 35 Hz to 85 Hz were identified from both FEA and EMA. Since the EMA identifies anti-resonances by graphically reading from measured FRF data, the accuracy of anti-resonances will largely depend on the frequency resolution of the FRF data. Another

possible error source is the noise encountered in the measurement, which is particularly important due to the low response level in the vicinity of an anti-resonance. The first anti-

Table 4.3.3a - Modification Results of Relocating The 1-st Resonance of Grid Structure

EMA					
Original ω_1 (Hz)	Desired ω_1 (Hz)	Result ω_1 (Hz)	Desired $\Delta\omega_1$(Hz)	Result $\Delta\omega_1$(Hz)	Error (%)
44.41	40.00	39.93	4.41	4.48	1.58
FEA					
Original ω_1 (Hz)	Desired ω_1 (Hz)	Result ω_1 (Hz)	Desired $\Delta\omega_1$(Hz)	Result $\Delta\omega_1$(Hz)	Error (%)
44.73	40.00	40.00	4.73	4.73	.000
Node		Element Part. Ratio		Modification Results (KG)	
3		1		0.217	
4		1		0.217	
16		1		0.217	
17		1		0.217	

resonances from FEA and EMA were 72.5 Hz and 72 Hz respectively. Points 10, 11, 12 and 20 were selected for applying the modifications. These modifications were realised by rigidly attaching 4 brass blocks to the corresponding points. The results are given in **Table 4.3.3b**.

From **Table 4.3.3a** and **Table 4.3.3b**, it is noted that the proposed methods for relocating resonance and anti-resonance have been successfully applied to the grid structure. The analytical results (those from the finite element model) produced no errors. However, when analytical results are applied to the real structure, small errors appeared due to the original discrepancies between the analytical model and the real structure. As described in Chapter 2, these discrepancies are mainly from the modelling idealisation. **Figure 4.3.3a**

Table 4.3.3b - Modification Results of Relocating Anti-Resonance of FRF $\alpha(8,16)$ of Grid Structure

EMA					
Original ω_a (Hz)	Desired ω_a (Hz)	Result ω_a (Hz)	Desired $\Delta\omega_a$(Hz)	Result $\Delta\omega_1$(Hz)	Error (%)
71	66.65	65	5.57	6	0.5
FEA					
Original ω_1 (Hz)	Desired ω_1 (Hz)	Result ω_1 (Hz)	Desired $\Delta\omega_1$(Hz)	Result $\Delta\omega_1$(Hz)	Error (%)
72.49	66.45	66.45	5.57	5.57	0.00

Node	Element Part. Ratio	Modification Results (KG)
10	1	0.186
11	1	0.186
12	1	0.186
21	1	0.186

and **Figure 4.3.3b** show the experimental FRF curves of the original structure and modified structure which shows that the first natural frequency has been relocated as predicted from 44.41 Hz to 39.93 Hz. **Fig.4.3.3c** and **Figure 4.3.3d** show the experimental FRF curves of the original structure and the modified structure and shows that the first anti-resonance of $\alpha(8,16)$ has been relocated as predicted from 71 Hz to 65 Hz.

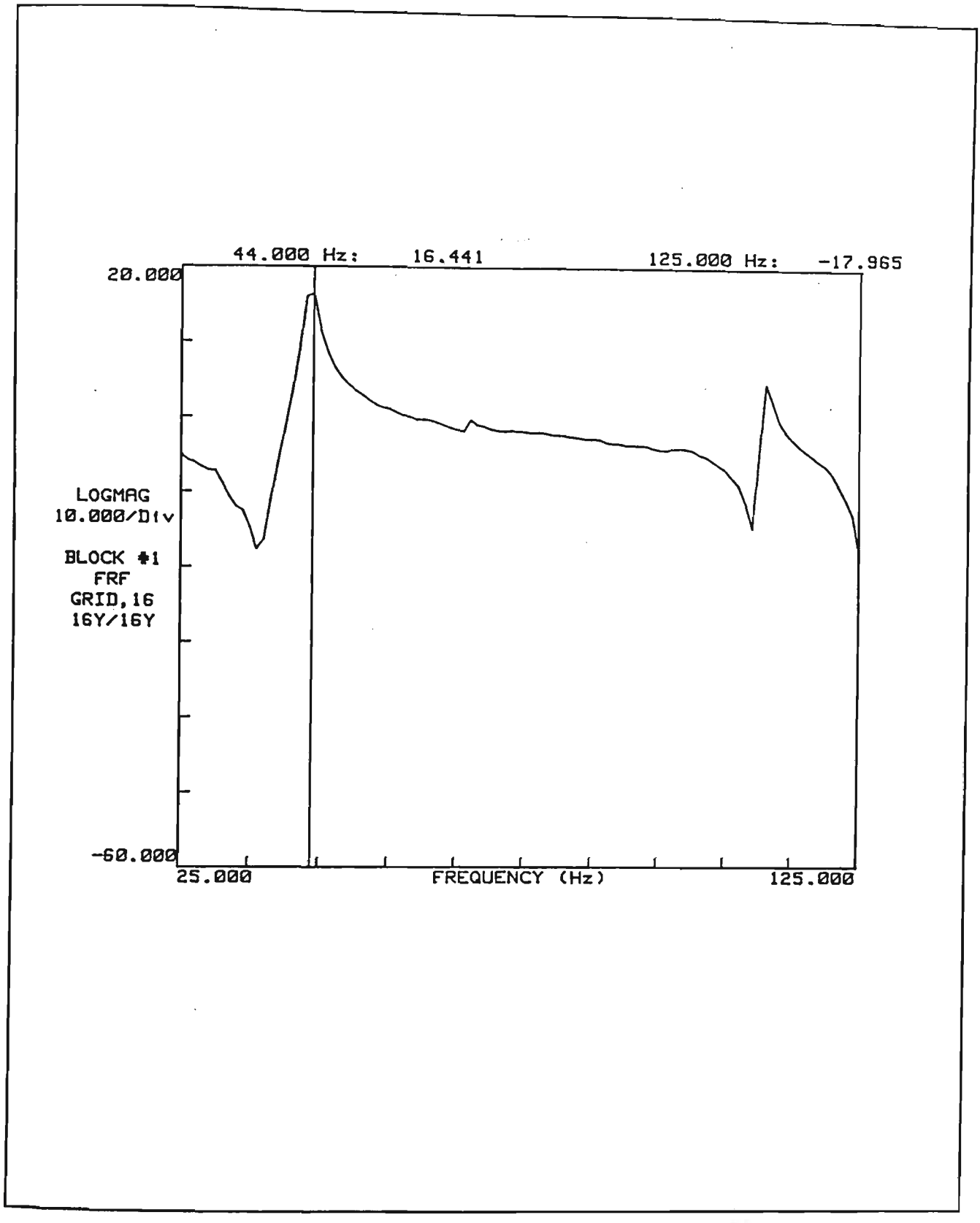


Figure 4.3.3a - FRF of Original Cross-Stiffened Grid Structure, $\alpha(16,16)$

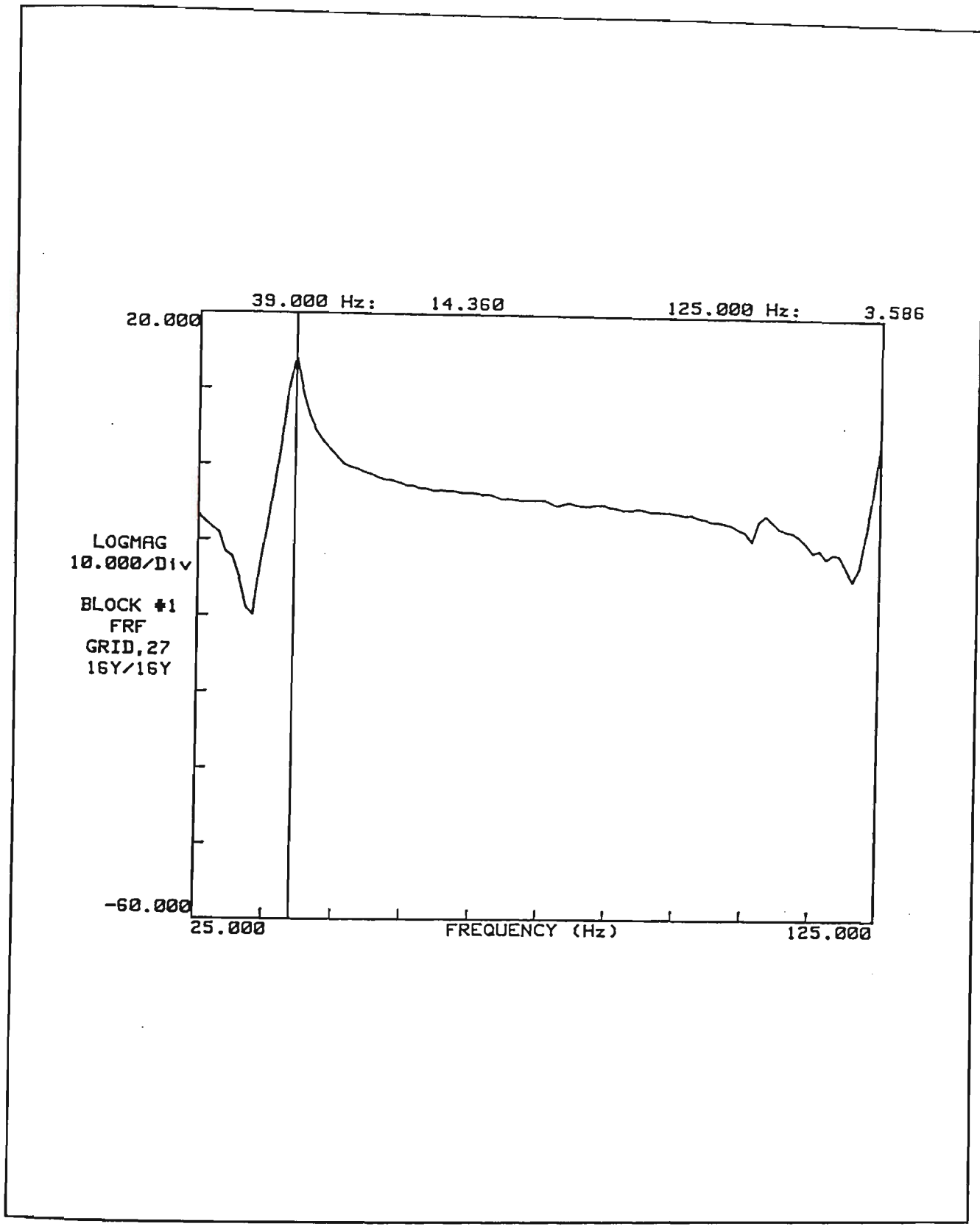


Figure 4.3.3b - FRF of Modified Cross-Stiffened Grid Structure, $\alpha(16,16)$

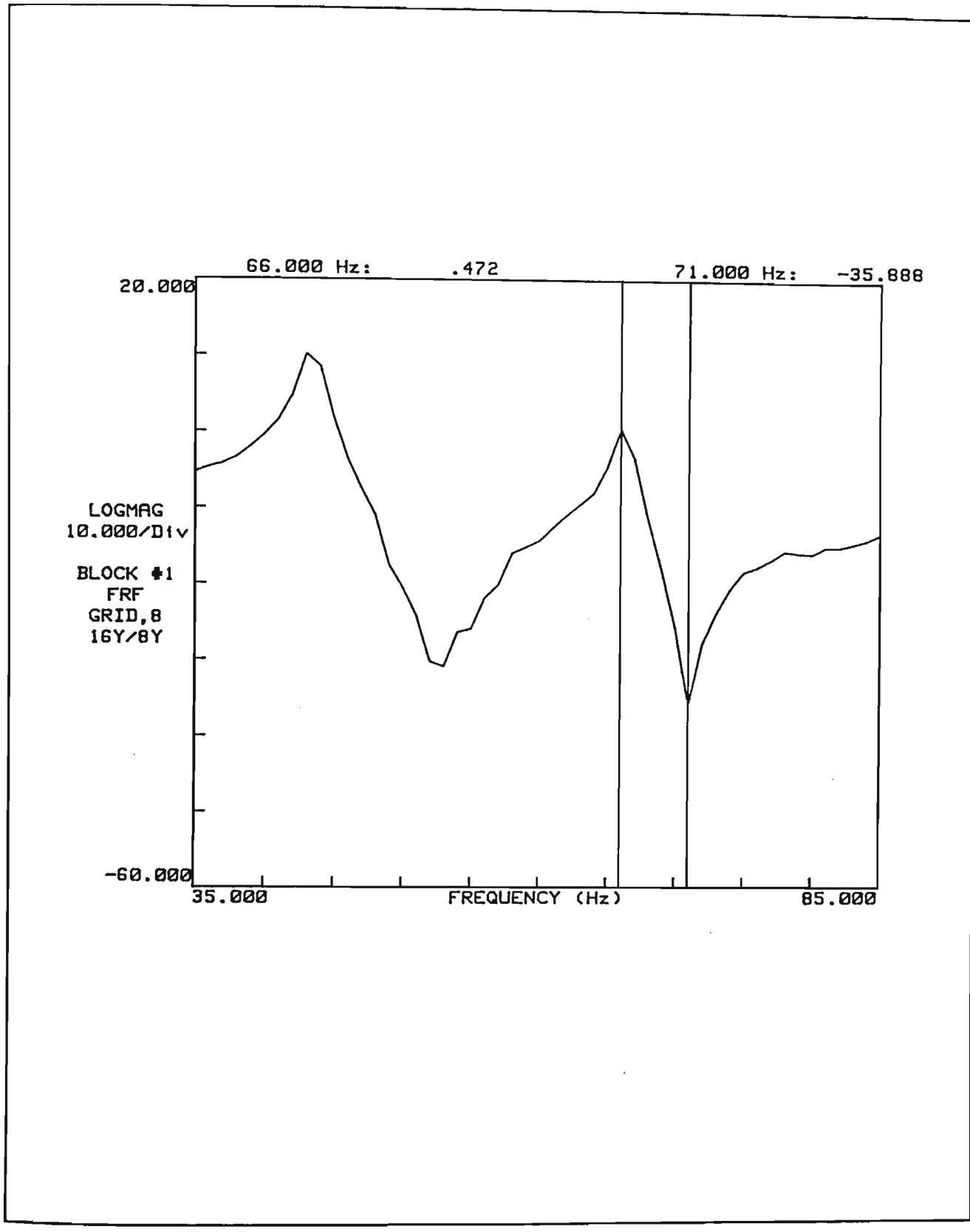


Figure 4.3.3c - FRF of Original Cross-Stiffened Grid Structure, $\alpha(8,16)$

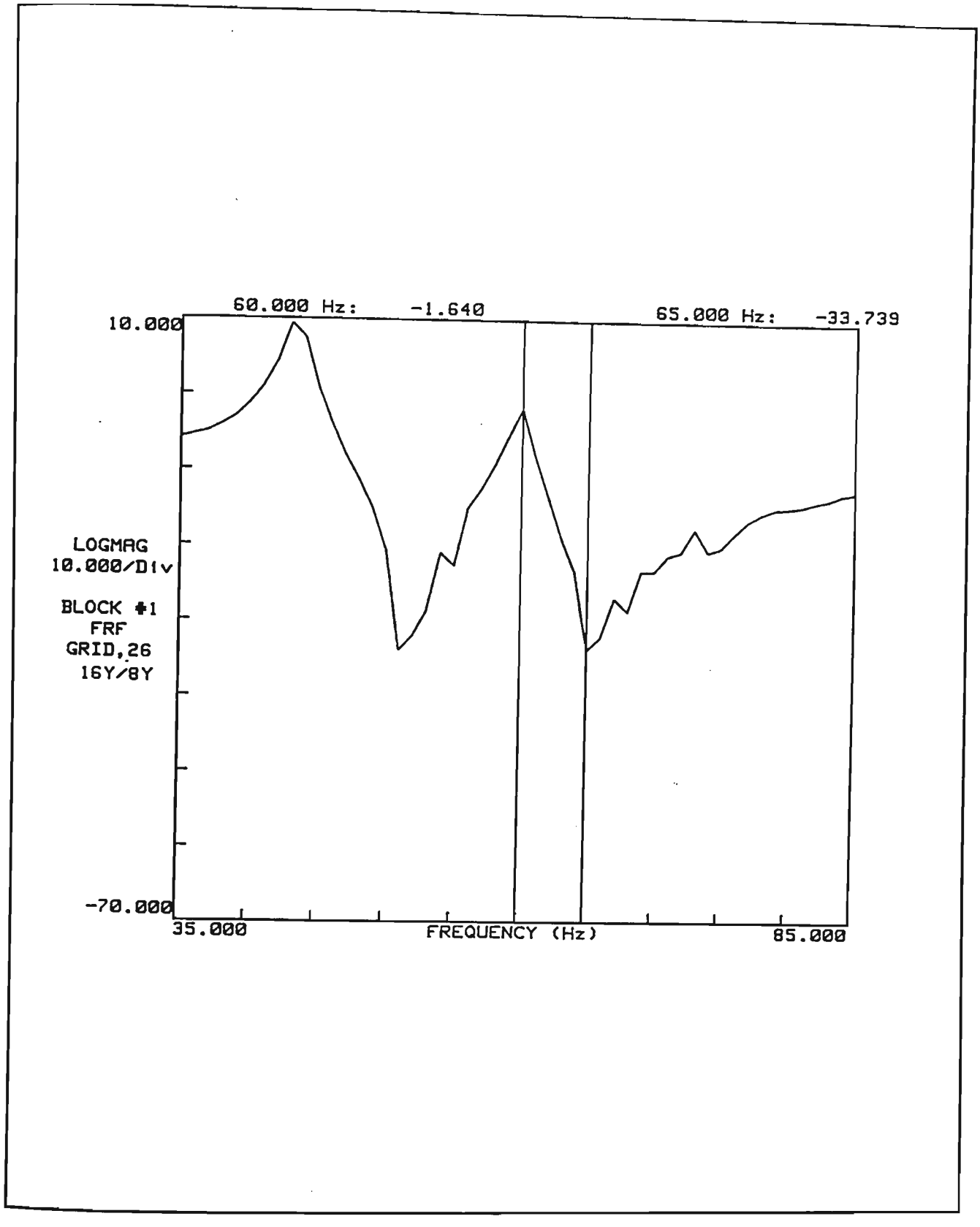


Figure 4.3.3d - FRF of Modified Cross-Stiffened Grid Structure, $\alpha(8,16)$

CHAPTER 5

STRUCTURAL DYNAMIC OPTIMISATION BY LOCAL STRUCTURAL MODIFICATION

The previous Chapters have developed and verified the procedures for relocating a resonance and an anti-resonance of a structural system by local structural modifications. The purpose of this Chapter is to introduce a method for dynamic optimisation by structural modification. For a linear vibratory system, its dynamic properties are usually characterised by its frequency response characteristics whose resonance and anti-resonance phenomena are of major concern. The objective of system dynamic optimisation entails avoiding the resonance response to a specified excitation within a frequency range of interest.

