Bi-directional Evolutionary Method for Stiffness and Displacement Optimisation

by

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Thesis submitted in fulfilment of the requirement of the degree of Master of Engineering to Victoria University of Technology

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Melbourne, Australia

February 1999

In memory of my grandmother and dedicated to my parents.

纪念我的祖母，并献给我的父亲和母亲。

ACKNOWLEDGEMENTS

This thesis is a result of my work over the past 18 months with help from many people. I wish to express my deepest gratitude to my supervisor, Assoc. Prof. Y.M. Xie, for having introduced me to an interesting research topic. Without his valuable guidance and limitless patience, this thesis would not have been possible. I am also thankful to my nominated co-supervisor, Assoc. Prof. C. Perera, for his support and inspiration.

Grateful acknowledgements are extended to Prof. G.P. Steven and Dr O.M. Querin at the University of Sydney, for their advice and encouragement at all time. Thanks are also due to Mr I. Campbell, Assoc. Prof. Ö. Turan and Assoc. Prof. G. Thorpe for their assistance.

I would like to take this opportunity to express my appreciation to:

- School of the Built Environment, Victoria University of Technology, which has provided financial support and facilities for the research.
- The Australia Research Council (ARC) for funding the research project.
- G+D Computing Pty. Ltd. for providing the computer software.
- The International Affairs Branch and Postgraduate Studies Unit for their kind assistance.

I am also indebted to my previous supervisors, Profs. W.J. Yi and P.S. Shen, at Hunan University, P.R. China. It was from them that I got to know the real meaning of a researcher and engineer.

Special thanks go to Nha and Dhayanthi, for their stimulating work on related projects, and Jyoti, Mahesh, Jin, Li , Anne, Bill and Stephen for their friendship.

Last, but not the least, my gratitude is due to my family: my grandmother, my parents, my sister, brother-in-law and my nephew, as well as all my friends. Their love and attention in all these years have provided me with great strength and resource.

CERTIFICATE OF RESEARCH

This is to certify that except where specific reference to other investigation is made, the work described in this thesis is the result of the candidate's own investigations.

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DECLARATION

This is to certify that neither this thesis, nor any part of it, has been presented or is being concurrently submitted in candidature for any other degree at any other university.

Candidature

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SUMMARY

This thesis presents a method for structural optimisation called bi-directional evolutionary structural optimisation (BESO). It is an extension of the systematic research on the evolutionary method. The basic concept of evolutionary structural optimisation (ESO) is that by slowly removing the inefficient material, the structure evolves towards an optimum. BESO extends the concept by allowing for the efficient material to be added while the inefficient material is removed. The formulation of BESO is motivated to improve the reliability and efficiency of the ESO method.

The BESO method for topological optimisation of 2D continua subject to stiffness and displacement constraints is the major task of this thesis. The theoretical aspects are explored by following the optimality criteria algorithm for problems of discrete design variables. These aspects include the optimality criteria, sensitivity analysis, displacement extrapolation and evolutionary procedure. The bi-directional evolutionary procedure is incorporated with the finite element analysis to realise an automatic optimisation process.

A wide range of examples are tested by using the proposed BESO procedure. Different design conditions are considered including stiffness optimisation and single or multiple displacement optimisation under single and multiple loading conditions. The solution reliability and parametric effect are further studied to improve the BESO performance. The comparison of results by BESO and ESO are attempted and the satisfactory agreement demonstrates the validity of the proposed procedure. Two major conclusions are derived from the work in this thesis. The first one is that BESO is as effective as ESO, and the second one is that BESO can be computationally more efficient in most cases.

CONTENTS

Chapter 1

Introduction

1.1 Structural Optimisation

Structural optimisation aims at finding the best design of a structure with the minimum weight or cost while satisfying requirements on strength, stiffness, reliability or functionality. It is motivated by the quest to make the most of material available to produce structures of high performance and low cost. Optimal designs can bring significant economic and ecological benefits, particularly in the present context of growing manufacturing or construction demand based on scarce funds and resources.

The history of optimisation theory could be dated back centuries. Focusing on the mathematical aspects of the concept, the original theoretical framework has been elegantly established using analytical approaches. Since then, though the development is more engineering-orientated, the coverage is yet limited to very idealised cases, e.g. the fully stressed design of some simple structural components. Due to the associated mathematical complexities, the topic of structural optimisation has remained academic interest rather than a practical design technique for quite a long period.

Significant advancement in structural optimisation has been made in the last three decades. This may be mainly attributed to three factors. First, various numerical methods based on the analytical principles have been proposed. These methods are free from the sophisticated mathematical derivation and emphasise more on the aspect of efficient algorithms. Second, structural discretising techniques provide the numerical basis for the algorithms. Among those techniques, the finite element method (FEM) is the most popularly used tool for structural analysis. Third, the availability of powerful digital computers has facilitated the combination of optimisation algorithms and structural analysis techniques to create automated design capabilities. In contrast to the traditional trial-and-error design routine, it was recognised during this period that structural optimisation can be effectively included in the design process. Its applications have been extended to a wide range of fields such as civil, marine, mechanical, automobile and aerospace engineering.

Despite the increasing application of structural optimisation, it has not enjoyed the same level of popularity as the finite element method. This may be due to the variety of optimisation problems in terms of structural system and design constraint. Unlike the finite element method, there is no set procedure in structural optimisation that can be followed for different kinds of problems. Furthermore, as most optimisation methods involve repeated structural analysis and sensitivity calculation, the computational cost can be prohibitively high, particularly for large size structures.

The formulation of the evolutionary structural optimisation (ESO) method has effectively reduced the gap between structural optimisation and finite element analysis. Compared to traditional methods, ESO is characterised by its simple concept and easy adaptability. The basic idea is that by slowly removing inefficient material from a structure, the residual shape evolves towards an optimum. The integration of this idea with the finite element analysis has resulted in a powerful design tool able to address a wide range of optimisation problems. At this stage, ESO has been applied to optimal designs with constraints on stress, stiffness, frequency and buckling load under conventional loading or thermal conditions. The structural systems under consideration include plane and spatial trusses, frames and 2D and 3D continua.

Further advancement was made when the concept of bi-directional evolution was introduced in ESO. While based on the same idea of gradual evolution of the structure, bi-directional evolutionary structural optimisation (BESO) differs from the classical ESO in two ways. First, the efficient material can be added to the structure while the inefficient material is removed to modify the structure. Second, the initial design can be of any size which defines loading and boundary conditions instead of an over-sized domain. Attempts have been made to apply BESO to stress optimisation. The results are in good agreement with those of ESO. Furthermore, BESO has shown great potential for reducing the solution effort.

The work conducted in this thesis involved formulating BESO for stiffness and displacement optimisation. The mathematical aspects were first addressed and the optimisation procedure presented. When combined with finite element analysis, the procedure was programmed and run on digital computers, thus the optimisation proceeded automatically. Computer code was developed to solve various topology optimisations with stiffness and displacement constraints. Comparison of results obtained by BESO and other alternative methods are presented. As two closely related evolutionary techniques, ESO and BESO are further compared in terms of design performance and computational efficiency.

1.2 Aims and Scope of the Research

The aim of the thesis is to investigate the theory and application of BESO for 2D continuous structural systems with stiffness and displacement constraints. The specific objectives are to:

- Explore the general mathematical representation of the evolutionary concept for structural optimisation.
- Investigate topology optimisation subject to stiffness and displacement constraints. Formulate optimisation algorithms using optimality criteria techniques, accounting for different load cases as well as multiple displacement constraints.
- Propose procedures for stiffness and displacement optimisation and develop the computer code linked to the finite element analysis software.
- Conduct numerical tests and compare the results to those obtained by alternative methods.

1.3 Significance of the Research

There is a need to deepen our understanding of the bi-directional evolutionary technique. This will contribute to the improvement and maturity of the evolutionary method in particular, and the advancement of structural optimisation synthesis in general. The techniques tested in the thesis can also provide engineers with a valuable design tool of benefit to the relevant engineering and industrial communities.

1.4 Layout of the Thesis

The thesis consists of seven chapters:

Chapter 1 outlines the general background of structural optimisation and the basic concept of ESO as well as aims and significance of the thesis.

Chapter 2 reviews the history and status of structural optimisation. Different optimisation methods are described and their advantages and limitations are discussed. Among those discussions, approaches to the topology optimisation of 2D and 3D continua are emphasised. The latest development in stiffness and displacement optimisation techniques will be reviewed in detail.

Chapter 3 describes the state-of-the-art of the evolutionary structural optimisation (ESO) method. Basic concepts and procedures for stress and sensitivities approaches are briefly outlined. The background and current results of the bi-directional evolutionary structural optimisation (BESO) method are presented in more detail. The BESO procedure for stress optimisation is described.

Chapter 4 presents the theoretical basis of stiffness and displacement optimisation. The mathematical aspects of BESO are explored by following the optimality criteria procedure. These aspects include the sensitivity analysis, optimality criteria and scaling of design. They are investigated for various cases where alternative loading conditions and multiple displacement constraints are considered. Calculation of the sensitivity number and displacement extrapolation are two major focuses. The procedure of BESO for stiffness and displacement optimisation is proposed for the computer implementation.

Chapter 5 conducts numerical tests conducted on the basis of chapter 4. It is organised according to the design objective (stiffness and displacement) and thus includes two major parts. Examples of stiffness optimisation under single and multiple loading conditions are presented in the first part. Displacement optimisation under the same conditions is set out in the second part, with single and multiple displacement constraints included. Each example is studied by both BESO and ESO. Their results and solution times are compared and the advantages and disadvantages of the two methods are summarised.

Chapter 6 investigates various numerical aspects of BESO method. Measures for improving the reliability of results are first proposed. They include solving problems concerned with sharp changes in structural behaviour, singularity in stiffness matrix and maintenance of design symmetry. Parametric studies on the effect of the initial design, modification ratio and addition ratio are conducted with several examples. Guidelines for parameter selection are given towards the end.

Chapter 7 summarises the results of BESO and reaches general conclusions regarding the effectiveness and efficiency of BESO. Further investigations of the BESO method are recommended.

Chapter 2

Overview of Structural Optimisation

This chapter reviews the development of the theory and application of structural optimisation. The mathematical background is first described, followed by classic methods using the differential calculus and calculus of variations. Numerical methods are reviewed and their algorithms and features are briefly presented. Topology optimisation is introduced in more detail and stiffness and displacement optimisation techniques are highlighted . The chapter concludes by summarising the present situation and future direction of structural optimisation.

2.1 Mathematical Statement

The mathematical interpretation of structural optimisation is related to solving the function extremum. Optimisation involvs with determining the extremum (most often, the minimum) of functions subject to certain constraints (Haftka and Gürdal 1992), i.e.

Minimise $f(\mathbf{x})$. (2.1a)

Such that $g_j(\mathbf{x}) = 0,\ j = 1,\ldots,n_e$, (2.1b)

$h_j(\mathbf{x}) \geq 0,\ j = n_e + 1,\ldots,n_g$, (2.1c)

$\underline{\mathbf{x}} \leq \mathbf{x} \leq \overline{\mathbf{x}}$, (2.1d) where $\mathbf{x}$ is the vector of *design variables* and $f(\mathbf{x})$ is the *objective function*. $g_j(\mathbf{x})$ and $h_j(\mathbf{x})$ are equality and inequality *constraints*, thus the problem is called constrained optimisation. In contrast, those problems without constraints are called unconstrained optimisation. Equation (2.1d) is the side constraint where $\underline{\mathbf{x}}$ and $\overline{\mathbf{x}}$ are the lower and upper bounds of design variables. Design variables, objective functions and constraints constitute the fundamental concepts of structural optimisation.

From the engineering point of view, the *objective function* $f(\mathbf{x})$ is usually chosen to be the criterion/criteria representing the structural volume, weight, cost, performance, serviceability or their combination.

Constraints $g_j(\mathbf{x})$ or $h_j(\mathbf{x})$ can be divided into behavioural constraints and geometrical constraints.

Behavioural constraints imposed on the structural response include:

- Static behaviour: maximum stress, maximum displacement or mean compliance.
- Dynamic behaviour: natural frequency or dynamic response.
- Stability behaviour: buckling load .

Geometrical constraints are related to the non-structural aspects, such as functionality or fabrication. They can be:

- Requirements of the number of structural components.
- Restriction on cross-sectional dimensions.

- Limitation of structural boundaries or holes.

Design variables are independent quantities which define a structure system and can be modified during the optimisation process. They can assume continuous or discrete values. According to the physical significance and the type of design variables, structural optimisation can be divided into three broad categories (Kirsch 1989):

Size optimisation: the design variables can be the thickness of plates or shells, cross sectional properties of bars, beams or columns, either being the section area or the moment of inertia, etc.

Shape optimisation: mainly deals with modification of structural geometry. Geometrical variables can be the coordinates of member joints in discrete structures, the length and location of supports of beam structures or the height of shell structures. They can be either continuous or discrete quantities.

Topology optimisation: for discrete skeletal structures such as trusses, frames or honeycombs, topology optimisation is also known as layout optimisation. It is used to determine the pattern of member connection as well as the number and spatial sequence of nodes and elements. Both size and geometrical variables may be involved. For continuous structures, the optimal topology design is concerned with finding the optimum profile of external and internal boundaries. Topology optimisation is usually accompanied by size and shape optimisations and is the most difficult and challenging task among the three, as will be discussed in later sections.

Many researches have reviewed the development of structural optimisation (Schmit 1981; Vanderplaats 1982). We shall start with classic methods and their significance in mathematical exploration of this field.

2.2 Classic Methods

2.2.1 Differential Calculus

The optimisation problem was noticed as early as several centuries ago. Systematic investigations started when the *differential calculus* was introduced in the 17th century. Conditions for existence of extreme values are stated as that the first order of derivative of objective functions with respect to the design variable is equal to zero , i.e.

$$\nabla f_i(\mathbf{x}) = 0, \quad i = 1,2,...,n. \tag{2.2}$$

The solution vector $\{x_1, x_2, ..., x_n\}$ to the system of equations constitutes the extreme points.

The above situation can only be applied to very simple cases of *unconstrained* optimisation. However, constrained problems are most often encountered in practice. For *equality constrained* optimisation, there are two techniques for deriving the necessary conditions. First, if the constraint equation can be solved to obtain the relationship between dependent design variables, the constrained problems are transformed into unconstrained ones. Second, in cases where constraints are implicit functions of design variables, a general method called *Lagrangian multiplier* technique can be used. Employing the same denotations as in equations (2.1), an auxiliary function making use of the Lagrangian multiplier λ_j is formulated as follows:

$$L(\mathbf{x},\lambda) = f(\mathbf{x}) + \sum_{j=1}^{n_e} \lambda_j g_j \tag{2.3}$$

with the necessary conditions of an extremum expressed as

$$\begin{aligned} &\frac{\partial L}{\partial x_i} = 0, \quad i = 1,\ldots,n, \\ &\frac{\partial L}{\partial \lambda_j} = 0, \quad j = 1,\ldots,n_e. \end{aligned} \tag{2.4}$$

Optimisation is to solve the above system of equations with altogether $n+n_e$ unknowns. The number of Lagrangian multipliers n_e is equal to that of constraints. The purpose of the multiplier is to link the objective functions and the constraints and to determine the relative weight of each constraint.

For a general class of problems with both equality and inequality constraints, the necessary condition for an extremum is summarised as the Kuhn-Tucker conditions. They can be simply expressed as follows:

$$\nabla f_i(\mathbf{x}) + \sum_{j=1}^{n_e} \lambda_j \nabla g_j(\mathbf{x}) + \sum_{j=n_e+1}^{n_g} \lambda_j \nabla h_j(\mathbf{x}) = 0, \quad i = 1,2,\ldots,n, \tag{2.5}$$

The complementary slackness conditions are needed to considered in the above equation and the Lagrangian multipliers for inequality constraints λ_j $(j = n_e+1,\ldots, n_g)$ are required to be greater than zero.

2.2.2 Calculus of Variations

Calculus of variations is a generalisation of the differentiation theory. It deals with optimisation problems having an objective function f expressed as a definite integral of a functional F defined by an unknown function y and some of its derivatives (Haftka and Gürdal 1992).

The objective function can be defined as

$$f = \int_a^b F(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}) dx, \tag{2.6}$$

where y is directly related to the design variable x. Optimisation is to find the form of function $y = y(x)$ instead of individual extreme points.

Analogous to the case of differential calculus, the necessary condition for an extremum is the vanish of the first order of variation:

$$\delta f = \int_a^b \left(\frac{\partial F}{\partial y} + \frac{\partial F}{\partial y'} + \ldots \right) dx = 0. \tag{2.7}$$

Apply boundary conditions, after arrangement, equation (2.7) can be finally expressed in form of *Euler-Lagrange Equation* as follows:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0, \tag{2.8a}$$

with the natural boundary conditions (*x=a* and *x=b*):

$$\left[\frac{\partial F}{\partial y'}\right]_{x=a} = 0, \text{ and } \left[\frac{\partial F}{\partial y'}\right]_{x=b} = 0. \tag{2.8b}$$

The differential calculus and calculus of variations emphasise the analytical exploration of optimisation problems. Their earliest application to structural design might be due to Maxwell (1895) in designing the least weight layout of frameworks. The later research on the optimal topology of trusses by Michell (1904) was well known as Michell type structures. Except for those results, the application of classical analytical methods is very limited because of the mathematical complexity and impractical idealisations, which may lead to meaningless solution in some cases. Nonetheless, analytical methods are of fundamental importance in that they explore the mathematical nature of optimisation and provide the lower bound optimum against which the results by alternative methods can be checked.

2.3 Main Approaches to Structural Optimisation

2.3.1 Mathematical Programming

Mathematical programming (MP) was one of the most popular optimum search techniques which was formulated in 1950s (Heyman 1951). It is a step-by-step search approach involving repeated processes. It starts from an initial design defined by a selected set of design variables. A better design is searched in the direction of gradient of behaviour functions, which is in the form of Lagrangian auxiliary functions as given in equation (2.3). At each step, the value of behaviour function of a new structural design is evaluated. Design variables are modified gradually until the objective function achieves convergence.

At the earlier stage, the mathematical programming method is limited to linear problems where the objective functions and constraints are linear functions of design variables. In 1960s, *nonlinear programming* (NLP) was integrated with finite element analysis as first suggested by Schmit (1960). Since then, numerous algorithms of nonlinear programming techniques have appeared such as feasible direction (Zoutendijk 1960), gradient projection (Rosen 1961) and penalty function methods (Fiacco and McCormick 1968). On the other hand, approximation techniques have also been studied to use the standard linear programming to address nonlinear problems, such as sequential linear programming (Arora 1993).

2.3.2 Optimality Criteria

Optimality criteria (OC) method was analytically formulated by Prager and co-workers in 1960s (Prager and Shield 1968; Prager and Taylor 1968). It is later developed numerically and become a wide accepted structural optimisation method (Venkayya *et al.* 1968). It also adopts concepts of objective functions and constraints but differs from MP in the redesign steps. While the optimum is searched gradually by using direct numerical algorithms in MP, OC method defines *a prior* criterion and the optimum is achieved when the criterion is satisfied. Defining such a criterion may take advantage of the special design condition and structural behaviour.

In general, most of OC algorithms consist of four fundamental steps: structural analysis, stating criteria, scaling and resizing.

The Kuhn-Tucker condition constitutes the optimality criteria:

$$\sum_{j=1}^{n_c} e_{ij}\lambda_j = 1, \quad i = 1,2,\dots,n, \tag{2.9}$$

where $e_{ij} = \frac{\partial g_j}{\partial x_i} \Big/ \frac{\partial f_i}{\partial x_i}$. λ_j is the Lagrangian multiplier, which is usually set as unity in the case of single constraint. For multiple constraints, Lagrangian multipliers need to be solved to identify the active constraints as well as speed up the convergence. e_{ij}, defined as the ratio between the sensitivity of constraints and that of objective functions, is known as Lagrangian energy density. Equation (2.9) provides the physical insight of the OC method that in an optimal design, the weighted sum of Lagrangian energy density is the same for all structural elements.

The optimal criterion of equation (2.9) can be transferred to a recurrence formula to develop an iteration algorithm as follows:

$$x_i^{h+1} = x_i^h \left[\sum_{j=1}^{n_c} e_{ij}\lambda_j \right]^{1/\alpha}, \tag{2.10}$$

where h and h+1 represent the design cycles. α in the exponent is the over-relaxation factor which controls the step size.

While resizing can proceed without the scaling step (Khan and Willmert 1981; Zacharopoulos *et al.* 1984), some improvement on OC algorithm such as generalised compound scaling method proves to be very effective (Grandhi *et al.* 1992).

A special form of OC method is the fully stressed design (FSD) technique for truss structures (Gellatly and Berke 1971). The basic idea behind FSD is that the structure where each member sustains its allowable stress σ_i^U under at least one loading condition has the minimum or near minimum weight. It is a rather intuitive approach as the recurrence formula for updating the design variable is derived from some approximated physical relationships, e.g.

$$x_i^{h+1} = x_i^h \frac{\sigma_i^V}{\sigma_i^U}, \tag{2.11}$$

which is known as stress-ratio approximation.

The mathematical programming and optimal criteria methods are the two best established and widely accepted optimisation techniques. They are equivalent in problem formulations but different in solution algorithms. Mathematical programming is featured by its mathematical elegance and generality. It is less problem specific and is particularly suitable for problems of multiple constraints problem. However, the computational cost increases dramatically when a large number of constraints or design variables are considered. This limits its application to large size structures. In contrast, optimality criteria method is less size dependent and offers a high convergence speed. Though the convergence may become unstable, especially in the case of inappropriately defined initial designs, the relatively low computational cost of OC makes it particularly appealing for large structural systems.

It is worth noting that these two methods are reconciled, to a large extent, by formulating *dual MP* methods (Fleury 1979), which can be interpreted as *generalised OC* methods. In dual methods, the constrained primary minimisation problem is transformed into the maximisation of a quasi-unconstrained dual function which is only related to the Lagrangian multipliers. When the primal problem is convex, explicit and mathematically separable, use of dual methods is very effective by introducing some intermediate design variables. Based on the use of reciprocal design variable, the convex linearisation method (CONLIN) (Fleury and Braibant 1986) was well developed and was later generalised as the method of moving asymptotes (MMA) (Svanberg 1987). As dual methods search the optimum direction in the space of Lagrangian multipliers instead of that of the primal design variables, it can save considerable computing efforts when the number of constraints is smaller than that of design variables.

2.3.3 Genetic Algorithms

Genetic algorithms (GA) were originally developed in 1970s (Holland 1975). In recent years, it has been extended to the field of structural optimisation (Goldberg 1989). The principle of genetic algorithms uses Darwinian's theory of survival of the fittest. The procedure consists of reproduction, crossover and mutation. In the beginning, an initial population of designs (individuals) is randomly created, with design variables represented by a code of bit strings. The fitness of each individual is evaluated according to a fitness function. Those fittest members are allowed to reproduce and cross among themselves, resulting in a new generation with member having higher degree of most favourable characteristics than the parent generation. This process repeats iteratively until the best individual of the population reaches a near-optimum solution.

Genetic algorithms may not be as efficient as traditional MP or OC methods because they are quite computationally intensive. Nonetheless, they still serve as reliable and robust techniques for their merits. Compared to the gradient-based search methods, genetic algorithms search the solution more extensively in that in involves a set of a candidate solutions (individuals). They work on the objective function itself rather than its derivatives and are more likely to converge to a global optimum instead of a local one. Furthermore, genetic algorithms transfer design variables into a code representation, typically, into a binary bit-string, which is integer in nature. Therefore, it is highly potential for problems involving a mix of continuous, discrete and integer design variables. This makes genetic algorithms very suitable for composite structures (Nagendra *et al.* 1993; Le Riche and Haftka 1994; Kogiso *et al.* 1994).

2.4 Topology Optimisation

Most of the previous work is limited to size optimisation where the structural layout is not allowed to change during optimisation. Shape and topology optimisations have attracted increasing interests recently. Compared to the fixed-layout optimisation, the optimal topology design can result in far more significant improvement on structure performance and bring substantial savings in material costs.

The earlier efforts are devoted to the optimal layout design of discrete structures. The topic has its origin in Michell's least weight design of truss structures (Michell 1904). His work has been further developed later by Prager and Rozvany and the results are well known as layout theory (Prager and Rozvany 1977; Rozvany 1989). In recent years, more work has been done on topology optimisation of 2D and 3D continua with the emergence of many efficient and robust algorithms. The following sections are to introduce different methods in each structure subdivision.

2.4.1 Discrete Structures

There are many publications reviewing the history and advancement in this field. According to the survey by Topping (1993), methods for the optimal layout design can be grouped into three categories according to the choice of design variables:

Geometric approach

Both the coordinates of joints and cross-sectional properties are taken as design variables in this approach. The earliest work might be due to Schmit in optimising a three-bar truss using a steepest descent-alternative mode nonlinear algorithm (Schmit 1960). In this approach, the number of joints and connecting members is fixed unless some joints coalesce during optimisation, which changes the structural configuration. One major problem associated with the geometric approach is the inclusion of mixed design variables. Size and geometrical variables can be of different order in magnitude, which adds difficulties to the overall convergence. This leads to the formulation of the hybrid approach.

Hybrid approach

The hybrid approach divides size and geometrical variables into two design spaces. Accordingly, there are two steps in updating design variables. For example, in studying a truss structure subject to multiple load cases (Vanderplaats and Moses 1972), first, the element is resized by stress-ratio methods while the topology keeps unchanged. Then the optimal position of element nodes is determined next.

Ground structure approach

In contrast to the foregoing two methods, the ground structure approach only deals with size variables. A ground structure consists of a dense set of nodes and a large number of potential connections between those nodes. The number and position of the nodes are fixed while the number and size of connecting elements are altered. Size variables are still continuous, but if the section area of some elements reduces to zero during optimisation, these elements are deleted from the structure and the topology image changes accordingly.

Combined with the mathematical programming and optimality criteria algorithms, the ground structure approach is now widely used in layout optimisation. The effect of the initial grid definition on the final optimum has also been studied (Dorn *et al.* 1964).

2.4.2 Continuous Structures

The traditional method for topology optimisation of 2D and 3D continua can be the boundary variation approach. A great deal of literature has appeared regarding the mathematical model, description of boundary shapes, generation of finite element mesh and solution strategies (Bennett and Botkin 1986; Haftka and Grandhi 1986).

The description of boundary shapes is essential to the boundary variation approach. There are three ways to represent the boundary, namely, the boundary nodes, polynomials and splines. In the survey by Ding (1986), different ways are compared in respects of design variable selection, numerical accuracy and optimal shape. For the numerical implementation of the boundary variation approach, a capability of automated mesh refinement is indispensable. The refinement can be undertaken either globally by re-dividing the whole structure or locally by introducing additional elements or increasing the order of finite elements. On the basis of the above boundary description and mesh regeneration techniques, some conventional solution strategies such as mathematical programming, optimality criteria and genetic algorithms (Yang 1988; Kita and Tanie 1997) are used to solve the optimum problem.

In comparison to the boundary variation method, there is another class of methods using the ground structure approach. As also discussed in the last sub-section, a ground structure is an over-populated structure universe consisting of a large number of potential structural elements. The optimisation is to determine the elements occupied by the material, i.e. the density distribution. A remarkable advantage of the ground structure approach is that the design domain is fixed thus the problem of the mesh re-generation can be avoided. Using the ground structure description, there have appeared three kinds of methods for the continuum topology optimisation, namely, the homogenisation method, density function method and heuristic methods.

