Approximation of the Sum of Reciprocal of Imaginary Parts of Zeta Zeros

APPROXIMATION OF THE SUM OF RECIPROCAL OF IMAGINARY PARTS OF ZETA ZEROS

MEHDI HASSANI

ABSTRACT. In this paper, we approximate γ_n, where $0 < \gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \cdots$ are consecutive ordinates of nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$; the Riemann zeta function. Then we obtain explicit bounds for the summation $\sum_{0<\gamma\leq T} \frac{1}{\gamma}$.

1. INTRODUCTION

The Riemann zeta-function is defined for $Re(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and extended by analytic continuation to the complex plan with one singularity at $s = 1$; in fact a simple pole with residues 1. This was one of the results which B. Riemann obtained in his only paper on the theory of numbers [10], another one is functional equation which stated symmetrically as follows:

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \tag{1.1}$$

where $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt$ is a meromorphic function of the complex variable s, with simple poles at $s = 0, -1, -2, \cdots$ (see [8]). Riemann made a number of wonderful conjectures. For example, he guessed that the number $N(T)$ of zeros ρ of $\zeta(s)$ with $0 < \Im(\rho) \leq T$ and $0 \leq \Re(\rho) \leq 1$, satisfies the following relation:

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

This conjecture of Riemann proved by H. von Mangoldt more than 30 years later [4, 7]. Some immediate corollaries of above approximate formula, which is known as Riemann-van Mangoldt formula, are

$$\mathcal{A}(T) = \sum_{0<\gamma\leq T} \frac{1}{\gamma} = O(\log^2 T),$$

and $\gamma_n \sim \frac{2\pi n}{\log n}$ when $n \to \infty$, where $0 < \gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \cdots$ are consecutive ordinates of nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$, which follow by partial summation from Riemann-van Mangoldt formula and using the obvious inequality $N(\gamma_n - 1) < n \leq N(\gamma_n + 1)$, respectively [7]. In this paper, we make some explicit approximation of γ_n and $\mathcal{A}(T)$.

1991 *Mathematics Subject Classification.* Riemann Zeta function.
Key words and phrases. 11S40.

2. Approximation of γ_n

In 1941, Rosser [11] introduced the following approximation of $N(T)$:

$$|N(T)-F(T)| \le R(T) \qquad (T \ge 2), \tag{2.1}$$

where

$$F(T)=\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}+\frac{7}{8}, \tag{2.2}$$

and

$$R(T)=0.137\log T+0.443\log\log T+1.588. \tag{2.3}$$

In this paper, using Rosser's result, we approximate γ_n, and then $\mathcal{A}(T)$, explicitly. Using (2.1) and $N(\gamma_n)=n$, we have:

$$(F-R)(\gamma_n) \le n \le (F+R)(\gamma_n).$$

Both of the functions $(F\pm R)(T)$ are increasing for $T\ge 14$, thus,

$$(F+R)^{-1}(n) \le \gamma_n \le (F-R)^{-1}(n),$$

holds for every $n\ge 1$. Unfortunately, finding an explicit formula for the inverses $(F\pm R)^{-1}(T)$ isn't possible and we must replace error term R by another one. Let

$$Y(T)=\frac{25}{147}T.$$

For every $T\ge 14$, we have $R(T)\le Y(T)$, and the functions $(F\pm Y)(T)$ are increasing for $T\ge 18$. Since $\gamma_2\simeq 21.02$, we obtain:

$$(F+Y)^{-1}(n) \le \gamma_n \le (F-Y)^{-1}(n) \qquad (n\ge 2).$$

Now, we are able to find inverses $(F\pm Y)^{-1}(T)$; considering Lambert W function $W(x)$, defined by $W(x)e^{W(x)}=x$ for $x\in[-e^{-1},+\infty)$, for every $n\ge 2$ we yield that:

$$\frac{1}{4}\frac{(8n-7)\,\pi}{W\left(\frac{1}{8}\,(8n-7)\,e^{-1+\frac{50}{147}\pi}\right)} \le \gamma_n \le \frac{1}{4}\frac{(8n-7)\,\pi}{W\left(\frac{1}{8}\,(8n-7)\,e^{-1-\frac{50}{147}\pi}\right)}, \tag{2.4}$$

which holds also for $n=1$. To make some explicit bounds, independent of Lambert W function, we use the following bounds

$$\log x-\log\log x < W(x) < \log x,$$

which the left hand side holds true for $x>41.19$ and the right hand side holds true for $x>e$ [5]. Thus, we obtain:

$$\gamma_n < \frac{2\pi(n-\frac{7}{8})}{\log(n-\frac{7}{8})-\log\left(\log(n-\frac{7}{8})-(1+\frac{50}{147}\pi)\right)-(1+\frac{50}{147}\pi)}, \tag{2.5}$$

which holds for $(n-\frac{7}{8})e^{-(1+\frac{50}{147}\pi)}>41.19$ or equivalently for $n>\frac{7}{8}+41.19e^{1+\frac{50}{147}\pi}\approx 326.83$, and by computation for $13\le n\le 326$, too. Also, we obtain:

$$\frac{2\pi(n-\frac{7}{8})}{\log(n-\frac{7}{8})-(1-\frac{50}{147}\pi)} < \gamma_n, \tag{2.6}$$

which holds for $(n-\frac{7}{8})e^{-(1-\frac{50}{147}\pi)}>e$ or equivalently for $n>\frac{7}{8}+e^{2-\frac{50}{147}\pi}\approx 3.41$, and by computation for $n=1$ and $n=3$, too.

3. Approximation of $\mathcal{A}(T)$

We note that:

$$\mathcal{A}(T) = \mathcal{G}(N) = \sum_{n=1}^{N} \frac{1}{\gamma_n},$$

in which

$$N = \max\{n : \ \gamma_n \leq T\} = N(T).$$

Now, we are ready to make explicit bounds for $\mathcal{A}(T)$.

3.1. **Upper Bound.** Consider (2.6) and the following inequality[1]:

$$\mathcal{G}(N_0) < \frac{4}{\pi} \sum_{n=1}^{N_0} \frac{\log\left(\frac{1}{8}(8n-7)\, e^{-1+\frac{50}{147}\pi}\right)}{8n-7} \qquad (N_0 = 9996).$$

For every $N \geq N_0$, we have

$$\begin{aligned} \mathcal{G}(N) &< \frac{4}{\pi} \sum_{n=1}^{N} \frac{\log\left(\frac{1}{8}(8n-7)\, e^{-1+\frac{50}{147}\pi}\right)}{8n-7} \\ &= \frac{4}{\pi} \sum_{n=1}^{N} \frac{\log(8n-7)}{8n-7} + c_1 \Psi\left(N + \frac{1}{8}\right) - c_1 \Psi\left(\frac{1}{8}\right), \end{aligned}$$

where $c_1 = \frac{25}{147} - \frac{1+3\log 2}{2\pi} \approx -0.3200403161$, and $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ with $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$, is digamma function [8]. For every $x \geq 1$, it is known that

$$\log\left(x - \frac{1}{2}\right) < \Psi(x) \leq \log\left(x - 1 + e^{-c}\right), \tag{3.1}$$

where $c \simeq 0.5772156649$ is Euler constant [1]. In other hand, we have:

$$\sum_{n=1}^{N} \frac{\log(8n-7)}{8n-7} < \sum_{n=2}^{N} \frac{\log(8n)}{8(n-1)} = \frac{\log 8}{8} H(N-1) + \frac{1}{8} \sum_{n=2}^{N} \frac{\log n}{n-1},$$