5.1 FRF and 'POLE-ZERO' Cancellation Theory

5.1.1 Expressions for FRF

(i) Characteristic Polynomial Expression of System Receptance FRF - For an undamped NDOF vibratory structural system, its receptance FRF matrix is expressed as,

$$|\alpha(\omega)| = \left[-\omega^2|M|+|K|\right]^{-1} \tag{5.1.1}$$

the individual term $|\alpha(\omega)|$ can be computed using the following equation,

$$\alpha_{ij}(\omega) = \frac{\det\left[-\omega^2 adj[M]_{ij}+adj[K]_{ij}\right]}{\det\left[-\omega^2[M]+[K]\right]} \tag{5.1.2}$$

Equation (5.1.2) can also be written as the ratio of two characteristic polynomials,

$$\alpha_{ij}(\omega) = \frac{a_0+a_1\omega^2+a_2\omega^4+ \ldots \ldots +a_{n-1}\omega^{2(n-1)}}{b_0+b_1\omega^2+b_2\omega^4+ \ldots \ldots +b_{n-1}\omega^{2(n-1)}+b_n\omega^{2n}} \qquad (5.1.3)$$

Since equations (5.1.2) and (5.1.3) are derived from the mass and stiffness matrices of the system, they will be available once the system is fully identified by FEM. As well, from equations (5.1.2), note that the system frequency response property can be determined in the physical domain in terms of the system mass and stiffness distributions. However, equation (5.1.3) appears just another mathematical expression of the receptance FRF. The $\alpha_{ij}(\omega)$ given in equation (5.1.3) offers no explicit physical significance.

(ii) Partial Fraction Expression of System Receptance FRF - The partial fraction expression of an FRF has been developed in Chapter 3 which is in terms of the system's modal properties,

$$\alpha_{ij}(\omega) = \sum_{r=1}^{n} \frac{{}_rA_{ij}}{\omega_r^2-\omega^2} = \sum_{r=1}^{n} \frac{\phi_{ir}\phi_{jr}}{\omega_r^2-\omega^2} \qquad (5.1.4)$$

Mathematically, equation (5.1.4) can be expanded from equation (5.1.3). The conditions on which this partial fraction expansion depends are that (i) the eigenvalues are distinct, and (ii) the order of ω in the numerator of equation (5.1.3) is less than the corresponding order of the denominator. These two conditions are met for most linear vibratory systems.

Equation (5.1.4) relates the system frequency response properties to the system modal properties. Using equation (5.1.4) ensures it will be easier to understand the nature of resonance response and how different modes contribute to the system response to certain excitation. Equation (5.1.4) also indicates that if the structure is driven by a single-point force, a mode which has a nodal coordinate at the excitation point can not be excited. This means that if ϕ_{jr} or ϕ_{ir} is equal or close to null, then the r-th term on the right hand side of equation (5.1.4), which represents the contribution of the r-th mode to the receptance, will be eliminated. The total frequency response of the system is usually dominated by the contribution of the mode whose natural frequency is closest to the excitation frequency. Therefore, by eradicating this mode, a low vibration level results.

(iii) POLE-ZERO Expression of System Receptance FRF - POLEs and ZEROs of the system transfer function originate from control theory. If the system matrix for the undamped structural system is obtained by Laplace transform, it may be expressed as a function of Laplace variable s,

$$|Z(s)| = \left[s^2|M| + |K|\right] \tag{5.1.5}$$

The transfer matrix |H(s)|, which is defined as the inverse of the system matrix [Z(s)], is constructed by the individual transfer function $H_{ij}(s)$ between input i and output j, where,

$$H_{ij}(s) = \frac{Laplace\ transform\ of\ output\ i}{Laplace\ transform\ of\ input\ j} \tag{5.1.6}$$

Thus,

$$H_{ij}(s) = \frac{\det\left[s^2 adj[M]_{ij} + adj[K]_{ij}\right]}{\det\left[s^2 adj[M] + adj[K]\right]} \tag{5.1.7}$$

The POLEs of the system are defined as those s which result in a null denominator in equation (5.1.7), whereas the ZEROs of the system are those s which lead to null numerator in equation (5.1.7).

For harmonically excited undamped linear vibratory systems, the Laplace variable s may be replaced by iω (ω is the frequency of excitation). Thus, equation (5.1.7) becomes,

$$H_{ij}(i\omega) = \alpha_{ij}(\omega) = \frac{\det\left[-\omega^2 adj[M]_{ij} + adj[K]_{ij}\right]}{\det\left[-\omega^2 adj|M| + adj[K]\right]} \tag{5.1.8}$$

Note that for such systems, the system transfer function and FRF are effectively the same. Each transfer function (or FRF) describes the dynamic properties of the structure between the input at a particular DOF and the response at another DOF and is a function of Laplace variable s (or frequency ω).

Factorising the numerator and denominator of equation (5.1.8) gives the singular points in the form of ZEROs of the FRF, that is, when $\alpha_{ij}(\Omega_p)=0$, and the poles of the FRF, occur where $\alpha_{ij}(\omega_q)=\infty$.

$$\alpha_{ij}(\omega) = \chi \frac{(\omega^2-\Omega_1^2)(\omega^2-\Omega_2^2) \ \ldots \ \ldots \ (\omega^2-\Omega_m^2)}{(\omega^2-\omega_1^2)(\omega^2-\omega_2^2) \ \ldots \ \ldots \ (\omega^2-\omega_n^2)} = \chi \frac{\prod_{p=1}^{m} (\omega^2-\Omega_p^2)}{\prod_{q=1}^{n} (\omega^2-\omega_q^2)} \qquad (5.1.9)$$

The square root of the ZEROs of the FRF are in fact the anti-resonance frequencies of the FRF, which are also the eigen-frequencies of the respective 'virtual' system. The square root of the POLEs are in fact the resonance frequencies of the system. For an undamped structural system, the system matrix is symmetrical and positive-definite. Therefore, the POLEs of the system will be real and positive. However, the ZEROs will be real and positive or complex conjugate pairs. Only those positive ZEROs correspond to the anti-resonances of the system.

5.1.2 Drive Point FRF and Transfer FRF

(i) Drive Point FRF - An FRF whose response and excitation coordinates are identical is defined as a 'drive point FRF'. Considering the definition of a 'virtual' system (see Chapter 3), it is noticed that the 'virtual' system corresponding to a drive point FRF is equivalent to a system which is obtained by constraining the 'question' coordinate of the original system. Such a system will always possess vibration modes whose number is one less than that of the original system. The resonance frequencies of such a system will be the anti-resonances of the FRF of the original system. In addition, all driving point FRFs have an anti-resonance between any two neighbouring resonances, as discussed from the partial fraction form of the FRF in Chapter 3.

(ii) Transfer FRF - An FRF whose response and excitation coordinates are different is defined as a transfer FRF. The general form of a transfer FRF depends on the 'Modal Constants'. The definition of 'Modal Constant' is given in Section 3.1. Since the 'mass' and 'stiffness' matrices of the 'virtual system' corresponding to a transfer FRF are obtained by deleting one row and one column of the mass and stiffness matrices of the original system, normally, the symmetry and positive definiteness or semi-positive definiteness properties associated with the original mass and stiffness matrices will be

destroyed. Therefore, the eigen-values of such a 'virtual' system will generally exist in the forms of positive, negative and complex conjugates. However, only positive eigen-values have a clear physical meaning, since their square roots represent the anti-resonance frequencies of the FRF. For an FRF without any anti-resonance, the corresponding 'virtual' system will only have negative and complex conjugate pair eigen-values. This type of transfer FRF is also called 'transmissibility FRF'[91].

5.1.3 'POLE-ZERO' Cancellation and Nodal Coordinate Creation

(i) 'POLE-ZERO' Cancellation - As discussed above, the square roots of the POLEs in an FRF are the resonance frequencies of the system, and the square roots of the ZEROs in the FRF are the anti-resonance frequencies. Considering the 'POLE-ZERO' form of an FRF given in equation (5.1.9),

$$\alpha_{ij}(\omega) = \chi\frac{(\omega^2-\Omega_1^2)(\omega^2-\Omega_2^2)\ \ldots\ \ldots\ (\omega^2-\Omega_m^2)}{(\omega^2-\omega_1^2)(\omega^2-\omega_2^2)\ \ldots\ \ldots\ (\omega^2-\omega_n^2)} = \chi\frac{\prod_{p=1}^{m}(\omega^2-\Omega_p^2)}{\prod_{q=1}^{n}(\omega^2-\omega_q^2)} \qquad (5.1.9)$$

Note that if one of the POLEs can be made to coincide with one of the ZEROs, the corresponding terms in the numerator and denominator of the equation will eliminate each other. This phenomenon is also known as 'POLE-ZERO' cancellation. For instance, if the r^{th} POLE equals the s^{th} ZERO, the equation will become,

$$\alpha_{ij}(\omega) = \chi\frac{\prod_{p=1,p\neq s}^{m}(\omega^2-\Omega_p^2)}{\prod_{q=1,q\neq r}^{n}(\omega^2-\omega_q^2)} \qquad (5.1.10)$$

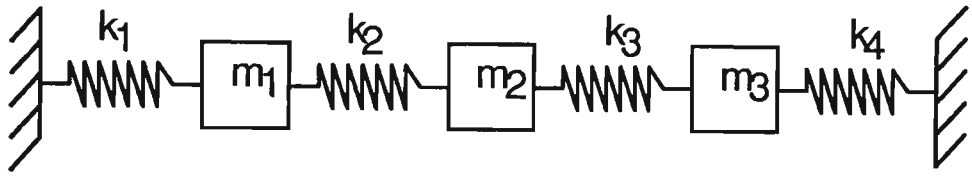
The 'POLE-ZERO' cancellation effect can be visualised by investigating the FRF curves of a 3-DOF system, see **Figure 5.1.2**. For drive point FRFs, for instance, $\alpha_{11}(\omega)$ and $\alpha_{22}(\omega)$, normally, their FRF curves will exhibit 3 resonance peaks and 2 anti-resonance troughs as shown in the upper graph for $\alpha_{11}(\omega)$. However, only 2 resonance peaks and one anti-resonance trough are seen from $\alpha_{22}(\omega)$ (lower graph). This is because the second

resonance of the system coincides with one of its anti-resonances. As a result, they eliminate each other. In addition, it is worth mentioning that, for the same system, different FRFs will have different anti-resonance frequencies. Therefore, the 'POLE-ZERO' cancellation will not occur in all FRFs simultaneously.

(ii) Nodal Coordinate Creation - It has been found that the 'POLE-ZERO' cancellation is always accompanied by the creation of a nodal coordinate. Comparing equations (5.1.4) with (5.1.9), if the r^{th} resonance peak is eliminated by 'POLE-ZERO' cancellation, it is equivalent to eliminating the r^{th} term in the partial fraction series of equation (5.1.4). This means the r^{th} modal constant of the corresponding FRF is equal to null. Considering that the r^{th} modal constant is defined as the product of two elements in the r^{th} mode shape which are relevant to the excitation and response coordinates, the null modal constant implies that either the excitation or the response coordinates is a nodal coordinate for that vibration mode. It is difficult to ascertain which coordinate will be the nodal coordinate for the transfer FRF, but provided that if the excitation or response coordinate is a nodal coordinate for a vibration mode, the resonance with respect of this vibration mode will not manifest itself. This is also in accordance with the fact that if the excitation force acts on the nodal coordinate of one mode, this vibration mode will not be excited, and the total response of the system will be determined by the contribution of the other vibration modes. However, if the 'POLE-ZERO' cancellation occurs in a drive point FRF, as the modal constant becomes the square of one element of the mode shape, the nodal coordinate can be readily determined.

5.2 Dynamic Optimisation By Local Structural Modification

The dynamic optimisation of a vibratory system is aimed at searching for feasible design configurations which satisfy a set of functional requirements, for the design that performs best given pre-defined criteria. A typical structural dynamic optimisation problem is to find a set of p variables, expressed mathematically as:



$m_1 = m_2 = m_3 = 10$ KG

$k_1 = k_2 = k_3 = k_4 = 50$ KN/M

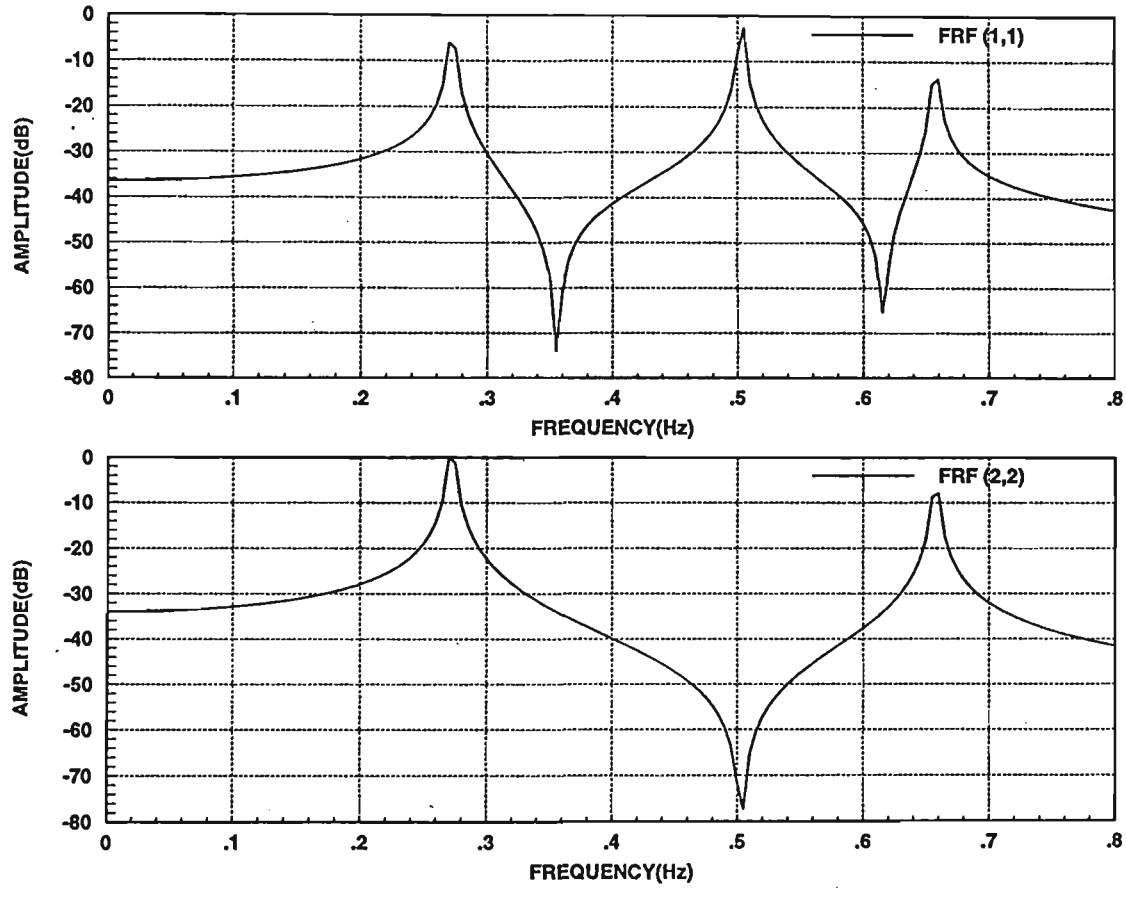


Figure 5.1.2 - 'POLE-ZERO' Cancellation

$$\{\delta\} = \{\delta_1\ \delta_2\ \delta_3\ \ldots\ \ldots\ \delta_p\} \tag{5.2.1}$$

which satisfy a set of q of constraints, such that

$$\begin{aligned} g_i(\delta) &\geq 0 \qquad i = 1,2,\ldots\ \ldots q_g \\ h_j(\delta) &= 0 \qquad j = 1,2,\ldots\ \ldots q_h \qquad\qquad q = q_g + q_h \end{aligned} \tag{5.2.2}$$

for a pre-defined criterion.

5.2.1 Expansion of Non-Resonance Frequency Range By Dynamic Optimisation

One of the methods used for averting the resonance is to relocate the natural frequencies of a system beyond the working frequency range. It is vital to modify the structure so as to effect the desired changes without at the same time introducing new unwanted effects. For example, an unwanted effect could be that a proposed modification intended to relocate a natural frequency beyond the working frequency range, introduces an adjacent natural frequency into the working frequency range. This emphasises the need to search for a structural modification by quantitative means, to preserve some of the system dynamic properties, such as resonance frequency or anti-resonance frequency, after the modification.

For a modified system, which is characterised by changing the magnitude and distribution of mass and stiffness of the original system, it should usually be expected that all of its natural frequencies (and, of course, their corresponding mode shapes) will be affected, even if different modes are affected to differing extents. It is difficult to predict how a modification aimed at changing a certain vibration mode will affect the neighbourhood modes, particularly if the structural modification involves both mass and stiffness. It is important to 'fix' a certain mode when the modification is carried out for other modes.

(i) Modification by a Semi-Definite System - As stated in Chapter 3, the equation of motion of the modified system, which is differs from the original system by the matrices $\Delta[M]$ and $\Delta[K]$, may be written as,

$$-\omega^{*2}([M]+\Delta[M])\{Y^*\}+([K]+\Delta[K])\{Y^*\} = \{0\} \tag{5.2.3}$$

where

$$\Delta[M] = \sum_{r=i,\ j,\ k} \delta m_r \{e_r\}\{e_r\}^T \tag{5.2.4a}$$

$$\Delta[K] = \sum_{p,\ q=i,\ j,\ k} \delta k_{pq} \{e_{pq}\}\{e_{pq}\}^T \tag{5.2.4b}$$

Without loss of generality, the coordinates which are involved in the modification are assumed to be the coordinates i, j, k. Thus, the mass modification of local coordinate r (r=i, j, k) is represented as δm_r, the stiffness modification between coordinate p (p=i, j, k) and coordinate q (q=i, j, k) is represented as δk_{pq}.

If the free vibration of the original system can be started off accurately by its u^{th} mode, say, $\{\psi\}_u$, the equation of motion of the system can be expressed as,

$$-\omega_u^2[M]\{\psi\}_u+[K]\{\psi\}_u = \{0\} \tag{5.2.5}$$

Mathematically, comparing equation (5.2.5) with equation (5.2.3), it may be observed that if the modified system also satisfies equation (5.2.5), then,

$$-\omega_u^2\Delta[M]\{\psi\}_u+\Delta[K]\{\psi\}_u = \{0\} \tag{5.2.6}$$

In other words, if equation (5.2.6) is satisfied, the u^{th} mode of the original system will be preserved after structural modification. However, other modes will be modified.