Homogenisation Method

This method is featured by the composite material representation of structure and equivalent homogenisation coefficients (Bendsøe and Kikuchi 1989).

First, the element of a discretised structure is modelled as porous media composed of solid materials and voids at the microscopic level. Take a square element with a rectangular cavity for example, as shown in Fig. 2.1. The characteristics of the cell can be represented by the spatial coordinates a, b. In the case where the hole is allowed to rotate, the rotation θ is also taken as a design variable. The states of the porous media can be described as follows:

$a = b = 0$, void.

$0 < a < 1$,
$0 < b < 1$, composite material with void.

$a = b = 1$, solid.

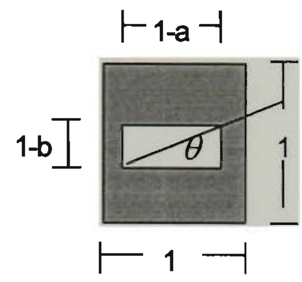


Fig. 2.1: Porous media.

The equivalent properties of the porous media are computed using the homogenisation theory (Babuska 1976 and 1977). Those properties, namely, the elastic modulus and mass density, are discontinuous within the cell. However, their equivalence can be derived as functions of spatial coordinates a, b. Then the constitutive relationship can be determined from the equivalent elastic coefficients together with a matrix of rotation related to θ. By this means, these three spatial coordinates are updated gradually to find the optimal material distribution.

The homogenisation method has been successfully applied to 2D and 3D continua for both static and dynamic problems with weight constraint (Tenek and Hagiwara 1993; Ma *et al.* 1995). The effect of different cell models on computing results has also been examined. In algorithm aspects, the homogenisation method employs traditional mathematical programming or optimality criteria as search techniques. On one hand, it carries on their advantages such as the rigorous theoretical basis and good convergence behaviour. On the other hand, difficulties associated with those traditional methods are magnified in the homogenisation method. As shown in Fig 2.1, each element has three design variables and their sensitivity analysis can be very time-consuming for large size structural systems.

Density Function Method

This method was proposed for stiffness and frequency optimisations of continuous structures (Yang and Chuang 1994). The design objective is to minimise the mean compliance subject to the weight constraint. The essence of the method is an empirical relationship between the elasticity modulus and mass density with the latter used as design variables. The density function method has yielded similar results to those of the homogenisation method. However, it is strongly dependent on the empirical relationship assumption.

Heuristic Methods

The term 'heuristic methods' refers to those addressing structural optimisation problems in a less mathematical but more intuitive way. It is proposed in comparison to those conventional methods where either mathematical programming or optimality criteria algorithm is followed as a solution routine. As the name suggests, heuristic methods do not involve much complex mathematical formulation. Instead, they are derived from simple concepts or natural laws. Those methods fall into two categories, as discussed below.

Adaptive biological growth method was first formulated by Mattheck (1997). This technique has its philosophical origin in nature. It is motivated to simulate the growth process of biological species that adapt themselves to the environment. The simple and natural principle of this method is that in optimisation, the shape of structure evolves to reach a uniform stress state. This is easily achieved by adding material in over-stressed areas and removing in under-stressed areas. Two strategies have been suggested for changing the material distribution.

The first one is called soft kill option (SKO). The structure is first analysed by the finite element method to obtain the element stress. Then the Young's modulus E is adjusted and set the value of the stress. This means that the stronger areas will sustain more load than the weaker areas in an updated structure. The new structure represented by non-homogenous material (different modulus) is re-analysed and stress is redistributed. This process repeats until there is not much change in Young's modulus. During this course, the less loaded area becomes softer and softer until the modulus reduces to near-zero, then the element is 'killed' and removed from the structure.

The second technique is related to some fictitious temperature fields. The element stress is transformed into nodal temperatures. The finite element analysis is then performed to find the nodal thermal displacement, which represents the expansion or shrinkage of the element. By this means, the structure shape changes and grows to an optimum design.

Evolutionary structural method (ESO) is based on the simple idea that by slowly removing inefficient material from a structure, the residual shape evolves towards an optimum (Xie and Steven 1993). It shares similarities with SKO in principles. While the SKO method kills a low-stress element softly by gradually changing the Young's modulus, the evolutionary method removes this element immediately at one step. At each single iteration, only a small number of elements are removed in order to ensure a smooth transient between generations.

The research on ESO is quite extensive and covers problems with stress, stiffness/displacement, frequency and buckling load constraints (Xie and Steven 1997; Chu 1997; Manickarajah 1998). In these cases, the material efficiency is measured by the element stress as well as sensitivity number. The calculation of sensitivity numbers and evolutionary procedure will be detailed in the next chapter.

It is natural to extend the theoretical basis of ESO by allowing material to be added as well as removed. This new approach is called *bi-directional evolutionary structural optimisation* (BESO) (Querin 1997). An attractive characteristic of BESO is that the evolution can start from a very simple initial design instead of an over-populated domain. BESO with the stress constraint has been investigated for 2D and 3D continua (Young *et al.* 1998).

As discussed above, heuristic methods are simple in concept and flexible in implementation. They are easily programmed on computers. This is particularly true when the finite element analysis software has become a common design tool.

In order to have a clear view of the recently developed methods for topology optimisation and allow a quick comparison, Table 2.1 summarises the main characteristics of some techniques presented in the previous sections:

Table 2.1: Comparison of methods for topology optimisation.

Items	Methods	Homogenisation Method	Density Function Method	SKO	ESO
Constraints	Uniform Stress			✓	✓
	Stress Concentration			✓	✓
	Displacement				✓
	Compliance	✓	✓		✓
	Frequency	✓	✓		✓
	Buckling Load				✓
Structural System	1D Discrete Structure				✓
	2D Continuum	✓	✓	✓	✓
	3D Continuum	✓	✓	✓	✓
	Plate in Bending	✓			✓
Loading Conditions	Single	✓			✓
	Multiple	✓			✓
Algorithms	FEM	✓	✓	✓	✓
	MP/OC	✓	✓		

2.5 Stiffness and Displacement Optimisation Techniques

Stiffness and displacement requirements can be due to the consideration of structural serviceability. For example, the lateral displacement of a high rise building or the deflection of a bridge has to be within a prescribed limit. Most of optimisation methods discussed in the last two sections can be applied to stiffness and displacement optimisation. To some extent, investigations on the optimal design with stiffness and displacement constraints have been the starting point of other more complicated problems such as the system stability and dynamic response. This is due to that the displacement is the basic form of structure response in finite element analysis, which is fundamentally formulated by using the displacement method. Further, the virtual load method allows for the expression of displacement in form of the structural strain energy. This greatly facilitates the generation of algorithms for stiffness optimisation, displacement optimisation and even stress optimisation. In optimising some truss structures, it is a common practice to convert stress constraints to displacement constraints by introducing a pair of virtual loads on the concerned structural component. Most often, the stiffness and displacement optimisation can be treated as a linear problem if the design variable is appropriately selected. The use of reciprocal variables greatly simplifies the mathematical formulation and helps to develop a highly effective algorithm (Berke 1970). Nonetheless, complexities are added when the structure is imposed on multiple displacement constraints. It is normally unlikely that all constraints are active at an optimum. While the presumption of each constraint as active one makes the algorithm inefficient, failure to include all potential active constraints may influence the function convergence. For this reason, considerable efforts have been devoted to the determination of active constraints as well as estimation of Lagrangian multipliers, and many methods have been suggested such as the recurrence relations, linear equations and Newton-Raphson methods (Taig and Kerr 1973; Rizzi 1976; Austin 1977).

While most earlier investigations concentrate on the design of continuous variables, the optimisation involving discrete design variables has seen significant progress recently. This is driven by the engineering design practice where the structural component can be selected from a set of sizes. Discrete problems can be solved in two steps. First, the problem is treated as a continuous variable optimisation and the solution is found;

Second, the discrete solution is proposed on the basis of the continuous solution. Many techniques have been suggested for the second step, such as rounding-up, Lagrangian relaxation, pseudo-discrete section selection and branch and bound methods (Ringerts 1988; Sandgren 1990; Chan *et al.* 1994). Huang and Arora (1995) have provided a comparison of different methods and pointed out that the computation of mixed variable problems can be substantially higher than that of continuous problems.

More recent development in stiffness and displacement optimisation may be the topology design of compliant mechanisms (Ananthasuresh 1994). Compliant mechanisms are widely used for transferring the force or motion thus may have requirements both on stiffness and flexibility. The optimisation aims at designing a mechanism so that the output displacement at a certain node is maximised and at the same time the global stiffness is ensured. Two forms of objective functions are proposed to account for the above two contradicting design requirements, namely, the weighted linear combination (Ananthasuresh 1994) and ratio of displacement to mean compliance (Frecker *et al.*). It is found that the optimality criteria derived from these two objective functions take the same form, which states that the ratio of virtual potential energy to the strain energy is equal for each element. Saxena and Ananthasuresh (1998) have investigated the convergence behaviour of the objective function and proposed an algorithm based on the combination of optimality criteria and mathematical programming.

2.6 Summary

Over the past several decades, structural optimisation has grown from an abstract mathematical concept to a practical engineering design tool. It has become a fusion of multi-disciplinary subjects covering mathematics, mechanics, engineering, structural analysis and computer graphics. During this course, its applications have been extended to various fields such as aeronautical, mechanical, automobile, civil and marine engineering.

As far as the algorithm is concerned, mathematical programming (MP) and optimality criteria (OC) seem to have reached their mature stage. Most of the recent work is the refinement of these methods, with focus put on special considerations arising from different problems such as structural stability and dynamic behaviour. Along with the extension of traditional techniques, heuristic methods play an increasingly important role particularly for topology optimisation. These methods are simple in concept and easy for computer implementation. They are able to deal with almost all of the corresponding problems solved by traditional methods.

A most significant factor contributing to the advancement of structural optimisation can be the availability of high capacity digital computers. From the historical point of view, the progress in the field of structural optimisation was relatively slow before 1950s. The development has accelerated in 1960s when a variety of numerical algorithms were implemented on powerful yet inexpensive computers. Computer aided design has become an indispensable feature of structural optimisation. Researches have been conducted world-wide to develop the optimisation software tailored to different fields of industries, such as TSO (Lynch *et al.* 1977) developed by Air Force Wright Aeronautical Laboratories, STARS (Wellen and Bartholomew 1990) by Royal Aerospace Establishment and CAOS (Rasmussen 1990) by Technical University of Denmark. This process is still under way and points the trend of future exploration, especially those in shape and topology optimisation.

Chapter 3

State-of-the-Art of Evolutionary Structural Optimisation

3.1 Introduction

The concept of evolution adopted in structural optimisation has been suggested frequently during the past 15 years. The synthesis of evolutionary structural optimisation (ESO) method has been developed since it is first used for the optimal design of uniform stress structures (Xie and Steven 1993). ESO is based on the idea that by systematically removing the inefficient material, a structure can evolve towards an optimum.

This concept is clearly reflected in the fully stressed design (FSD). A fully stressed design is a highly idealised optimum where every part of the structure sustains its allowable stress so as to make the best use of material strength. The optimum can be obtained from an initial design by repeatedly removing the inefficient, i.e. low-stressed material. Such a technique can be called *stress approach* as it uses the element stress, e.g. von Mises stress, as the driving criterion in the evolution process.

In comparison to the stress approach, there is another kind of optimisation problems using the sensitivity number as the driving criterion. Stiffness/displacement, frequency and buckling load optimisation can be grouped into this category. The ESO method for these problems is called ***sensitivity approach***. We shall introduce in this chapter these two approaches, followed by an new technique called bi-directional evolutionary structural optimisation (BESO). The chapter concludes with a summary of features of the evolutionary method.

3.2 Stress Approach

ESO is a numerical method combined with the finite element analysis (FEA). It progresses in an iterative manner. The procedure of stress approach can be outlined as follows:

1. Define the design domain which the structure is allowed to occupy. Set up a finite element mesh to fully cover the domain.

2. Perform finite element analysis to obtain the stress distribution.

3. As the design is over-sized and far from an optimum, the element stress level σ_e can be quite different within the design domain. The lightly stressed elements are not efficiently used and can be removed. An inequality is defined to identify those inefficient elements as follows:

$$\sigma_e < RR_i \, \sigma_{\max} , \tag{3.1}$$

where $\sigma_{\max}$ is the maximum element stress and RR_i is the current *rejection rate*. Remove elements satisfying the inequality and the structure is updated.

4. Repeat steps 2 and 3 using the same value of RR_i until no elements satisfy the inequality. This means that the structure has reached a steady state corresponding to the current RR_i. To proceed the evolution, the rejection rate is assigned a new value by the following recurrence equation:

$$RR_{i+1} = RR_i + ER, \quad i = 0,1,\ldots,n, \tag{3.2}$$

where ER is the *evolutionary rate*.

5. Steps 2 to 4 are repeated and steady states corresponding increasing rejection rates are obtained progressively.

6. The evolution terminates when the stress limit is exceeded or a prescribed amount of material is reached.

On the basis of the original formulation, many forms of variations of ESO have been proposed for different problems using the stress approach.

- *Uniform surface stress*: elements can only be removed from the structural boundary and no inner cavity is produced. The structure evolves to a shape where the surface stress is uniformly distributed. This technique is called nibbling ESO (Xie and

Steven 1997).

- *Reduction of stress concentration*: shapes of cut-out, hole, joint, etc. are optimised in order to reduce the maximum stress. Nibbling ESO techniques are also adopted (Xie and Steven 1997).
- *Intelligent cavity creation* (ICC): non-structure constraints are imposed apart from the uniform stress requirement. The optimum has a prescribed number of cavities (Kim 1998).
- *Thermal stress optimisation*: to obtain the optimum design of uniform stress under the thermal load conditions (Li *et al.* 1997).
- *Elastic contact*: the contacting profile of several separate bodies is optimised to reduce the maximum contact stress (Li *et al.* 1998).
- *Nonlinear problems*: structures with material and geometric nonlinearities are investigated where the strain energy density is used as the evolution criterion (Querin *et al.* 1996).

3.3 Sensitivity Approach

3.3.1 Sensitivity Analysis

Apart from the strength requirement, a structure may also needs to comply with requirements on displacement/stiffness, frequency or buckling load. The sensitivity analysis is to study the effect of material elimination on the above structural behaviour. Derivations in this section are based on the work by Chu (1997), Manickarajah (1998) and Xie and Steven (1996).

The static behaviour of a discretised structure is governed by the following equilibrium equation:

$$\mathbf{K}\mathbf{u} = \mathbf{P}, \tag{3.3}$$

where $\mathbf{K}$ is the global stiffness matrix, $\mathbf{u}$ is the displacement vector and $\mathbf{P}$ is the load vector.

Suppose that the *i*th element is removed from a structure, the mean compliance, defined by $C = \frac{1}{2}\mathbf{P}^T\mathbf{u}$, will have a change equal to

$$\alpha_i = \Delta C = \mathbf{u}_i{}^{\mathrm{T}}\mathbf{K}_i\mathbf{u}_i, \tag{3.4}$$

where $\mathbf{K}_i$ is the element stiffness matrix and $\mathbf{u}_i$ is the element displacement vector. α_i is called stiffness sensitivity number.

For dynamic problems, the equation for free vibration is

$$(\mathbf{K} - \omega_j^2\mathbf{M})\mathbf{u}^{(j)} = \mathbf{0}, \tag{3.5}$$

where $\mathbf{M}$ is the global mass matrix, ω_j is the circular frequency of the *j*th mode shape and $\mathbf{u}^{(j)}$ is the corresponding eigenvector.

The eigenvalue sensitivity due to an element removal is

$$\alpha_i^{(j)} = \Delta(\omega_j^2) = \mathbf{u}_i^{(j)T}(\omega_j^2\mathbf{M}_i - \mathbf{K}_i)\mathbf{u}_i^{(j)}, \tag{3.6}$$

where $\mathbf{M}_i$ is the element mass matrix and $\mathbf{u}_i^{(j)}$ is the element eigenvector. It is assumed in equations (3.5) and (3.6) that the eigenvector $\mathbf{u}^{(j)}$ has been normalised with respect to the global mass matrix $\mathbf{M}$.

The buckling behaviour of a structure is represented by the following eigenvalue problem:

$$(\mathbf{K} + \lambda_j\mathbf{K}_g)\mathbf{u}^{(j)} = \mathbf{0}, \tag{3.7}$$

where $\mathbf{K}_g$ is the geometric stiffness matrix, λ_j is the *j*th the eigenvalue and $\mathbf{u}^{(j)}$ is the corresponding eigenvector.

In the case of size optimisation, suppose that the *i*th element has a stiffness change $\Delta\mathbf{K}_i$ due to resizing. Perform the similar mathematical derivations to those in frequency sensitivity, the sensitivity of the fundamental eigenvalue is found to be

$$\alpha_i = \Delta\lambda_1 = \mathbf{u}_i{}^T\Delta\mathbf{K}_i\mathbf{u}_i, \tag{3.8}$$

where the effect of size modification on the geometric stiffness matrix has been ignored.

The sensitivity number represents the contribution of element modification to the concerned structural behaviour. In stiffness optimisation, for example, we usually want to reduce the mean compliance. Therefore, eliminating the elements with the smallest absolute value of sensitivity number will be the most effective. Similarly, in buckling or frequency optimisation, if we want to increase the frequency or buckling load (a common situation), elements with the largest sensitivity number can be removed.

3.3.2 Evolution Procedure

1. Construct a finite element model considering all supports and loads.
2. Conduct the finite element analysis to obtain the structural response. They can be the displacement in static problems and eigenvalue and eigenvector in eigenvalue problems.
3. Calculate the sensitivity number α_i for each element using equation (3.4), (3,6) or (3.8).
4. Remove elements according to the sensitivity number and the optimisation requirement so that the structure evolves towards a desired direction.
5. Repeat steps 2~4 until the structure reaches the prescribed weight or the change in structure behaviour becomes negligible.

3.4 Aspects in Computer Implementation

Software Interface

As discussed above, the structural optimisation task is usually divided into two parts, namely, the structural analysis and design modification. For most of optimisation techniques, the first part utilises numerical methods such as the finite element or boundary element method. The second part is fulfilled by using algorithms based on mathematical programming or optimality criteria.

There are two ways in implementing the optimisation task as a whole design process on computers. The first is that the designer writes a program that includes both parts. This is impractical and unnecessary because many structural analysis software packages have been developed and become easily accessible. Most often, the designer is provided with both analysis and optimisation modules and the task is reduced to develop the interface between them. However, this is not so easy a task. Most optimisation algorithms need to repeatedly evaluate the derivatives of objective functions and constraints using the structural analysis results. At this point, it is very difficult to transform the structural analysis package into a subroutine called by the design updating program.

The interface between structural analysis and optimisation is relatively simple in ESO as the two modules are physically independent. For sensitivity approaches, for example, the input to the optimisation code can be either displacements or mode shapes as the output of the finite element analysis. The FEA is performed by using a standard commercial software STRAND6 (G+D Computing 1993) in which the output is identical in form thus can be processed similarly in the optimisation code.

Element Status

As far as the structural description is concerned, ESO can be interpreted as a ground structure approach because if defines an over-populated structural universe. As discussed in Chapter 2, in a ground structure the position of nodes and elements are fixed and only the number of elements is changing in optimisation. In ESO, The initial finite element mesh is used throughout the evolution and the element property number is used to declare the existence and absence of an element. For example, in the beginning, each element within the design domain is assigned a non-zero property number according to its physical material properties such as the Young's modulus, Poisson's ratio and plate thickness. If an element is eliminated during the process, its property number is switch to zero. This means that in the current structure, this element does not physically exist thus is ignored in assembling the global stiffness and/or mass matrices.

3.5 Bi-directional Evolutionary Structural Optimisation (BESO)

3.5.1 Background

ESO is an iterative method and hundreds of runs of finite element analysis may be needed before the optimum is reached. For this reason, the size of the finite element model becomes an important factor which can heavily affect the solution time. To ensure that there are adequate elements left after repetitive structural modifications, an over-sized initial FE model is required in ESO. For some structures divided by a fine FE mesh, or 3D problems, the computational cost of ESO can be very high. Another concern of ESO is that elements removed in previous iterations cannot be recovered later. This requires that the number of elements removed at each iteration should be very small. Otherwise, elements can be deleted prematurely and the evolution may be misled.

The bi-directional evolutionary structural optimisation (BESO) method provides an answer to the above two problems. BESO allows for removing inefficient elements as well as *adding efficient ones*. Therefore, it is more flexible in choosing the initial design and recovering the inappropriately removed element. By defining a small and simple initial design, BESO can significantly reduce the size of the finite element model thus improve the computing efficiency.

3.5.2 Procedure

The implementation of BESO for stress optimisation is straightforward as the element stress is used to determine the material efficiency and inefficiency. The procedure is outlined as follows:

1. Specify the maximum allowable physical domain and discretise it with a finite element mesh.
2. Specify the *initial design* which contains the connecting elements defining the loading and supporting conditions. Elements other than connecting elements are

assigned a property number 0. There are many initial designs satisfying the boundary conditions and it is natural to use the simplest one with the smallest number of elements. Variations of initial designs have effect on the final solution and this point will be investigated in Chapter 6.

3. Carry out the finite element analysis and calculate the element stress. The element removal or addition is determined by the following two inequalities:

$$\sigma_e < RR_i \, \sigma_{\max} \qquad \text{and} \qquad (3.9a)$$

$$\sigma_e > IR_i \, \sigma_{\max} , \qquad (3.9b)$$

where IR is called *inclusion rate.* All elements are checked against expression (3.9a) to decide if it is under-stressed. The element is removed on satisfaction of this expression. Additionally, the *boundary elements* are checked against (3.9b). Take a 4-node square element for example, the boundary elements are featured by at least one free edge and can be easily tracked during optimisation. If a boundary element satisfies expression (3.9b), it means that the element is over-loaded and is strengthened by adding elements around its free edges.

4. Repeat step 3 until a steady state is reached. Update the removal rate and inclusion rate by the following recurrence formulas:

$$RR_{i+1} = RR_i + ER, \quad i = 0,1,\ldots,n, \qquad (3.10a)$$

$$IR_{i+1} = IR_i - ER, \quad i = 0,1,\ldots,n. \qquad (3.10b)$$

5. Repeat steps 2~4 until the rejection rate becomes, say, as large as 25%, or some prescribed perform index (Querin 1997) reaches the minimum.

The extension of BESO to other categories using the sensitivity approach is the main task of this thesis. Stiffness optimisation using the BESO procedure will be explored in length in subsequent chapters.

3.6 Summary

The strength of evolutionary method lies in its simplicity and generality, which can be attributed to two factors. Firstly, it employs the finite element method as the structural analysis tool, so a wide range of structural systems can be covered. Secondly, it uses the element stress or sensitivity number to drive the evolution. Those driving criteria are similar in form thus the evolution procedure can be also similar. In fact, a common procedure exists for different kinds of problems and only the calculation of driving criteria is different. In cases where the solution cost and robustness become a concern for large scale structures, the bi-directional ESO (BESO) serves as an alternative technique.

Chapter 4

BESO for Stiffness and Displacement Optimisation-Theory

This chapter deals with the theoretical basis of BESO method. Principles of optimality criteria are followed to explore the mathematical interpretation of stiffness and displacement optimisation including the optimality condition, sensitivity analysis and scaling techniques. The bi-directional evolutionary algorithm is proposed and programmed to the computer code which contains displacement extrapolation, sensitivity calculation and element modification. The code is linked to the finite element analysis software to realize a computer-aided-design process.

The term 'design variable' will be frequently used in this chapter. For simplicity in description, the design variable x_i is chosen as a non-dimensional quantity. For truss structures, it is defined as $x_i = A_i / A_{oi}$ where A_{0i} is the bar area. For 2D continua under plane stress or plate bending conditions, $x_i = t_i / t_{oi}$ is assumed where t_{0i} represents the plate thickness.

Stiffness optimisation is first studied, followed by the more complex displacement optimisation where problems of single and multiple constraints are addressed. Optimisation under single and multiple loading conditions is investigated.

4.1. Stiffness Optimisation

For a system modeled by finite elements, the static behavior is represented by the following equilibrium equation:

$$\mathbf{K}\mathbf{u} = \mathbf{P}, \tag{4.1}$$

where $\mathbf{K}$ is the global stiffness matrix, $\mathbf{u}$ is the displacement vector and $\mathbf{P}$ is the load vector.

The overall stiffness of a structure can be indirectly evaluated by the mean compliance, which is defined as

$$C = \frac{1}{2}\mathbf{P}^T\mathbf{u} = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} = \sum_{i=1}^{n}(\frac{1}{2}\mathbf{u}_i{}^T\mathbf{K}_i\mathbf{u}_i) = \sum_{i=1}^{n}C_i\,, \tag{4.2}$$

where $\mathbf{K}_i$ and $\mathbf{u}_i$ are the stiffness matrix and displacement vector of the ith element. $C_i = \frac{1}{2}\mathbf{u}_i{}^T\mathbf{K}_i\mathbf{u}_i$, is the element strain energy. Based on such a definition, designing the stiffest structure is equivalent to minimising the mean compliance C .

4.1.1 Sensitivity Analysis

A typical optimality criterion includes two components, namely, sensitivity of objective function and constraint, and Lagrangian multipliers. Therefore, before formulating the criteria, we shall first investigate the derivatives of the mean compliance and structural weight, i.e. sensitivity analysis.

Differentiating equation (4.2) with respect to the ith design variable results in

$$\frac{\partial \mathbf{K}}{\partial x_i}\mathbf{u}+\mathbf{K}\frac{\partial \mathbf{u}}{\partial x_i}=\frac{\partial \mathbf{P}}{\partial x_i}, \tag{4.3}$$

Assume that the load vector does not change with the design variable, thus

$$\frac{\partial \mathbf{u}}{\partial x_i}=-\mathbf{K}^{-1}\frac{\partial \mathbf{K}}{\partial x_i}\mathbf{u}. \tag{4.4}$$

Referring to equation (4.2), the derivative of the mean compliance is

$$\frac{\partial C}{\partial x_i}=\frac{1}{2}\mathbf{P}^T\frac{\partial \mathbf{u}}{\partial x_i}=-\frac{1}{2}\mathbf{P}^T\mathbf{K}^{-1}\frac{\partial \mathbf{K}}{\partial x_i}\mathbf{u}=-\frac{1}{2}\mathbf{u}^T\frac{\partial \mathbf{K}}{\partial x_i}\mathbf{u}. \tag{4.5}$$

Suppose the design variable has a small change and becomes x_i'. Using the first order Taylor series, the mean compliance will change as follows:

$$\Delta C=\sum_{i=1}^{n}\frac{\partial C}{\partial x_i}(x_i'-x_i)=-\frac{1}{2}\mathbf{u}^T\sum_{i=1}^{n}\left(\frac{\partial \mathbf{K}}{\partial x_i}(x_i'-x_i)\right)\mathbf{u}. \tag{4.6}$$

Assume that the stiffness matrix is a linear function of the zth order of the design variable, i.e.

$$\mathbf{K}(c\,\mathbf{x}^z) = c\mathbf{K}(\mathbf{x}^z), \tag{4.7}$$

where c is an arbitrary constant.