---

[1]We generate this numerical inequality, because the inequality (2.6) isn't true for $n = 2$. To compute the value of N_0, which is best possible value, we used numerical data concerning zeros of $\zeta(s)$, due to A. Odlyzko [9] and the following program in Maple software worksheet:

```
restart:
with(stats):
N:=9996:
x:=array(1..N):
fp:=fopen("zeros1.txt",READ):
g:=0:
for i from 1 by 1 to N do g:=g+1/describe[mean](fscanf(fp,"%f",x[i])) end do:
fclose(fp):
G(N)=g;
```

G.A. Pirayesh helped me to write above program, which I deem my duty to thank him for his kind helps.

where $H(N)=\sum_{n=1}^{N}\frac{1}{n}$ and for every $N\geq 1$, we have $H(N)\leq c+\log(N-1+e^{1-c})$ (see [1]). Also, we have:

$$\sum_{n=2}^{N}\frac{\log n}{n-1}<\int_{1}^{N}\frac{\log t}{t-1}dt=-\text{dilog}(N),$$

where $\text{dilog}(x)$ is Dilogarithm function, defined by $\text{dilog}(x)=\int_1^x\frac{\log t}{1-t}dt$ for $x>0$ (see [16]). It is known that [6] for every $x>1$, the inequalities

$$\mathcal{D}(x,N)<\text{dilog}(x)<\mathcal{D}(x,N)+\frac{1}{x^N}$$

holds true for all $N\in\mathbb{N}$, with

$$\mathcal{D}(x,N)=-\frac{1}{2}\log^2 x-\frac{\pi^2}{6}+\sum_{n=1}^{N}\frac{\frac{1}{n^2}+\frac{1}{n}\log x}{x^n}.$$

Therefore, we have

$$-\frac{1}{2}\log^2 x-\frac{\pi^2}{6}+\frac{1+\log x}{x}<\text{dilog}(x), \tag{3.2}$$

and using this, we obtain

$$\mathcal{G}(N)<\frac{1}{4\pi}\log^2 N+\left(\frac{\log 8}{2\pi}+c_1\right)\log N+\left(\frac{c\log 8}{2\pi}+\frac{\pi}{12}-c_1\Psi\left(\frac{1}{8}\right)\right)+E_1(N),$$

where,

$$E_1(N)=\frac{\log 8}{2\pi}\log\left(1+\frac{e^{1-c}-2}{N}\right)+c_1\log\left(1-\frac{3}{8N}\right)-\frac{1+\log N}{2\pi N}<-\frac{\log N}{2\pi N}.$$

Thus

$$\mathcal{G}(N)<\frac{1}{4\pi}\log^2 N+c_2\log N+c_3-\frac{1}{2\pi}\frac{\log N}{N}, \tag{3.3}$$

for every $N\geq 9996$, with $c_2=\frac{\log 8}{2\pi}+c_1\approx 0.0109130841$ and $c_3=\frac{c\log 8}{2\pi}+\frac{\pi}{12}-c_1\Psi\left(\frac{1}{8}\right)\approx -2.231824968$. Also, it holds true for $4905\leq N\leq 9995$, by computation. Remembering $N=N(T)$, and using (2.1), we obtain the following explicit upper bound:

$$\begin{aligned}\mathcal{A}(T) &< \frac{1}{4\pi}\log^2\left(F(T)+R(T)\right)+c_2\log\left(F(T)+R(T)\right)\\ &+ c_3-\frac{1}{2\pi}\frac{\log\left(F(T)+R(T)\right)}{F(T)+R(T)}\qquad (N(T)\geq 4905).\end{aligned}\tag{3.4}$$