The above relationship can be easily proved mathematically, and the physical meaning is explained as follows:

Considering the sparsity of the mass and stiffness incremental matrices Δ[M] and Δ[K], equation (5.2.6) can be reduced to,

It is evident that equation (5.2.7) is also the equation of motion of a 3-DOF semi-definite

$$-\omega_u^2\begin{bmatrix}\delta m_i & 0 & 0\\ 0 & \delta m_j & 0\\ 0 & 0 & \delta m_k\end{bmatrix}\begin{Bmatrix}\psi_i\\ \psi_j\\ \psi_k\end{Bmatrix}_u+\begin{bmatrix}\delta k_{ij}+\delta k_{ik} & -\delta k_{ij} & -\delta k_{ik}\\ -\delta k_{ji} & \delta k_{ij}+\delta k_{jk} & -\delta k_{jk}\\ -\delta k_{ik} & -\delta k_{jk} & \delta k_{ik}+\delta k_{jk}\end{bmatrix}\begin{Bmatrix}\psi_i\\ \psi_j\\ \psi_k\end{Bmatrix}_u=\begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix} \qquad (5.2.7)$$

mass-spring system subjected to free undamped vibration whose mass and stiffness coefficients are $\delta m_{r,\,(r=i,j,k)}$ and $\delta k_{pq,\,(p,q=i,j,k)}$, respectively. Using the concept of mass and stiffness modification ratios (see Chapter 3) and sub-matrix method (see [36]), equation (5.2.7) can be re-written as,

$$(-\omega_u^2\lambda\sum_{r=i,j,k}\{e_r\}^T\eta_r\{e_r\}+\lambda\sum_{p,q=i,j,k}\{e_{pq}\}^T\kappa_{pq}\{e_{pq}\})\{\psi\}_{u,s}=\{0\} \qquad (5.2.8)$$

Equation (5.2.8) can also be recast as,

$$(-\omega_u^2\sum_{r=i,j,k}\{e_r\}^T\eta_r\{e_r\}+\sum_{p,q=i,j,k}\{e_{pq}\}^T\kappa_{pq}\{e_{pq}\})\{\psi\}_{u,s}=\{0\} \qquad (5.2.9)$$

if this initially at rest 3-DOF semi-definite system is attached to the original system, which is vibrating accurately in its u^{th} mode, on the corresponding coordinates determined by the s^{th} sub-vector of the u^{th} mode, it will be equivalent to starting off this semi-definite system in a mode identical to the s^{th} sub-vector. Therefore, the vibration of the combined system will still vibrate harmonically under the u^{th} mode of the original system. It can then be concluded that the u^{th} mode of the original system will not be affected if system is modified by attaching a semi-definite system which possesses a mode identical to a sub-set of the u^{th} mode of the original system. This can also explained using the pseudo force method (details can be obtained from Brandon *et al*[43]). The pseudo force method states that the free vibration of a modified structural system can be considered as a harmonically forced vibration of the original system whose the excitation force may be replaced by the variations of internal constrain force caused by the local structural modification. This can also be visualised from equation (3.2.4), where the mode shape and natural frequency of the modified structural system may be treated as the harmonic response subjected to the pseudo force with the natural frequency of the modified system. If the inertia part and potential part of the right hand side (which represents the pseudo force) of equation (3.2.4) balance with each other at one of the natural frequencies of

original system, the natural mode of the original system corresponding to this natural frequency will not be affected. Otherwise, the system must self-adjust to achieve a new balance which will be reflected in creating new natural modes.

It has to be pointed out that no definite physical significance is required for such a semi-definite system. The mass and stiffness coefficients of such a system may be negative, which implies subtraction of mass or/and stiffness from the original system. In other words, the semi-definite system is only an 'assumed' system.

(ii) Determination of Mass and Stiffness Modification Ratios - From equation (5.2.8), note that the semi-definite system mentioned above can be determined by adjusting the mass and stiffness modification ratios. For certain structural modifications which are expressed in terms of the mass and stiffness modification ratios and scaling factors, the dynamic optimisation problem may be defined as below:

To determined a set of variables that satisfy the objective function,

$$G(\lambda\ \eta_1\ \eta_2\ \ldots\ \ldots\ \eta_p\ \kappa_1\ \kappa_2\ \ldots\ \ldots\ \kappa_q) = 0 \tag{5.2.10}$$

subjected to the constraints,

$$g_i(\eta_1\ \eta_2\ \ldots\ \ldots\ \eta_p\ \kappa_1\ \kappa_2\ \ldots\ \ldots\ \kappa_q) = 0 \qquad i=1,2,\ldots\ \ldots p+q \tag{5.2.11}$$

where, the objective equation (5.2.10) is for relocating a natural frequency from ω_n to ω_n^*, as discussed at Chapter 3, the constraint equation (5.2.11) is for searching the modification ratios which can preserve a natural mode after structural modification. Re-writing equation (5.2.9), yields,

$$-\omega_u^2 \sum_{r=i,j,k} \eta_r \{e_r\}^T \{e_r\} \{\psi\}_{u,s} + \sum_{p,q=i,j,k} \kappa_{pq} \{e_{pq}\}^T \{e_{pq}\} \{\psi\}_{u,s} = \{0\} \tag{5.2.12}$$

Notice that ω_u and $\{\psi\}_u$ are the known eigen-pair of the original system which need to be preserved. To avoid a trivial solution, presetting η_i, equation (5.2.11) can be re-arranged to become,

$$-\omega_u^2 \sum_{r=j,k} \eta_r \{e_r\}^T \{e_r\} \{\psi\}_{u,s} + \sum_{p,q=i,j,k} \kappa_{pq} \{e_{pq}\}^T \{e_{pq}\} \{\psi\}_{u,s} = \omega_u^2 \eta_i \{e_i\}^T \{e_i\} \{\psi\}_{u,s} \tag{5.2.13}$$

Let,

$$-\omega_u^2 \{e_r\}^T \{e_r\} \{\psi\}_{u,s} = \{A\}_r \,, \qquad (r = i,j,k) \tag{5.2.14a}$$

and

$$\{e_{pq}\}^T \{e_{pq}\} \{\psi\}_{u,s} = \{B\}_{pq} \,, \qquad (p,q = i,j,k) \tag{5.2.14b}$$

define vectors

$$\{R\} = \{\eta_i \ \eta_j \ \eta_k \ \kappa_{ij} \ \kappa_{jk} \ \ldots \ \ldots \ \kappa_{pq}\}^T \tag{5.2.15a}$$

$$\{d\} = \omega_u^2 \eta_i \{e_i\}^T \{e_i\} \{\psi\}_{u,s} \tag{5.2.15b}$$

and matrix

$$[D] = [\{A_i\} \ \ldots \ \ldots \ \{A_k\} \ \{B_{ij}\} \ \ldots \ \ldots \ \{B_{kk}\}] \tag{5.2.16}$$

equation (5.2.13) can then be recast into,

$$[D]\{R\} = \{d\} \tag{5.2.17}$$

Since the total number of mass and stiffness modification ratios need to be identified from equation (5.2.17) is problem dependent, the coefficient matrix [D] is normally rectangular. This will make matrix |D| rank deficient. The singular value decomposition (SVD) method is employed to solve equation (5.2.17), which lead to solution,

$$\{R\} = [V][\Sigma]^{-1}[U]^T\{d\} \tag{5.2.18}$$

where

$$[D] = [U][\Sigma][V]^T \tag{5.2.19}$$

For an under-determined problem, that is, the number of unknowns in equation (5.2.17) is greater than the number of constrained equations, the solution of equation (5.2.18) is not unique. Using the SVD method, the minimum Euclidean norm solution is given. The under-determined problem is normally characterised by the fact that the row dimension of [D] is less than its column dimension. For an over determined problem, i.e., the number of unknowns in equation (5.2.17) is less than the number of constrained equations, no exact solutions are given. However, using the SVD method, the least square solution will be obtained. This type of problem is normally characterised by the notion that the row dimension of [D] is smaller than its column dimension.

(iii) Expanding the Non-Resonance Frequency Range - Once the mass and stiffness modification ratios have been identified, the optimisation procedure can be carried out according to the mass and stiffness modification ratio thus obtained. For a structural modification problem without involving the optimisation, the mass and stiffness modification ratios can be given arbitrarily, as discussed in Chapter 3. Here, the mass and stiffness modification ratios can be utilised as a set of optimum values which lead to certain dynamic properties. Substituting the mass and stiffness modification ratios into equation (3.2.6), and re-arranging, it yields:

$$[\alpha(\omega_r^*)](\omega_r^{*2}\sum_{r=i,j,k}\{e_r\}^T\eta_r\{e_r\}-\sum_{p,q=i,j,k}\{e_{pq}\}^T\kappa_{pq}\{e_{pq}\})\{\psi\}_r^* = \frac{1}{\lambda}\{\psi\}_r^* \qquad (5.2.20)$$

From equations (5.2.9) and (5.2.20), it is noticed that the u^{th} vibration mode (including the natural frequency and mode shape) which need to be preserved is independent to the variation of the scaling factor λ, since the λ obtained from equation (5.2.20) will not effect the u^{th} mode, as indicated by equation (5.2.8). If the r^{th} mode is adjacent to the u^{th} mode, using these modification ratios, the r^{th} resonance can be relocated to a desired location (subject to the practical considerations) without altering the u^{th} mode. This is of particular significance for creating a non-resonance region in a wide frequency range.

5.2.2 POLE-ZERO Cancellation

(i) Determination of Mass and Stiffness Modification Ratio For Unaltered Anti-

Resonance - In general, for an arbitrary structural modification, the relocation of resonances is usually accompanied by the change of anti-resonances. However, if the optimisation procedure using a semi-definite system can be applied to the 'virtual' system , one of its eigen frequencies (an anti-resonance frequency for the original system) is expected to be unaltered, the pole-zero cancellation can be achieved by relocating a resonance frequency to match this anti-resonance frequency.

Considering the 'virtual' system corresponding to receptance FRF $\alpha(\omega)_{uv}$, the mass and stiffness modification ratios can be optimised by the following equation,

$$-\omega_a^{*2}\sum_{r=j,k}\eta_r\{e_r\}_u\{e_r\}_v^T\{\psi\}_a^* + \sum_{p,q=i,j,k}\kappa_{pq}\{e_{pq}\}_u\{e_{pq}\}_v^T\{\psi\}_a^* = \omega_a^{*2}\eta_i\{e_i\}_u\{e_i\}_v^T\{\psi\}_a^* \tag{5.2.21}$$

By following procedure in Section (5.2), the mass and stiffness modification ratios can be determined.

(ii) Elimination of Resonance Peak - Once the mass and stiffness modification ratios are identified by equation (5.2.21), the resonance frequency adjacent to the preserved anti-resonance can be relocated to meet it by solving for the appropriate scaling factor λ. Substituting the mass and stiffness modification ratios and ω_a into equation (5.2.20) yields,

$$[\alpha(\omega_a)](\omega_a^2\sum_{r=i,j,k}\{e_r\}^T\eta_r\{e_r\} - \sum_{p,q=i,j,k}\{e_{pq}\}^T\kappa_{pq}\{e_{pq}\})\{\psi\}_r^* = \frac{1}{\lambda}\{\psi\}_r^* \tag{5.2.22}$$

By solving this generalised eigen equation, the modification which will lead to the 'POLE-ZERO' cancellation can be determined.

5.3 Numerical and Experimental Verification

In this Section, results of two numerical and one experimental studies will be reported to illustrate and verify the method developed in this Chapter. The first study is numerical and is carried out on a mass-spring system to check the validity of the methods. The second

study is numerical and is carried out on a truss system to illustrate the general procedure. The third study is experimental and is carried out on a grid structure by concentrated mass modification to demonstrate the applicability of the method for practical engineering problems.

5.3.1 A 7-DOF Mass-Spring System

A 7-DOF mass-spring system which appeared in Li *et al*[85] is considered in this Section, see **Figure 5.3.1a**. The receptance FRF between coordinates 5 and 6 (mass 5 and mass 6) was examined. **Table 5.3.1a** gives the first three resonances and anti-resonance. To eliminate the second resonance peak by relocating it to coincide with the first anti-resonance, the mass constants of coordinates 1, 2, 3 and relevant stiffness constants k_{12}, k_{23} are considered as modification objectives. The results of using the method in Section 5.2.2 are given in **Table 5.3.1b**. **Figure 5.3.1b** gives the comparison of the FRF curve of the original system and that of the modified system. It can be observed from the figure that the second resonance and the first anti-resonance have eliminated each other. The first three resonances and anti-resonances of the modified system are given in **Table 5.3.1c**. **Table 5.3.1d** shows that coordinate 5 of the modified system has been made a nodal coordinate for the seconde mode after 'POLE-ZERO' cancellation. The general procedure is summarised as follows,

(1) determine the coordinates of interest for excitation and response;
(2) identify the resonances and anti-resonances of the FRF selected in (1);
(3) determine the resonance and anti-resonance for cancellation;
(4) determine the mass and stiffness (element) modification ratios;
(5) determine modification amounts with the identified modification ratio from (4) by using methods developed in Section 3.2.1 and Section 3.2.2,
(6) determining whether the proposed modification is acceptable.

Table 5.3.1a - The 1-st Three Resonances and Anti-Resonances of FRF $\alpha(5,16)$ of Original Mass-Spring System

No.	Resonance (Hz)	Anti-Resonance (Hz)
1	0.26338	0.42935
2	0.43932	0.48059
3	0.50816	0.50329

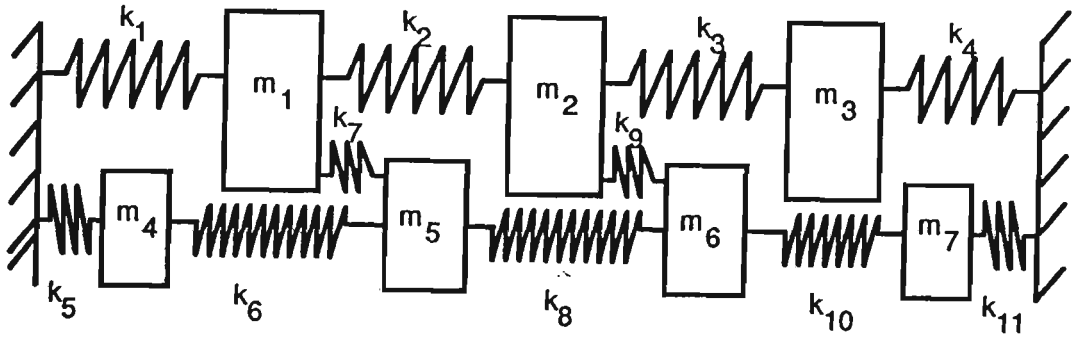
Table 5.3.1b - Dynamic Optimisation of the Mass-Spring System*

Location	Mass (Stiffness) Modification Ratio	Variations
m_1	1.0000 (pre-set)	4.047 KG
m_2	-0.4606	-1.864 KG
m_3	0.0000	0.000 KG
k_2	-6.2123	-25.14 NM^{-1}
k_3	0.0000	0.000 NM^{-1}

* The Minimum Euclidean Norm Solution

Table 5.3.1c - The 1-st Three Resonances and Anti-Resonances of Optimised Mass-Spring System

Mode	Resonance (Hz)	Anti-Resonance (Hz)
1	0.26175	***0.42935***
2	***0.42935***	0.50329
3	0.48325	0.55772



$m_1 = 20KG$ $m_2 = m_6 = 15KG$ $m_3 = m_5 = m_7 = 10KG$ $m_4 = 5KG$

$k_1 = k_3 = k_4 = k_6 = 100$ KN/M $k_2 = k_8 = 70$ KN/M

$k_5 = k_7 = k_9 = k_{10} = k_{11} = 50$ KN/M

Figure 5.3.1a - A 7-DOF Mass-Spring System

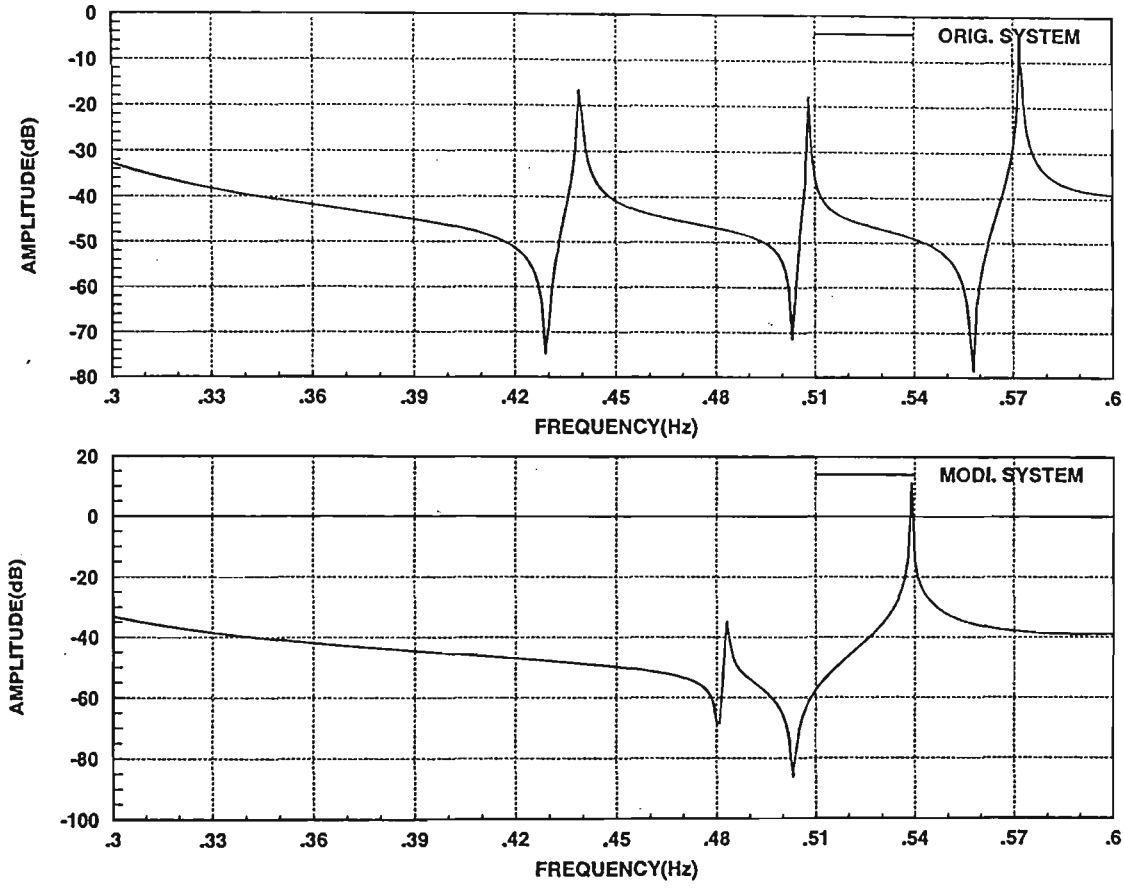


Figure 5.3.1b - FRF Comparison of the Mass-Spring System, $\alpha(5,6)$ (Upper: Original System, Lower: Modified System)

Table 5.3.1c - The 2-nd Mode Shape (Mass Normalised) Comparison of the Mass-Spring System

Coordinate (Mass)	Before Dynamic Optimisation	After Dynamic Optimisation
1	-0.10578	-0.13776
2	-0.11576	-0.06098
3	-0.09350	-0.04793
4	0.01700	0
5	0.01903	0
6	0.09036	0.09839
7	0.18979	0.18071

5.3.2 A Planar Truss System

The planar truss system which was used in Section 4.4.1 is considered for further illustrating the general procedure. According to the procedures stated in Section 5.3.1, the receptance FRF between X coordinate of node 6 and X coordinate of node 10 of the truss system was investigated and the first resonance was selected for elimination. The results are given in **Table 5.3.2a**, and the comparison of the original and modified FRF curves are given in **Figure 5.3.2a**. Since the X coordinate of node 4 had been created as a nodal coordinate, the resonance peak in receptance FRF between X coordinate of node 6 and Y coordinate of node 9 had been eliminated as well, as shown in **Figure 5.3.2b**.