If one element is removed from the structure, making use of equations (4.6) and (4.7), the change in the mean compliance due to such a removal is

$$\Delta C = -\frac{1}{2}\mathbf{u}_i^T \frac{\partial \mathbf{K}_i}{\partial x_i}\mathbf{u}_i(0-1) = \frac{z}{2}\mathbf{u}_i^T \frac{\mathbf{K}_i}{x_i}\mathbf{u}_i. \tag{4.8}$$

The change due to an element addition is similar to equation (4.8) but different in sign. Therefore,

$$\Delta C = \frac{z}{2}\mathbf{u}_i^T\mathbf{K}_i\mathbf{u}_i = zC_i. \quad \text{(for element removal)} \tag{4.9a}$$

$$\Delta C = -\frac{z}{2}\mathbf{u}_i^T\mathbf{K}_i\mathbf{u}_i = -zC_i. \quad \text{(for element addition)} \tag{4.9b}$$

It is noted that ΔC is always positive for removed elements and negative for added elements.

The above sensitivity analysis is based on the first order derivative. For structures of z=1 such as trusses and 2D continua under plane stress conditions, the first order approximation is sufficiently accurate. In the case of z >1 which occurs to plate bending problem (z=3), it is desirable to employ higher order derivatives. However, those derivatives are complicated in form and its computing cost can be unnecessarily high (Haftka and Gürdal 1992). As far as ESO and BESO are concerned, it is found from the numerical experience that the first order derivative is also reliable for plate bending elements (Chu 1997). For this reason, the sensitivity analysis in this thesis employs the linear approximation.

As for the weight constraint, it changes as follows:

$$\Delta W = -W_i \quad \text{(for element removal)} \tag{4.10a}$$

$$\Delta W = W_i \quad \text{(for element addition)} \tag{4.10b}$$

4.1.2 Optimality Criteria

For optimisation using the sensitivity approach, the evolutionary method is a gradient based search technique. BESO for stiffness optimisation can be mathematically formulated by following the optimality criteria procedure.

The problem of stiffness optimisation with a prescribed weight W^* can be stated as

$$\text{Minimise} \quad f = C(\mathbf{x}) = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} \tag{4.11a}$$

$$\text{Subject to} \quad g = W^* - \sum_{i=1}^{n} W_i x_i = 0, \tag{4.11b}$$

$$x_i \in \{0,1\}. \tag{4.11c}$$

The design variable is chosen from a set $\{0,1\}$, which declares the absence or presence of an element.

The Lagrangian function is

$$\begin{aligned} L(\mathbf{x},\lambda) &= f-\lambda g \\ &= C(\mathbf{x})-\lambda(W^*-\sum_{i=1}^{n}W_i x_i), \end{aligned} \tag{4.12}$$

where λ is the Lagrangian multiplier.

In conventional OC method, the optimality criterion for problem of continuous design variables is

$$\frac{\partial L}{\partial x_i}=\frac{\partial f}{\partial x_i}-\lambda\frac{\partial g}{\partial x_i}=0,\quad i=1,\ldots,n. \tag{4.13}$$

However, the design variable is discrete in the evolutionary method. So it is necessary to replace the derivative in equation (4.13) with the function increment, i.e.

$$\Delta L_i=\frac{\partial L}{\partial x_i}\Delta x_i=\frac{\partial f}{\partial x_i}\Delta x_i-\lambda\frac{\partial g}{\partial x_i}\Delta x_i=0,\quad i=1,\ldots,n, \tag{4.14}$$

where ΔL_i denotes the increment in the Lagrangian function due to the change in the ith design variable.

Recalling equations (4.9) and (4.10), for a removed element:

$$\frac{\partial f}{\partial x_i}\Delta x_i=\Delta C=zC_i, \tag{4.15a}$$

$$\frac{\partial g}{\partial x_i}\Delta x_i=-\Delta W=W_i,\quad i=1,\ldots,n, \tag{4.15b}$$

and for an added element:

$$\frac{\partial f}{\partial x_i}\Delta x_i = \Delta C = -zC_i, \quad (4.16a)$$

$$\frac{\partial g}{\partial x_i}\Delta x_i = -\Delta W = -W_i, \quad i = 1,\ldots,n. \quad (4.16b)$$

Substituting equations (4.15) and (4.16) into equation (4.14) results in

$zC_i - \lambda W_i = 0$, or

$$\lambda = \frac{zC_i}{W_i}. \quad (4.17)$$

The constant z can be omitted as it is the same for all structural elements, therefore,

$$\lambda = \frac{C_i}{W_i} > 0. \quad (4.18)$$

Equation (4.18) represents the optimality criterion of the evolutionary algorithm. It is consistent with the well known condition regarding the overall stuffiness optimisation. That is, at an optimum the ratio of element strain energy to its weight is the same for all structural elements (Venkayya *et al.* 1973; Morris 1982).

Equation (4.18) can also be interpreted as an effectiveness parameter of the *i*th element. For continuous problems, such a parameter is defined as the ratio of derivatives of the objective function and that of the constraint. The resizing is conducted so as to "*increase the utilisation of the more effective variables and decrease that of the less effective ones*" (Haftka and Gürdal 1992). BESO does not involve the resizing procedure. Instead, it employs element removal and addition to modify the structure. Similar to the principle adopted in resizing techniques, the structure is updated in such a way that the most effective elements are added and the least effective ones are removed.

At this point, equation (4.18) serves as an indicator of element efficiency. Introducing $C_i = \frac{1}{2}\mathbf{u}_i{}^T\mathbf{K}_i\mathbf{u}_i$ to equation (4,18), adding a negative sign for added elements and omitting the coefficient '1/2', one obtains that

$$\alpha_i = \frac{\mathbf{u}_i^T\mathbf{K}_i\mathbf{u}_i}{W_i}. \qquad \text{(for element Removal)} \qquad (4.19a)$$

$$\alpha_i = -\frac{\mathbf{u}_i^T\mathbf{K}_i\mathbf{u}_i}{W_i}. \qquad \text{(for element addition)} \qquad (4.19b)$$

α_i is called the sensitive number. To minimise the compliance it will be most effective to remove and add elements with the smallest sensitivity number.

Equation (4.18) can be re-written as

$C_i = \lambda W_i$, then

$$\sum_{i=1}^{n} C_i = \lambda \sum_{i=1}^{n} W_i. \qquad (4.20)$$

Therefore, at the optimum, the Lagrangian multiplier is

$$\lambda = \frac{C}{W*}, \qquad (4.21)$$

which is identical to the conclusion obtained by Morris (1982).

Substitute equation (4.21) into the Lagrangian function equation (4.12), it is found that

$$L(\mathbf{x}) = \frac{W \times C}{W^*}. \tag{4.22}$$

4.1.3 Scaling of Design

4.1.3.1 Objective Compliance

Most of optimality criteria algorithms involve scaling procedure. It is to move an intermediate design to the constraint boundary and to help to trace the change in objective function. For exterior approximation problems where the solution is searched out of the feasible region, scaling technique is also used to convert an infeasible design to the feasible region.

Scaling techniques in stiffness/displacement constraint have been proposed by many researchers (Kirsch 1993; Morris 1982). For the problem as stated in equation (4.11), scaling is performed so that the structure can satisfy the weight specification. Based on the previous definition of design variables, the structure weight is proportional to the design variable. Therefore, a scaling parameter can be easily defined as

$$\mu = \frac{W^*}{W} = \frac{x_i'}{x_i}, \quad i = 1,2,\ldots,n \tag{4.23}$$

where x_i and x_i' denote the design variables before and after scaling, respectively. All the design variables are scaled equally by the same parameter μ. Therefore, the stiffness matrix of a scaled design is

$$\mathbf{K}(\mathbf{x'}^z) = \mathbf{K}((\mu\mathbf{x})^z) = \mu^z\mathbf{K}, \tag{4.24}$$

where we assume that the stiffness matrix is the linear function of the *z*th order of design variables.

The displacement vector is

$$\mathbf{u}(\mathbf{x'}) = \mathbf{K}^{-1}(\mathbf{x'}^z)\mathbf{P} = \mu^{-z}\mathbf{K}^{-1}\mathbf{P} = \mu^{-z}\mathbf{u}(\mathbf{x}) \tag{4.25}$$

and the objective function is

$$C(\mathbf{x'}) = \frac{1}{2}\mathbf{P}^T\mathbf{u}(\mathbf{x'}) = \mu^{-z}(\frac{1}{2}\mathbf{P}^T\mathbf{u}(\mathbf{x})) = \mu^{-z}C = \left(\frac{W}{W*}\right)^z C. \tag{4.26}$$

Equation (4.26) represents the compliance of a feasible design which possesses the current topology while having the target weight $W*$. This is defined as the *objective compliance*:

$$C_{obj} = \left(\frac{W}{W*}\right)^z C. \tag{4.27}$$

It is worth noting that equation (4.27) reduces to equation (4.22) when $z=1$ is assumed.

Equation (4.23) is not valid if the structural weight is not proportional to the design variable, e.g. a frame structure where the moment of inertia is taken as the design variable. In such a case, an iterative procedure is required to approximate the scaling parameter μ and the subsequent objective compliance. This problem has been discussed by Morris (1982).

4.1.3.2 Generalised Sensitivity Number

In stiffness optimisation, the structural weight and mean compliance are always changing in opposite directions, i.e. increasing weight will cause decrease in structural strain energy and vice versa. Their overall effect can be combined in equation (4.27), which can serve two purposes. First, it evaluates the extent of optimisation of a candidate design. A smaller value means a better solution. Secondly, it transfers the constrained optimisation problem in form of (4.11) to an equivalent unconstrained one where only the objective compliance C_{obj} is to be minimised.

For simplicity, assume that z =1 in the objective compliance. Suppose an element is modified in the structure, either being removed or added, differentiating equation (4.27) and omitting the effect of W^* yields:

$$\Delta C_{obj} = W(\Delta C) + (\Delta W)C. \tag{4.28}$$

Define

$$\eta_i = \frac{\Delta C_{obj}}{CW} = \frac{\Delta C}{C} + \frac{\Delta W}{W}. \tag{4.29}$$

At the optimum where C_{obj} assumes the minimum, each element should have $\eta_i = 0$. When the design is away from the optimum, however, it is to find the element with the minimum of η_i (η_i <0) so that its removal or addition will cause the largest decrease in the objective compliance. In this sense, η_i serves as a generalised sensitivity number. As derived before, using the sensitivity numbers α_i defined in equations (4.19a) and (4.19b), the structural elements are divided into two groups as potentially added and removed elements. Sensitivity numbers are compared in each group separately. In contrast, the generalised sensitivity number η_i make the effect of element removal and addition comparable. It is a common indicator for all elements.

At this point, η_i is more suitable for locating modified elements. However, numerical accuracy problems may arise in the practical implementation. As shown in equation (4.9), computing C_i requires information on the element displacement. For potentially added elements, the displacement is undefined and can only be calculated approximately. C_i based on the approximated displacement is considerably larger than that of existing elements. For this reason, the generalised sensitivity number η_i is only of theoretical significance. In implementation of BESO, α_i is used and removed and added elements are compared separately. This point will be further discussed in later sections on displacement approximation.

4.1.4 Multiple Loading Conditions

In previous sections, some basic concepts in BESO such as optimality criterion, sensitivity number and objective compliance are presented. They are used for problems of single loading condition. A structure may be subject to different environments and support several load cases. The above derivation can be extended to multiple load cases with only minor changes.

Assume there are l load cases applied independently. Re-define the objective function in equation (4.11a) as

$$f = \sum_{k=1}^{l} a_k C^K(\mathbf{x}). \tag{4.30}$$

Where $C^k(\mathbf{x})$ is the mean compliance associated with the kth load case and a_k is the weighting coefficient. As a special case, f is defined as the average compliance, i.e. assuming $a_k = 1/l$.

Accordingly, the sensitivity number for multiple loading conditions becomes

$$\alpha_i = \frac{\pm\sum_{k=1}^{l} a_k C_i^K}{W_i} = \sum_{k=1}^{l} a_k \alpha_i^k, \tag{4.31}$$

which is the weighted sum of individual sensitivity numbers, where the positive and negative signs are for removed and added elements, respectively.

Likewise, the objective compliance is

$$C_{obj} = \left(\frac{W}{W*}\right)^{z} \sum_{k=1}^{l} a_k C^k(\mathbf{x}) = \sum_{k=1}^{l} a_k C_{obj}^k \tag{4.32}$$

4.2 Displacement Optimisation

In many situations, it is required that the displacement at a particular location of a structure is within an allowable limit. Displacement optimisation can be approached in a similar way as stiffness optimisation.

Optimisation with a single displacement constraint is first investigated in this section, followed by multiple constraint problems including multiple load cases and multiple displacement constraints.

4.2.1 Sensitivity Analysis

A usual approach to dealing with an individual displacement is to use a *virtual load vector* $\mathbf{Z}^{(1)}$, which has all its components being equal to zero except the one corresponding to the constrained displacement component. The non-zero component is given a unit value and the unit virtual load is of the same direction as the displacement constraint. The superscript in $\mathbf{Z}^{(1)}$ means the first displacement constraint.

The constrained displacement can be expressed as

$$u^{(1)} = \mathbf{Z}^{(1)T}\mathbf{u} = \mathbf{Z}^{(1)T}\mathbf{K}^{-1}\mathbf{P} = \mathbf{u}^{(1)T}\mathbf{K}\mathbf{u} , \tag{4.33}$$

where $\mathbf{u}^{(1)}$ is the displacement response to the virtual load, defined by:

$$\mathbf{K}\mathbf{u}^{(1)} = \mathbf{Z}^{(1)} . \tag{4.34}$$

Equation (4.33) can be re-written as

$$u^{(1)} = \sum_{i=1}^{n} \mathbf{u}_i^{(1)T}\mathbf{K}_i\mathbf{u}_i = \sum_{i=1}^{n} C_i^{(1)} , \tag{4.35}$$

where $\mathbf{u}_i^{(1)}$ is the element displacement vector due to the virtual load. $C_i^{(1)}$ is known as the element virtual strain energy.

Suppose that an element is removed from the structure. For simplicity, z=1 in equation (4.7) and the linear approximation of displacement are assumed. Using equation (4.4), the displacement will change as follows:

$$\Delta u^{(1)} = \mathbf{Z}^{(1)T}\frac{\partial \mathbf{u}}{\partial x_i}\Delta x_i = \mathbf{Z}^{(1)T}\mathbf{K}^{-1}\frac{\partial \mathbf{K}}{\partial x_i}(0-1)\mathbf{u} = \mathbf{u}_i^{(1)T}\mathbf{K}_i\mathbf{u}_i = C_i^{(1)} . \tag{4.36}$$

Similarly, an added element will cause changes in displacement as

$$\Delta u^{(1)} = \mathbf{Z}^{(1)T}\frac{\partial \mathbf{u}}{\partial x_i}\Delta x_i = -\mathbf{u}_i^{(1)T}\mathbf{K}_i\mathbf{u}_i = -C_i^{(1)} . \tag{4.37}$$

4.2.2 Single Displacement Constraint

4.2.2.1 Optimality Criteria

The optimisation problem with a single displacement constraint u^* can be stated as

Minimise $\quad f = W = \sum_{i=1}^{n} W_i x_i$. (4.38a)

Subject to $\quad g = u^* - u \geq 0$, (4.38b)

$\quad x_i \in \{0,1\}$. (4.38c)

The Lagrangian function is

$$L(\mathbf{x}, \lambda) = W - \lambda(u^* - u) . \tag{4.39}$$

The optimality criterion is

$$\frac{1}{\lambda} = \frac{(\partial g/\partial x_i)\,\Delta x_i}{(\partial f/\partial x_i)\Delta x_i} = \frac{-\Delta u^{(1)}}{\Delta W} . \tag{4.40}$$

Substituting equations (4.10) and (4.37) into equation (4.40) leads to

$$\frac{1}{\lambda} = \frac{C_i^{(1)}}{W_i} . \tag{4.41}$$

which is similar to equation (4.18). In fact, they are equivalent as dual problems. Like in stiffness optimisation, the sensitivity number is defined as

$$\alpha_i = \frac{\mathbf{u}_i^{(1)T}\mathbf{K}_i\mathbf{u}_i}{W_i}. \qquad \text{(for element Removal)} \qquad (4.42a)$$

$$\alpha_i = -\frac{\mathbf{u}_i^{(1)T}\mathbf{K}_i\mathbf{u}_i}{W_i}. \qquad \text{(for element addition)} \qquad (4.42b)$$

$\alpha_i^{(1)}$ can be either positive or negative.

The Lagrangian multiplier can be calculated at the optimum as

$$\frac{1}{\lambda} = \frac{u^*}{W}. \qquad (4.43)$$

Substituting it to equation (4.39) results in the following Lagrangian function

$$L(\mathbf{x}) = \frac{u \times W}{u^*}. \qquad (4.44)$$

4.2.2.2 Scaling of Design

The scaling technique used here is to scale the whole structure so that it maintains the current topology and at the same time the constrained displacement is equal to the limit.

Based on the same assumption as in stiffness optimisation, it is found that

$$W(\mathbf{x}') = \mu W = \left(\frac{u}{u^*}\right)^{\frac{1}{z}} W = W_{obj}, \qquad (4.45)$$

where W_{obj} is called objective weight.

4.2.3 Multiple Constraints

4.2.3.1 Optimality Criteria

The multiple constraint optimisation is much more difficult than single optimisation. Firstly, the active constraint is to be identified out of a series of constraints. This is usually conducted by using design scaling techniques. Secondly, in single constraint problems, it is possible to use a relative value for the Lagrangian multiplier, which is often set to unity. This is not the case for multiple constraints. In fact, the calculation of Lagrangian multiplier is an important task of optimisation involving multiple constraints.

The Lagrangian function of multiple constraint problems can be expressed as

$$L(\mathbf{x},\lambda) = W - \sum_{j=1}^{m} \lambda_j (u^{(j)} - u^{(j)}*) . \tag{4.46}$$

Where $u^{(j)}*$ is the displacement limit of the jth constraint and λ_j is the corresponding Lagrangian multiplier.

The optimality criterion can be derived as

$$1 = \frac{\sum_{j=1}^{m} \lambda_j (\partial g_j / \partial x_i) \Delta x_i}{(\partial f / \partial x_i) \Delta x_i} = \frac{\sum_{j=1}^{m} \lambda_j C_i^{(j)}}{W_i} , \tag{4.47}$$

where $C_i^{(j)}$ is the virtual energy of the ith element associated with the jth displacement constraint. Equation (4.47) is the same as the result based on the reciprocal approach (Morris 1982). It states that at the optimum, the weighted sum of the ratio of virtual strain energy to the element weight is equal to unity. Lagrangian multipliers are the weighing factors.

From equation (4.47), the sensitivity number of multiple constraint problems is derived as

$$\alpha_i = \frac{\pm\sum_{j=1}^{m} \lambda_j C_i^{(j)}}{W_i} = \sum_{j=1}^{m} \lambda_j \alpha_i^{(j)}, \qquad (4.48)$$

where $\alpha_i^{(j)}$ is the individual sensitivity number calculated from equation (4.42).

4.2.3.2 Calculation of Lagrangian Multipliers

Computing the Lagrangian multiplier plays a key role in the multiple constraint optimisation. In conventional optimality criteria algorithms, Lagrangian multipliers are required for recurrence relations to resize the design variable. In BESO, as stated in equation (4.48), Lagrangian multipliers are weighting factors of the sensitivity number of an individual load case.

Different methods for calculating the Lagrangian multiplier have been surveyed by Morris (1982). Improvement on these conventional methods is suggested by Chu (1997) for the use of ESO. In summary, there are three types of approaches:

1. Recurrence relations

$$\lambda_j^{h+1} = \lambda_j^h \left(\frac{u}{u^*} \right)_h^{1/b}, \tag{4.49}$$

where h represents the iteration cycle.

It is necessary to re-write the optimality criterion equation (4.47) as follows in order to discuss the features of the above recurrence relation:

$$1 = \frac{\sum_{j=1}^{m} \lambda_j C_i^{(j)}}{W_i} = \frac{\mathbf{u}_i^T \mathbf{K}_i \sum_{j=1}^{m} \lambda_j \mathbf{u}_i^{(j)}}{W_i}. \tag{4.50}$$

As shown in equation (4.50), once the Lagrangian multiplier is available, the sum of element virtual energy as the numerator can be computed easily. The individual strain energy of a single constraint needs not to be computed. Instead, a unit virtual load vector weighted by the Lagrangian multiplier $F = \sum_{j=1}^{m} \lambda_j \mathbf{Z}^{(j)}$ is applied to determine the overall virtual displacement. Furthermore, the recurrence relation can gradually reduce the Lagrangian multiplier of a passive constraint. Therefore, it is not necessary to differentiate between the active and passive constraints at one iteration. However, this may cause problems when a passive constraint become potentially active in later stage. A remedy to this drawback is to define a ratio relation, as discussed below.

2. Ratio relation

$$\lambda_j^{h+1} = \left(\frac{u}{u^*}\right)_h^{1/b}, \qquad (4.51)$$

in which the Lagrangian multiplier is only related to the current displacement. This makes sense as the effect of displacement greater than the constraint will dominate and the contribution of those far below the limit can be very small. Also, once their contribution becomes more significant, such a change can be easily picked up and reflected in the Lagrangian multiplier.

The exponent b is a step size parameter. When a design is in the feasible region ($u^{(j)} < u^{(j)}*$), a value of $b > 1$ can be used to decrease the difference between contributions of constraints and $b < 1$ will magnify the difference. The converse is valid when the design is in the infeasible region.

3. Linear equations

The following linear equations have been proposed by Chu (1997):

$$\sum_{p=1}^{m} \lambda_p \sum_{i=1}^{n} \left(\frac{(\alpha_i^{(j)} \operatorname{sign} u^{(j)})(\alpha_i^{(p)} \operatorname{sign} u^{(p)})}{W_i} \right) = u^{(j)}* \qquad (j = 1,\ldots,m), \qquad (4.52)$$

where

$\operatorname{sign} u^{(j)} = 1$ when $u^{(j)} > 0$ and
$\operatorname{sign} u^{(j)} = -1$ when $u^{(j)} < 0$.

The Lagrangian multiplier can be calculated by solving a system of equations as given above. The major advantage of this method is that it takes account of the interdependence among constraints. However, the computational cost is much higher than that of the other two relations.

In this thesis, Lagrangian multipliers are calculated according to the ratio relation of equation (4.51) and $b=1$ is assumed.

4.2.3.3 Scaling of Design

The scaling factor is defined as

$$\mu = \max_{j=1,m}\left(\frac{u^{(j)}}{u^{(j)*}}\right)^{\frac{1}{z}} . \tag{4.53}$$

The objective weight is

$$W_{obj} = \mu W . \tag{4.54}$$

That is, the design is scaled according to the most critical constraint.

4.2.3.4 Multiple Loading Conditions

The Lagrangian function for multiple loading conditions is

$$L(\mathbf{x},\lambda) = W - \sum_{j=1}^{m} \lambda_j^k (u^{(j)k} - u^{(j)*}) \quad k = 1,\ldots,l, \tag{4.55}$$

where $u^{(j)k}$ is the displacement under the kth loading condition.

From equation (4.48), the sensitivity number for multiple loading conditions becomes

$$\alpha_i = \frac{\pm \sum_{k=1}^{l} \sum_{j=1}^{m} \lambda_j^k C_i^{(j)k}}{W_i} = \sum_{k=1}^{l} \sum_{j=1}^{m} \lambda_j^k \alpha^{(j)k}, \tag{4.56}$$

where $C_i^{(j)k}$ is the virtual strain energy associated with the jth constraint due to the kth load case. λ_j^k is the corresponding Lagrangian multiplier which is calculated from $\lambda_j^k = \frac{u^{(j)k}}{u^{(j)*}}$.

The scaling factor μ is defined as

$$\mu = \max_{\substack{j=1,m \\ k=1,l}} \left(\frac{u^{(j)k}}{u^{(j)*}} \right)^{\frac{1}{z}}. \tag{4.57}$$

An alternative approach to the multiple constraint problem is to consider the most critical loading condition. Out of l load cases, there exists one case where the concerned displacement is closest to the given limit and this load case is identified as the most critical one for a certain constraint j. Evidently, if a scaled design satisfies the jth displacement constraint, the constraint can be automatically satisfied by the other less critical load cases. At an iteration, the most critical cases are determined for each constraint and the subsequent analysis is based on these cases. The problem can be greatly simplified by this means.

4.3 Calculation of Sensitivity Number

4.3.1 Displacement Extrapolation

A basic concept in sensitivity number α_i is the element strain energy, as given in equations (4.19) and (4.42). The utilisation of the virtual load method generalises the calculation of these two kinds of sensitivities. They can be computed at element level as only the information of the concerned element is needed. This makes computation very economical and efficient. Also, in the case of multiple loading conditions, multiple displacement constraints or their combination, the overall sensitivity is the weighed sum of sensitive numbers of individual constraints. The weighting coefficients can be Lagrangian multipliers in equations (4.48) and (4.56) or some preset values in equation (4.32), depending on the nature of the problem.

The sensitivity analysis is performed after the finite element analysis. For existing elements, the sensitivity can be calculated readily as the information on both stiffness matrix and displacement is available. For those potentially added elements, extra work is needed as some of the nodal displacements are undefined.

First, though the term 'potentially added elements' has already been used occasionally in previous sections, it is necessary to give it a clear definition here. Take a plane stress problem for example, where four-node rectangular elements are adopted, as shown in Fig. 4.1. The shaded area represents the current structure with the boundary depicted by darker lines. Elements can be added along the structural boundary, either internal or external one. Those elements, partly drawn by dashed lines, are called 'potentially added elements'.

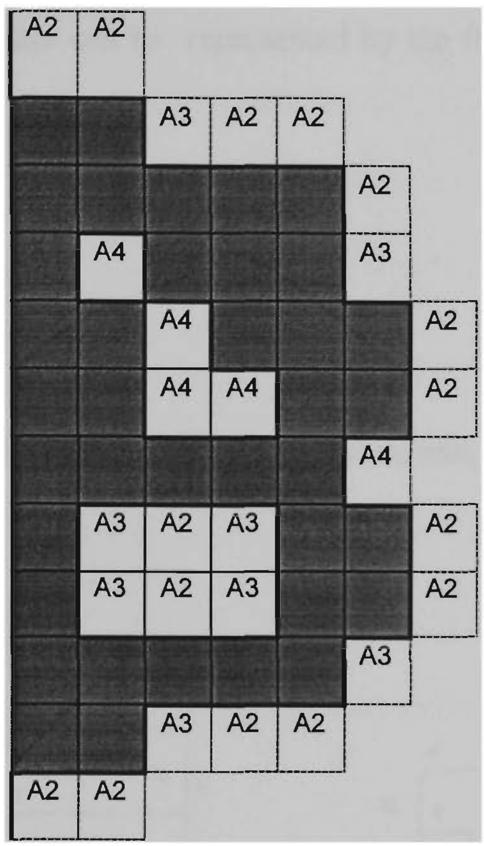


Fig. 4.1: Potentially added elements.

The potentially added elements can be divided into three types according to the number of undefined nodal displacements.

A2: connected with one boundary element and having two undefined nodal displacements.

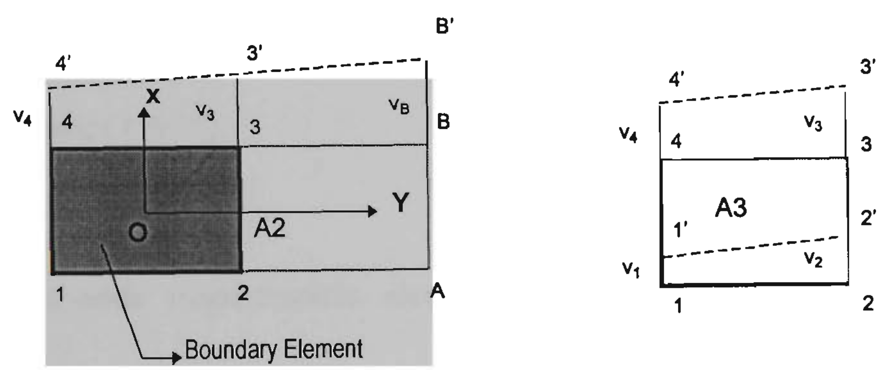
A3: connected with two boundary elements and having one undefined nodal displacements.