5.3.3 Cross-Stiffened Grid Structure

The cross-stiffened grid structure which was given in Section 4.1.3 and Section 4.3 will be investigated to demonstrate how to achieve a 'POLE-ZERO' cancellation by applying the concentrated mass modification to a practical engineering structure. Since the mass matrix of the 'virtual' system corresponding to a certain FRF, say, the one related to coordinates i and j, is derived from deleting the i^{th} row and j^{th} column of the mass matrix of the original system, any concentrated mass modification applied on coordinates i and j will not influence the anti-resonance in the corresponding FRF. However, the resonances of the

Table 5.3.2a - Dynamic Optimisation of the 32-Element FE Truss System

No. Element	No. Node		Elemental Part. Ratio	Modified C.S.A (M^2)
8	4	5	-0.7200E-1	19.53E-4
9	4	6	-0.5311E+1	7.059E-4
10	4	7	+0.1746E+0	20.11E-4
11	5	6	+0.2069E+0	20.19E-4
12	5	7	+0.1145E+0	19.72E-4
13	6	7	+0.1827E-0	20.13E-4
14	6	8	-0.2944E-2	19.70E-4
15	6	9	-0.1274E+1	16.67E-4
17	7	9	+0.4664E-1	19.81E-4
18	8	9	-0.1927E+0	19.24E-4
31	6 (Con. Mass)		+0.7044E+1	32.69KG
32	8 (Con. Mass)		-1 (Pre-set)	27.33KG

original system will be affected. This is based on the assumption that the concentrated mass modification applied on the i^{th} and j^{th} coordinates offers no effects on the other coordinates. For the grid structure, if, say, only the translational DOF along Y axis are of concern, the assumption holds. Therefore, the principle described above can be used to relocate a natural frequency to coincide with an anti-resonance frequency in order to achieving a 'POLE-ZERO' cancellation.

The same structure and experimental method described in Section 4.3 were used and FRF between coordinate locations 16 and 19 was investigated. Using FEM and methods developed in Section 3.2, the mass modification was predicted. Thus, a 0.266KG brass block was rigidly attached to location 19 to create a nodal coordinate in the 2-nd mode at location 16, which means that a 'POLE-ZERO' cancellation has been achieved in the FRF relating locations 16 and 19. **Figure 5.3.3a**. **Figure 5.3.3b** gave the mode shapes of the original and modified grid structures. **Figure 5.3.3c** and **Figure 5.3.3d** show the FRF curves of the original and optimised structure. As well, the curves of receptance FRF

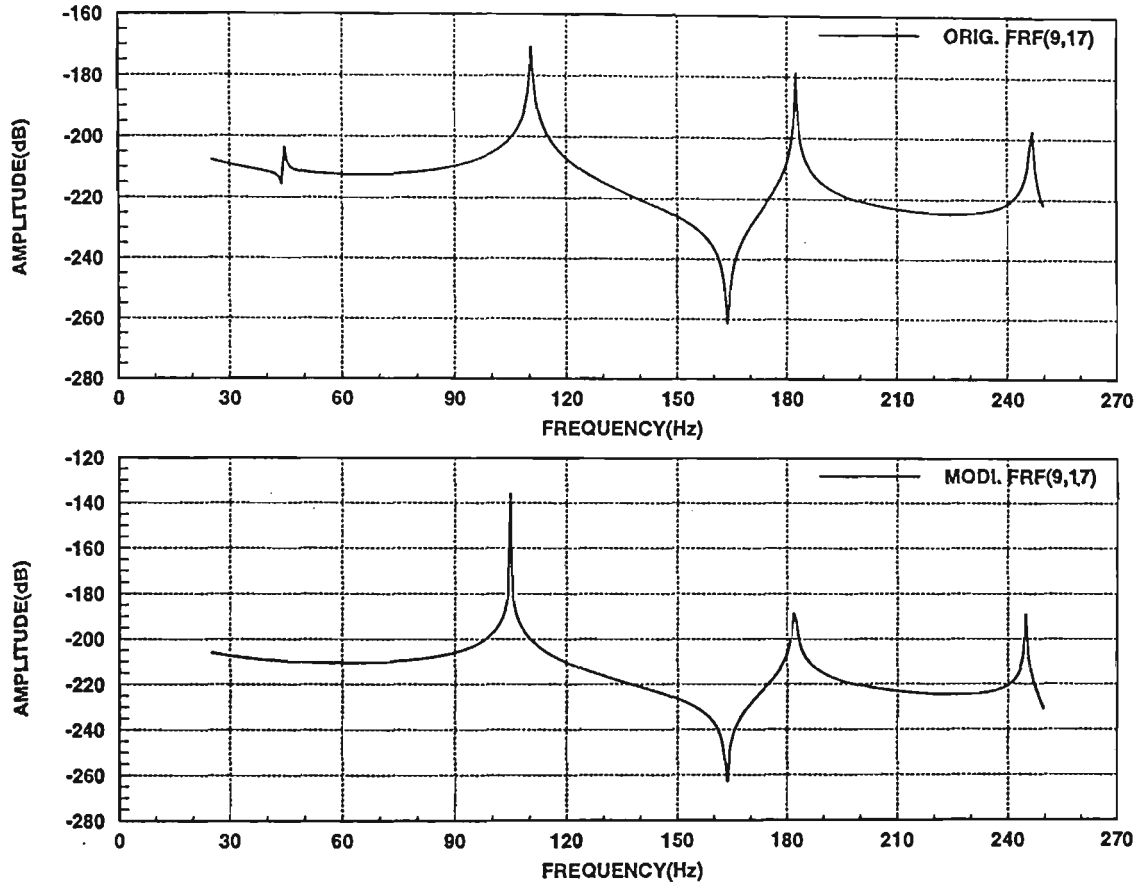


Figure 5.3.2a - FRF Comparison of Truss Structure $\alpha(9,17)$ (Upper: Original, Lower: Optimised)

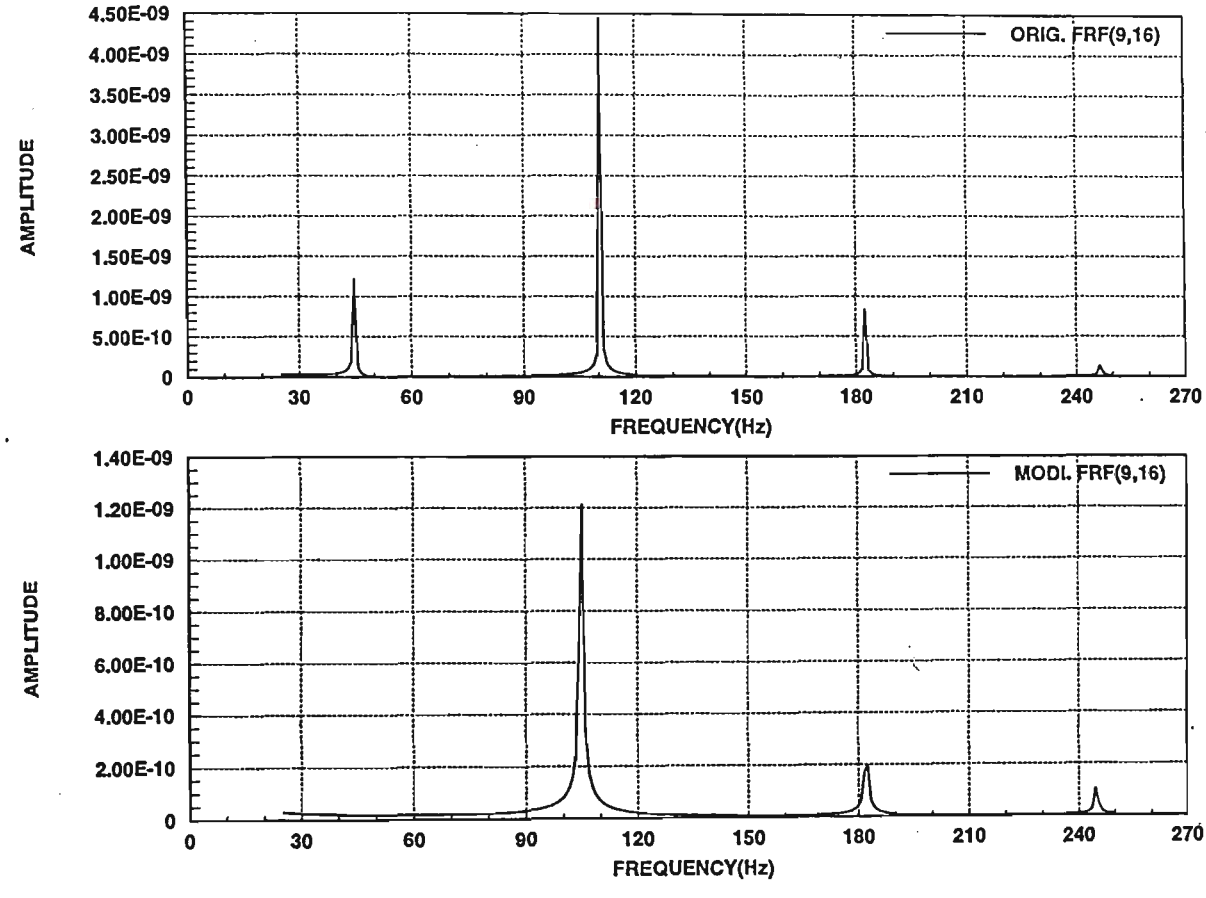


Figure 5.3.2b - FRF Comparison of The Truss Structure, $\alpha(9,16)$ (Upper: Original, Lower: Optimised)

relating coordinate locations 1 and 16 of the original and modified structures were given in **Figure 5.3.3e** to exhibit the effect of vibration response reduction by eliminating the 2-nd resonance after modification.

5.4 Discussions and Concluding Remarks

The numerical example used in Section 5.3 represents an under-determined problem, that is, the number of the unknowns is greater than the number of constraint equations. Therefore, the solution is not unique. Using the SVD method, the minimum Euclidean norm solution is given. It should be noted that even if the final solution is obtained, it may not be realisable, since the location and allowable modification is limited by practical considerations. As well, it should be noted that it is impossible to 'fix' a natural frequency and then to relocate an anti-resonance to achieve a 'POLE-ZERO' cancellation. This is because the realisation of 'POLE-ZERO' cancellations is always accompanied by the creation of a nodal coordinate, which implies a fundamental change (from nodal coordinate to nodal coordinate) in the mode shape. However, this change contradicts the condition that the resonance is 'fixed', and means the corresponding mode shape should not be changes after modification. In fact, it has been found that attempts to relocate an anti-resonance peak to eliminate a resonance will always result in an adjacent resonance being relocated to the very location where 'POLE-ZERO' cancellation occurs. Thus, the appearance of this new resonance overrides the achieved 'POLE-ZERO' cancellation. However, the proposed method can still be used to 'fix' a resonance and relocate an adjacent resonance as far as possible to create a wide frequency range without resonance.

In the event of 'over-determined' problem, one where, the number of structural parameters to be determined is less than that of constraint equations, the solution can only be obtained in the least square sense. This implies that a resonance will be relocated close to an anti-resonance. So far as structural dynamic optimisation is concerned, this can lead to a response in the vicinity of the relocated resonance being greatly reduced.

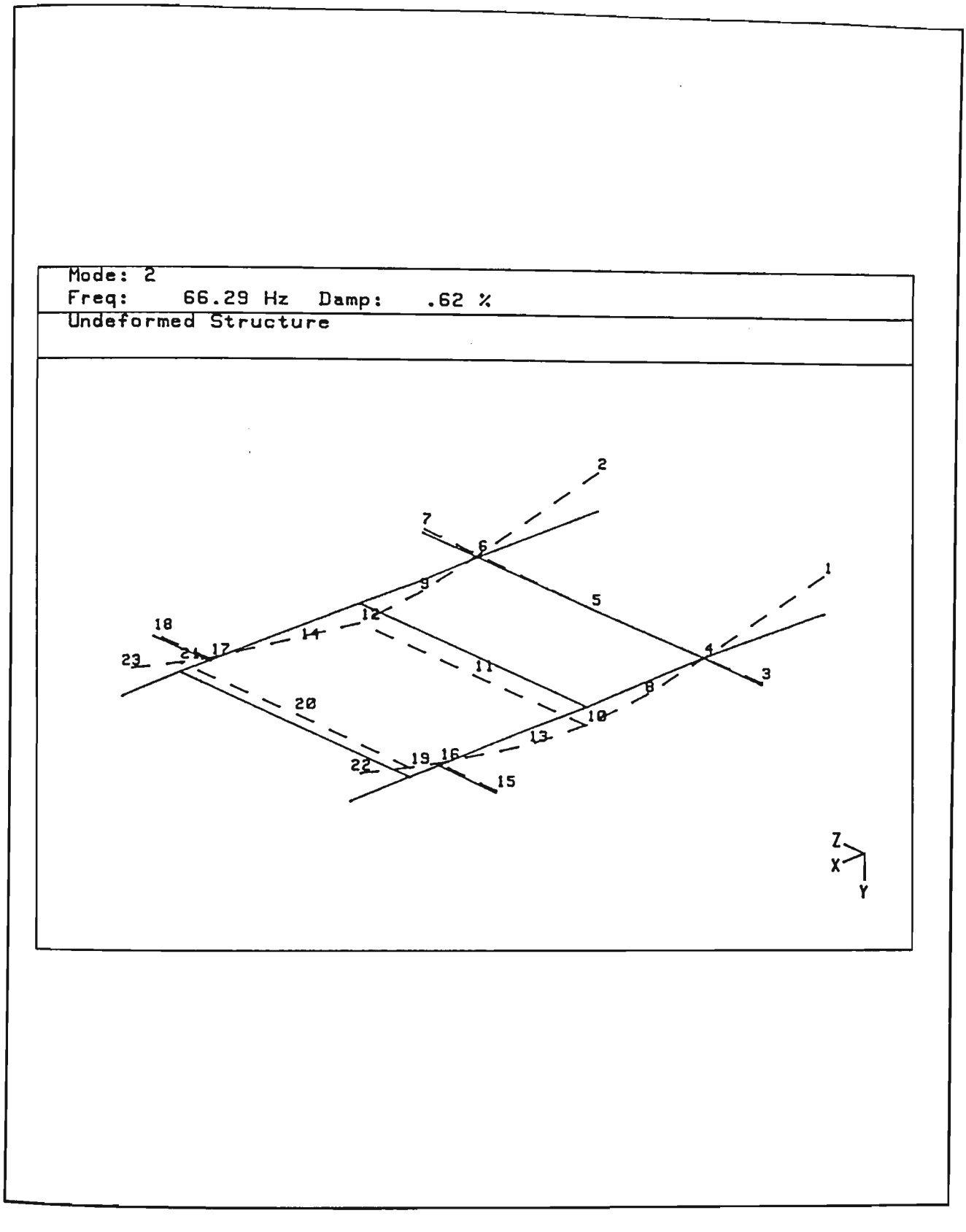


Figure 5.3.3a - The 2-nd Mode Shape of Original Cross-Stiffened Grid Structure (EMA)

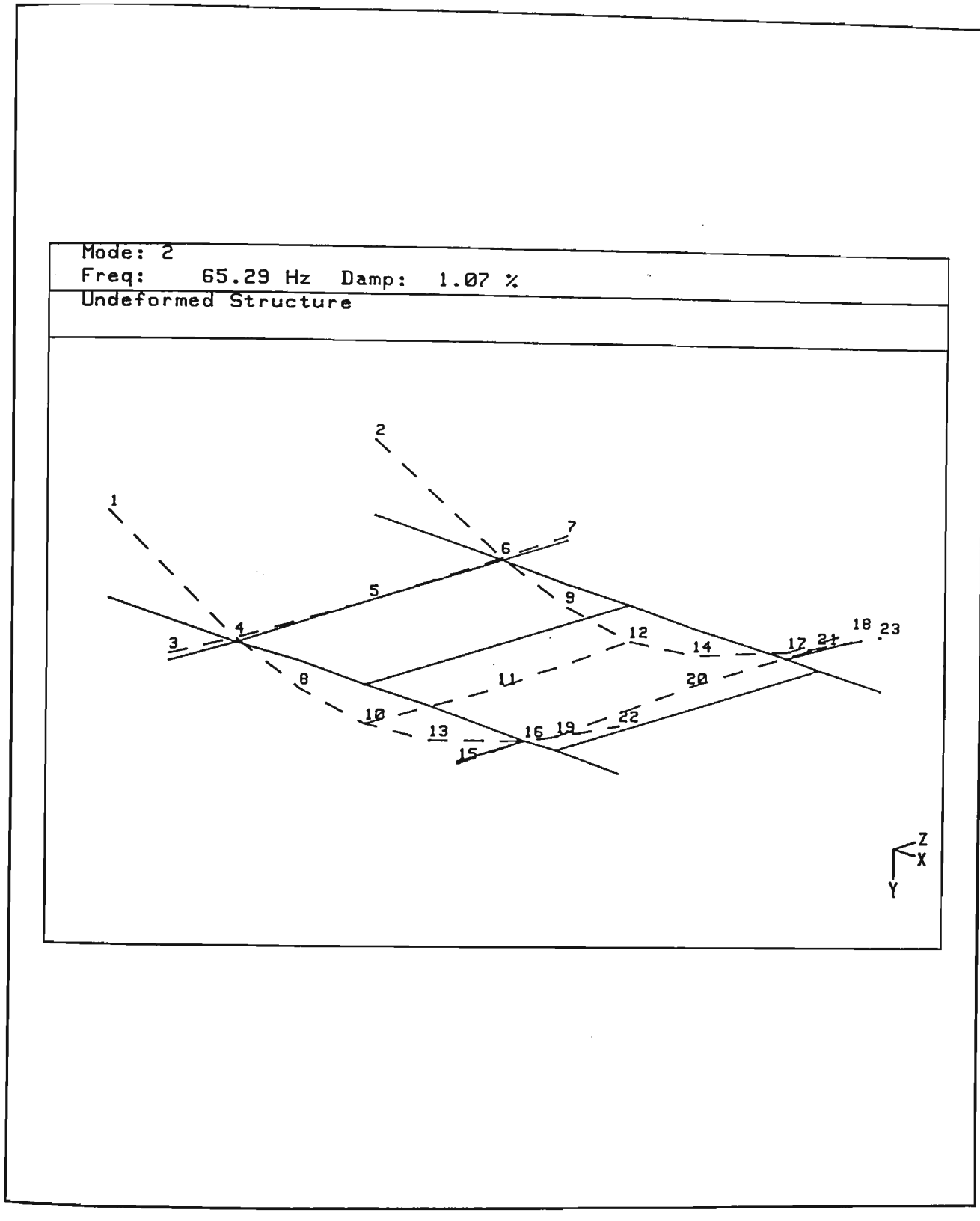


Figure 5.3.3b - The 2-nd Mode Shape of Optimised Cross-Stiffened Grid Structure (Location 16 had been made to be Nodal Coordinate)