A4: connected with three or four elements and all nodal displacements are known.

The undefined displacements can be approximated by extrapolation. The displacement field of a four-node element can be represented by the following bi-linear polynomial (Zienkiewicz 1977):

$$u = a_0 + a_1 x + a_2 y + a_3 xy$$
$$v = b_0 + b_1 x + b_2 y + b_3 xy, \tag{4.58}$$

where u and v are horizontal and vertical displacements, as shown in Fig. 4.2(a).



(a) Element type A2 (b) Element type A3

Fig. 4.2: Displacement extrapolation.

The two displacement components vary linearly along the element edge, as illustrated by line 4'3'. Extending such a linear distribution to the adjacent element A2, the displacement of node B is easily found to be

$$v_B = 2v_3 - v_4 \ . \tag{4.59}$$

The displacement of node 2 is calculated in the same way.

For element of type A3 with one nodal displacement unknown, it is assumed that the element is shaped as a parallelogram after deformation, as shown in Fig. 4.2(b). Therefore,

$$v_3 = v_2 + v_4 - v_1 \ . \tag{4.60}$$

The above graphical demonstration of displacement extrapolation is consistent with more rigorous mathematical derivations. Without losing generality, we briefly discuss here the shape function approach. More details on this topic can be found in the work by Zienkiewicz (1977).

For a four-node isoparametric element, displacement within the element can be expressed as

$$u = \sum_{i=1}^{n} N_i u_i, \quad v = \sum_{i=1}^{n} N_i v_i, \tag{4.61}$$

where n=4 , and u_i and v_i are displacements of nodes 1, 2, 3 and 4, as shown in Fig. 4.3. N_i is the shape function, defined with respect to the local coordinate system $\xi O \eta$. The shape function is expressed as the following equations:

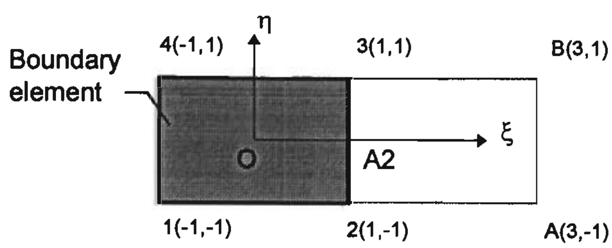


Fig. 4.3: A four-node isoparametric element.

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta) \qquad (4.62)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta).$$

The displacement extrapolation is based on the assumption that a point outside a reference element is defined using the same coordinate system as this element. Therefore, as shown in Fig. 4.3, nodes A and B have coordinates (3,-1) and (3,1) respectively, in the $\xi O\eta$ system defined for the boundary element. Substituting the coordinates (ξ, η) of nodes A and B to equation (4.62) leads to:

Node A: $N_1 = -1$, $N_2 = 2$ and $N_3 = N_4 = 0$. (4.63a)

Node B: $N_3 = 2$, $N_4 = -1$ and $N_1 = N_2 = 0$. (4.63b)

Substituting the above values into equation (4.61). The same expression as equation (4.59) is obtained.

It is worth noting that the above derivation based on shape functions is also applicable for four-node elements under plate bending conditions. The displacements in *xy* plane *u* and *v* are simply replaced by the deflection *w* and rotations with respects to the two axes θ_x and θ_y .

Further, it is also noted that the shape function approach can be applied to any type of finite element. Though the final displacements may not assume such simple forms as equations (4.59) and (4.60), the concept and procedure of the displacement extrapolation are applicable in general.

The underlying principle of the displacement extrapolation is that a potentially added element is to comply with the compatibility condition. However, the static equilibrium condition is not satisfied within the concerned element. For this reason, the element strain energy is over-estimated based on the fictitious displacement field. This point is reflected in the value of the sensitivity number. Take the respective maxima of sensitivity numbers of existing elements and potentially added elements for example. The difference between them can be as large as one order. This makes it necessary to group these two kinds of elements separately, as discussed in section 4.1.3.2 about the generalised sensitivity number η_i .

4.3.2 Modified Sensitivity Number for Eliminating Checkerboard Patterns

In optimising 2D continua, the checkerboard pattern is often observed where the solid and void elements distribute in an alternative manner similar to a checkerboard. Designs with checkerboard patterns are not desired in practice because of the highly unsmooth internal profile.

The reason due to the checkerboard pattern can be the numerical instability. Sigmund and Petersson (1998) has discussed the methods for preventing the formation of checkerboard patterns. The simplest one uses some post-processing techniques to smooth the resulted topology. Also, some techniques are suggested to suppers the appearance of checkerboard right in the solution process, such as the patch and filter techniques introduced to the homogenisation method (Bendsøe *et al.* 1993, Díaz and Bendsøe 1992). In the evolutionary method, it is found that checkerboard pattern is more likely to occur to four-node elements. Use of higher order element like eight-node isoparametric elements can avoid this problem. However, a higher order element means much higher more computational cost. Therefore, some alternative methods based on four-node element have been proposed.

The first one can be grouped into the post-processing techniques. First, some appropriate elements are recovered to the structure so that the checkerboard pattern can be eliminated temporarily. This will usually affect the objective function in a negative way. Therefore, the second step is to perform shape optimisation on the newly defined design. Elements are only removed from the boundary so that no further holes are produced. This strategy has been used for stiffness optimisation in ESO (Chu 1997). Another strategy is similar to the filter technique used in the homogenisation method and the effect of the neighbor elements on the specific element sensitivity is accounted for, as suggested by Manickarajah (1998). The extra work needed is to calculated an *average sensitivity number*. The steps are outlined briefly as follows:

1. The element sensitivity number is calculated using equation (4.19) or its variations. This is referred to as the initial sensitivity number.

2. This initial sensitivity number is assigned to the four nodes of the element and is called 'nodal sensitivity number'. A node adjoined by several elements has the same number of nodal sensitivity numbers. The nodal sensitivity numbers are summed up and an average is worked out.

3. Then an element has four average nodal sensitivity numbers. Again, they are summed up and the resulting average is called the *modified sensitivity number*. It is this number rather than the initial one that is used to decide the element modification in subsequent procedures.

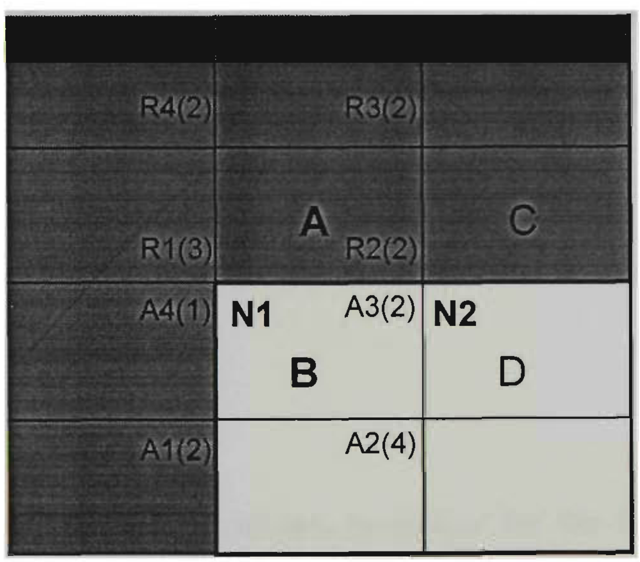


Fig. 4.4: Nodal sensitivity number.

The above pre-processing strategy is used in this thesis due to its simplicity and computing efficiency. When it is applied, the existing element and potentially added element are treated separately. For example, we shall calculate the sensitivity number for elements A and B in Fig. 4.4. The letter beside each node, e.g. A1(2) means the nodal sensitivity of the *first* node of an *additional* element B and the node connects *two* additional elements. There are two common nodes N1 and N2 in these two elements. For N2, it has two nodal sensitivities, R2(2) and A3(2). R2(2) is calculated by taking average of two existing elements A and C, whereas A3(2) is obtained as an average of two additional elements B and D. These two numbers are later assigned to their corresponding existing and potentially added elements.

4.4 Procedure of BESO

4.4.1 Basic Concepts

Like most of numerical methods, BESO carries out optimisation in an iterative and progressive manner. In each iteration, BESO is made up of three parts, namely, structural analysis, sensitivity number calculation and element modification. The procedure is very general in that only the second part varies with the nature and property of constraints.

We shall start with some basic concepts necessary for the further discussion of the BESO procedure.

1. Full design

The full design is a design domain that the structure is allowed to occupy, which is normally an over-sized area. This is a common feature of the ground structure approach. The reason is that there should be adequate elements remaining to represent the final optimum after repeated element elimination. The weight of the full design is denoted by W_0.

2. Initial design

The initial design is the one from which the evolution starts with. BESO can start optimisation from an arbitrary design within the full design. Preferably, it uses the simplest one connecting the loads and supports. An initial design can be specified either manually or automatically. The former is adopted in this thesis.

3. Maximum design

A maximum design is relevant to the objective compliance C_{obj} or objective weight W_{obj}. A structure grows from an initial design gradually. During this course, the objective compliance or objective weight decreases, which means the structure is evolving towards a better solution. There is a certain point at which C_{obj} or W_{obj} starts increasing. This means that the growth in structural weight on longer makes a better design. Therefore, it is possible to make C_{obj} or W_{obj} decrease again by reducing the structural weight. The design corresponding to this changing point is referred to as the maximum design and it has the largest weight $W_{\max}$ out of the whole evolution.

At this point, it is noted that C_{obj} or W_{obj} is not always decreasing smoothly. It keeps a decreasing trend but some small increase may happen occasionally. So a maximum design is not decided immediately after an increase is seen in C_{obj} or W_{obj}. Instead, if they have increased for, say, 10 continuous iterations, it is assumed that the maximum design has reached.

4. Modification ratio, addition ratio and removal ratio

As an iterative method, it is important to maintain a smooth change between designs of consecutive iterations. At a single iteration, the number of modified elements can be decided by the *modification ratio* and the *reference structure*. A reference structure can be the full design or the current structure. For example, if the current design with, say, 1000 elements is chosen as the reference design and modification ratio MR =1% is used, there will be 1000 ×1%, namely, ten elements modified at that iteration. In BESO, there are another two parameters, namely, *addition ratio* (AR) and *removal ratio* (RR). If, say, AR=0.6 and MR=1-0.6=0.4 are assumed, among the ten modified elements, six are added, four are removed and the net increase in the element number is two.

4.4.2 Evolutionary Procedure

For stiffness (case 1) and displacement optimisation (case 2), BESO is carried out in the following steps.

1. Construct a finite element mesh in the *full design*. Apply boundary and loading conditions. All elements are assigned a property number 0.

2. Specify the *initial design* within the full design. Elements in the initial design change their property numbers from 0 to 1. This means that in the current structure, only these elements physically exist and are used for the structural analysis. Elements with a property number 0 are absent. They will be ignored when the global stiffness matrix is assembled.

3. Perform the finite element analysis on the current structure to obtain the displacement response.

4. Identify the potentially added elements, namely, A2, A3 and A4 in Fig. 4.1. Assign different property numbers, say, 2, 3 and 4 to corresponding elements. Computer the nodal displacement by equations (4.59) and (4.60) in the case of four-node elements of plane stress problem.

5. Calculate the initial sensitivity number by using equation (4.19) or its variations for all existing and potentially added elements. Further calculate the modified sensitivity number by the procedure outlined in section 4.3.2.

6. Decide the number of modified elements according to the modification ratio MR, addition ratio AR and reference structure. Compare sensitivity numbers for existing and potentially added elements respectively. Remove elements with the smallest values of α_i and change their property numbers from 1 to 0. At the same time, add those elements with the smallest α_i and their property numbers change from 2,3 or

4 to 1. Change the property numbers of all the rest potentially added elements to 0. $AR > 0.5$ is used so that the structural weight increases.

7. Repeat steps 3 to 6 until a *maximum design* with $W = W_{\max}$ is reached. Set $AR < 0.5$ so that removed elements outnumber added ones thus the structure reduces its weight.

8. Repeat steps 3 to 7 until certain conditions are satisfied. For stiffness optimisation (case 1), the iteration stops when the prescribed weight is reached. For displacement optimisation (case 2), the evolution terminate when all the displacement constraints are satisfied.

Further, in stiffness optimisation, it may happen that the weight of the maximum design $W_{\max}$ is smaller than the prescribed weight W^*. In this situation (referred to as case 1A), steps 7 and 8 are changed as follows:

7. Repeat steps 3 to 6 until the structure arrives at the prescribed weight W^*. Set $AR = 0.5$ so that the weight keeps constant. During this course, the objective weight W_{obj} may decrease slightly. The program terminates when W_{obj} begins increasing for, say, 5 consecutive iterations. If there is no clear increasing trend, the program terminates when W_{obj} becomes oscillating. The design with the minimum W_{obj} is chosen as the optimum.

It is seen that the change of structural weight is divided into two stages: before the maximum design is reached, the weight increases and after the maximum design it decreases (or keeps constant). These two stages are referred to as *ascending* and *descending* stages, respectively. Different values of AR are assigned in these two stages.

For the clarity of description, it is assumed that all elements are of the same size and the material has homogenous density. If these assumptions do not hold true in some cases, only the value of addition ratio needs to be adjusted to accommodate factors of the element size and density whereas the procedure is the same.

The above procedure has been programmed into a computer code called BESODSP, which is link to the FEA software STRAND6. The flowchart of BESODSP is given at the end of this chapter as Fig 4.5.

The input of STRAND6 includes all physical, geometrical and load properties. The constrained displacement is treated as a load case where a unity load in the same direction as the constraint is imposed on the concerned node. Suppose that there are altogether NTOT load cases, the number of real load cases NREAL is entered into the input of BESODSP. Then the rest NTOT-NREAL (NTOT>NREAL) cases will be automatically identified as the virtual loads, i.e. constrained displacements. In the case of NTOT=REAL (no constrained displacement), the code processes the problem as a stiffness optimisation.

Both ESO and BESO can be performed by running the code BESODSP. A variable IBESO is set aside to identify the method to be used, with IBESO=1 representing BESO and IBESO=0 representing ESO.

4.4.3 Discussions on Different Structural Systems

There are three points worth noting regarding the topology optimisation of different types of structures using BESO procedure.

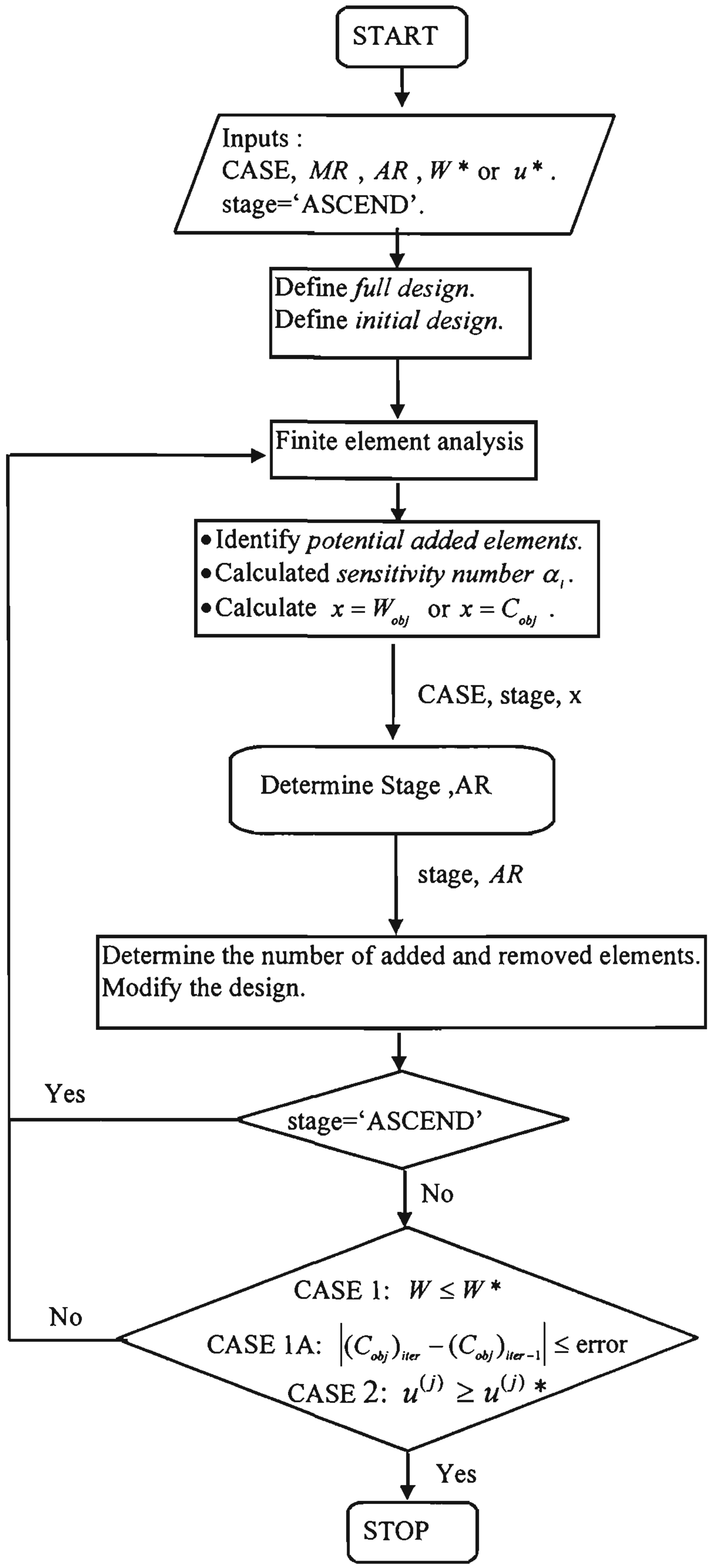
First, the potentially added element is a key point of BESO. It would be advantageous if there are plenty of potentially added elements available to choose from in a structure. At this point, continuous structures (2D or 3D) can be more suitable for BESO than truss or frame structures. Further, the latter has relatively larger size of elements, which may lead to inaccurate results in the displacement extrapolation. For this reason, the application of BESO focuses on 2D continua structures at this stage.

Second, BESO procedure involves the concept of objective compliance or objective weight, which is used to determine the maximum design. Its definition is based on scaling techniques as discussed in section 4.1.3. For 2D problems such as plane stress, thin plate bending, the scaling parameter can be calculated very easily. In cases where the definition of objective compliance or objective weight is not available, the maximum design can be specified before hand as a structure of a relatively large weight, say, 60%. This is to ensure a large design variable space.

Thirdly, though the procedure of BESO is discussed for structure of plane stress conditions in this thesis, it is not restricted to this kind of problem. In fact, BESO can be equally applied to other 2D or 3D elements. Only the modification on displacement extrapolation is needed.

4.5 Summary

The theory of topology optimisation with stiffness and displacement constraints using the BESO method is presented in this chapter. Though BESO is most suitable for continuous structures, its concept and procedure are general regardless of the type of structural systems. Compared to ESO, BESO is still in its initial developing stage, so the work is first conducted on simple cases, e.g. 2D plane stress problems. The next chapter will investigate the capability of BESO through some numerical examples.


START
Inputs :
CASE, MR , AR , W^* or u^* .
stage='ASCEND'.
Define *full design*.
Define *initial design*.
Finite element analysis
• Identify *potential added elements*.
• Calculated *sensitivity number* α_i.
• Calculate $x = W_{obj}$ or $x = C_{obj}$.
CASE, stage, x
Determine Stage ,AR
stage, AR
Determine the number of added and removed elements.
Modify the design.
Yes
stage='ASCEND'
No
CASE 1: $W \le W^*$
No
CASE 1A: $\left|(C_{obj})_{iter} - (C_{obj})_{iter-1}\right| \le$ error
CASE 2: $u^{(j)} \ge u^{(j)*}$
Yes
STOP

Chapter 5

BESO for Stiffness and Displacement Problems-Applications

5.1. Introduction

The previous chapter has explored the mathematical formulation and basic procedure of the bi-directional evolutionary structural optimisation (BESO) method. In this chapter we shall investigate its application to the topology optimisation of 2D continua through some numerical examples. Stiffness and displacement optimisations are conducted under multiple load and multiple constraint conditions in these examples. Most of the examples are chosen from the literature of ESO and other alternative methods. Comparisons of different methods are attempted and ESO severs as the major benchmark. Results are discussed in terms of structural response (compliance or displacement), optimal topology and computational efficiency. Conclusions regarding the performance of BESO and ESO are given towards the end of the chapter.

First of all, it is necessary to clarify a few points. All examples are tested by running the computer code BESODSP together with the FEA software STRAND6 on a Pentium 200 PC with 32 MB of RAM. The same denotations and parameters are used as those in Chapter 4. The 2D continua are under plane stress conditions and the four-node linear square element is used to discretise the structure. In interpreting the results. the terms 'structural weight', 'constrained displacement', 'objective compliance' and 'objective weight' will appear frequently. For simplicity, they are used as relative quantities, which have been scaled with respect to corresponding values of the full design, W_0 , u_0 , C_{obj}^0 and W_{obj}^0 .

As for the reference structure and two parameters, namely, the modification ration *MR* and addition ration *AR*, they are chosen from Table 5.1.

Table 5.1: Parameters and factors in ESO and BESO.

		MR	*AR*	Reference Structure
ESO		1.0%	--	Current structure
BESO	Ascending	1.5%	0.66	Full design
	Descending	1.5%	0.25	Current structure

The following aspects are considered in choosing the above parameters and reference:

Firstly, in ESO and the descending stage of BESO, we relate the number of modified elements to the current structure rather than the full design. This is more viable as it takes account of changes in the space of design variables.

Secondly, as BESO starts from a very small structure, the evolution may be unnecessarily slow if the number of modified element is related to the current structure. For this reason, we use the full design as reference in the ascending stage. Additionally, we may have chosen the addition ratio *AR*=(1-0.25)=0.75 for this stage. However, the design in the ascending stage has shorter boundary line and there are not sufficient elements available to be add, so a smaller addition ratio AR =0.66 is assumed instead.

Thirdly, MR=1.0% and MR=1.5% are assumed for ESO and BESO respectively. We may have used the same modification ratio for both methods to allow for the same or similar level of accuracy in the sensitivity analysis. However, unlike the case of ESO where the evolution speed is only controlled by MR, the speed of BESO is determined by MR as well as AR. For this reason, we increase the modification ratio slightly for BESO so that the speed of two methods can be comparable. The performance of BESO is affected by the selection of modification ratios. This aspect will be investigated in the next chapter.

Finally, there are another two 'rules' used for BESO:

- During the first 15 iterations, only element addition is allowed to avoid the disconnection of structure due to the element removal.
- Suppose there are totally N_e elements and N_p potential added elements in the current structure, the number of actually added elements N_{add} is finally decided by the following equation:

$$N_{add} = \min(N_e \times MR, N_p \times 0.75) \tag{5.1}$$

The reason is quite simple. If we allowed too high a proportion of potential added elements to be introduced, it would be meaningless to use the sensitivity number to compare their efficiency. The two parameters, namely 15 iterations and 0.75, are decided according to the numerical experience.

5.2 Stiffness Optimisation

5.2.1 Single Loading Condition

Example 1: three-point loaded beam

The structure studied here was a simply supported beam as shown in Fig. 5.1. Three concentrated loads were applied downward simultaneously at points of 1/4, 1/2, 3/4 of the beam span. The Young's modulus E=207 GPa and the Poisson's ratio ν=0.3 were assumed.

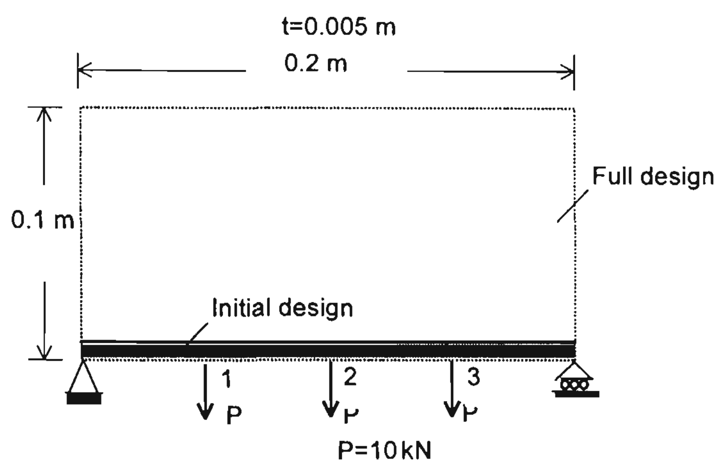


Fig. 5.1: Three-point loaded beam.

The full design was a rectangle which was divided into 100×50, totalling 5000 square elements. The initial design for BESO consisted of two rows of elements between the supports, as shown in Fig. 5.2. The bottom row was treated as a non-design area, that is, elements in this area were not allowed to change during optimisation. The dots in the background represented the finite element mesh. The design objective was to minimise the compliance with three weight limits: $W^* = 50\%$, 40% and 30%, respectively.

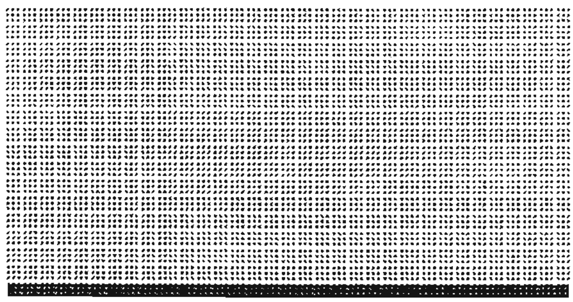


Fig. 5.2: FE model and initial design
(example 1).

Following the BESO procedure outlined in Chapter 4, the structure grows from the initial design to the desired optima. The objective compliance C_{obj} decreases sharply from an initial value of 108.11 to 1.03 at the 30th iteration. Fig. 5.3 illustrates the history of the objective compliance and structural weight from the 30th iteration onward. It can be seen that a decreasing tendency is kept in the objective compliance throughout out the evolution. A slight local increase is seen from point A to B, which corresponds to a peak (point C) in the weight diagram. The weight history is thus divided into ascending and descending stages at point C.

Fig. 5.4 displays changes of structural topologies. Fig. 5.4(d) is the maximum design at point C. Final optimal topologies of prescribed weights are shown through (e) to (g). Their corresponding topologies during the ascending stage are given as (a)~(c). Optimal topologies obtained by ESO are shown in Fig. 5.5. While they are higher than their BESO counterparts, the overall shapes are similar.