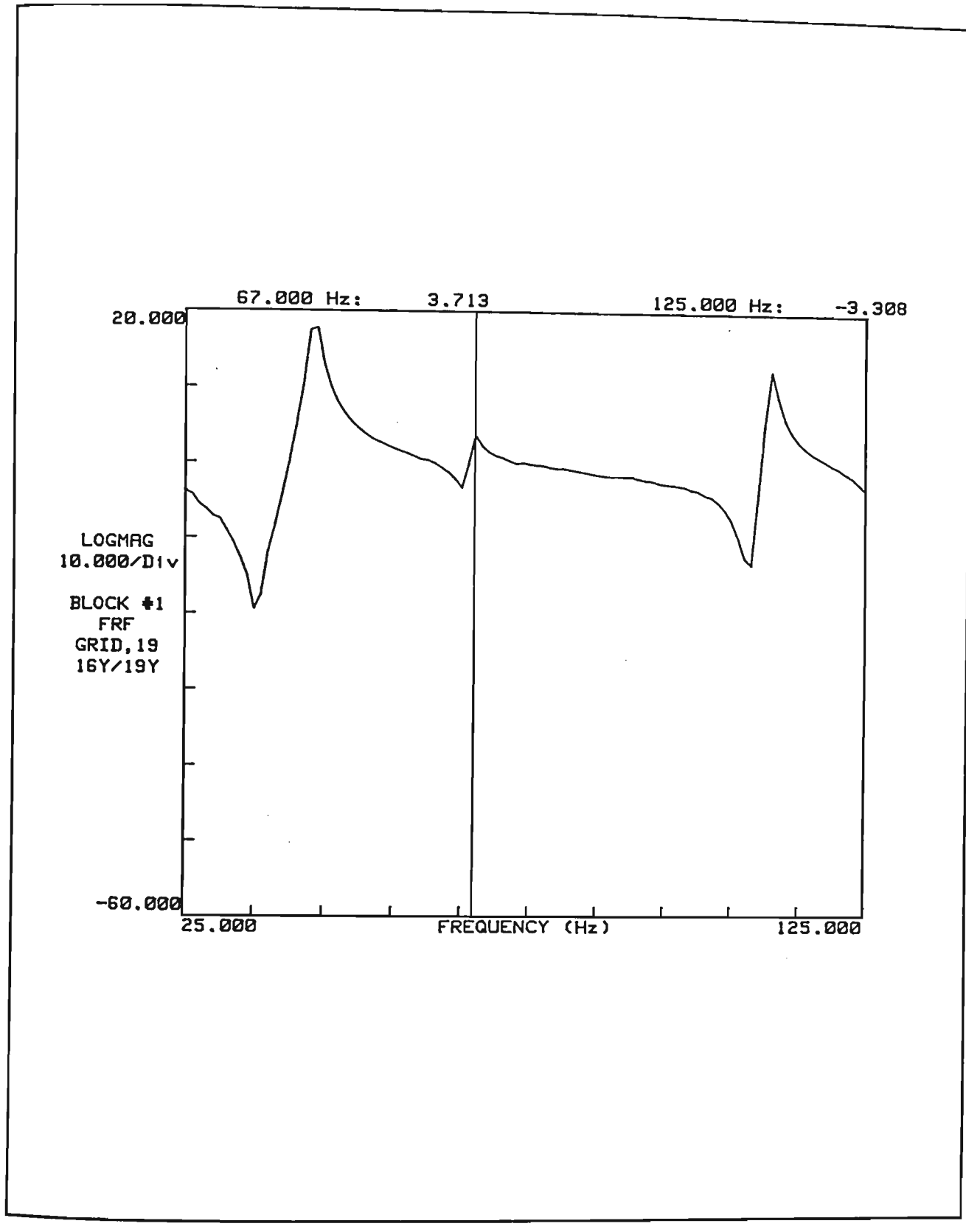


Figure 5.3.3c - FRF of Original Cross-Stiffened Grid Structure, $\alpha(16,19)$

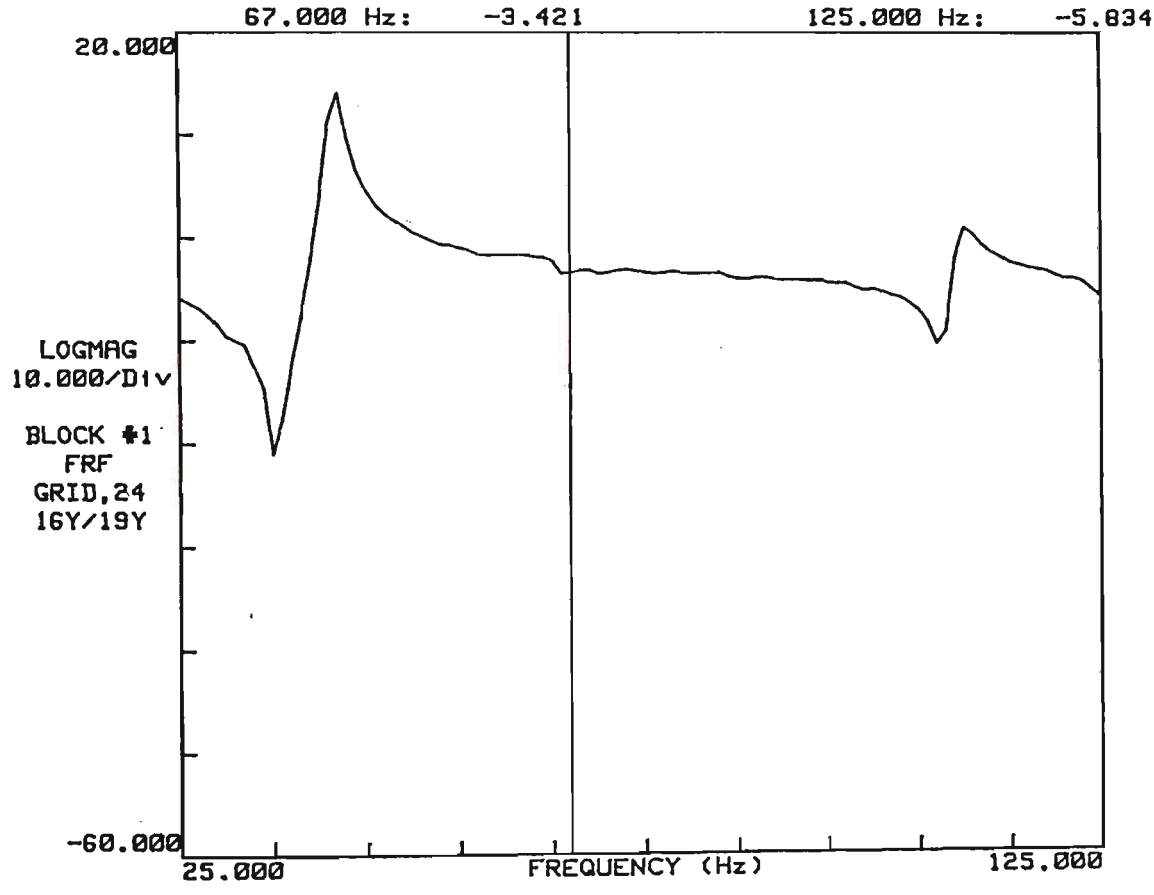


Figure 5.3.3d - FRF of Optimised Cross-Stiffened Grid Structure, $\alpha(16,19)$

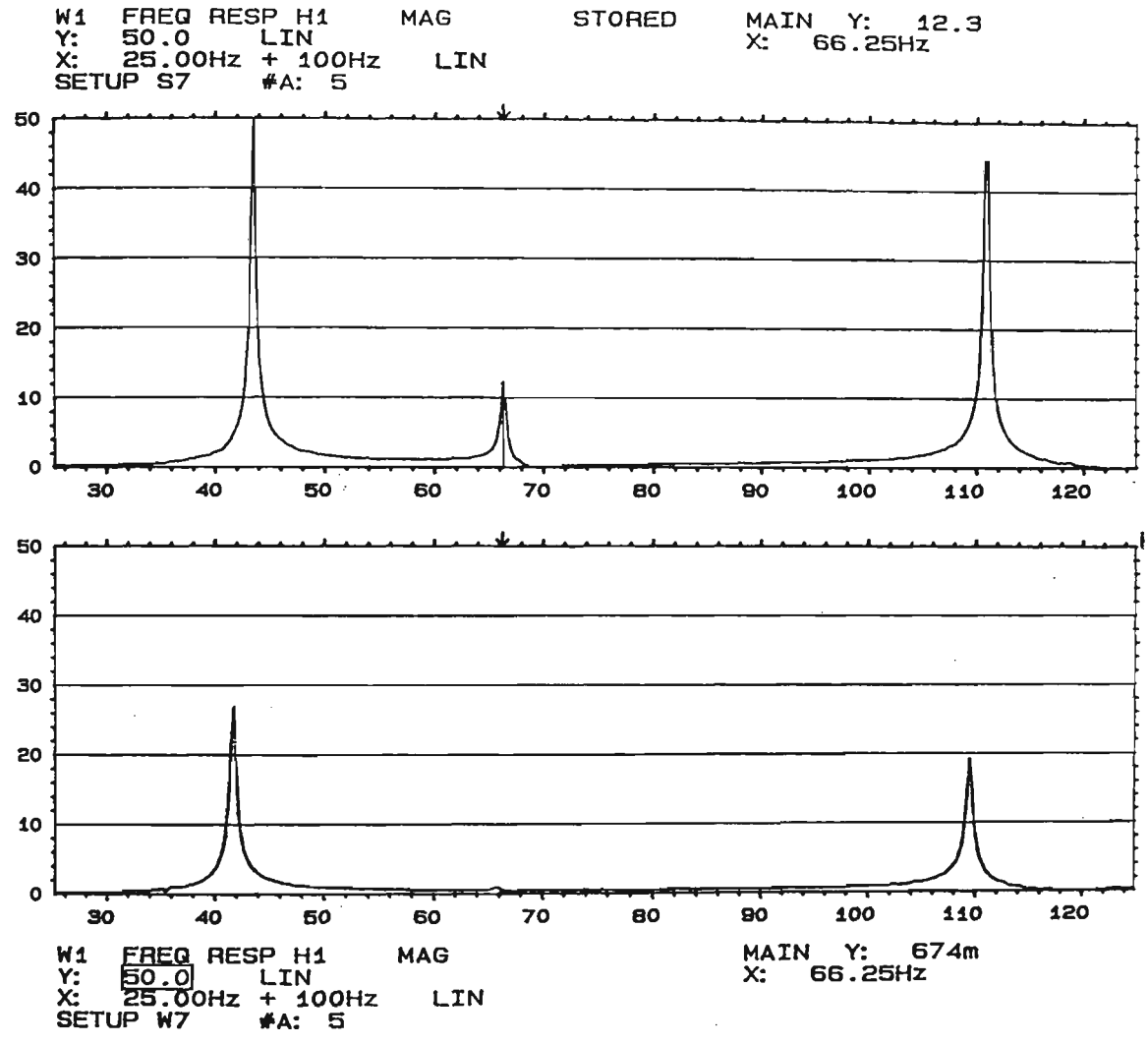


Figure 5.3.3e - FRF Comparison of Cross-Stiffened Grid Structure, $\alpha(1,16)$ (Upper: Original, Lower: Optimised)

CHAPTER 6

CONCLUSIONS AND FUTURE WORK

The goal of the research presented in this thesis was to investigate and develop effective methods and algorithms for vibration reduction by local structural modification. Significant contributions of this research include: the development of a unique method for vibration reduction; the initial investigation of a method for relocating anti-resonance and a method of structural dynamic optimisation by using the local structural modification technique for vibration reduction; development of an algorithm to solve a high order eigen problem. The purpose of this Chapter is to discuss the merits and limitations of the developed methods and algorithm and to provide suggestions for future research.

6.1 Discussions and Concluding Remarks

There are four important conclusions of this research which should be emphasised. First, the numerical and experimental examples indicated that the methods developed are capable of relocating an unwanted resonance to a desired location so that the vibration within a frequency range of interest can be greatly reduced. Since only those response properties related to the modification locations of the original system are required to determine the new response properties, the size of the problem can be greatly reduced by considering only the locations and DOFs selected for structural modification. Thus, the method is computationally efficient. As well, the element participant ratios (mass and stiffness modification ratios) provide multiple options for the application of this method and the accompany algorithm. By assigning the element participant ratios, users can decide to which extent a certain part of a structure should be modified to achieve desired dynamic

properties.

Secondly, a method has been developed to relocate the anti-resonance by applying the structural modification on a 'virtual' system which is defined according to the 'real' structural system. The numerical and experimental examples indicated that this method was able to relocate an anti-resonance to a desired location. Thus, it will lead to a significantly lower response in the vicinity of a frequency due to a given sinusoidal excitation. This is of considerable significance for a dynamic system whose regular input force is permanently located and whose vibration of a given location is of major concern, particularly when this location experiences a resonance response. Again, the method provides users with various possible solutions but the final decision may depend on practical considerations.

Thirdly, an algorithm was developed for the finite element implementation of the methods. The development of this algorithm enables the methods applied to the finite element model of a structural system with definite physical meanings by characterising the structural modification with locally altered structural physical parameters (thickness, width etc.). This algorithm is particularly useful when a finite element model constructed by higher order elements is dealt with thus making the implementation of the methods more straightforward and readily for practical application.

Compared with other methods, which (i) the modifications are represented by addition or subtraction of lumped masses while no stiffness variation is allowed in or/and of rib stiffeners while no mass changes are considered, or (ii) the modification is characterised such that the variations of the structural physical parameters are obtained from the perturbation or sensitivity analysis method, the primary merits of methods developed in this research can be summarised as follows:

(i) New methods have been developed in which structural vibration reduction can be analysed at the design stage by using local structural modification techniques.

(ii) Structural modification can be carried out locally to obtain desired resonance and anti-resonance features which can lead to a significant reduction of vibration level.

Since no iterative process is required, the procedure is computationally efficient.

(iii) Structural modification is realised with respect to the structural physical parameters directly and the parameter changes are limited only by practical considerations.

Finally, based on the above methods, a structural dynamic optimisation procedure has been designed for eliminating a resonance at some structural locations, by relocating the resonance to coincide with an anti-resonance and for expanding a no-resonance frequency range by 'fixing' a resonance and relocating its adjacent resonance. The effectiveness of this procedure has been validated by both numerical and experimental examples.

Since these methods are mainly based on the analytical models, the accuracy of the results from these methods will be influenced by the accuracy of the models adopted. In particular, care must be taken when dealing with a large, complex structure to ensure the most efficient use of the proposed methods.

6.2 Future Work

The research has provided methods for vibration reduction via local structural modification, and it also opens up some areas from which significant outcomes can be achieved by further investigation. Considering the limitations of the methods stated above, attention should first be given to the major cause(s) which induce inaccuracy in the proposed methods. It is recognised that major inaccuracy in the analytical model results from miss-modelling of the structural joints and boundary conditions, and experimental data are always considered accurate. If the structural modification and optimisation procedure can be based on the experimental data, and only those locations irrelevant to the joints and boundary conditions are utilised in conjunction with the experimental data, then the results obtained can be expected to be more reliable. The sub-structure synthesis and separation method and advanced data acquisition and analysis technology may be employed for this purpose. However, the availability of a adequate experimental modal data or response data of the original structure required by the structural modification procedure will still be a main barrier. A more efficient and accurate dynamic reduction

method becomes vital to make possible that the structural modification can be realised in a desired range of locations whose required data can easily be obtained experimentally.

Anti-resonance, as one of the important structural dynamics properties, should be further investigated. In particular, its effects on the vibration reduction should be explored further. To create an anti-resonance at desired locations becomes one of the targets of future research. As well, damping properties should be considered and an appropriate damping model may be the key to perfecting the technique.

For the structural dynamic optimisation problem, it has been noted that the linear programming and matrices derivative theories developed in recent years provide powerful tools for solving the multi-variate problems of linear models. Introducing these theories into the structural modification and dynamic optimisation techniques is also recommended for the future research.

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APPENDICES

APPENDIX I

THE BOOLEAN MAPPING MATRIX

The Boolean mapping matrix was used in **Chapter 3** to transform the local mass and stiffness matrices and incremental matrices to the global mass and stiffness matrices. A basic mathematical description will be given in this section along with a numerical example to explain how Boolean mapping matrix works for the transformation. For more detailed discussions readers can refer to [88].

As known, to construct a finite element model, a structure should have been discretised with a finite number of elements and these elements must satisfy the compatibility conditions, which implies that the elements can fit continuously together to form the complete model.

Provided that all elements have been disconnected from the original model, two different types of nodal point (at where the elements joint to each other) will be defined, namely, the global nodal points labelled $\{x^i\}_{i=1}^{G}$ and local nodal points labelled $\{x_e^N\}_{N=1}^{Ne}$, e=1,2,...,E. Since the compatibility conditions are satisfied, there must exist a simple correspondence between the label $\{x_e^N\}$ used to count local nodal points and label$\{x^i\}$ used to count global nodal points. When such a correspondence exists, it is said that node x_e^N of an element is coincident with node x^i of the connected model.

Mathematically, a mapping $|\beta|$ of local labels into global one is defined to described this correspondence as,

$$|\beta_e| : \{x_e^N\}_{N=1}^{N_e} \rightarrow \{x^i\}_{i=1}^{G} \qquad \text{(A1.1)}$$

Equivalently, since the correspondence is linear, it can be affected by the transformation,

$$x^i = \sum_{N=1} N_e |\beta_e|_N^i x_e^N \qquad e \text{ fixed} \tag{A1.2}$$

where $\beta\{e\}_N^i$ is an array of binary numbers(1 and 0) defined by,

$$|\beta_e|_N^i = \begin{cases} 1 & \textit{if node N of element e is coincident} \\ & \textit{with node i of connected model} \\ 0 & \textit{if otherwise} \end{cases} \tag{A1.3}$$

For each element e, the numbers $\beta\{e\}_N^i$ constitute elements of a rectangular (N_e by G) matrix $\beta\{e\}$ which is called a Boolean matrix.

Considering a one dimensional finite element mass-spring model shown in **Figure A1.1** which is constructed by three finite elements, the formulation of a Boolean matrix and transformation procedure will be given as follows:

The node numbering scheme indicated inside each element describes the local nodes, whereas that indicated externally labels the global nodes. Here G =6, N_1 =4, N_2 =3, and N_3 =4. From observing that,

$$\begin{bmatrix} x_{(1)}^1 \\ x_{(1)}^2 \\ x_{(1)}^3 \\ x_{(1)}^4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \end{bmatrix} = \begin{bmatrix} x^6 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{A1.4}$$

Hence, the Boolean matrix for element 1 is given by,

$$|\beta_1| = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{A1.5}$$

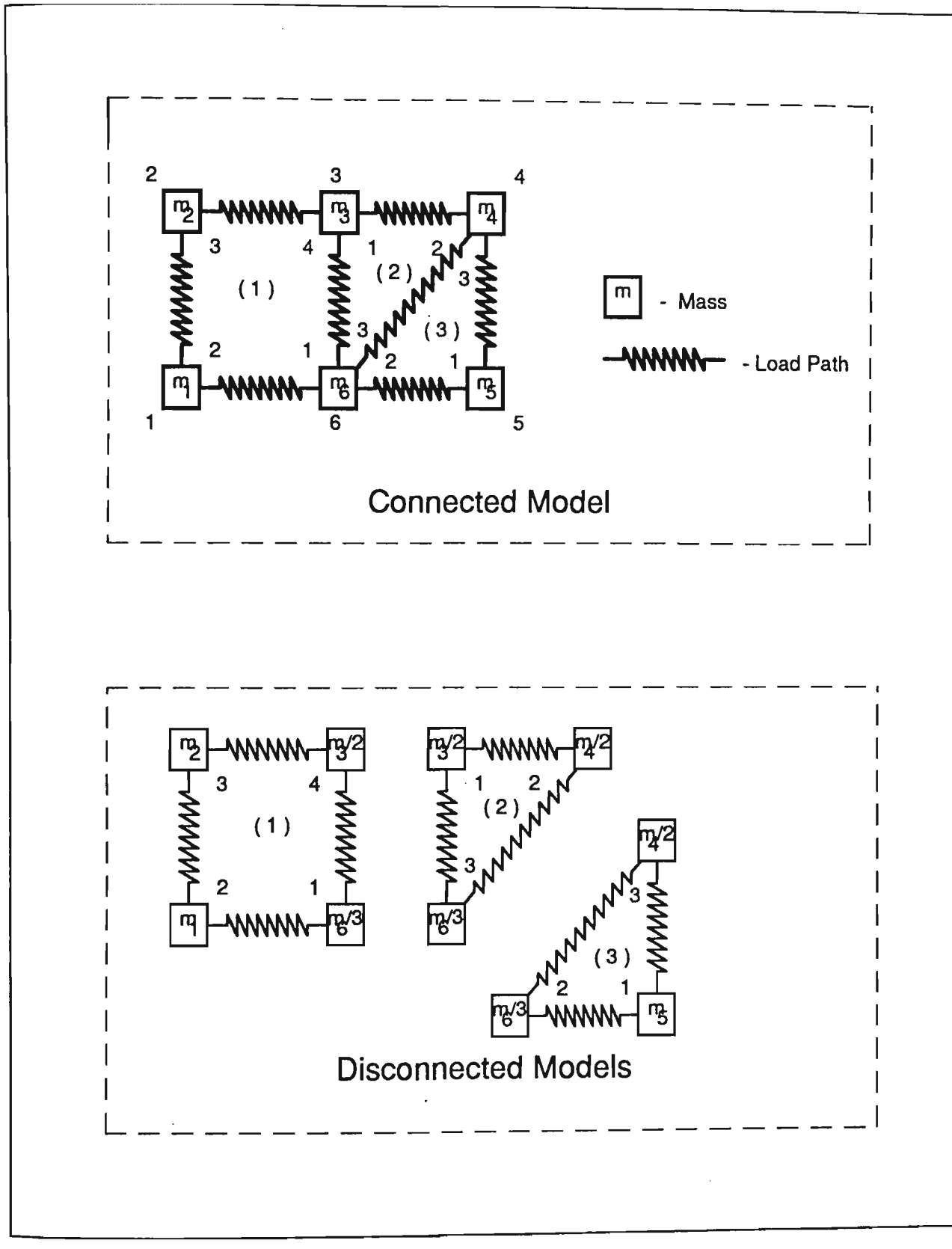

Figure A1.1 - 3-Element Mass-Spring System

Likewise,

$$[\beta_2] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{A1.6}$$

$$[\beta_3] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{A1.7}$$

Thus, using Boolean mapping matrices, the local mass matrices, which are

$$[M_1^{(e)}]_g = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & \frac{m_3}{2} & 0 \\ 0 & 0 & 0 & \frac{m_6}{3} \end{bmatrix} \tag{A1.8}$$

$$[M_2^{(e)}]_g = \begin{bmatrix} \frac{m_3}{2} & 0 & 0 \\ 0 & \frac{m_4}{2} & 0 \\ 0 & 0 & \frac{m_6}{3} \end{bmatrix} \tag{A1.9}$$

$$[M_3^{(e)}]_g = \begin{bmatrix} \frac{m_4}{2} & 0 & 0 \\ 0 & m_5 & 0 \\ 0 & 0 & \frac{m_6}{3} \end{bmatrix} \tag{A1.10}$$

will be transformed into global matrices as,

$$[M_1^{(g)}]_g = [\beta_1]^T[M_1^{(e)}]_g[\beta_1] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3/2 & 0 \\ 0 & 0 & 0 & m_6/2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_3/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_6/2 \end{bmatrix} \qquad \text{(A1.11)}$$

and

$$[M_2^{(g)}]_g = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_3/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_4/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_6/3 \end{bmatrix} \qquad \text{(A1.12)}$$

$$[M_3^{(g)}]_g = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_4/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_6/3 \end{bmatrix} \qquad \text{(A2.13)}$$

The global mass matrix for the connected model thus become,

$$[M] = \sum_{i=1}^{3} |\beta_i|^T [m_i^{(e)}] |\beta_i| = \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_6 \end{bmatrix} \quad \text{(A1.14)}$$

Following the same procedure, the global stiffness matrix can be obtained.

APPENDIX II

THE SINGULAR VALUE DECOMPOSITION

The singular value decomposition (SVD) technique is a powerful mathematical tool in dealing with problems characterised by a set of linear equations which is over-determined or under-determined. It is beyond the scope of this thesis to present a detailed explanation with a rigorous mathematical discussion of this technique. However, it is necessary to outline a simple description to highlight its specific applications in solving a set of over-determined or under-determined linear algebra equations which is encountered in the structural dynamic optimisation procedures as discussed in Chapter 5.

The focus here is the case where the coefficient matrix of a set of linear algebra equations is real and either the number of its row is greater than that of its column or otherwise. A Fortran subroutine can be found in [89], and a full review of this technique can be referred to [90].

For the over-determined problems defined by Equation (5.2.17), which implies more equations than unknowns, the singular value decomposition of the coefficient matrix can be expressed as,

$$[D]_{m\times n} = [U]_{m\times n}[\Sigma]_{n\times n}[V]^T_{n\times n} \qquad (m>n) \tag{A2.1}$$

where

$[U]_{m\times n}$ orthonormal matrix with its columns called left singular matrix

$[V]_{n\times n}$ orthonormal matrix with its columns called right singular matrix

$[\Sigma]_{n\times n}$ a matrix which contains singular values of [A], and singular values are ordered so that : $\sigma_1 \geq \sigma_2 \geq \ldots \ldots \geq \sigma_n \geq 0$ and

$$[\Sigma]_{n\times n} = \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_n \end{bmatrix}_{n\times n} \tag{A2.2}$$

Since $[U]_{m\times n}$ and $[V]_{n\times n}$ are orthonormal matrices, they satisfy

$$[U]^T_{n\times m}[U]_{m\times n} = [V]_{n\times n}[V]^T_{n\times n} = [\ I\ ]_{n\times n} \tag{A2.3}$$

The singular values are the non-negative square-roots of the eigenvalues of the matrix $[A]^T[A]$. Because $[A]^T[A]$ is symmetric and non-negative definite, its eigenvalues are always real and non-negative and therefore, singular values are always real and non-negative. The left and right singular matrices are the corresponding eigenvectors matrices of $[A]^T[A]$ and $[A][A]^T$. Therefore, the least square solution of equation (5.2.17) is given as,

$$\{R\}_{n\times 1} = [V]_{n\times n}[\Sigma]^{-1}_{n\times n}[U]^T_{n\times m}\{d\}_{m\times 1} \tag{A2.4}$$

where

$$[\Sigma]^{-1}_{n\times n} = \begin{bmatrix} \sigma_1^{-1} & & & 0 \\ & \sigma_2^{-1} & & \\ & & \ddots & \\ 0 & & & \sigma_n^{-1} \end{bmatrix}_{n\times n} \tag{A2.5}$$

In general, the diagonal matrix consisted of the singular values of matrix [D] will not be singular, and no $\sigma_{i\ (i=1,2,\ldots,n)}$ need to be set to zero. Occasionally, there might be column degeneracies in [D]. In this case, some unusually small singular values need to be set to zero and excluded from matrix $[\Sigma]^{-1}$ to prevent matrix $[\Sigma]^{-1}$ from becoming ill-conditioned

(noting that the corresponding column vectors in [V] and [U] should be excluded as well). The {R} vector thus obtained will give minimum residual r where,

$$r \equiv |[D]\{R\}-\{d\}| \tag{A2.6}$$

For the over-determined problems defined by Equation (5.2.17), which implies fewer equations than unknowns, no unique solution will be obtained. However, using SVD method, basis vectors which span the solution space will be given. Augmenting [D] matrix with zeros to be a square matrix and applying SVD to the augmented [D] matrix, following the same procedure stated above, a particular solution will be given as vector {R}. As well, the columns of [V] corresponding to zeroed σ_i are the basis vectors whose linear combinations, added to the particular solution, span the solution space. The particular solution $\{R\}_p$ satisfies minimum $|\{R\}|^2$ (minimum Euclidean norm) for all possible {R}.