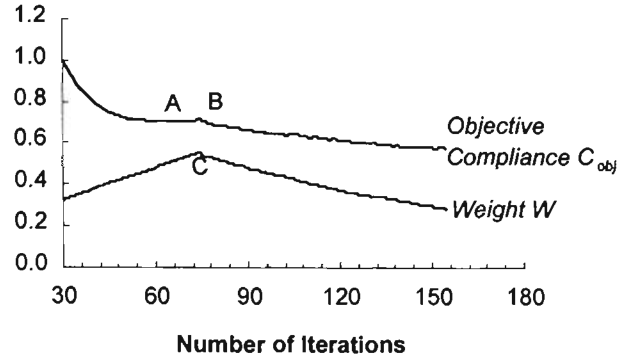
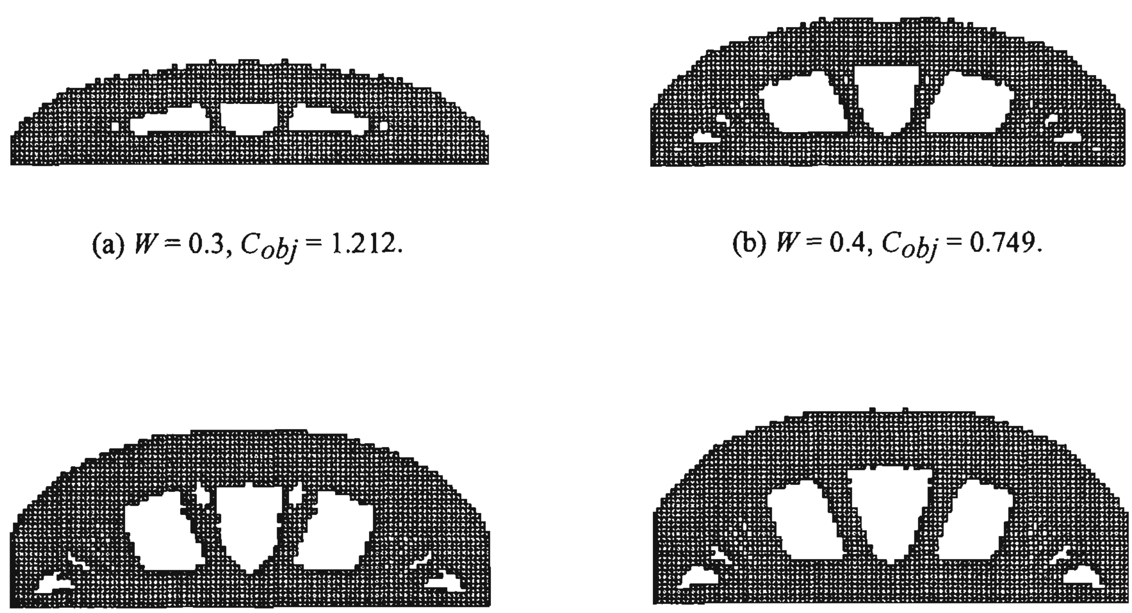


Fig. 5.3: Evolutionary history of objective compliance and structural weight.

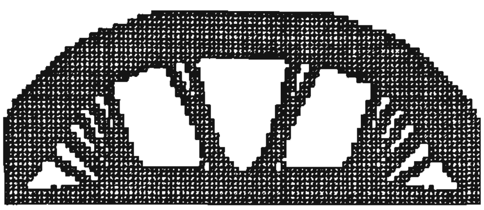


(a) $W = 0.3$, $C_{obj} = 1.212$.

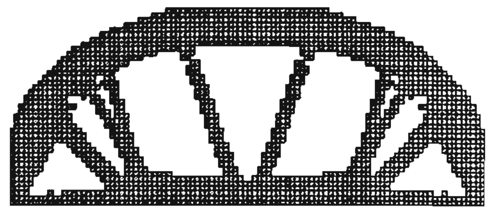
(b) $W = 0.4$, $C_{obj} = 0.749$.

(c) $W = 0.5$, $C_{obj} = 0.706$.

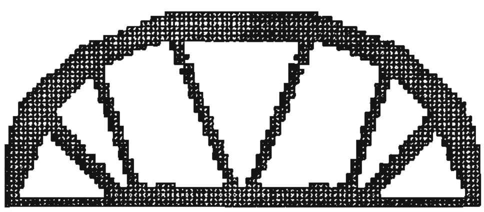
(d) $W = W_{max} = 0.55$, $C_{obj} = 0.706$.



(e) $W^* = 0.5$, $C_{obj} = 0.673$.

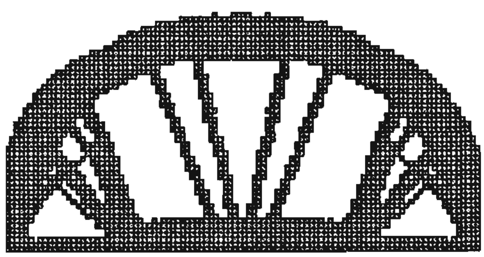


(f) $W^* = 0.4$, $C_{obj} = 0.636$.

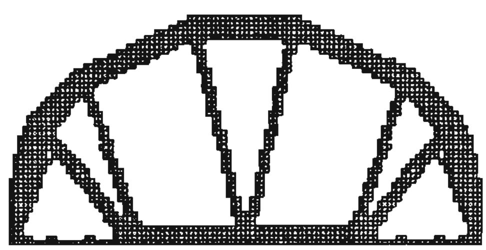


(g) $W^* = 30\%$, $C_{obj} = 0.584$.

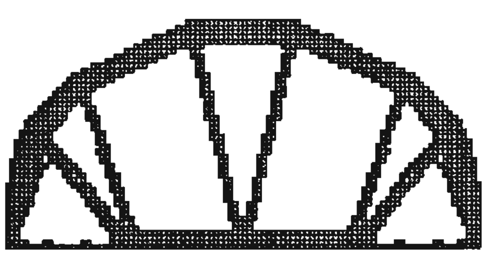
Fig. 5.4: Topologies in evolution (BESO).



(a) $W^* = 50\%$, $C_{obj} = 0.665$.



(b) $W^* = 40\%$, $C_{obj} = 0.623$.



(c) $W^* = 30\%$, $C_{obj} = 0.579$.

Fig. 5.5: Optimal topologies (ESO).

Table 5.2 summarises the result of the objective compliance. ESO yielded a better solution in this example and required less solution time. The solution time is determined by two factors, namely, the time required in one step as well as the total number of steps. In one step, the time is allocated to the structural analysis and element modification. The computing time in these two parts can be roughly estimated by orders of n^3 and n^2, respectively, where n is the number of finite elements. The structural analysis requires more time than the element modification and its proportion increases with the dimension of the finite element problem. In this respect, BESO has more potential in saving computing effort. On the other hand, BESO usually requires more design cycles before it arrives at an optimum than ESO. In this example, the second factor dominates and the strength of BESO is offset by a large number of iterations.

Table 5.2: Results of ESO and BESO (Example 1).

W^*		C_{obj}	Iteration	Time (Hour)
50%	ESO	0.665	68	2.5
	BESO	0.673	86	2.5
40%	ESO	0.623	85	3
	BESO	0.636	112	3.5
30%	ESO	0.579	115	3.8
	BESO	0.584	149	4.5

This example has been studied by the homogenisation method (Díaz and Bendsøe 1992). Where similar topologies have been obtained.

Example 2: angle piece

The mechanism of an angle piece was represented by the loading and boundary conditions shown in Fig. 5.6. Loads on the left-hand hole were to be transmitted to the supports located around the two right-hand holes. Physical properties were $E = 200$ GPa and $\nu = 0.3$. There was no weight limitation but the optimum with the minimum objective compliance was to be sought out of the rectangular area.

The full design was divided into a mesh of 96×80, as shown in Fig. 5.7, and the dark elements made up of the initial design.

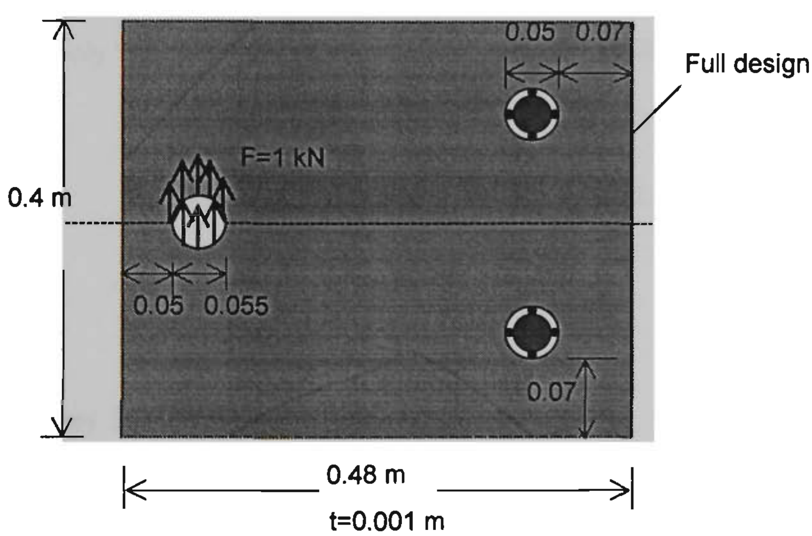


Fig. 5.6: Design domain of an angle piece.

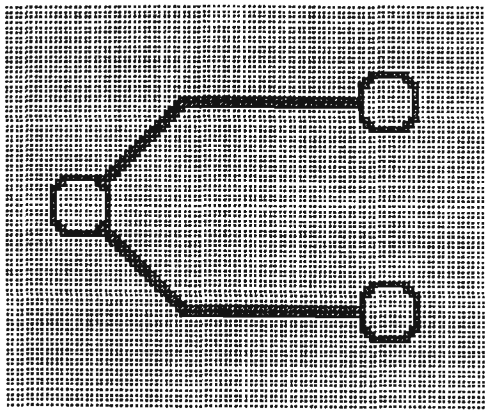
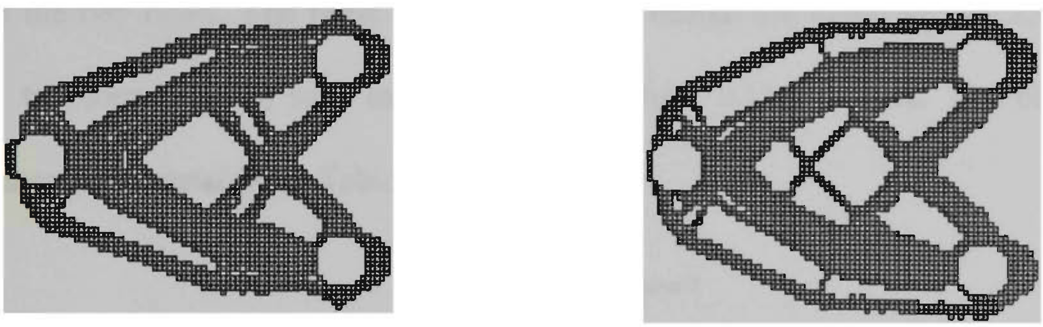


Fig. 5.7: FE model and initial design
(Example 2).



(a) $W = 0.283$, $C_{obj} = 0.545$, iteration = 97. (b) $W = 0.283$, $C_{obj} = 0.571$, iteration = 70.

Fig. 5.8: Optimal topologies: (a) BESO; (b) ESO.

Optimal topologies by ESO and BESO are given in Fig. 5.8. The two designs had the same weight 0.283, whereas the one obtained by BESO had a much smaller objective compliance. The times required by BESO and ESO were 3 and 4.5 hours, respectively. The maximum design had a relatively low weight (0.419), so BESO only needs marginally more iterations than ESO and it saved computing time. For such a large structure (totally 7416 elements), the savings in solution effort due to BESO were significant.

Example 3: lever arm

The shape in Fig. 5.9 models the loading and supporting conditions of a lever arm. The structure was fixed in the hole on the left and loads were applied at point A on the right. Two load cases were considered to respectively simulate the tension and bending conditions. The best optimum under each individual load case was the design objective.

The full design was discretised by a mesh of 76×38. There are 4776 elements excluding those in the two holes. The finite model and initial design are given in Fig. 5.10. The optimal topology of each load case is shown in Figs. 5.11 and 5.12. The objective compliance is summarised in Table 5.3.

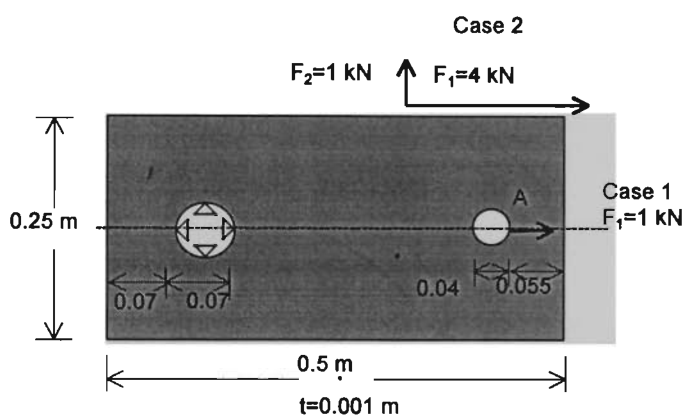


Fig. 5.9: Design domain of a lever arm.

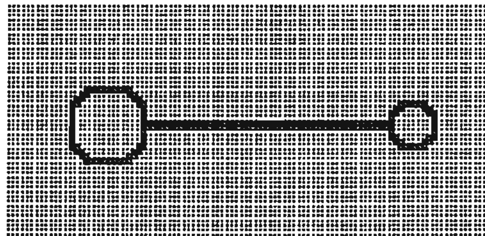


Fig. 5.10: FE model and initial design (Example 3).

(a) $W = 0.123$, $C_{obj} = 0.842$. (b) $W = 0.112$, $C_{obj} = 0.839$.

Fig. 5.11: Optimal topologies for case 1: (a) BESO; (b) ESO.

(a) $W = 0.145$, $C_{obj} = 0.366$. (b) $W = 0.147$, $C_{obj} = 0.366$.

Fig. 5.12: Optimal topologies for case 2: (a) BESO; (b) ESO.

Table 5.3: Best optima by ESO and BESO (Example 3).

	W	C_{obj}	Iteration	Time (Hour)
		Case 1		
ESO	0.123	0.842	198	4
BESO	0.112	0.839	89	1.2
		Case 2		
ESO	0.145	0.366	184	3.5
BESO	0.147	0.366	126	1.5

The values of objective compliance by the two methods were similar. BESO was by far the more efficient solution for this example. This is because the maximum designs only occupy a very small proportion of the full design, namely, 29% in case 1 and 36% in cases 2.

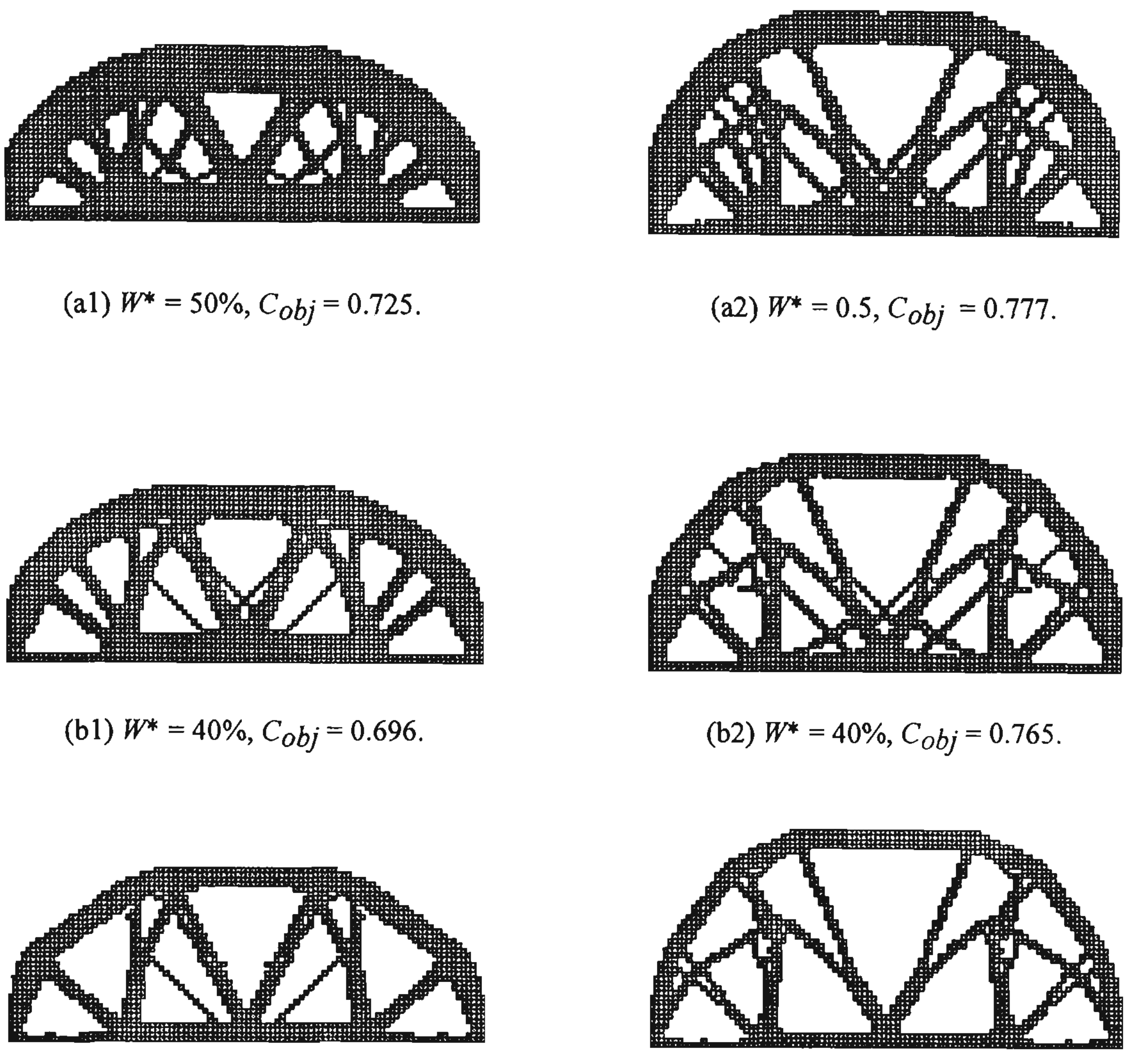
Examples 2 and 3 have been solved by Mattheck (1997) using the soft kill option (SKO) for uniform stress design.

5.2.2 Multiple Loading Conditions

Example 4: three-point loaded beam

The same full design, initial design and finite element setting as for example 1 were used here. However, three loads were applied independently, one at a time. The optimisation was to minimise the average compliance with a given weight, namely, $C = C^1 + C^2 + C^3$. The weight limits were prescribed as $W^* = 50\%$, 40% and 30%, the same as in the single load condition optimisation.

Fig. 5.13 compares the corresponding optimal topologies for different weights. As also shown in example 1, topologies obtained by the two methods share similarities in outer shape and some internal configurations. BESO produced designs of smaller height and larger size of arches and spokes.



(a1) $W^* = 50\%$, $C_{obj} = 0.725$.

(a2) $W^* = 0.5$, $C_{obj} = 0.777$.

(b1) $W^* = 40\%$, $C_{obj} = 0.696$.

(b2) $W^* = 40\%$, $C_{obj} = 0.765$.

(c1) $W^* = 30\%$, $C_{obj} = 0.696$.

(c2) $W^* = 30\%$, $C_{obj} = 0.821$.

Fig 5.13: Optimal topologies: (a1)~(c1); ESO: (a2)~(c2).

As shown in Table 5.4, the objective compliance obtained by BESO are considerably smaller than that by ESO. A study of the evolutionary history reveals that the weight of the maximum design was relatively small with W_{max}=0.562. Its objective compliance $C_{obj} = 0.754$, was much lower than that of ESO with the same weight. After the maximum design, BESO has a smaller compliance than ESO all along, as shown in Fig. 5.14. Also, BESO requires less solution time. The major reason can be that the finite element analysis under the three load cases consumed substantially more time in ESO.

Table 5.4: Results of BESO and ESO (Example 4).

W^*		C_{obj}	Iteration	Time (Hour)
50%	ESO	0.777	65	4.0
	BESO	0.725	91	3.3
40%	ESO	0.765	85	4.8
	BESO	0.696	119	4.3
30%	ESO	0.821	110	6.0
	BESO	0.696	155	5.6

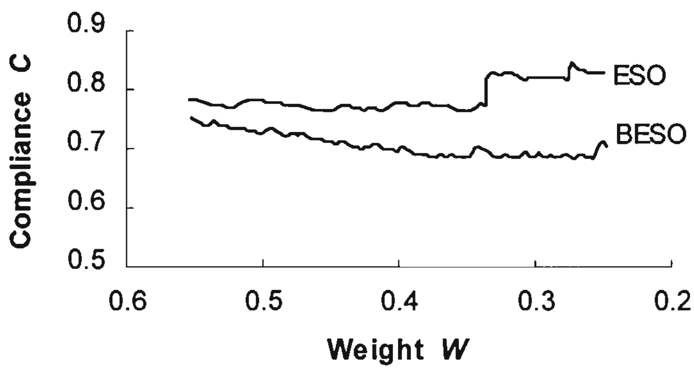


Fig. 5.14: Changes in mean compliance vs. structural weight (descending stage).

Example 5: square plate

In this example, a full design was given as a square plate in Fig. 5.15. The central hole was prevented from all movements. Two load cases, P and Q, were applied independently at the four corners. The Young's modulus E=200 GPa and the Poisson's ratio ν=0.3 were assumed. The weight limits W^*=50%, 40%, 30% and 20% were specified.

Fig. 5.16 gives the finite element setting and the initial design. A mesh of 50×50 was used. Final optimal designs by BESO and ESO are given in Fig. 5.17. The corresponding shapes are almost identical. They also agree well with the results by the homogenisation method (Díaz and Bendsøe 1992). BESO results in lower values of objective compliance for most of design cases. It also saves solution time, as shown in Table 5.5.

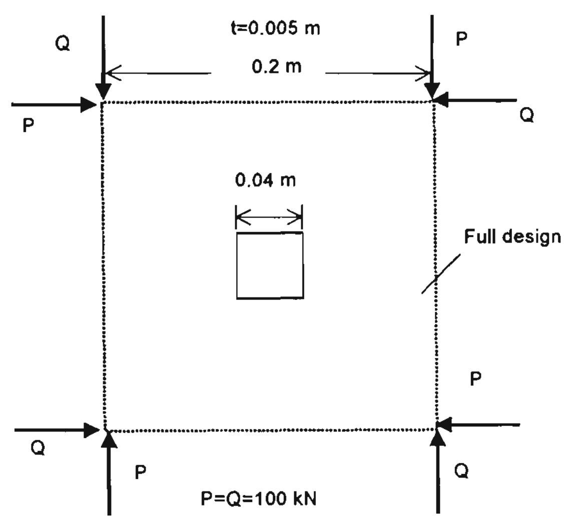


Fig. 5.15: Design domain.

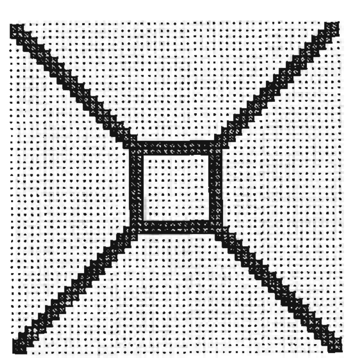
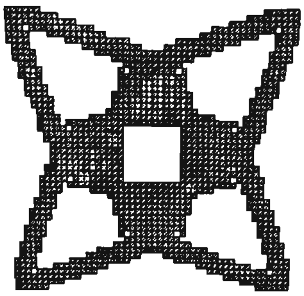
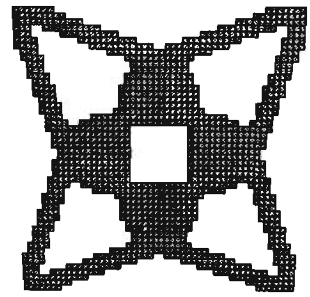


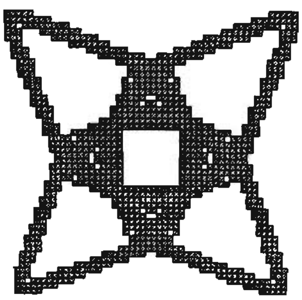
Fig. 5.16: FE model and Initial design (Example 5).



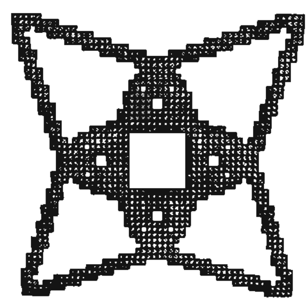
(a1) $W^* = 50\%$, $C_{obj} = 0.710$.



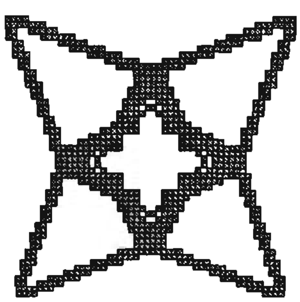
(a2) $W^* = 50\%$, $C_{obj} = 0.707$.



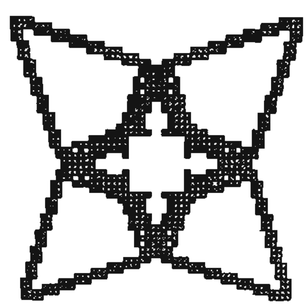
(b1) $W^* = 40\%$, $C_{obj} = 0.715$.



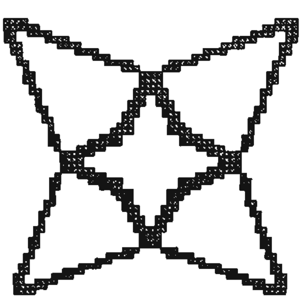
(b2) $W^* = 40\%$, $C_{obj} = 0.726$.



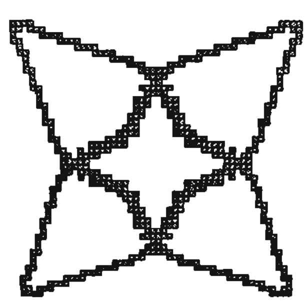
(c1) $W^* = 30\%$, $C_{obj} = 0.760$.



(c2) $W^* = 30\%$, $C_{obj} = 0.756$.



(d1) $W^* = 20\%$, $C_{obj} = 0.882$.



(d2) $W^* = 20\%$, $C_{obj} = 0.912$.

Fig. 5.17: Optimal topologies: BESO: (a1)~(d1); ESO: (a2)~(d2).

Table 5.5: Results of BESO and ESO (Example 5).

W^*		C_{obj}	Iteration	Time (Min.)
50%	ESO	0.707	59	100
	BESO	0.710	88	50
40%	ESO	0.726	74	120
	BESO	0.715	118	60
30%	ESO	0.756	99	150
	BESO	0.760	147	80
20%	ESO	0.912	127	160
	BESO	0.882	178	90

5.3 Displacement Optimisation

5.3.1 Single Displacement Constraint

Example 6: Michell type structure

A Michell type structure is shown in Fig. 5.18. We intended to design the lightest structure while the displacement at the mid-span (point A) was constrained. The full design was a rectangle which had an initial displacement $u = 0.36$ mm at point A. The displacement limits were $u^* = 0.6$, 0.8 and 1.0 mm, respectively. The Young's modulus E=207 GPa and the Poisson's ratio $v = 0.3$ were assumed.

A finite element mesh of 80×40 was used. The initial design consists of one row of elements at the bottom, as shown in Fig. 5.19. Optimal topologies corresponding to each displacement limit are given in Fig. 5.20.

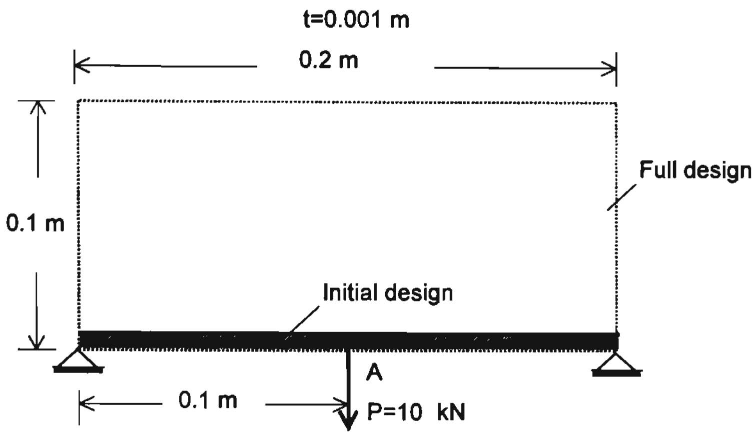


Fig. 5.18: Michell type structure.

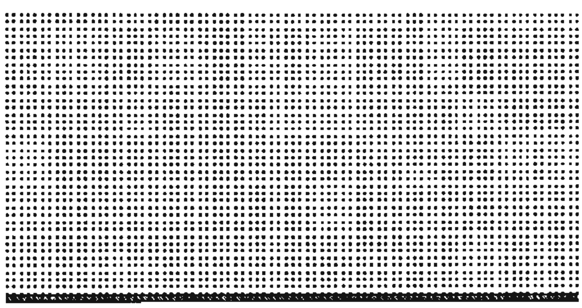
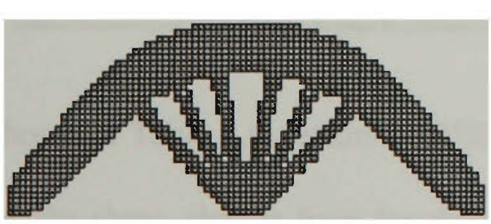
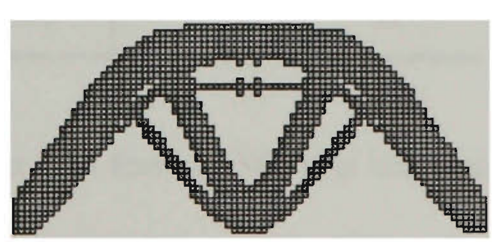


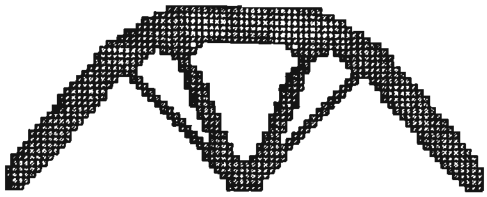
Fig. 5.19: FE model and initial design
(Example 6).



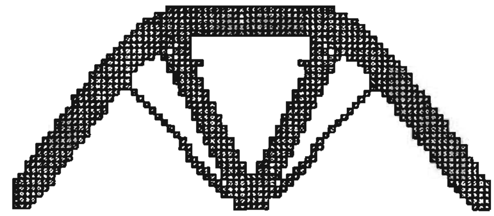
(a1) $u^* = 0.6$ mm, $W_{obj} = 0.576$.



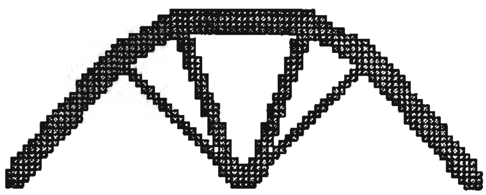
(a2) $u^* = 0.6$ mm, $W_{obj} = 0.590$.



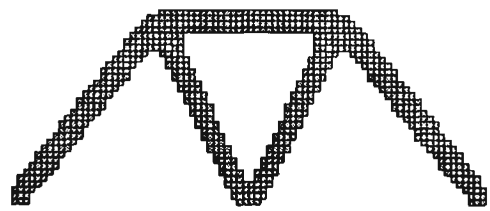
(b1) $u^* = 0.8$ mm, $W_{obj} = 0.577$.



(b2) $u^* = 0.8$ mm, $W_{obj} = 0.551$.



(c1) $u^* = 1.0$ mm, $W_{obj} = 0.548$.



(c2) $u^* = 1.0$ mm, $W_{obj} = 0.557$.

Fig. 5.20: Optimal topologies: BESO: (a1)~(c1); ESO: (a2)~(c2).

Table 5.6: Results of BESO and ESO (Example 6).

u^* (mm)		W	W_{obj}	Iteration	Time (Min.)
0.6	ESO	0.358	0.590	66	52
	BESO	0.345	0.576	66	36
0.8	ESO	0.252	0.551	89	63
	BESO	0.268	0.577	86	46
1.0	ESO	0.206	0.557	102	68
	BESO	0.198	0.548	113	57

As shown in Table 5.6, results of BESO are better than that of ESO. It is instructive to notice the shape change from topologies (b1) to (c1) and (b2) to (c2). While all of the four spokes change in size in BESO, ESO modifies the structure in such a way that the two outer spokes disappear thus the remaining two have much larger dimension than their BESO counterparts. If we allow material to be added as well as removed in shape b2, it will most likely to evolve to c1 instead of c2, since the former represents a better solution. This reflects one possible weakness of ESO that it may remove elements prematurely. As for computing efficiency, BESO is also more advantageous.

Example 7: cantilever beam

Another example for displacement optimisation is a cantilever beam as shown in Fig. 5.21. The beam was fixed at its left-hand side. A load $P = 3$ kN acted in the middle of the right-hand free end (point A), where the displacement constraint was imposed. The full design had the displacement at point A u_0 =0.33 mm. The displacement limits were u^* =0.5, 0.6 and 0.7 mm, respectively. The Young's modulus E=207 GPa and the Poisson's ratio ν=0.3 were assumed.

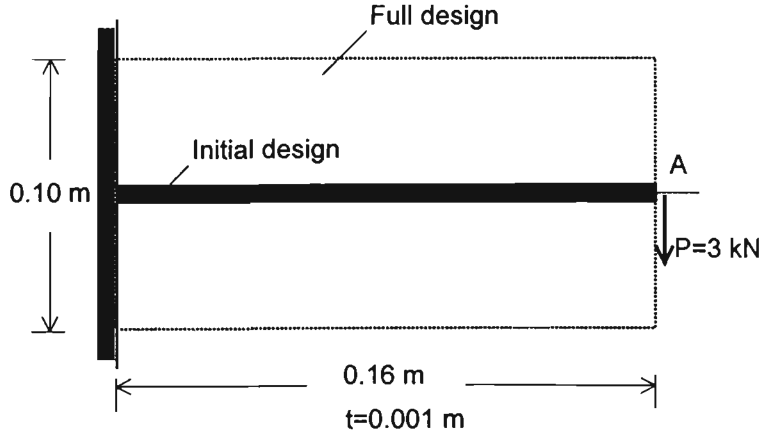


Fig. 5.21: A cantilever beam.

The full design consisted of 48×30, totalling 1440 elements. Fig. 5.22 gives the initial design including two rows of elements linking the load and supports. The final optima obtained by ESO and BESO are given in Fig. 5.23. They are similar to the results of the homogenisation method (Suzuki and Kikuchi 1991).

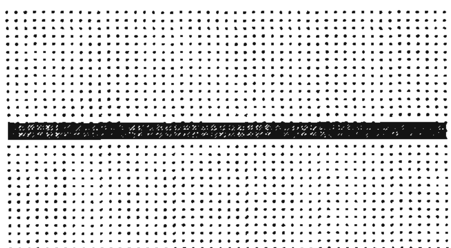
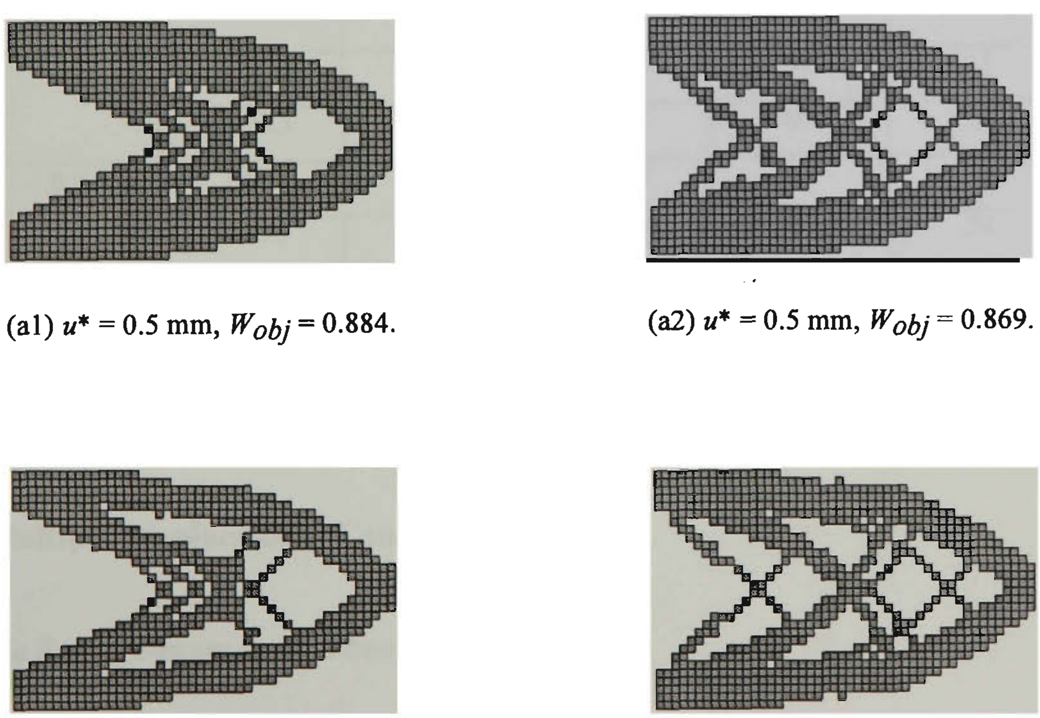


Fig. 5.22: FE model and initial design
(Example 7).

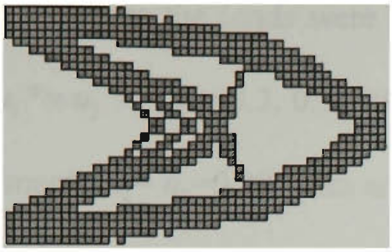


(a1) $u^* = 0.5$ mm, $W_{obj} = 0.884$.

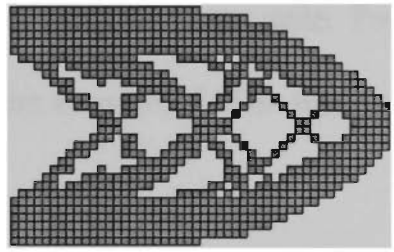
(a2) $u^* = 0.5$ mm, $W_{obj} = 0.869$.

(b1) $u^* = 0.6$ mm, $W_{obj} = 0.847$.

(b2) $u^* = 0.6$ mm, $W_{obj} = 0.892$.



(c1) $u^* = 0.7$ mm, $W_{obj} = 0.868$.



(c2) $u^* = 0.7$ mm, $W_{obj} = 0.873$.

Fig. 5.23: Optimal topologies: BESO: (a1)~(c1); ESO: (a2)~(c2).

Numerical results are provided in Table 5.7. For such a small finite element problem, as computing loads of structural analysis and optimisation are close, the efficiency of ESO and BESO is comparable.

Table 5.7: Results of BESO and ESO (Example 7).

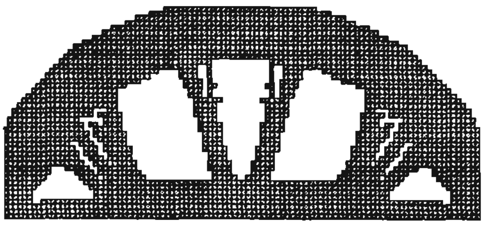
u^* (mm)		W	W_{obj}	Iteration	Time (Min.)
0.5	ESO	0.572	0.869	54	15
	BESO	0.597	0.884	86	15
0.6	ESO	0.494	0.892	68	20
	BESO	0.468	0.847	122	24
0.7	ESO	0.410	0.873	83	23
	BESO	0.415	0.868	140	27

5.3.2 Multiple Displacement Constraints

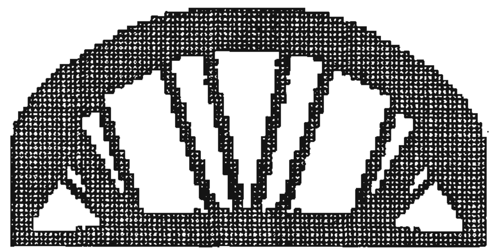
Example 8: three-point loaded beam

This example used the same full design, initial design, finite element setting and load condition as for example 1. Multiple displacement constraints were imposed on points

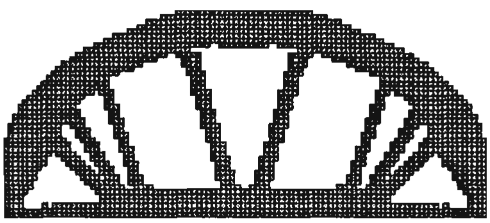
1, 2 and 3, where the loads were applied. They were equal in magnitude. Four cases where $u_1{}^* = u_2{}^* = u_3{}^*$=0.3, 0.35, 0.4 and 0.5 mm were considered. The full design had displacements $u_1 = u_3$=0.207 mm and u_2=0.227 mm.



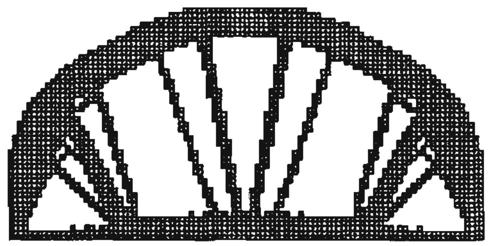
(a1) u^* = 0.3 mm, W_{obj} = 0.711.



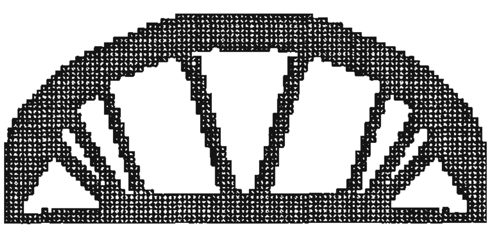
(a2) u^* = 0.3 mm, W_{obj} = 0.701.



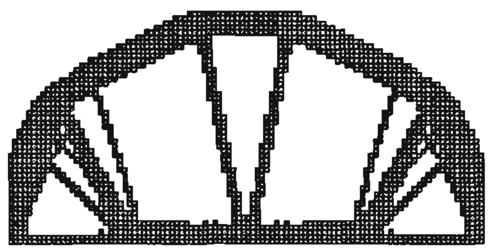
(b1) u^* = 0.35 mm, W_{obj} = 0.634.



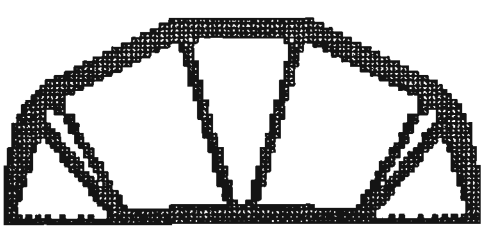
(b2) u^* = 0.35 mm, W_{obj} = 0.644.



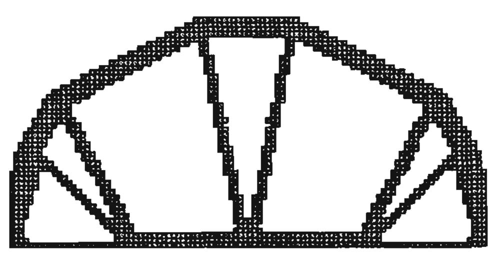
(c1) u^* = 0.4 mm, W_{obj} = 0.572.



(c2) u^* = 0.4 mm, W_{obj} = 0.626.



(d1) u^* = 0.5 mm, W_{obj} = 0.553.



(d2) u^* = 0.5 mm, W_{obj} = 0.552.

Fig. 5.24: Optimal topologies: BESO: (a1)~(d1); ESO: (a2)~(d2).

Optimal designs are compared in Fig. 5.24 and results are given in Table 5.8. BESO consumes less computing time.

Table 5.8: Results of BESO and ESO (Example 8).

u^* (mm)		W	W_{obj}^*	Iteration	Time (Hour)
0.3	ESO	0.530	0.701	61	4.5
	BESO	0.538	0.711	94	3.5
0.35	ESO	0.418	0.644	84	6
	BESO	0.414	0.634	127	5
0.4	ESO	0.357	0.626	96	6.5
	BESO	0.324	0.572	155	6
0.5	ESO	0.254	0.552	127	8
	BESO	0.251	0.553	186	7.2

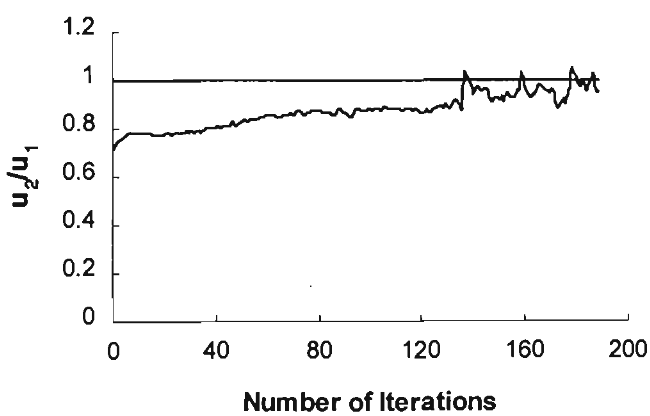


Fig. 5.25: Displacement history.

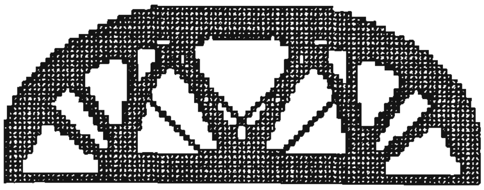
It was also noted that the displacements u_1 (u_3) and u_2 became closer during optimisation as a result of equal constraints. The change of displacement ratio u_2 / u_1 is displayed in Fig. 5.25. It increased steadily in the first 120 iterations. In the later evolution, there were some jumps where u_2 exceeded u_1. This only happened occasionally and most of time, u_2 was the dominant constraint and its Lagrangian multiplier was taken a larger value.

5.3.3 Multiple Displacement Constraints under Multiple Load Cases

Example 9: three-point loaded beam

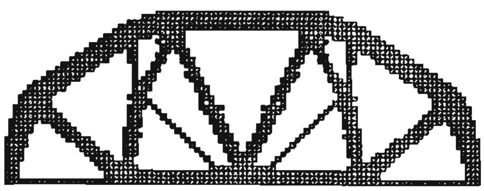
Here, we used example 1. It was assumed that three loads were applied independently. Equal displacement constraints were imposed simultaneously at points 1, 2 and 3 under each load case. Conceptually, the sensitivity analysis using equation (4.56) should take account of the contribution of altogether 3×3 individual sensitivity numbers. However, as discussed in Chapter 4.2.3.4, we can identify a critical load case from the outset in this example. As the three loads are equal in magnitude, the one applied at the location of one constrained displacement will always be the most critical load for that constraint. As a result, the final sensitivity number only consists of three components. This simplification can significantly reduce the computing efforts in finite element analysis.

Topologies corresponding to displacements $u_1^* = 0.2$ and 0.3 mm are shown in Fig. 5.26. The objective weight and solution time for BESO and ESO are compared in Table 5.9. BESO proves to be a better method in this case as it results in smaller objective weights as well as requiring less solution time.

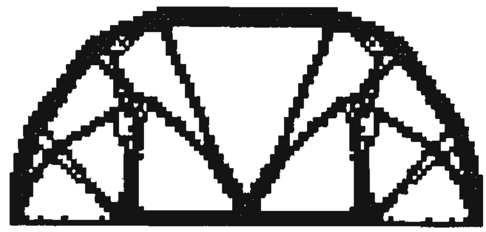


(a1) $u^* = 0.2$ mm, $W_{obj} = 0.768$.

(a2) $u^* = 0.2$ mm, $W_{obj} = 0.826$.



(b1) $u^* = 0.3$ mm, $W_{obj} = 0.759$.



(b2) $u^* = 0.3$ mm, $W_{obj} = 0.873$.

Fig. 5.26: Optimal topologies: BESO:(a1)~(b1); ESO: (a2)~(b2).

Table 5.9: Results of BESO and ESO (Example 9).

u^* (mm)		W	W_{obj} *	Iteration	Time (Hour)
0.2	ESO	0.424	0.826	80	4.5
	BESO	0.401	0.768	119	3.3
0.3	ESO	0.299	0.873	111	6.0
	BESO	0.259	0.759	172	5.5

With regarding to the above stiffness and displacement optimisation problems, three aspects are worth noting:

Firstly, it is instructive to compare different optimisations performed on the three-point loaded beam, as discussed in examples 1, 4, 8 and 9. A common feature of these problems is that the 2D continuum gradually reduces to a truss structure as the amount of material available decreases. This phenomenon has been observed in most of the results by using ESO and the homogenisation method, which is consistent with the analytical solution for truss layout optimisation (Michell 1904; Rozvany 1993). Further, optimal topologies under multiple load conditions, either with single or multiple constraints, possess more intricate architecture compared to single load optimisation. This point was also observed by Díaz and Bendsøe (1992).

Secondly, The change of sensitive number α_i was recorded in evolution. While the largest value of α_i only changes slightly from iteration to iteration, the smallest one increases rapidly as the evolution proceeds. Further, as stated in equations (4.18) and (4.41), the sensitivity number is uniformly distributed among all elements at an optimum. This highly idea state can hardly be achieved by the evolutionary method. This is mainly because of the discrete nature of design variables. For the 2D continua studied in this chapter, the high order of structural indeterminacy adds to the difficulty. In optimisation of some skeletal structures, however, the optimum was found to be a determinate truss where all the component had the same sensitivity number, i.e., the element strain energy was uniform (Chu 1997).

Thirdly, the two evolutionary methods BESO and ESO yielded broadly similar and comparable results. Nonetheless, differences between them in structural behaviour as well as optimal topology were also observed. Apart from reasons related to computing accuracy associated with a numerical method, their differences can also be due to the property of the ground structure approach. In this approach, as the final solution is a subset of the initial structural universe, the definition of a ground structure does affect the optimal solution. The effect of ground structure in ESO method has been studied by Chu (1997). BESO and ESO use different ground structures. Though a full design are defined in BESO, not all its element participate in the optimisation. From this point of view, the difference between BESO and ESO is unavoidable. This is also understandable because most numerical methods aim to find a design sufficiently close to the absolute optimum. Most often, these near-optimal solutions are not unique.

5.4 Summary

The BESO method was tested on some stiffness and displacement optimisation problems. Its feasibility was established by the satisfactory agreement of the results of BESO and those of ESO, which also demonstrats the validity of the evolutionary algorithm. Differences in optimal topologies obtained by two methods were observed, which reflects the feature of the evolutionary method as a ground structure approach. BESO and ESO were also compared in terms of computing cost. While BESO may need more iterations to reach an optimum solution, it can be computationally more efficient than ESO in most cases as a result of using a much smaller finite element model. This is particularly true when a large finite element problem is defined, multiple load cases or multiple displacement constraints are specified for optimisation.

Chapter 6

Further Studies on Various Aspects of BESO

The algorithm reliability and parametric sensitivity are important factors in evaluating an iterative numerical method. These two aspects regarding the BESO method are investigated in this chapter. It contains two major parts. The first part addresses some common problems encountered in the computational implementation of BESO. The effect of parametric variation on BESO performance is studied in the second part.

Initially, the problem of sharp change is studied and it is solved by using a post-processing strategy. Other considerations such as the singular element and maintenance of symmetry are also discussed. The parameter study in the latter part includes the effect of the initial design, modification ratio (*MR)* and addition ratio (*AR*). As the final result is closely related to the maximum design, differences may exist as a result of using different initial designs. The modification ratio (*MR*) controls the evolution speed, functioning similarly to the move limit in mathematical programming and the step size in the optimality criteria method. A smaller *MR* makes the evolution smoother but is computationally more expensive. The addition ratio (*AR*) is another speed related parameter as it controls the net weight change between two consecutive evolution cycles.

Numerical tests were conducted on some of the examples studied in Chapter 4. Unless otherwise specified, parameters and factors were same as those in Table 4.1.

6.1 Considerations in Numerical Aspects

6.1.1 Processing of Sharp Changes

As the evolutionary algorithm uses a step-by-step procedure, it is required in BESO and ESO that the convergence of the objective function is gradual and smooth. A necessary condition for this requirement is that the number of elements modified at a single iteration is small enough. However, this is not sufficient as occasional sharp changes in structural behaviour are observed in many examples, despite the use of a very small modification ratio.

The occurrence of sharp changes is mainly due to the assumption used in the sensitivity analysis that the displacement is the same before and after the element modification. Using the same denotations as those in Chapter 4 , equation (4.1) can be re-written in an incremental form:

$$(\mathbf{K} + \Delta\mathbf{K})(\mathbf{u} + \Delta\mathbf{u}) = \mathbf{P}, \text{i.e.} \tag{6.1}$$

$$\mathbf{K}\Delta\mathbf{u} + \Delta\mathbf{K}\mathbf{u} + \Delta\mathbf{K}\Delta\mathbf{u} = 0, \text{ then}$$

$$\Delta\mathbf{u} = -(\mathbf{K} + \Delta\mathbf{K})^{-1}\Delta\mathbf{K}\mathbf{u}, \tag{6.2}$$

where $\Delta\mathbf{K}$ and $\Delta\mathbf{u}$ are changes in the stiffness matrix and in the displacement, respectively.

Compared to equation (4.19), the above derivation does not ignore the contribution of $\Delta\mathbf{K}\Delta\mathbf{u}$ and thus can be more reliable. Suppose that there are altogether $n1$ elements modified (for simplicity, only consider removed elements) in the current structure, the mean compliance will change as

$$\Delta C = \frac{1}{2}\mathbf{P}^T\Delta\mathbf{u} = \frac{1}{2}\mathbf{P}^T(\mathbf{K} - \sum_{i=1}^{n1}\mathbf{K}_i)^{-1}\sum_{i=1}^{n1}\mathbf{K}_i\mathbf{u} = \frac{1}{2}\mathbf{u}'^T(\sum_{i=1}^{n1}\mathbf{K}_i)\mathbf{u} = \frac{1}{2}\sum_{i=1}^{n1}\mathbf{u}_i'^T\mathbf{K}_i\mathbf{u}_i\ , \tag{6.3}$$

where $\mathbf{u}'$ is the displacement vector of the new structure after element removal. In contrast, the change in the mean compliance calculated from the sensitivity number α_i is the summation of contribution of each removed element, i.e.

$$\Delta C = \frac{1}{2}\sum_{i=1}^{n1}\mathbf{u}_i^T\mathbf{K}_i\mathbf{u}_i\ . \tag{6.4}$$

Equation (6.4) can be a reliable approximation of equation (6.3) when the difference in the displacement field of the new and old structures is negligible. But this does not always hold true, particularly when the effect of all participating elements on the displacement field is combined. In some situations, it may happen that each removed element has a very small sensitivity number as calculated from equation (4.19), while their combination may cause significant changes in the displacement field. In this case, equation (6.4) under-estimates such changes and some elements can be inappropriately removed. In BESO, these elements are most likely to be recovered in later steps and the excessive displacement can be alleviated to a certain extent. The same cannot happen in ESO. As elements cannot be brought back, the following evolution based on the incorrectly modified structure will gradually deviate from the true direction and cannot reach an optimum.

A post-processing strategy is used in ESO and BESO to remedy the problem of sharp changes. Removed elements are firstly recovered to the design and the later evolution is based on this new design. To avoid immediate elimination of these recovered elements in the successive iterations, they are assigned as non-design elements and this status will be kept continuously for, say, ten iterations. Beyond this point, they are free to be modified.