APPENDIX III

```
      PROGram  SYSTEM


C     |
C     |  *  EXAMPLE RPOGRAM DESIGNED FOR SOLVING STRUCTURAL
C     |     DYNAMIC MODIFICATION PROBLEM OF A PLANAR TRUSS SYSTEM
C     |
C     |
C     |
C     |  *  AUTHOR ... YUEQIANG LI
C     |
C     |
C     |
C     |  *  DEPARTMENT OF MECHANICAL ENGINEERING
C     |     VICTORIA UNIVERSITY OF TECHNOLOGY
C     |
C     |
C     |
C     |
C     |                  * FILES *
C     |
C     |  *  TRUSS.DAT ... ... ... .. INPUT FILE FOR STRUCTURAL
C     |                     GEOMETRICAL MAPPING &
C     |                     PHISICAL PROPERTIES
C     |
C     |  *  TRUSS.OUT ... ... ... .. OUPUT FILE OF CONSTRUCTED
C     |                     ELEMENTAL MASS AND STIFFNESS
C     |                     MATRICES
C     |
C     |  *  SYSTEM.TRUSS.RES ... ... OUTPUT FILE FOR GLOBAL MASS
C     |                     AND STIFFNESS MATRICES,
C     |                     EIGEN SOLUTIONS
C     |
C     |  *  SYSTEM.TRUSS.MODIF.DAT . INPUT FILE FOR MODIFICATION
C     |                     DATA
C     |
C     |  *  TRUSS.MODIF.RES ... ...  OUTPUT FILE FOR STRUCTURAL
C     |                     MODIFICATION
C     |


C     --------- MAIN PROGRAM STARTS HERE ----------


      PARAMETER(ID=30)
      CHARACTER  CH*80,SM*12
      REAL SYSM(ID,ID),SYSK(ID,ID),X(ID),Y(ID),
     +     WN(ID),CSA(ID,ID),EL(ID,ID),RSPHI(ID,ID)
      INTEGER DOF
      OPEN (3,FILE='SYSTEM.TRUSS.RESU')
      OPEN (4,FILE='SYSTEM.TRUSS.MODIF.DAT')
```

```
      CALL TRUSS(SYSM,SYSK,DOF,CSA,EL,X,Y,ID)

      WRITE(*,*)'ELEMENTAL RESULTS IN TRUSS.OUT'

      CALL EIGENSOL(SYSM,SYSK,DOF,ID,WN,RSPHI)

      WRITE(*,*)'GLOBAL RESULTS & EIGEN SOLUTION IN SYSTEM.TRUSS.RESU'

       WRITE(*,*) 'PERFORM STRUCUTRAL DYNAMIC MODIFICATION? (Y/N)'
       READ(4,*) SM
       WRITE(*,*) SM
       CLOSE (UNIT=4)
       CALL SDM(CSA,EL,WN,RSPHI,X,Y,DOF,ID,SM)
      WRITE(*,*)'MODIFICATION  RESULTS IN TRUSS.MODIF.RES'

       END

C-------------------------------------------------------


      SUBROUTINE TRUSS(SYSM,SYSK,DOF,CSA,EL,X,Y,ID)


      PARAMETER (ID1=30)
      CHARACTER  CH*80
      REAL STIFF(ID1,ID1),SMAS(ID1,ID1),SYSK(ID,ID),SYSM(ID,ID),
     +      NSUP(ID1),NPOSS(ID1),NXSUP(ID1),NYSUP(ID1),ST(4,4),SM(4,4),
     +      TEMPK(ID1,ID1),TEMPM(ID1,ID1),CSA(ID,ID),
     + EL(ID,ID),X(ID),Y(ID)
      INTEGER LEMS,NFIXD,FIXD,DOF,NON
C
      OPEN (1,FILE='TRUSS.DAT')
      OPEN (2,FILE='TRUSS.OUT')

       READ(1,*)NON,LEMS,NFIXD
      DOF=2*NON
      READ(1,*) (NPOSS(II),NXSUP(II),NYSUP(II),II=1,NFIXD)
      DO 32 II=1,NON
            READ(1,*) X(II),Y(II)
            WRITE(2,*) 'THE COORDIANTE OF NODE',ii,' is: ',X(II),Y(II)
        WRITE(2,*)
   32 CONTINUE
      WRITE(2,*)
      WRITE(2,*)'        **      '

      CALL NULL(STIFF,ID1,ID1)
      CALL NULL(SMAS,ID1,ID1)
      CALL NULL(CSA,ID1,ID1)
      CALL NULL(SYSM,ID,ID)
      CALL NULL(SYSK,ID,ID)

      DO 10 II=1,NFIXD*2
            NSUP(II)=0
10    CONTINUE
      FIXD=0
      DO 20 II=1,NFIXD
            IF (NXSUP(II).NE.0) THEN
              NSUP(II*2-1)=NPOSS(II)*2-1
              FIXD=FIXD+1
            END IF
            IF (NYSUP(II).NE.0) THEN
              NSUP(II*2)=NPOSS(II)*2
              FIXD=FIXD+1
            END IF
20    CONTINUE
```

```
      READ(1,*) E,RHO

      DO 30 LEM=1,LEMS
        WRITE(2,*)
            WRITE(2,111) LEM
        WRITE(2,*)
111         FORMAT(' ELEMENT NUMBER',I4,' UNDER COMPUTATION')
            READ(1,*) I, J,CSA(I,J)
       IF (I.EQ.J) GOTO 2000
        EL(I,J)=SQRT((X(J)-X(I))**2+(Y(J)-Y(I))**2)
        XL=EL(I,J)
        WRITE(2,*)' THE ELEMENT BETWEEN NODES ',I,J
        WRITE(2,*)' CROSS SECTIONAL AREA  ', CSA(I,J)
        WRITE(2,*)' ELEMENTAL LENGTH      ', EL(I,J)

        C=(X(J)-X(I))/XL
            S=(Y(J)-Y(I))/XL
            ST(1,1)=C**2
            ST(1,2)=C*S
            ST(1,3)=-C**2
            ST(1,4)=-C*S
            ST(2,1)=C*S
            ST(2,2)=S**2
            ST(2,3)=-C*S
            ST(2,4)=-S**2
            ST(3,1)=-C**2
            ST(3,2)=-C*S
            ST(3,3)=C**2
            ST(3,4)=C*S
            ST(4,1)=-C*S
            ST(4,2)=-S**2
            ST(4,3)=C*S
            ST(4,4)=S**2

            CN=CSA(I,J)*E/XL
            WRITE (2,*) 'ELEMENTAL STIFFNESS CONSANT :',CN
        WRITE (2,*)

            CALL SCPROD(ST,4,4,CN)
            CH='ELEMENTAL STIFFNESS MATRIX'
            CALL MATOUTF2(4,ST,CH,4)

            CALL NULL(SM,4,4)
            SM(1,1)=2.0
            SM(2,2)=2.0
            SM(3,3)=2.0
            SM(4,4)=2.0
            SM(1,3)=1.0
            SM(3,1)=1.0
            SM(2,4)=1.0
            SM(4,2)=1.0

            CN=RHO*CSA(I,J)*XL/6.0
            WRITE (2,*) 'ELEMENTAL MASS CONSTANT',CN
        CALL SCPROD(SM,4,4,CN)

        GOTO 2001

2000      WRITE(2,*)'CONCENTRATED MASS ON NODE ',i,CSA(I,J)*RHO
         SM(1,1)=CSA(I,J)*RHO
             SM(2,2)=CSA(I,J)*RHO
         SM(3,3)=0
         SM(4,4)=0

2001     CH='ELEMENTAL MASS MATRIX'

        CALL MATOUTF2(4,SM,CH,4)
```

```
          CALL ASSEMB(I,J,4,ST,STIFF,ID1)
          CALL ASSEMB(I,J,4,SM,SMAS,ID1)
        CALL NULL(ST,4,4)
        CALL NULL(SM,4,4)

   30 CONTINUE

      L1=1
      K=1
      DO 55 II=1,DOF
           IF (NSUP(K).NE.II) THEN
             DO 56 J=1,DOF
               TEMPK(L1,J)=STIFF(II,J)
               TEMPM(L1,J)=SMAS(II,J)
   56        CONTINUE
             L1=L1+1
           ELSE
             K=K+1
           END IF
   55 CONTINUE
      L1=1
      K=1
      DO 57 II=1,DOF
           IF (NSUP(K).NE.II) THEN
             DO 58 J=1,DOF
               SYSK(J,L1)=TEMPK(J,II)
               SYSM(J,L1)=TEMPM(J,II)
   58        CONTINUE
             L1=L1+1
           ELSE
             K=K+1
           END IF
   57 CONTINUE

      DOF=DOF-FIXD

      CH='THE DOF OF THE TRUSS SYSTEM :'
      WRITE(3,*) CH, DOF
      CH='THE SYSTEM MASS MATRIX'
      CALL MATOUT(DOF,SYSM,CH,ID)
      CH='THE SYSTEM STIFFNESS MATRIX'
      CALL MATOUT(DOF,SYSK,CH,ID)
      END


      SUBROUTINE NULL(A,M,N)
      REAL A(M,N)
      DO 1 I=1,M
           DO 1 J=1,N
             A(I,J)=0.0
    1 CONTINUE
      RETURN
      END


      SUBROUTINE SCPROD(A,M,N,CN)
      REAL A(M,N)
      DO 10 I=1,M
           DO 10 J=1,N
             A(I,J)=CN*A(I,J)
   10 CONTINUE
      RETURN
      END


      SUBROUTINE ASSEMB(I,J,NDF2,ST,A,ID1)
      REAL A(ID1,ID1),ST(NDF2,NDF2)
```

```
      NDF=NDF2/2
      I1=NDF*I-NDF
      J1=NDF*J-NDF
      DO 10 II=1,2
            DO 10 JJ=1,2
              MM=I1+II
              MN=I1+JJ
              NM=J1+II
              N9=J1+JJ
              A(MM,MN)=A(MM,MN)+ST(II,JJ)
              A(NM,MN)=A(NM,MN)+ST(II+NDF,JJ)
              A(MM,N9)=A(MM,N9)+ST(II,JJ+NDF)
              A(NM,N9)=A(NM,N9)+ST(II+NDF,JJ+NDF)
   10 CONTINUE
      RETURN
      END



            SUBROUTINE MATOUTF2(N,A,CH,NP)
           CHARACTER  CH*80
           REAL A(NP,NP)
           WRITE(2,*) '-----------------------------------------'
           WRITE(2,33) CH
   33    FORMAT (1X,A80)
           WRITE(2,*) '-----------------------------------------'
           WRITE(2,*)
           DO 333 II=1,N,6
            DO 222 I=1,N
            III=II+5
                     IF (III .GT. N) III=N
           WRITE(2,555) (A(I,J),J=II,III)
   555   FORMAT (1X,8E20.8)
   222   CONTINUE
           WRITE(2,*)
   333   CONTINUE
        END

C-------------------------------------------------------



       SUBROUTINE EIGENSOL(SYSM,SYSK,DOF,ID,WN,RSPHI)

      PARAMETER(ID2=30)
      CHARACTER CH*80
      REAL SYSM(ID,ID),SYSK(ID,ID),RSPHI(ID,ID),RLUMDA(ID2),
     +     RLUMDA1(ID2),WN(ID)
      COMPLEX A(ID2,ID2),B(ID2,ID2),XC(ID2,ID2),EIGV(ID2)
      INTEGER DOF
       OPEN (10,FILE='DATA1')

       CALL DAPREP(SYSK,SYSM,A,B,ID,ID2,DOF)
       CALL JACOBS(A,B,XC,EIGV,ID2,DOF)
       CALL UNDAMP(XC,EIGV,RSPHI,RLUMDA,ID,ID2,DOF)
       CALL EIGSRT(RLUMDA,RSPHI,ID2,DOF)
       CALL RESET(RLUMDA,RLUMDA1,ID2,DOF)
       CALL NATFRE(WN,RLUMDA1,ID2,DOF,ID)
        CH=' NATURALFREQ(Hz)'
       CALL VNOUT(WN,CH,ID,DOF)
        CH='THE MASS NORMALISED MODESHAPES ARE'
       CALL MATOUT(DOF,RSPHI,CH,ID)
       WRITE(10,*)(WN(I)/2/3.14159,I=1,DOF)
       WRITE(10,*)((RSPHI(I,J),J=1,DOF),I=1,DOF)
```

```
      END


      SUBROUTINE DAPREP(SYSK,SYSM,A,B,INP,NP,N)
      REAL SYSM(INP,INP),SYSK(INP,INP)
      COMPLEX A(NP,NP),B(NP,NP)
       DO 1 I=1,N
       DO 1 J=1,N
        A(I,J)=CMPLX(SYSK(I,J),0)
1      B(I,J)=CMPLX(SYSM(I,J),0)
      END


      SUBROUTINE JACOBS(A,B,X,EIGV,NP,N)
      PARAMETER(NPS=257,NSMAX=70,RTOL=1E-8)
      REAL DE(NPS),TOL
      COMPLEX A(NP,NP),B(NP,NP),X(NP,NP),EIGV(NP),D(NPS)
      COMPLEX F,G,AA,BB,CC,D1,D2,AK,AJ,BK,BJ,DLT,XJ,XK,CMM
      DO 110 I=1,N
      IF (A(I,I).NE.0.0D0.AND.B(I,I).NE.0.0D0) GOTO 100
      STOP
100   D(I)=A(I,I)/B(I,I)
110   DE(I)=ABS(D(I))
      DO 130 I=1,N
      DO 120 J=1,N
120   X(I,J)=(0.0D0,0.0D0)
130   X(I,I)=(1.0D0,0.0D0)
      IF (N.EQ.1) RETURN
      NSWEEP=0
      NR=N-1
140   NSWEEP=NSWEEP+1
      EPS=(0.01D0**NSWEEP)**2
150   FORMAT(1X,'EPS=',D16.10)
      DO 240 J=1,NR
      JJ=J+1
      DO 240 K=JJ,N
      AJK=ABS(A(J,K))
      AJJ=ABS(A(J,J))
      AKK=ABS(A(K,K))
      EPTOLK=AJK*AJK/AJJ*AKK
      BJK=ABS(B(J,K))
      BJJ=ABS(B(J,J))
      BKK=ABS(B(K,K))
      EPTOLM=BJK*BJK/BJJ*BKK
      IF ((EPTOLK.LT.EPS).AND.(EPTOLM.LT.EPS)) GOTO 240
      AA=A(J,J)*B(J,K)-B(J,J)*A(J,K)
      CC=A(J,K)*B(K,K)-B(J,K)*A(K,K)
      BB=B(J,J)*A(K,K)-A(J,J)*B(K,K)
      DLT=BB*BB/4.0D0-AA*CC
      D1=BB/2.0D0-SQRT(DLT)
      D2=BB/2.0D0+SQRT(DLT)
      IF (ABS(D2)-ABS(D1)) 170,170,160
160   F=CC/D2
      G=AA/D2
      GOTO 180
170   F=CC/D1
      G=AA/D1
180   IF (N-2.LE.0) GOTO 220
      JP1=J+1
      JM1=J-1
      KP1=K+1
      KM1=K-1
      DO 190 I=1,JM1
      AJ=A(I,J)
      AK=A(I,K)
      A(I,J)=AJ+G*AK
      A(I,K)=AK+F*AJ
```

```
      BJ=B(I,J)
      BK=B(I,K)
      B(I,J)=BJ+G*BK
190     B(I,K)=BK+F*BJ
      DO 200 I=KP1,N
      AJ=A(J,I)
      AK=A(K,I)
      A(J,I)=AJ+G*AK
      A(K,I)=AK+F*AJ
      BJ=B(J,I)
      BK=B(K,I)
      B(J,I)=BJ+G*BK
200     B(K,I)=BK+F*BJ
      DO 210 I=JP1,KM1
      AJ=A(J,I)
      AK=A(I,K)
      A(J,I)=AJ+G*AK
      A(I,K)=AK+F*AJ
      BJ=B(J,I)
      BK=B(I,K)
      B(J,I)=BJ+G*BK
210     B(I,K)=BK+F*BJ
220     AJ=A(J,J)
      AK=A(K,K)
      A(J,J)=AJ+2*G*A(J,K)+AK*G*G
      A(K,K)=AK+2*F*A(J,K)+AJ*F*F
      A(J,K)=0.0D0
      BJ=B(J,J)
      BK=B(K,K)
      B(J,J)=BJ+2*G*B(J,K)+BK*G*G
      B(K,K)=BK+2*F*B(J,K)+BJ*F*F
      B(J,K)=0.0D0
      DO 230 I=1,N
      XJ=X(I,J)
      XK=X(I,K)
      X(I,J)=XJ+XK*G
230     X(I,K)=XK+XJ*F
240     CONTINUE
      DO 250 I=1,N
      CAB=SQRT(B(I,I))
      IF (CAB.EQ.0.0D0) STOP 'ZERO DIAGONAL IN JACOBS'
250     EIGV(I)=A(I,I)/B(I,I)
      DO 260 I=1,N
      TOL=RTOL*DE(I)
      DE(I)=ABS(EIGV(I)-D(I))
      IF (DE(I).GT.TOL) GOTO 310
260     CONTINUE
      EPS=RTOL**2
      DO 270 J=1,NR
      JJ=J+1
      DO 270 K=JJ,N
      AJK=ABS(A(J,K))
      AJJ=ABS(A(J,J))
      AKK=ABS(A(K,K))
      EPSK=AJK*AJK/AJJ*AKK
      BJK=ABS(B(J,K))
      BJJ=ABS(B(J,J))
      BKK=ABS(B(K,K))
      EPSM=BJK*BJK/BJJ*BKK
      IF (EPSK.GT.EPS.OR.EPSM.GT.EPS) GOTO 310
270     CONTINUE
280     DO 290 I=1,N
      DO 290 J=1,N
      B(J,I)=B(I,J)
290     A(J,I)=A(I,J)
      DO 300 J=1,N
      CMM=SQRT(B(J,J))
```

```
      DO 300 K=1,N
300    X(K,J)=X(K,J)/CMM
      RETURN
310    CONTINUE
      DO 320 I=1,N
      D(I)=EIGV(I)
320    DE(I)=ABS(D(I))
      IF (NSWEEP.LT.NSMAX) GOTO 140
      GOTO 280
330    FORMAT(25HO*** ERROR SOLUTION STOP )
      END


      SUBROUTINE UNDAMP(XC,EIGV,RSPHI,RLUMDA,ID,ID2,N)
      REAL RSPHI(ID,ID),RLUMDA(ID2)
      COMPLEX XC(ID2,ID2),EIGV(ID2)
       DO 2 I=1,ID2
2       RLUMDA(I)=0
       DO 1 I=1,N
        RLUMDA(I)=REAL(EIGV(I))
        DO 1 J=1,N
        RSPHI(I,J)=REAL(XC(I,J))
1      CONTINUE
      END


      SUBROUTINE EIGSRT(D,V,NP,N)
      REAL D(NP),V(NP,NP)
       DO 13 I=1,N-1
            K=I
            P=D(I)
            DO 11 J=I+1,N
              IF(D(J).LE.P)THEN
                K=J
                P=D(J)
              ENDIF
11        CONTINUE
            IF(K.NE.I)THEN
              D(K)=D(I)
              D(I)=P
              DO 12 J=1,N
                P=V(J,I)
                V(J,I)=V(J,K)
                V(J,K)=P
12          CONTINUE
            ENDIF
13     CONTINUE
      RETURN
      END


      SUBROUTINE RESET(RLUMDA,RLUMDA1,NP,N)
      REAL RLUMDA(NP),RLUMDA1(NP)
      DO 1 I=1,N
       RLUMDA1(I)=RLUMDA(I)
       IF (ABS(RLUMDA(I+1)/RLUMDA(I)).GT.1E+4) THEN
        DO 2 J=1,I
2         RLUMDA1(J)=0
       END IF
1      CONTINUE
      END


      SUBROUTINE NATFRE(WN,RLUMDA1,NP,N,ID)
      REAL WN(ID),RLUMDA1(NP)
       DO 1 I=1,N
1       WN(I)=SQRT(RLUMDA1(I))
```

```
      END


            SUBROUTINE MATOUT(N,A,CH,NP)
           CHARACTER  CH*80
           REAL A(NP,NP)
           WRITE(3,*) '-------------------------------------------'
           WRITE(3,33) CH
33     FORMAT (1X,A80)
           WRITE(3,*) '-------------------------------------------'
           WRITE(3,*)
           DO 333 II=1,N,6
            DO 222 I=1,N
            III=II+5
                     IF (III .GT. N) III=N
           WRITE(3,555) (A(I,J),J=II,III)
555     FORMAT (1X,8E20.8)
222     CONTINUE
           WRITE(3,*)
333     CONTINUE
        IF(CH.EQ.'THE MASS NORMALISED MODESHAPES ARE')
     +    WRITE(3,*)'---- END OF RESULTS OUTPUT ----'
           RETURN
        END

        SUBROUTINE VNOUT(A,CH,NP,N)
        CHARACTER  CH*80
        REAL A(NP)
        WRITE(3,*) '-------------------------------------------'
            WRITE(3,100) CH
100     FORMAT (1X,A80)
           WRITE(3,*) '-------------------------------------------'
           WRITE (3,*)
           DO  1 I=1,N
             WRITE(3,110) I,CH,A(I)/2/3.14159
110     FORMAT(1X,'MODE',I2,A12, ' = ',F25.8)
1       CONTINUE
        END


C       ---------------------------------------------------------------


        SUBROUTINE SDM(CSA,EL,WN,RSPHI,X,Y,DOF,ID,SM)

        PARAMETER(ID3=30)
        CHARACTER CHM*80,CH*80,SM*12,WM*12
        REAL RSPHI(ID,ID),WN(ID),CSA(ID,ID),EL(ID,ID),X(ID),Y(ID)
        REAL DM(ID3,ID3),DK(ID3,ID3),RMF(ID3,ID3),ALF(ID3,ID3),
     +       A(ID3,ID3),B(ID3,ID3),H(ID3,ID3),AUX(ID3,ID3),
     +       WR(ID3),WI(ID3)
        INTEGER DOF
       DIMENSION IR(ID3),JR(ID3),NONM(ID3)
       OPEN(4,FILE='SYSTEM.TRUSS.MODIF.DAT')
       OPEN(5,FILE='TRUSS.MODIF.RES')

       READ(4,*) SM
       READ(4,*) WMD
        WRITE(5,*)
        WRITE(5,*)'  ## THE RESULTS FILE OF truss.sdm.ftn ##  '
        WRITE(5,*)

        CHM='THE FIRST FIVE NATURAL FREQUENCIES'
        CH ='NATURAL FREQUENCY'

        WRITE(5,*) '-------------------------------------------'
            WRITE(5,173) CHM
```

```
173    FORMAT (1X,A80)
         WRITE(5,*) '----------------------------------------'
         WRITE (5,*)
         DO  81 I=1,5
           WRITE(5,183) I,CH,WN(I)/2/3.14159
183    FORMAT(1X,'MODE',I2,A17, ' = ',F10.4)
81     CONTINUE
     WRITE(5,*)
     WRITE(5,*)'          **   **   **          .
     WRITE(5,185)WMD
185    FORMAT(1X,'DESIRED NATURAL FREQUENCY (Hz).....',F10.4)
     WRITE(5,*)
     WRITE(5,*)
     WRITE(5,*)'          **   **   **          .
     WRITE(5,*)'    STRUCTURAL PARAMETER FOR MODIFICATION:'
     WRITE(5,*) 'CROSS SECTIONAL AREA OR/AND CONCENTRATED MASS'
     WRITE(5,*)
     WMD=WMD*2*3.14159

     CALL BUILD1(DM,DK,EL,X,Y,IR,JR,RMF,NONM,ID
    +          ,ID3,WMD,NMDOF,LEMS,MNOD)
     CALL RECPTNCE(WN,RSPHI,ALF,IR,JR,ID,ID3,WMD,DOF,NMDOF)
     CALL FORMH(DK,DM,A,B,H,ALF,AUX,IR,JR,NMDOF,ID3)
      CALL ELMHES(AUX,NMDOF,ID3)
      CALL HQR(AUX,NMDOF,ID3,WR,WI)
      CALL RWM(CSA,RMF,WR,WI,NONM,NMDOF,LEMS,MNOD,ID,ID3)
     END



     SUBROUTINE BUILD1(DM,DK,EL,X,Y,IR,JR,RMF
    +               ,NONM,ID,ID3,WMD,NMDOF,LEMS,MNOD)

     PARAMETER(ID4=30)
     CHARACTER CH4*80
     REAL EL(ID,ID),X(ID),Y(ID)
     REAL DM(ID3,ID3),DK(ID3,ID3),RMF(ID3,ID3)
     DIMENSION IR(ID3),JR(ID3),NONM(ID3)

     REAL STIFF(ID4,ID4),SMAS(ID4,ID4),SM(4,4),ST(4,4),
    +    TEMPK(ID4,ID4),TEMPM(ID4,ID4)
     DIMENSION  NSUP(ID4),NPOSS(ID4),IR1(ID4),IR2(ID4),IR3(ID4),
    +                  NXSUP(ID4),NYSUP(ID4)
     INTEGER FIXD,EFIX
     READ(4,*) MNOD,LEMS,NFIXD
     WRITE(5,*)
     WRITE(5,*)'          **   **   **          .
     WRITE(5,*)' NODES INVOLEVED IN MODIFICATION..... ';  MNOD
     WRITE(5,*)' ELEMENTS INVOLVED IN MODIFICATION... ',  LEMS
     NMDOF=MNOD*2
     CALL NULL(STIFF,ID4,ID4)
     CALL NULL(SMAS,ID4,ID4)
     CALL NULL(RMF,ID3,ID3)
     CALL NULL(DK,ID3,ID3)
     CALL NULL(DM,ID3,ID3)
      DO 1000       IM=1,MNOD
1000    READ(4,*) NONM(IM)
     DO 82 II=1,NFIXD
82     READ(4,*) NPOSS(II),NXSUP(II),NYSUP(II)
       DO 1006 IDM=1,2*MNOD
         IR1(IDM*2-1)=2*NONM(IDM)-1
         IR1(IDM*2)=2*NONM(IDM)
1006     IR2(IDM)= IR1(IDM)
     DO 10 II=1,ID4
          NSUP(II)=0
  10 CONTINUE
    FIXD=0
```

```
      DO 20 II=1,NFIXD
            IF(NXSUP(II).NE.0) THEN
              NSUP(II*2-1)=NPOSS(II)*2-1
              FIXD=FIXD+1
            END IF
            IF(NYSUP(II).NE.0) THEN
              NSUP(II*2)=NPOSS(II)*2
              FIXD=FIXD+1
            END IF
   20 CONTINUE
      DO 1001 IRE=1,2*MNOD
       DO 1001 II=1,2*NFIXD
         IF (NSUP(II).NE.0.AND.IR2(IRE).GE.NSUP(II)) THEN
         IR1(IRE)=IR1(IRE)-1
      END IF
 1001  CONTINUE
      EFIX=0
       DO 1002 IRE=1,2*MNOD
         IF (IR1(IRE).EQ.0) THEN
         EFIX=EFIX+1
       END IF
 1002  CONTINUE
      NMDOF=2*MNOD-EFIX
      IJD=1
      DO 1003 IJ=1,NMDOF
        DO 1004 IRE=IJD,2*MNOD
           IF (IR1(IRE).NE.0) THEN
           IJD=IRE+1
            GOTO 1005
           END IF
 1004     CONTINUE
 1005      IR(IJ)=IR1(IRE)
          JR(IJ)=IR(IJ)
 1003   CONTINUE
      READ(4,*) E,RHO
       WRITE(5,*)
       WRITE(5,*)

      WRITE(5,*)'          **   **   **          '

     DO 40 LEM=1,LEMS
       WRITE(5,*)
       WRITE(5,*)'--------------------------------'
           WRITE(5,111) LEM
  111        FORMAT(' MODIFIED  ELEMENT NO.      ',I4)
           READ(4,*) I, J,RMF(NONM(I),NONM(J))
       IF(I.EQ.J) GOTO 3000
       WRITE(5,*)' ELEMENT BETWEEN NODES  ',NONM(I),',  ',NONM(J)
        WRITE(5,*)'--------------------------------'
       ER=RMF(NONM(I),NONM(J))
       XL=EL(NONM(I),NONM(J))
       WRITE(5,*)'ELEMENTAL RATIO OF MODIFI.  :',ER
       WRITE(5,*)'LENGTH OF ORIGINAL ELEMENT  :',XL
       C=(X(NONM(J))-X(NONM(I)))/XL
           S=(Y(NONM(J))-Y(NONM(I)))/XL

       ST(1,1)=C**2
           ST(1,2)=C*S
           ST(1,3)=-C**2
           ST(1,4)=-C*S
           ST(2,1)=C*S
           ST(2,2)=S**2
           ST(2,3)=-C*S
           ST(2,4)=-S**2
           ST(3,1)=-C**2
           ST(3,2)=-C*S
           ST(3,3)=C**2
```

```
          ST(3,4)=C*S
          ST(4,1)=-C*S
          ST(4,2)=-S**2
          ST(4,3)=C*S
          ST(4,4)=S**2

          CN=ER*E/XL
          CALL SCPROD(ST,4,4,CN)

          CALL NULL(SM,4,4)
          SM(1,1)=2.0
          SM(2,2)=2.0
          SM(3,3)=2.0
          SM(4,4)=2.0
          SM(1,3)=1.0
          SM(3,1)=1.0
          SM(2,4)=1.0
          SM(4,2)=1.0

          CN=WMD**2*ER*RHO*XL/6.0
          CALL SCPROD(SM,4,4,CN)
        GOTO 3001

3000    SM(1,1)=WMD**2*RHO*RMF(NONM(I),NONM(J))
        SM(2,2)=WMD**2*RHO*RMF(NONM(I),NONM(J))
        SM(3,3)=0
        SM(4,4)=0
3001    CALL ASSEMB(I,J,4,ST,STIFF,ID4)
          CALL ASSEMB(I,J,4,SM,SMAS,ID4)
        CALL NULL(ST,4,4)
        CALL NULL(SM,4,4)
  40 CONTINUE

     LR=1
     KR=1
     DO 50 II=1,MNOD*2
          IF (NSUP(KR).NE.IR2(II)) THEN
            DO 60 J=1,MNOD*2
              TEMPK(LR,J)=STIFF(II,J)
          TEMPM(LR,J)=SMAS(II,J)
  60        CONTINUE
            LR=LR+1
          ELSE
            KR=KR+1
          END IF
  50 CONTINUE

     LR=1
     KR=1
     DO 70 II=1,MNOD*2
          IF (NSUP(KR).NE.IR2(II)) THEN
            DO 80 J=1,MNOD*2
              DK(J,LR)=TEMPK(J,II)
              DM(J,LR)=TEMPM(J,II)
  80        CONTINUE
            LR=LR+1
          ELSE
            KR=KR+1
          END IF
  70 CONTINUE
     END


     SUBROUTINE PROD(A,B,M,N,CN)
     REAL A(M,N),B(M,N)
     DO 1 I=1,M
      DO 1 J=1,N
```

```
1     A(I,J)=0
     DO 2 I=1,M
      DO 2 J=1,N
2     A(I,J)=B(I,J)*CN
     END

      SUBROUTINE MATOUTF5(N,A,CH,NP)
          CHARACTER  CH*80
          REAL A(NP,NP)
          WRITE(5,*) '----------'
          WRITE(5,33) CH
33     FORMAT (1X,A80)
          WRITE(5,*) '----------'
          WRITE(5,*)
          DO 333 II=1,N,6
           DO 222 I=1,N
           III=II+5
                    IF (III .GT. N) III=N
          WRITE(5,555) (A(I,J),J=II,III)
555    FORMAT (1X,8E20.8)
222    CONTINUE
          WRITE(5,*)
333    CONTINUE
      END


      SUBROUTINE RECPTNCE(B,C,ALF,IR,JR,ID,ID3,OME,DOF,NMDOF)
     INTEGER DOF
     REAL B(ID),C(ID,ID),ALF(ID3,ID3)
     DIMENSION IR(ID3),JR(ID3)
     DO 1 N2=1,NMDOF
      DO 1 N3=1,NMDOF
       ALF(IR(N2),JR(N3))=0
        DO 2 N4=1,DOF
         ALF(IR(N2),JR(N3))=ALF(IR(N2),JR(N3))+
    +    C(IR(N2),N4)*C(JR(N3),N4)/(B(N4)*B(N4)-OME*OME)
2       CONTINUE
1     CONTINUE
       END

     SUBROUTINE FORMH(DK,DM,A,B,H,ALF,AUX,IR,JR,NMDOF,ID3)
     REAL DM(ID3,ID3),DK(ID3,ID3),ALF(ID3,ID3),A(ID3,ID3),
    +     B(ID3,ID3),H(ID3,ID3),AUX(ID3,ID3)
     DIMENSION IR(ID3),JR(ID3)
      DO 1 I=1,NMDOF
       DO 1 J=1,NMDOF
1       H(I,J)=ALF(IR(I),JR(J))
     CALL MUL(H,DK,A,NMDOF,ID,ID3)
     CALL MUL(H,DM,B,NMDOF,ID,ID3)
     DO 2 I=1,NMDOF
      DO 2 J=1,NMDOF
2      AUX(I,J)=B(I,J)-A(I,J)

     RETURN
     END

      SUBROUTINE MUL(H,SIG,C,NMDOF,ID,ID3)

      REAL H(ID3,ID3),SIG(ID3,ID3),C(ID3,ID3)
      DO 2 I=1,NMDOF
       DO 2 J=1,NMDOF
2        C(I,J)=0
      DO 1 I=1,NMDOF
       DO 1 J=1,NMDOF
        DO 1 K=1,NMDOF
         C(I,J)=C(I,J)+H(I,K)*SIG(K,J)
1     CONTINUE
```

```
      END

      SUBROUTINE RWM(CSA,RMF,WR,WI,NONM,NMDOF,LEMS,MNOD,ID,ID3)
      PARAMETER(ID7=30)
      REAL CSA(ID,ID),RMF(ID3,ID3),WR(ID3),WI(ID3)
      REAL CSAM(ID7,ID7)
      DIMENSION NONM(ID3)

      CALL NULL(CSAM,ID7,ID7)
      WRITE(5,*)
      WRITE(5,*)
           WRITE(5,*) '      *** THE MODIFICATION RESULTS ***      '
      DO 2 I=1,NMDOF
         C=0
        DO 1 I1=1,MNOD
         DO 1 J1=I1,MNOD
        IF(WI(I).EQ.0.AND.RMF(NONM(I1),NONM(J1)).NE.0) THEN
         IF (WR(I).NE.0.AND.ABS(RMF(NONM(I1),NONM(J1))/WR(I))/
     +    (CSA(NONM(I1),NONM(J1))+1E-10).LE.100) THEN
         IF ((RMF(NONM(I1),NONM(J1))/WR(I)+
     +    CSA(NONM(I1),NONM(J1))).GT.0) THEN
          CSAM(NONM(I1),NONM(J1))=RMF(NONM(I1),NONM(J1))/WR(I)+
     +    CSA(NONM(I1),NONM(J1))
         C=C+1
         END IF
         END IF
         END IF
 1      CONTINUE

        IF (C.EQ.LEMS) THEN
        WRITE(5,*)
         WRITE(5,*)' ----------------------------------------------
     +------------'
         WRITE(5,184)
         WRITE(5,*)' ----------------------------------------------
     +------------'
         WRITE(5,*)
184       FORMAT(1X,'    NODE','    NODE','     ORIGINAL C.S.A.'
     +,'     MODIFIED C.S.A.')

         DO 3 I1=1,MNOD
         DO 3 J1=I1,MNOD
         IF (CSAM(NONM(I1),NONM(J1)).NE.0 ) THEN
          WRITE(5,183)NONM(I1),NONM(J1),CSA(NONM(I1),NONM(J1))
     +,CSAM(NONM(I1),NONM(J1))
183       FORMAT(1X,I8,I8,E20.4,E20.4)
          END IF
3        CONTINUE
         GOTO 2
        END IF
2      CONTINUE
      END

      SUBROUTINE ELMHES(A,N,NP)

      REAL A(NP,NP)
      DO 17 M=2,N-1
        X=0.
        I=M
        DO 11 J=M,N
          IF(ABS(A(J,M-1)).GT.ABS(X))THEN
            X=A(J,M-1)
            I=J
          ENDIF
11      CONTINUE
        IF(I.NE.M)THEN
          DO 12 J=M-1,N
```

```
        Y=A(I,J)
        A(I,J)=A(M,J)
        A(M,J)=Y
12      CONTINUE
       DO 13 J=1,N
        Y=A(J,I)
        A(J,I)=A(J,M)
        A(J,M)=Y
13      CONTINUE
      ENDIF
      IF(X.NE.0.)THEN
       DO 16 I=M+1,N
        Y=A(I,M-1)
        IF(Y.NE.0.)THEN
         Y=Y/X
         A(I,M-1)=Y
         DO 14 J=M,N
          A(I,J)=A(I,J)-Y*A(M,J)
14        CONTINUE
         DO 15 J=1,N
          A(J,M)=A(J,M)+Y*A(J,I)
15        CONTINUE
        ENDIF
16      CONTINUE
      ENDIF
17    CONTINUE
      RETURN
      END


      SUBROUTINE HQR(A,N,NP,WR,WI)

      REAL A(NP,NP)
      REAL WR(NP),WI(NP)
      ANORM=ABS(A(1,1))
      DO 12 I=2,N
       DO 11 J=I-1,N
        ANORM=ANORM+ABS(A(I,J))
11     CONTINUE
12    CONTINUE
      NN=N
      T=0.
1     IF(NN.GE.1)THEN
       ITS=0
2      DO 13 L=NN,2,-1
        S=ABS(A(L-1,L-1))+ABS(A(L,L))
        IF(S.EQ.0.)S=ANORM
        IF(ABS(A(L,L-1))+S.EQ.S)GO TO 3
13      CONTINUE
       L=1
3      X=A(NN,NN)
       IF(L.EQ.NN)THEN
        WR(NN)=X+T
        WI(NN)=0.
        NN=NN-1
       ELSE
        Y=A(NN-1,NN-1)
        W=A(NN,NN-1)*A(NN-1,NN)
        IF(L.EQ.NN-1)THEN
         P=0.5*(Y-X)
         Q=P**2+W
         Z=SQRT(ABS(Q))
         X=X+T
         IF(Q.GE.0.)THEN
          Z=P+SIGN(Z,P)
          WR(NN)=X+Z
          WR(NN-1)=WR(NN)
```

```
          IF(Z.NE.0.)WR(NN)=X-W/Z
          WI(NN)=0.
          WI(NN-1)=0.
        ELSE
          WR(NN)=X+P
          WR(NN-1)=WR(NN)
          WI(NN)=Z
          WI(NN-1)=-Z
        ENDIF
        NN=NN-2
      ELSE
        IF(ITS.EQ.30)PAUSE 'too many iterations'
        IF(ITS.EQ.10.OR.ITS.EQ.20)THEN
          T=T+X
          DO 14 I=1,NN
            A(I,I)=A(I,I)-X
14        CONTINUE
          S=ABS(A(NN,NN-1))+ABS(A(NN-1,NN-2))
          X=0.75*S
          Y=X
          W=-0.4375*S**2
        ENDIF
        ITS=ITS+1
        DO 15 M=NN-2,L,-1
          Z=A(M,M)
          R=X-Z
          S=Y-Z
          P=(R*S-W)/A(M+1,M)+A(M,M+1)
          Q=A(M+1,M+1)-Z-R-S
          R=A(M+2,M+1)
          S=ABS(P)+ABS(Q)+ABS(R)
          P=P/S
          Q=Q/S
          R=R/S
          IF(M.EQ.L)GO TO 4
          U=ABS(A(M,M-1))*(ABS(Q)+ABS(R))
          V=ABS(P)*(ABS(A(M-1,M-1))+ABS(Z)+ABS(A(M+1,M+1)))
          IF(U+V.EQ.V)GO TO 4
15      CONTINUE
4       DO 16 I=M+2,NN
          A(I,I-2)=0.
          IF (I.NE.M+2) A(I,I-3)=0.
16      CONTINUE
        DO 19 K=M,NN-1
          IF(K.NE.M)THEN
            P=A(K,K-1)
            Q=A(K+1,K-1)
            R=0.
            IF(K.NE.NN-1)R=A(K+2,K-1)
            X=ABS(P)+ABS(Q)+ABS(R)
            IF(X.NE.0.)THEN
              P=P/X
              Q=Q/X
              R=R/X
            ENDIF
          ENDIF
          S=SIGN(SQRT(P**2+Q**2+R**2),P)
          IF(S.NE.0.)THEN
            IF(K.EQ.M)THEN
              IF(L.NE.M)A(K,K-1)=-A(K,K-1)
            ELSE
              A(K,K-1)=-S*X
            ENDIF
            P=P+S
            X=P/S
            Y=Q/S
            Z=R/S
```

```
          Q=Q/P
          R=R/P
          DO 17 J=K,NN
            P=A(K,J)+Q*A(K+1,J)
            IF(K.NE.NN-1)THEN
              P=P+R*A(K+2,J)
              A(K+2,J)=A(K+2,J)-P*Z
            ENDIF
            A(K+1,J)=A(K+1,J)-P*Y
            A(K,J)=A(K,J)-P*X
17          CONTINUE
           DO 18 I=L,MIN(NN,K+3)
            P=X*A(I,K)+Y*A(I,K+1)
            IF(K.NE.NN-1)THEN
              P=P+Z*A(I,K+2)
              A(I,K+2)=A(I,K+2)-P*R
            ENDIF
            A(I,K+1)=A(I,K+1)-P*Q
            A(I,K)=A(I,K)-P
18          CONTINUE
          ENDIF
19        CONTINUE
         GO TO 2
        ENDIF
       ENDIF
      GO TO 1
      ENDIF
      RETURN
      END

C  ---END OF PROGRAM---
```