The procedure is very simple and easy to program. At the beginning of each iteration, the displacement (or mean compliance) is checked against that of the last iteration. A difference as large as 10% is used to define the occurrence of a sharp change. If a sharp change arises, instead of going ahead with the optimisation as sensitivity analysis and element modification, the program turns to the next iteration directly after the elements are recovered. The example of the cantilever beam studied in Chapter 4 is used for illustration.

Sharp changes are seen in the later descending stage. Fig. 6.1 displays the constrained displacement vs. the structure weight. A jump is first seen from A1 to A2, then the displacement has a decrease from A2 to B1 because some elements are recovered gradually. There is another jump from B1 to B2, which is much larger in magnitude. At point C, the displacement is recovered slightly. Though BESO is capable of correcting some prematurely removed elements, results from A1 to D were not acceptable.

In contrast, the displacement change is smooth and gradual if the previously proposed recovery strategy is used, as shown by the black line in Fig. 6.1. Elements removed at A1 are recovered immediately. The subsequent evolution starting from the recovered structure still follows the previous track. In the later stage, a slight increase is seen in the structural weight because of frequent element recovery. The weight becomes stable in a range of 18%~20% and has no further decrease. This means that only a small number of elements remain which are necessary for the connection condition.

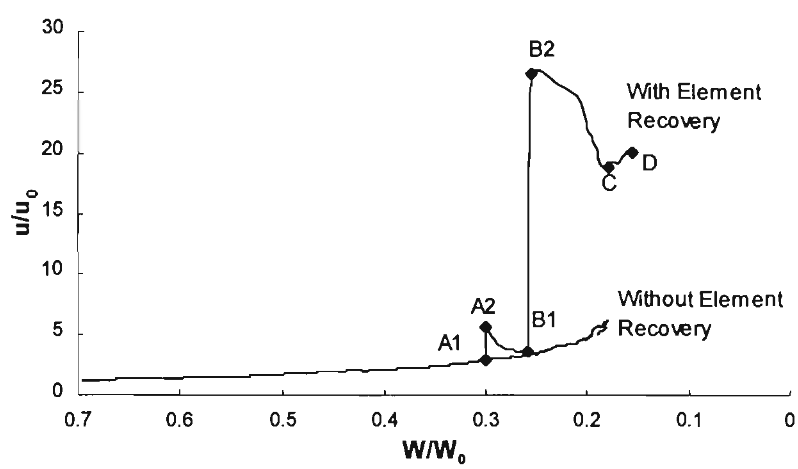
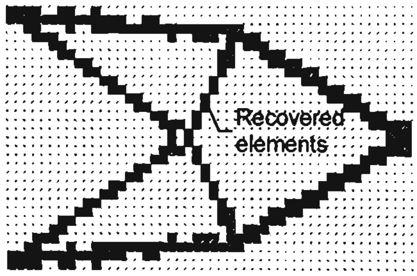
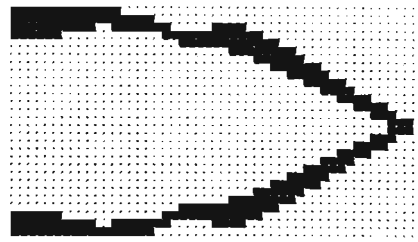


Fig. 6.1: Displacement changes.

The lightest design is attained at the 189th iteration, as shown in Fig. 6.2(a). Elements in grey are those recovered ones. Compared to the design of the same weight without being processed as shown in Fig. 6.2 (b), we can see that four inner bars are removed incorrectly, resulting in an excessively flexible design. The displacement at the free end is 3 mm and 77 mm, respectively.



(a) $W = 0.18\%$, $u = 3$ mm.



(b) $W = 0.18\%$, $u = 77$ mm.

Fig. 6.2: Designs of cantilever beam:
(a) with element recovery; (b) without element recovery.

6.1.2 Singularity of the Stiffness Matrix

Another problem that may arise in ESO and BESO is that the stiffness matrix may become singular during evolution as a result of element removal. This happens to a four-node element when its nodes are inadequately connected. As the element has no in-plane rotational stiffness corresponding to the drilling freedom, an element can rotate freely if there is only one node connected to other elements, and thus cause a singularity in the stiffness matrix.

Measures are taken to prevent as well as to remedy singular elements in the BESO method. Firstly, at the end of the previous iteration, all the elements are checked and those connected to other elements only at one node are removed. This measure, however, may miss the case where a group of elements are totally disconnected with the remaining structure. These kinds of elements can be detected quite easily as they have a very small sensitivity number (nearly equal to zero). Therefore, a second measure is taken in the current iteration to pick up the elements of near-zero sensitivity and exclude them from consideration.

6.1.3 Maintenance of Symmetry

For symmetric structures, it is often necessary to keep all new designs also symmetrical. Special consideration in determining the number of elements in relation to the number of the symmetric axes is required to avoid lose of the symmetry. If a structure has one symmetrical axis, for example, the symmetry can be maintained by simply modifying an even number of elements at each iteration. Similarly, if the structure is symmetrical with respect to n axes, the number of modified elements should be the multipliers of $2n$. However, this does not account for the case where there are elements located right on the axes. These elements have no symmetrical counterparts and the symmetry may be violated if they are modified incorrectly. Another factor affecting the symmetry is that the symmetric elements supposed to have the exactly same value of sensitivity number may actually assume slightly different values due to numerical errors.

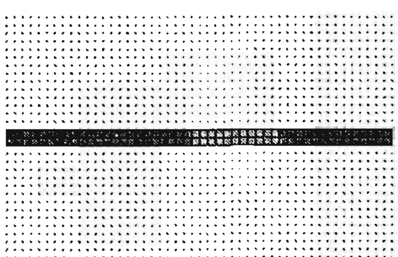
A most convenient way to accommodate all possibilities is to introduce a check process. At the end of the element modification, sensitivity numbers of the last removed and added elements are recorded as α_{ADD} and α_{REM}, respectively. Then all existing elements and potential added elements are checked in terms of sensitivity number. Elements that have the same value of sensitivity as α_{ADD} or α_{REM} are picked up and modified accordingly.

6.2 Parametric Study

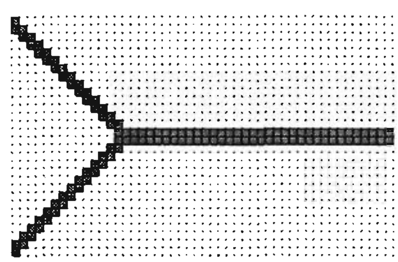
From the tests on BESO in Chapter 5, we note that optimum was found in the descending stage. The structure in the ascending stage is not fully developed as the number of design variables is still increasing. For this reason, the parametric study in this section focuses on the maximum design and the behaviour in the descending stage.

6.2.1 Effect of Initial Designs

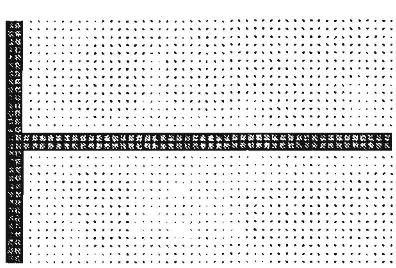
In BESO, elements connecting loads and supports define the initial design. There can be several different kinds of connections for some structures. Take the cantilever beam for example, using the same finite element mesh as in Chapter 5, we can define three initial designs, as shown in Fig. 6.3.



(a) case 1.



(b) case 2.



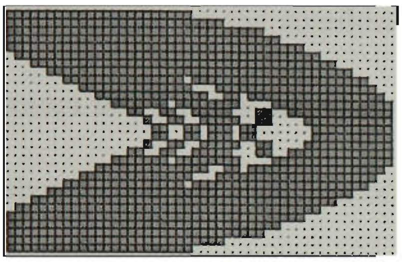
(c) case 3.

Fig. 6.3: Initial designs.

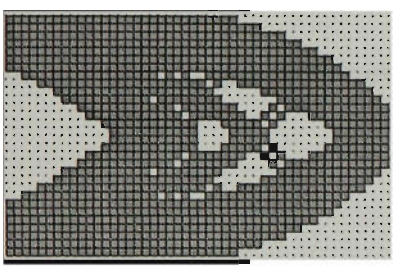
Case 1: Three supports in the middle of the left side are connected to the load.

Case 2: Six outer supports at the top and bottom are connected to the load.

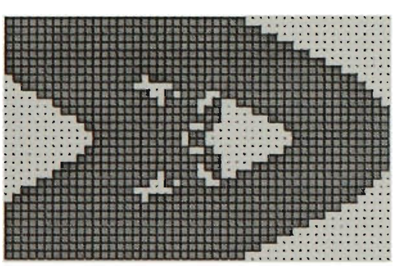
Case 3: In addition to the middle linkage used in case 1, all the supports are connected.



(a) Iteration = 78, W = 0.646, W_{obj} =0.878 (case 1).

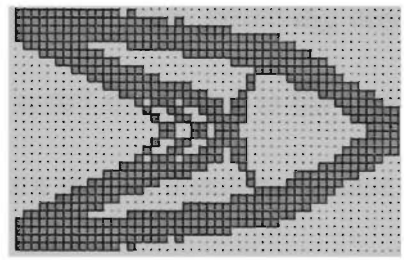


(b) Iteration = 87, W = 0.707, W_{obj} = 0.847 (case 2).

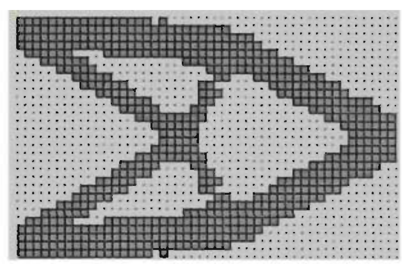


(c) Iteration = 90, W = 0.731, W_{obj} = 0.862 (case 3).

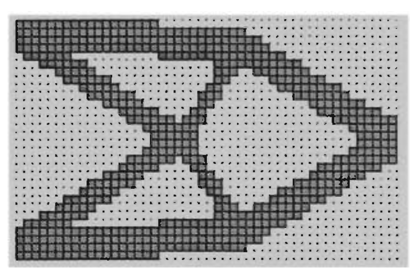
Fig. 6.4: Maximum designs.



(a) W =0.415, Iteration =140 (case 1).



(b) W = 0.401, Iteration = 161 (case 2).



(c) W = 0.384, Iteration =174 (case 3).

Fig. 6.5: Topologies with displacement u^* = 0.7 mm.

The evolution parameters used are MR=1.5%, AR = 0.66 for the ascending stage and AR = 0.25 for the descending stage.

Among the maximum designs of the three cases given in Fig. 6.4, case 3 has the largest weight and case 2 has the smallest objective weight. Topologies satisfying constrained displacement u^*=0.7 mm are given in Fig. 6.5. The result of case 3 is the best, with a weight about 3% lower than that of case 1. Case 2 has a similar topology to case 3.

The displacement change of the whole descending stage is given in Fig. 6.6. With the same structural weight, the displacement yielded in case 3 is the smallest in most iterations. Further, the results of cases 2 and 3 are very close. The above behaviour can be explained from two perspectives.

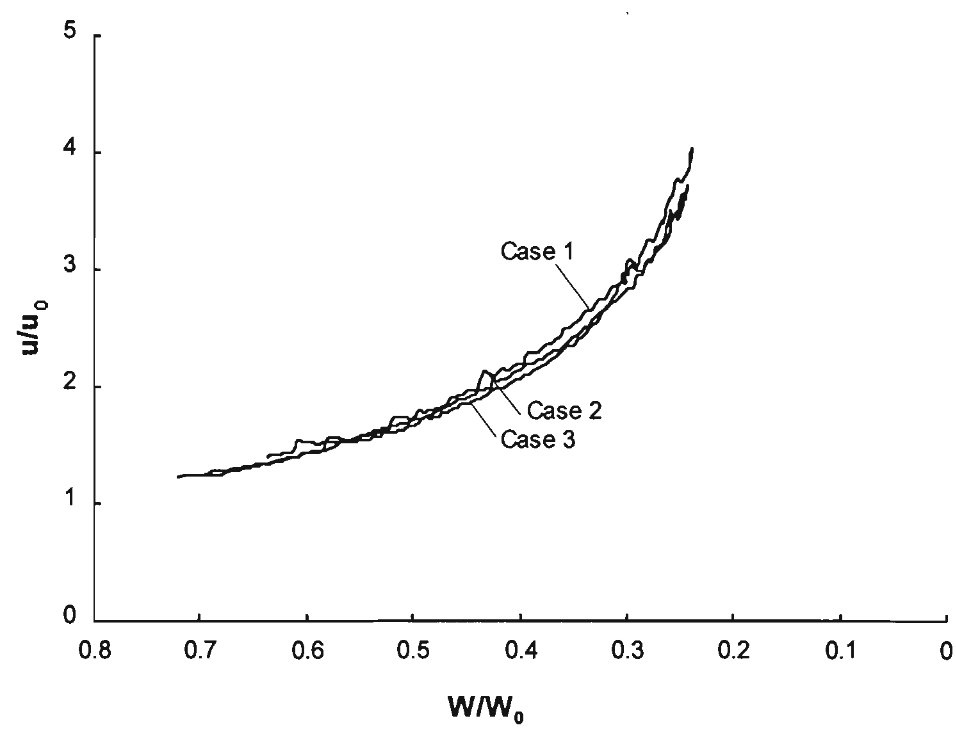


Fig. 6.6: Displacement change corresponding to different initial designs.

Firstly, the initial design of case 3 is the best representation of the given loading and supporting conditions in that all the supports are connected. Also, compared to case 1, cases 2 and 3 can better reflect the mechanical behaviour of a cantilever beam which is supposed to have a larger cross section at the fixed end than the free end.

Secondly, the space of design variables is determined by existing elements as well as potential added elements. While the number of existing elements is changing steadily throughout the evolution, the number of potential added elements can be very arbitrary. Apart from the relevance to the structural weight, the number of potential added element is also affected by the shape of design, especially the structural boundaries and holes. In this respect, cases 2 and 3 may be more advantageous because their initial design contains more potential added elements. Further, at the same level of structural weight, their potential added elements outnumber that of case 1 in most of iterations. This means that the optimisation is performed in a larger space so that there are more options to choose from.

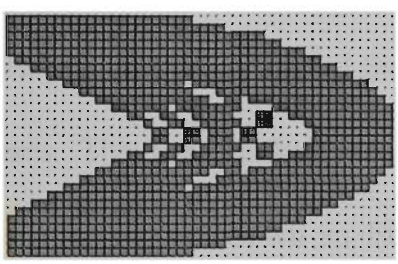
5.2.2 Effect of Modification Ratio (*MR*)

The modification ratio (*MR*) can affect the final results as well as the solution time. This sub-section is to investigate the effect of different value of *MR* on these aspects. Examples of the cantilever beam, the Michell type structure and the three-point loaded beam are tested. They represent structures of small, moderate and large size of finite element model, respectively. The same finite element setting and initial design as those in Chapter 5 are used.

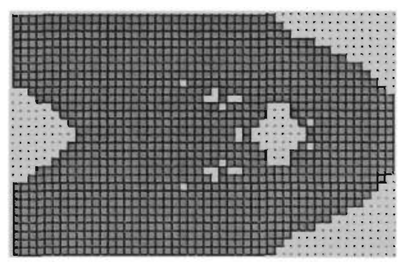
Three cases where MR =1.5% (case 1), 3.0% (case 2) and 5.0% (case 3) are studied. Addition ratios AR = 0.66 and AR = 0.25 are assumed for ascending and descending stage, respectively.

1. cantilever beam

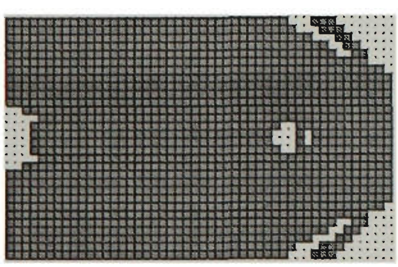
Fig. 6.7 gives the maximum designs of three cases. It is noted that as the modification ratio increases, both the structural weight and the objective weight increase. The maximum design for case 3 almost grows to the full design and contains few cavities.



(a) Iteration = 78, W = 0.646, W_{obj} = 0.878 (case 1).



(b) Iteration = 45, W = 0.814, W_{obj} = 0.892 (case 2).



(c) Iteration = 29, W = 0.916, W_{obj} = 0.937 (case 3).

Fig. 6.7: Maximum designs corresponding to different MR.

Fig. 6.8 gives the displacement change against the structural weight during the descending stage. It was found that results of three cases were very close when the structural weight was greater than 45%. The benefit of smaller modification ratios used in cases 1 and 2 was clear as the weight became smaller. As for these two cases, the result of case 1 was not always the better one. This may be due to the maximum design. A small maximum design may be unfavourable for the subsequent evolution because of the small space of design variables, as also discussed in the study of initial design.

Additionally, within 150 iterations, the smallest design was reached in case 2 instead of case 3, despite that the latter had the fastest decrease in structural weights. This is due to two causes. The first is that case 3 starts decreasing the weight from a very large maximum design, as shown in Fig. 6.7(c). The second is that the evolution has to stabilise on a relatively large weight to meet the requirement of suppressing sharp changes. Among the three cases, sharp changes appeared in case 3 most frequently.

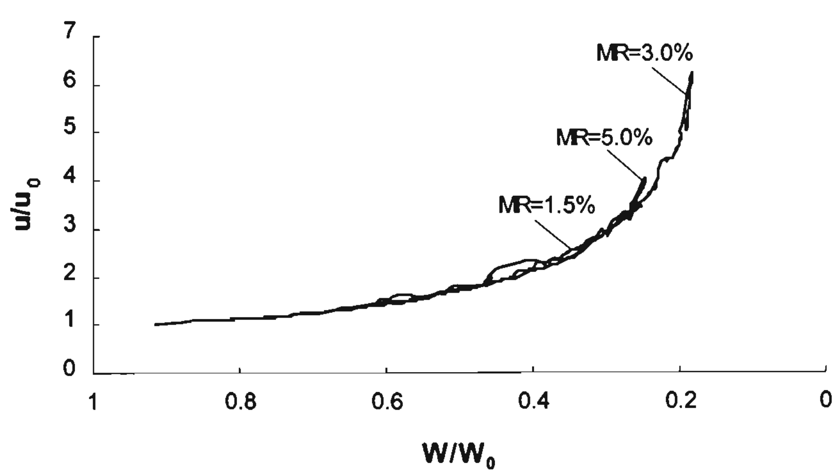
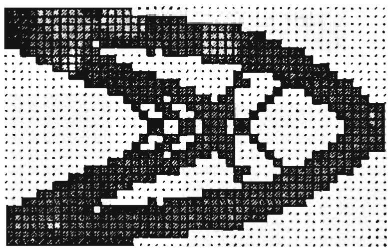
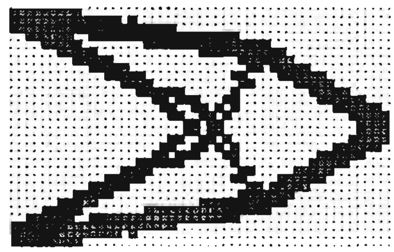


Fig. 6.8: Displacement change corresponding to different *MR* (cantilever beam).

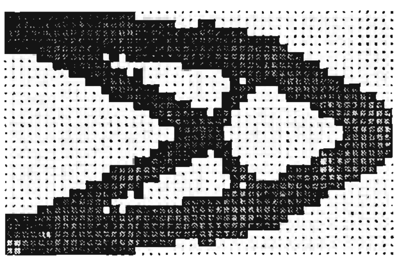
The two topologies with $u^* = 0.6$ mm and $u^* = 1.0$ mm are shown in Figs. 6.9 and 6.10. Topologies by different modification ratios were similar in the outer shape while the inner configuration was different. A tendency for the configuration to become simpler as the modification ratio increases can be observed.



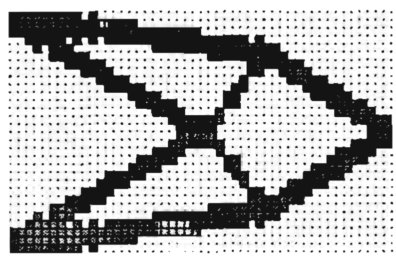
(a) $W_{obj} = 0.847$ (case 1).



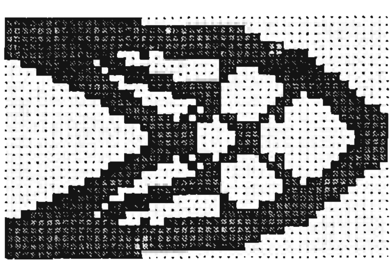
(a) $W_{obj} = 0.882$ (case 1).



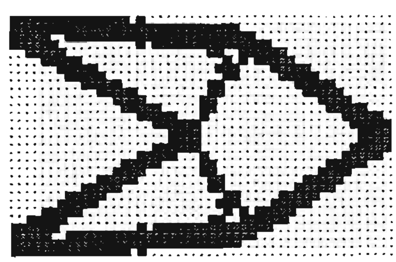
(b) $W_{obj} = 0.850$ (case 2).



(b) $W_{obj} = 0.857$ (case 2).



(c) $W_{obj} = 0.860$ (case 3).



(c) $W_{obj} = 0.882$ (case 3).

Fig. 6.9: Topologies with displacement $u^* = 0.6$ mm.

Fig. 6.10: Topologies with displacement $u^* = 1.0$ mm.

Table 6.1 compares the solution time of the three cases, as the modification ratio increases, the solution time to reach the design of the same displacement decreases. Also considering the result of displacements, MR =3.0% appears to be the most suitable parameter for this example.

Table 6.1: Results by different MR (cantilever beam).

u^* (mm)	MR	W_{obj}	Iteration	Time (Min.)
0.6	1.5%	0.847	122	26
	3.0%	0.850	79	18
	5.0%	0.860	54	17
1.0	1.5%	0.882	190	38
	3.0%	0.857	115	24
	5.0%	0.882	76	21

2. Michell type structure

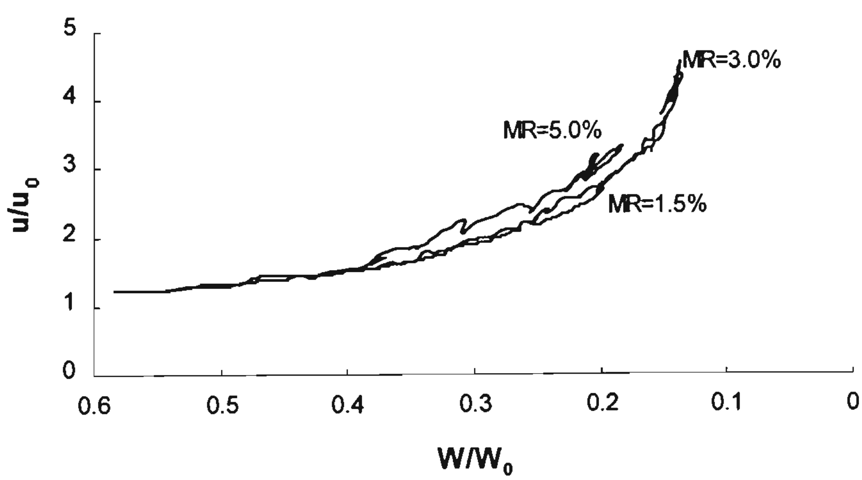
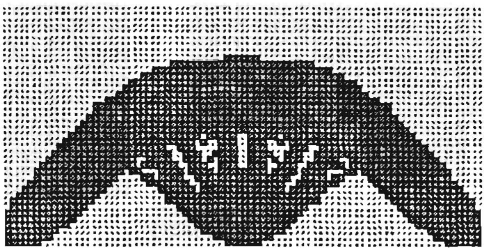
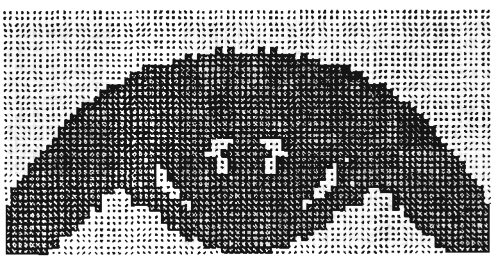


Fig. 6.11: Displacement change corresponding to different MR (Michell type structure).

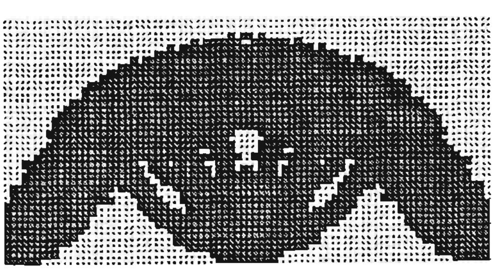
Similar phenomenon to the last example can be observed in Fig. 6.11. The effect of different modification ratios is clearly reflected when the structural weight is relatively small (around 40%). Case 3 is the most unstable one and results of cases 1 and 2 are very close. Case 2 obtains the smallest structure in 300 iterations.



(a) Iteration = 57, $W = 0.443$, $W_{obj} = 0.629$ (case 1).

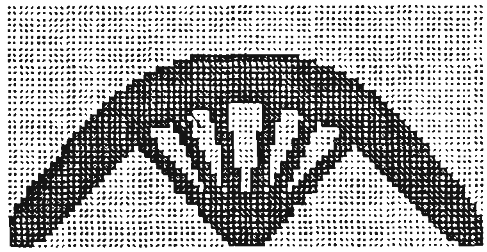


(b) Iteration = 41, $W = 0.536$, $W_{obj} = 0.679$ (case 2).

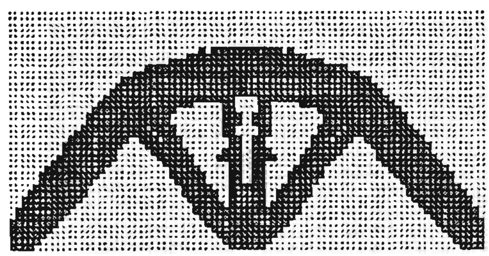


(c) Iteration =38, $W = 0.662$, $W_{obj} = 0.717$ (case 3).

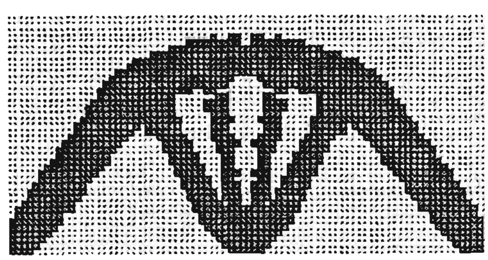
Fig. 6.12: Maximum designs corresponding to different *MR*.



(a) $W_{obj} = 0.576$ (case 1).



(b) $W_{obj} = 0.579$ (case 2).



(c) $W_{obj} = 0.622$ (case 3).

Fig. 6.13: Topologies with displacement $u^* = 0.6$ mm.

The maximum designs are given in Fig. 6.12. A higher value of *MR* leads to a larger design with a larger objective weight. Topologies corresponding to a displacement limit u^*=0.6 mm are given in Fig. 6.13. As the modification ratio increases, the structure grows higher, as a result of larger maximum design. Additionally, topologies shown in Figs. 6.13(b,c) contain fewer inner details than those in Fig. 6.13(a), with the number of spokes reducing from six to four, and the design possessing coarser boundaries.

The comparison of solution times is shown in Table 6.2. Case 3 is not as efficient as case 2 despite a larger value of modification ratio. Firstly, the finite element problem has a larger size in case 3 due to a larger maximum design. Secondly, many iterations are required to process the sharp changes. Case 2, with comparable displacement results to case 1 and the highest efficiency, is the best scheme for this example.