APPENDIX IV

```
1 % cat TRUSS.DAT
14        32        2              (No. of nodes, No. of elements, No. of fixed nodes)

1         1         1              (fixed nodes)
14        1         1

0         0                        (geometrical data)
1         0
1         1
2         0
2         1
3         0
3         1
4         0
4    1
5         0
5         1
6         0
6         1
7         0

68E9      2.8E3                    (E modulus, desity)

1         2         19.7E-4              (connectivity, c.s.a.)
1         3         19.7E-4
2         3         19.7E-4
2         4         19.7E-4
2         5         19.7E-4
3         4         19.7E-4
3         5         19.7E-4
4         5         19.7E-4
4         6         19.7E-4
4         7         19.7E-4
5         6         19.7E-4
5         7         19.7E-4
6         7         19.7E-4
6         8         19.7E-4
6         9         19.7E-4
7         8         19.7E-4
7         9         19.7E-4
8         9         19.7E-4
8         10        19.7E-4
8         11        19.7E-4
9         10        19.7E-4
9         11        19.7E-4
10        11        19.7E-4
10        12        19.7E-4
10        13        19.7E-4
11        12        19.7E-4
11        13        19.7E-4
12        13        19.7E-4
12        14        19.7E-4
13        14        19.7E-4
6         6          1E-02
8         8          1E-02

 -- end of file TRUSS.DAT ---
```

```
2 % cat SYSTEM.TRUSS.MODIF.DAT

'Y'                    (performing sdm)

80                     (desired natrual frequency)

4 6      2                 (No. of nodes, elements and fixed node involes modification)

4                      (nodes topology)
5
6
7

1        1        1          (fixed d.o.f)
14       1        1


68E9     2.8E3              (E modulus, density)

1        2        2          (connectivity, element part. ratio)
1        3    1
1        4        2
2        3        1
2        4        2
3        4        1

 -- end of file SYSTEM.TRUSS.MODIF.DAT --


3 % truss.sdm.bin

 ELEMENTAL RESULTS IN TRUSS.OUT
 GLOBAL RESULTS & EIGEN SOLUTION IN SYSTEM.TRUSS.RESU
 PERFORM STRUCUTRAL DYNAMIC MODIFICATION? (Y/N)
 Y
 MODIFICATION  RESULTS IN TRUSS.MODIF.RES


4 % cat  TRUSS.MODIF.RES

   ## THE RESULTS FILE OF truss.sdm.ftn ##

 ---------------------------------------------
 THE FIRST FIVE NATURAL FREQUENCIES (Hz)
 ---------------------------------------------

 MODE 1NATURAL FREQUENCY =   44.6858
 MODE 2NATURAL FREQUENCY =  110.5648
 MODE 3NATURAL FREQUENCY =  182.4096
 MODE 4NATURAL FREQUENCY =  246.6581
 MODE 5NATURAL FREQUENCY =  341.1596

        **   **   **
 DESIRED NATURAL FREQUENCY (Hz)..... 80.0000


        **   **   **
    STRUCTURAL PARAMETER FOR MODIFICATION:
 CROSS SECTIONAL AREA OR/AND CONCENTRATED MASS


        **   **   **
  NODES INVOLEVED IN MODIFICATION..... 4
  ELEMENTS INVOLVED IN MODIFICATION... 6


        **   **   **
```

MODIFIED ELEMENT NO. 1
ELEMENT BETWEEN NODES 4, 5

ELEMENTAL RATIO OF MODIFI. : 2.000000
LENGTH OF ORIGINAL ELEMENT : 1.000000

MODIFIED ELEMENT NO. 2
ELEMENT BETWEEN NODES 4, 6

ELEMENTAL RATIO OF MODIFI. : 1.000000
LENGTH OF ORIGINAL ELEMENT : 1.000000

MODIFIED ELEMENT NO. 3
ELEMENT BETWEEN NODES 4, 7

ELEMENTAL RATIO OF MODIFI. : 2.000000
LENGTH OF ORIGINAL ELEMENT : 1.414214

MODIFIED ELEMENT NO. 4
ELEMENT BETWEEN NODES 5, 6

ELEMENTAL RATIO OF MODIFI. : 1.000000
LENGTH OF ORIGINAL ELEMENT : 1.414214

MODIFIED ELEMENT NO. 5
ELEMENT BETWEEN NODES 5, 7

ELEMENTAL RATIO OF MODIFI. : 2.000000
LENGTH OF ORIGINAL ELEMENT : 1.000000

MODIFIED ELEMENT NO. 6
ELEMENT BETWEEN NODES 6, 7

ELEMENTAL RATIO OF MODIFI. : 1.000000
LENGTH OF ORIGINAL ELEMENT : 1.000000

*** THE MODIFICATION RESULTS ***

NODE	NODE	ORIGINAL C.S.A.	MODIFIED C.S.A.
4	5	0.1970E-02	0.2799E-01
4	6	0.1970E-02	0.1498E-01
4	7	0.1970E-02	0.2799E-01
5	6	0.1970E-02	0.1498E-01
5	7	0.1970E-02	0.2799E-01
6	7	0.1970E-02	0.1498E-01

NODE	NODE	ORIGINAL C.S.A.	MODIFIED C.S.A.
4	5	0.1970E-02	0.1043E+00
4	6	0.1970E-02	0.5315E-01
4	7	0.1970E-02	0.1043E+00
5	6	0.1970E-02	0.5315E-01
5	7	0.1970E-02	0.1043E+00
6	7	0.1970E-02	0.5315E-01