Table 6.2: Results by different *MR* (Michell type structure).

u^* (mm)	*MR*	W_{obj}	Iteration	Time (Min.)
0.6	1.5%	0.576	66	45
	3.0%	0.579	56	30
	5.0%	0.622	51	40

3. Three-point loaded beam

The same design conditions as in example 4 of Chapter 5 were used here. As seen from Fig. 6.14, the displacements of three cases were almost the same in the earlier descending stage. As the structure becomes smaller, case 3 yields larger displacement and sharp changes occur in many iterations. Within 150 iterations, case 2 obtains the smallest structure, the same as the previous two examples.

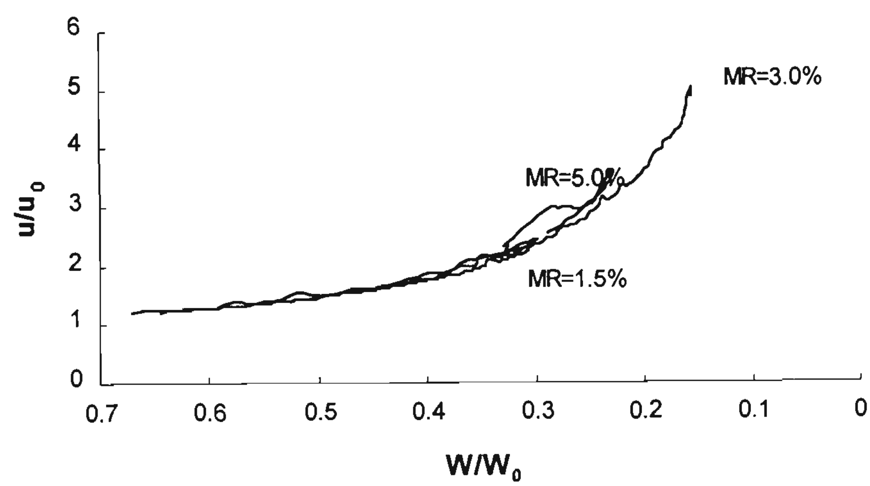
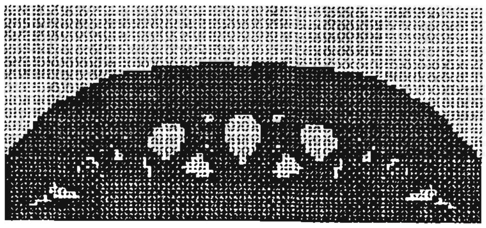
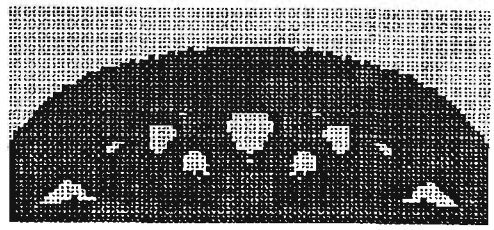


Fig. 6.14: Displacement change corresponding to different *MR* (three-point loaded beam).

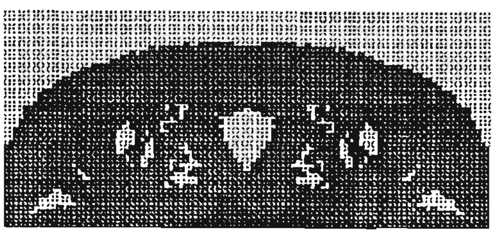
As shown in Fig. 6.15, the maximum design becomes taller with increasing modification ratio, as also observed in the Michell type structure. Topologies of two weights W^*=50% and W^*=30% are given in Figs. 6.16 and 6.17. The inner configuration becomes simpler and sections of spokes become larger as the modification ratio increases. Some noise-like small holes are observed in Fig. 6.17(c). This is because many removed elements are recovered in later iterations as a result of the sharp change processing.



(a) Iteration = 77, $W = 0.566$, $C_{obj} = 0.754$ (case 1).

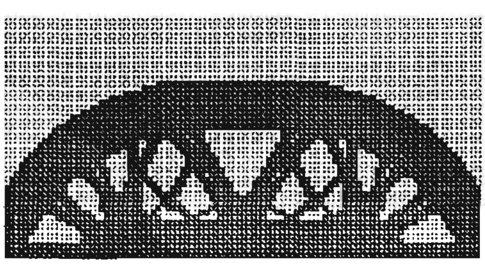


(b) Iteration = 56, $W = 0.645$, $C_{obj} = 0.779$ (case 2).

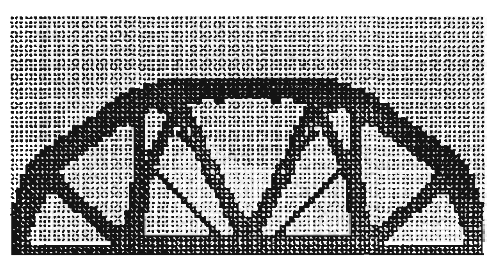


(c) Iteration = 45, $W = 0.671$, $C_{obj} = 0.802$ (case 3).

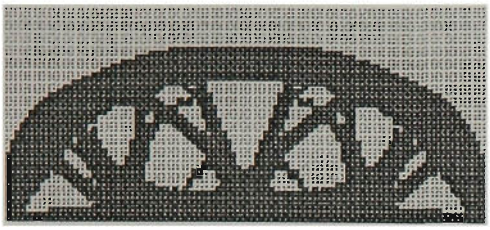
Fig. 6.15: Maximum designs corresponding to different *MR*.



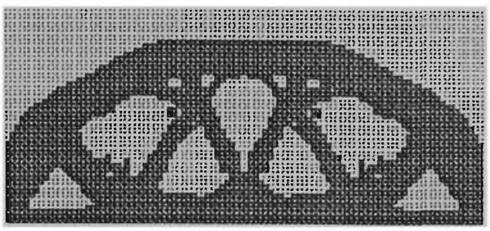
(a) $C_{obj} = 0.725$ (case 1).



(a) $C_{obj} = 0.696$ (case 1).

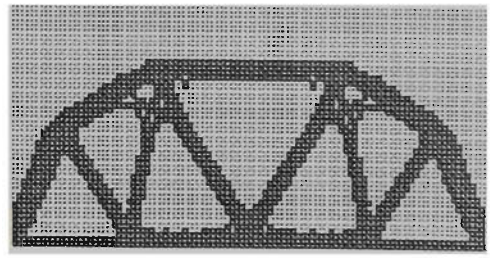


(b) C_{obj} = 0.734 (case 2).

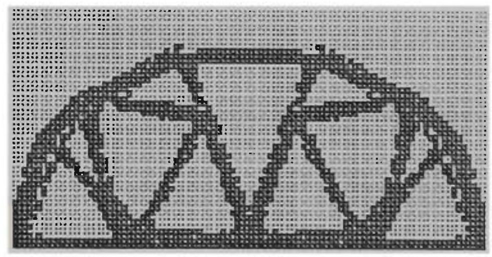


(c) C_{obj} = 0.766 (case 3).

Fig. 6.16: Topologies of weight W^* = 50% .



(b) C_{obj} = 0.703 (case 2).



(c) C_{obj} = 0.847 (case 2).

Fig. 6.17: Topologies of weight W^* = 30% .

Table 6.3: Results by different *MR* (three-point loaded beam).

W^*	*MR*	C_{obj}	Iteration	Time (Hour)
50%	1.5%	0.725	91	4.2
	3.0%	0.734	71	2.8
	5.0%	0.766	55	2.3
30%	1.5%	0.696	155	7.0
	3.0%	0.703	101	4.3
	5.0%	0.847	100	4.7

Table 6.3 lists the solution time of the three cases. For a larger structure (W^*=50%), use of a larger modification ratio can save the computational cost. This is not the case when

the structure weight is low (30%) as case 3 takes longer than case 2. As also mentioned in the last example, this is because processing of sharp changes consumes many iterations. For this reason, case 2 can be taken as the best scheme for this example.

In studying the effect of the modification ratio *MR*, a point to be kept in mind is that there does not exist a fixed value of modification ratio that is optimal for all problems. Though the selection of a suitable modification ratio can be problem specific, we may summarise some instructive points from the above three examples. In doing so, we divide the optimisation problem into two groups according to the target weight.

- Group 1: The target weight W^* is relatively high compared to the full design, say, $W^*>40\%W_0$, or the displacement limit is small, say, $u<2\ u_0$.
- Group 2. Cases other than group 1.

As for group 1, we observe that from the maximum design to a middle-weighted topology, results of different *MR* were similar and increasing the modification ratio can lower the computing cost. Therefore, a large *MR* (3.0% or 5.0%) can be used to obtain satisfactory results with reasonable computing efforts.

As for group 2, it is noted that the difference in results becomes distinct when the structure weight was relatively small. A smaller modification ratio *MR* can yield a better solution. Use of a large modification ratio may be disadvantageous in three ways. First, the displacement results may be unreliable. Second, the target weight may be too small to be reached because of suppression of sharp changes. Third, more iterations are needed to reach the same target weight due to processing the sharp changes. Taking all these factors into account, and compromising between the result reliability and computing efficiency, we may use a moderate value of modification ratio (3.0%) for this kind of problem.

Another point is that the maximum design plays an important role in affecting the result as well as the computational efficiency. A small maximum design may be unreliable due to inadequate design variables. For this reason, a too small *MR* (smaller than 1.5%) is not recommended. This is a case different from ESO where a smaller modification ratio almost always means improved results (Chu 1997).

6.2.3. Effect of Addition Ratio (*AR*)

In this investigation, the same addition ratio $AR = 0.66$ was assumed for the ascending stage thus the maximum designs were the same. Different values of addition ratio were used for the descending stage and their effects were studied. AR =0.25 was specified for case 1 and AR =0.33 for case 2. MR=1.5% was assumed in both cases.

1. Michell type structure

The change in displacement during 300 iterations is shown in Fig. 6.18. Results obtained by using different addition ratios were very close. Some slight differences are observed only at later iterations.

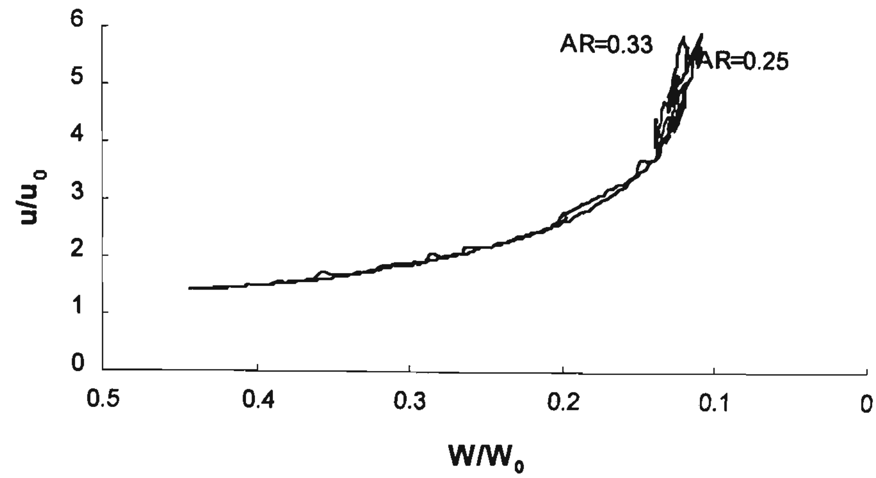
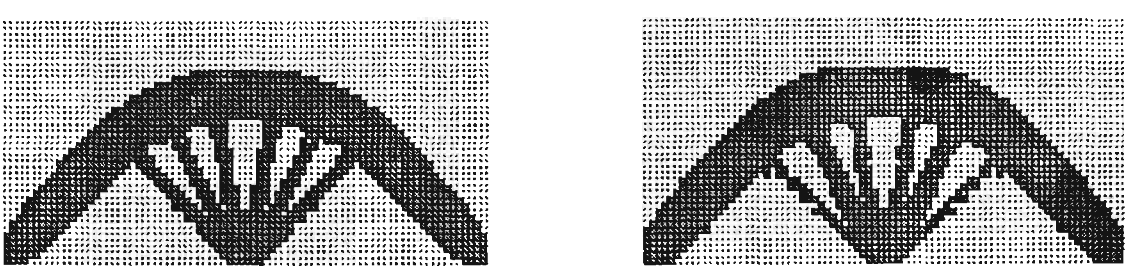


Fig. 6.18: Displacement change corresponding to different *AR* (Michell type structure).

Fig. 6.19 gives two topologies with the constrained displacement u^*=0.6 mm. Though they are not identical, their difference is not as distinct as that due to the effect of modification ratio (see Fig. 6.13). They are similar in the topology height, number of spokes and many structural details.



(a) $AR = 0.25$, $W_{obj} = 0.576$ (case 1). (b) $AR = 0.33$, $W_{obj} = 0.587$ (case 2).

Fig. 6.19: Topologies of different *AR*: $u^* = 0.6$ mm.

2. Three-point loaded beam

The same design conditions as in example 4 of Chapter 5 were used. The displacement in the two cases are very close, as shown in Fig. 6.20.

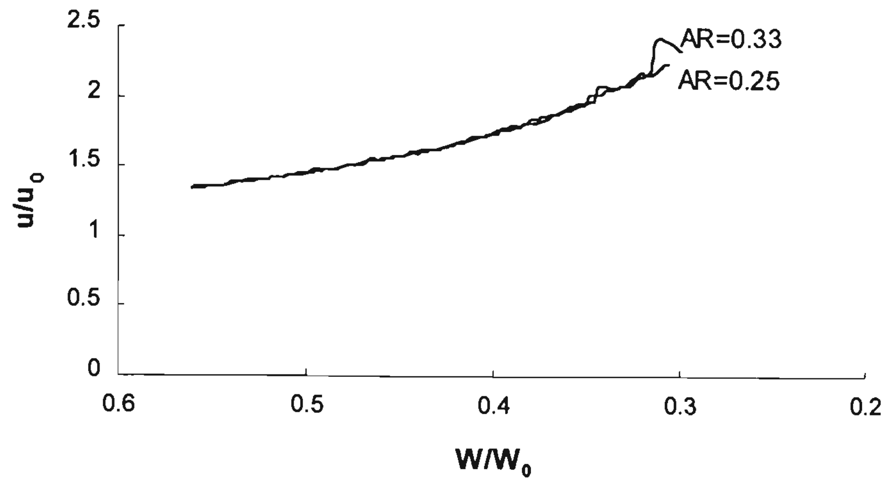
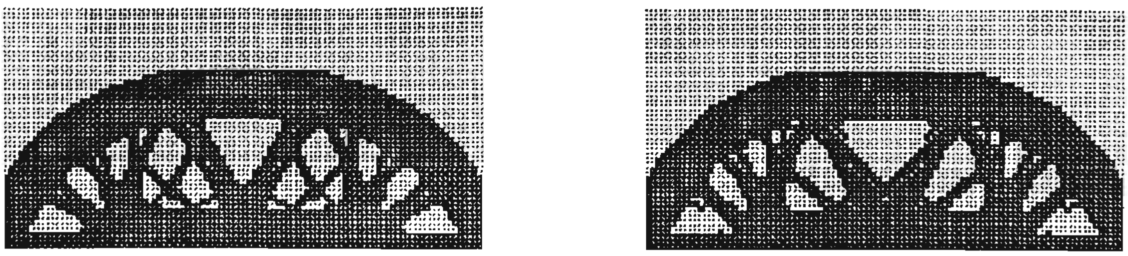


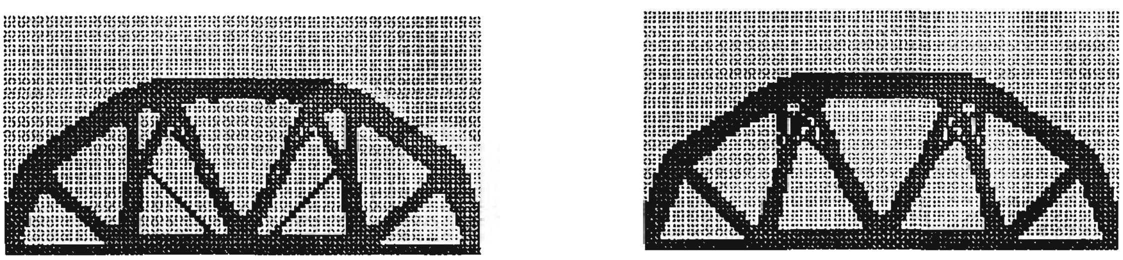
Fig. 6.20: Displacement change corresponding to different *AR* (three-point loaded beam).

Designs of W^*=50% and W^*=30% are shown in Figs. 6.21 and 6.22. Compared to topologies of the Michell structure in the last example, the difference in corresponding topologies can be easily observed. This may be because of consideration of multiple load conditions. The corresponding objective weights were very close, while case 2 required longer computing time, as shown in Table 6.4. The modification ratio decides the total change in element numbers between two cycles. At the same time, the addition ratio determines the net decrease and thus also affects the evolution speed. The net element decrease rates in two cases were (0.66-0.33)MR = 0.33MR and (0.75-0.25)MR = 0.5MR, respectively. Therefore, to reach the same weight case 2 involves more iterations than case 1.



(a) $AR = 0.25$, $C_{obj} = 0.725$ (case 1).   (b) $AR = 0.33$, $C_{obj} = 0.726$ (case 2).

Fig. 6.21: Topologies of different AR: $W^* = 50\%$.



(a) $AR = 0.25$, $C_{obj} = 0.696$ (case 1).   (b) $AR = 0.33$, $C_{obj} = 0.698$ (case 2).

Fig. 6.22: Topologies of different AR: $W^* = 30\%$.

Table 6.4: Results by different AR (three-point loaded beam).

W^*	AR	C_{obj}	Iteration	Time (Hour)
50%	0.25	0.725	91	4.2
	0.33	0.726	97	4.5
30%	0.25	0.696	155	7.0
	0.33	0.698	182	8.5

From the above case study, we can see that the topologies and numerical results change with the initial design as well as the two parameters. Different initial designs and parameters result in different maximum designs. To some extend, the maximum design can be viewed as a sort of ground structure for the BESO method, which can heavily affect the final solution. In studying the effect of the addition ratio, we use the same maximum design for different cases. This may explain the insensitivity of final results to the addition ratio. In contrast, different maximum designs were used in studying the effect of different initial designs and modification ratios. As a result, the difference in solution was more distinct than that using alternative addition ratios.

The parametric sensitivity reflects the numerical instability which is associated with most of the optimisation methods (Sigmund and Petersson 1998). This can be due to the nature of the optimisation problem. The displacement and stiffness optimisation is not convex thus the convergence may lead to a local minimum instead of a global one. In BESO, it is seen that while these solutions of 'local minimum' vary in topologies, they are similar in numerical values. Therefore, we can assume that these local solutions are sufficient close to the global minimum.

6.3 Conclusions

Strategies for suppressing sharp changes, processing singular elements, keeping symmetric properties are proposed in this chapter. They are included in the computer code BESODSP and are executed automatically without manual interruption.

At this stage, the initial design is specified by manually assigning appropriate property numbers in constructing the finite element model. It is possible to complete this task automatically. The automatic specification is more attractive when there are several alternative initial designs, which represent different load paths from the applying points to the supports. To enhance the performance of the BESO method, we can improve the algorithm in such a way that the 'best' initial design is picked up and used for optimisation. Such a best design may be featured by full interpretation of loading and boundary conditions, longest boundary lines and probably, smallest objective weight or objective compliance.

Another factor affecting BESO performance is the modification ratio (*MR*). Too small or large modification ratios may cause accuracy problems. A suitable range for cases of larger target weight can be 3%~5%. A value of 3% can be used for cases of small target weight. Designs obtained by using relatively larger modification ratio assume less complicated topologies. This can be convenient for the subsequent image processing for practical purpose. A modification ratio smaller than 1.5% is not recommended, mainly because the resultant maximum design may be too small. BESO is less sensitive to the addition ratio AR than to the modification ratio. A value of $AR = 0.66$ for the ascending stage and $AR = 0.25$ for descending stage are suggested from the solution efficiency point of view.

Chapter 7

Conclusions and Recommendations

7.1 Conclusions

Basic theories of the bi-directional evolutionary structural optimisation (BESO) method have been presented in this thesis. The simple concept of BESO is that by slowly removing inefficient elements as well as adding efficient ones, the resultant structure gradually evolves towards an optimum. BESO can be mathematically interpreted as an optimality criterion algorithm with discrete design variables. Its advantage is that a small initial design is defined as the starting point of evolution thus the computation can be more efficient compared to the ESO method. The optimisation constraints under investigation were stiffness and displacement. A wide range of design problems were studied including single and multiple constraints under one and several alternative load cases. This chapter summarises the main conclusions of this study.

Sensitivity Numbers and Displacement Extrapolation

The sensitivity number α_i and displacement extrapolation play essential roles in BESO. On one hand, as the sensitivity number is derived on the basis of optimality criteria (OC) procedure, BESO is generally valid regardless of the structural system. On the other hand, as an initial exploration of the bi-directional method, the study in this thesis focuses on the 2D continuum under plane stress condition. The displacement extrapolation is easy for this kind of finite element model. Results on 2D continua may serve as a precursor for future investigation on some more complicated structural systems.

The physical meaning of sensitivity number is the element strain energy. For potential added elements, it is an approximated value based on an extrapolated displacement field. While the displacement field is kinematically admissible, the equilibrium conditions within the elements are not satisfied. For this reason, the sensitivity number for potential added elements is over-estimated and has a much larger value than that for the existing elements.

A generalised sensitivity number η_i is formulated for structural systems where the definition of the objective weight or objective compliance is available. η_i indicates the change in the Lagrangian function due to element modification. It makes the effect of element removal and addition comparable thus the structural weight can automatically change to minimise the Lagrangian function. However, due to the accuracy problem in the displacement extrapolation of potential added elements, the generalised sensitivity number is only of theoretical importance. In practice, we still use the sensitivity number α_i to evaluate the element efficiency.

Evolutionary Procedures

The procedure of BESO is basically similar to that of ESO, i.e. the structural modification following the finite element analysis. Their difference is displayed at two levels. At a single iteration level, extra work is needed in BESO to identify the potential added elements as well as to calculate their sensitivity numbers. At the whole evolution level, unlike the case in ESO where the structure weight keeps decreasing all along, BESO procedure contains two stages, namely, the ascending stage (weight increases) and descending stage (weight decreases). These two stages are controlled by assigning appropriate values to the addition ratio AR, with $AR>0.5$ for the ascending stage and $AR<0.5$ for the descending stage, respectively. They are divided at a point where the objective weight W_{obj} or objective compliance C_{obj} reaches the first minimum value. The optimal solution is normally sought in the descending stage.

Numerical Tests and Parametric Studies

A number of examples of 2D plane stress problems were studied using the BESO and ESO procedures. Comparisons were conducted according to three criteria, namely, the structural results (objective stiffness or objective compliance), final optimal topology and solution time for reaching the optimum.

1. Structural Results

- The differences in structural results obtained from the two methods are within 3%. This demonstrates the validity of the evolutionary algorithm as an integrated system.
- Solutions of BESO are better in most cases, despite a larger modification ratio (1.5% in BESO and 1.0% in ESO). This points to the strength of the concept of bi-directional evolution, which allows for the prematurely removed elements to be recovered.
- The strength of BESO is achieved at a cost of more design cycles, which may also reflects a property of less stable convergence. A major reason for convergence instability can be that the space of design variables is smaller and is arbitrarily changing during the optimisation. This potential weakness may explain the disadvantage of BESO in some examples.

2. Optimal Topology

The change of optimal topologies observed in both ESO and BESO has the similar tendencies:

- When the target weight is relatively large (say, 50%), the final topology has a 2D continua representation. As the target weight decreases, a topology assuming skeletal form is resulted.
- Using the same full design and initial design, the topology for multiple load cases has more complicated configurations.

Further comparison between ESO and BESO reveals:

- that the optimal topologies are similar in the outer shape.

- that BESO yields topology with simpler inner architecture and larger connections.
- that in truss topology, the arc height obtained by BESO is smaller than that obtained by ESO.

Different topologies with close numerical results can be explained by the ground structure approach where the finial optimal design is dependent on the ground structure.

3. Solution Time

The total solution time is mainly determined by the dimension of finite element problems as well as the total design cycles. On one hand, as the computation for structural modification is nominal compared to that for structural analysis, BESO is more likely to save solution time in one iteration. On the other hand, BESO normally requires more iterations as it experiences two stages. Based on these factors, the following conclusions are reached:

- In most cases, BESO can be computationally more efficient mainly than ESO as a results of smaller finite element models.
- For problems with multiple constraints or multiple load conditions, the savings in solution time by using BESO are particularly significant.
- In cases where the maximum structure occupies a high proportion of the full design (say, greater than 70%), BESO needs many more iterations than ESO and may be less efficient.

The performance of BESO is affected by the initial structure, the modification ratio MR and addition ratio AR.

The initial design decides the dimension of the design space. A definition fully interpreting the support and load conditions and having longer boundaries can be more advantageous.

The modification ratio *MR*, together with the reference structure, controls the total number of modified elements at one design cycle. While a large modification ratio should be avoided for the sake of smooth transition, too small a value of *MR* (say, smaller than 1.5%) is not recommended because it may lead to a very small maximum design. BESO may yield unfavourable results from a small maximum design because of fewer design variables. Different topologies are obtained using different modification ratios. Using a larger *MR* results in topologies of simpler inner configuration and coarser boundary. A range of modification ratio *MR* 1.5%~3.0% is recommended for the trade-off between the solution accuracy and the computing efficiency.

Another parameter, the addition ratio *AR* controls the net change in elements between two design cycles. Its effect on optimal results is not as critical as that of modification ratio. The numerical results and topologies are similar for different values of *AR*. *AR*=0.66 for the ascending and *AR*=0.25 for the descending stage are recommended mainly for the consideration in computing efficiency.

7.2 Recommendations for Further Investigation

As mentioned in Chapter 4, the extension of BESO to other types of structural systems can be one area for further study. Some modifications on the displacement extrapolation and objective weight are needed to account for a particular kind of finite element. Among these investigations, BESO for plate bending problems can be the most straightforward if the four-node isoparametric element is used. BESO for skeletal structures such as plane and spatial trusses and frames remains a challenge. This is because some major modifications to the basic concept of displacement extrapolation may be required.

Stiffness and displacement constrained problems can be the simplest case using the sensitivity approach. Theoretically, the procedure of BESO can be applied to buckling load and dynamic frequency constraints equally. Two conditions add to the difficulties for these eigenvalue optimisation problems. The first is that the objective function or constraint is not linear either in the space of design variables or that of reciprocal variables. This means that we may not be able to define a Lagrangian function of an explicit form as in the stiffness and displacement optimisation. The second is that the mode shapes may swap their orders unpredictably during optimisation, which is referred to as the multi-modal phenomenon. The multi-modal problem may slow down or force an end to the evolution. As for the dynamic study, ESO has solved the natural frequency optimisation, and more efforts can be devoted to dynamic behaviour in harmonic or transient response for forced vibration so as to address more practical engineering design problems.

Also, more practical designs can be achieved if multiple criteria are included simultaneously. A typical example is the design of a compliant mechanism, as mentioned in Chapter 2. BESO can be extended to optimise a compliant mechanism and the major modification can be determination of the weight of an individual criterion. Further, the criteria can be more comprehensive and include stress, stability and frequency. Solving of multiple problems may help to handle some real-life design tasks.

Additionally, as a means of computer aided design (CAD), BESO can be programmed in such a way that it allows for an entirely automatic process to achieve its full potential. If we introduce such functions as automatic mesh generation, the effort of the designer can be reduced to specifying the loading and support conditions. The computer can generate the element mesh as well as define the initial design. In the event of several alternative initial designs, the 'best' one can be automatically selected for further optimisation.

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