3.2. **Lower Bound.** Consider (2.5), which holds true for $n\geq 13$, and $\mathcal{G}(12)\approx 0.3731710458$. For every $N\geq 13$ we have

$$\begin{aligned}\mathcal{G}(N) &> \mathcal{G}(12)+\frac{4}{\pi}\sum_{n=13}^{N}\frac{\log(n-\frac{7}{8})-\log\left(\log(n-\frac{7}{8})-(1+\frac{50}{147}\pi)\right)-(1+\frac{50}{147}\pi)}{8n-7}\\ &= \frac{4}{\pi}\sum_{n=13}^{N}\left\{\frac{\log(8n-7)}{8n-7}-\frac{\log\left(147\log(n-\frac{7}{8})-147-50\pi\right)}{8n-7}\right\}\\ &+ c_4\Psi\left(N+\frac{1}{8}\right)+c_5,\end{aligned}$$

where

$$c_4 = \frac{4}{\pi}\left(-\frac{3}{8}\log 2 + \frac{1}{8}\log 147 - \frac{1}{8} - \frac{25}{588}\pi\right) \approx 0.1340756439,$$

and

$$c_5 = \mathcal{G}(12) - c_4\left(\frac{13236224754014816}{1220833367678925} + \Psi\left(\frac{1}{8}\right)\right) + \frac{6618112377007408}{1220833367678925\pi} \approx 1.769772.$$

Easily, we have:

$$\sum_{n=13}^{N} \frac{\log(8n-7)}{8n-7} = \frac{1}{8}\sum_{n=13}^{N} \frac{\log\left(n-\frac{7}{8}\right)}{n-\frac{7}{8}} + \frac{\log 8}{8}\sum_{n=13}^{N} \frac{1}{n-\frac{7}{8}},$$

and

$$\sum_{n=13}^{N} \frac{\log\left(n-\frac{7}{8}\right)}{n-\frac{7}{8}} > \int_{13-\frac{7}{8}}^{N+1-\frac{7}{8}} \frac{\log t}{t}dt = \frac{1}{2}\log^2\left(N+\frac{1}{8}\right) + c_6,$$

with

$$c_6 = -\frac{9}{2}\log^2 2 + 3\log 2\log 97 - \frac{1}{2}\log^2 97 \approx -3.113184782,$$

and

$$\sum_{n=13}^{N} \frac{1}{n-\frac{7}{8}} = \Psi\left(N+\frac{1}{8}\right) + c_7,$$

with

$$c_7 = -\left(\frac{13236224754014816}{1220833367678925} + \Psi\left(\frac{1}{8}\right)\right) \approx -2.453465877.$$

In other hand, we have:

$$\begin{aligned}
\sum_{n=13}^{N} \frac{\log\left(147\log(n-\frac{7}{8}) - 147 - 50\pi\right)}{8n-7} &< \sum_{n=13}^{N} \frac{\log\left(147\log(n-\frac{7}{8})\right)}{8n-7} \\
&= \frac{1}{8}\log 147\Psi\left(N+\frac{1}{8}\right) + c_8 \\
&+ \sum_{n=13}^{N} \frac{\log\left(\log(8n-7) - \log 8\right)}{8n-7},
\end{aligned}$$

where

$$c_8 = -\frac{1}{8}\log 147\left(\frac{13236224754014816}{1220833367678925} + \Psi\left(\frac{1}{8}\right)\right) \approx -1.530482008.$$

Also, we have:

$$\begin{aligned}
\sum_{n=13}^{N} \frac{\log\left(\log(8n-7) - \log 8\right)}{8n-7} &< \sum_{n=13}^{N} \frac{\log\log(8n-7)}{8n-7} < \int_{8(12)-7}^{8N-7} \frac{\log\log t}{t}dt \\
&= (\log\log(8N-7) - 1)\log(8N-7) + c_9,
\end{aligned}$$

where

$$c_9 = -\log\log 89\log 89 + \log 89 \approx -2.251270867.$$

Therefore, combining all of above inequalities and considering (3.2), for every $N \geq 13$, we obtain:

$$\begin{aligned}\mathcal{G}(N) &> \frac{1}{4\pi}\log^2\left(N+\frac{1}{8}\right)-\frac{4}{\pi}\left(\log\log(8N-7)-1\right)\log(8N-7)\\ &+ c_{10}\log\left(N-\frac{7}{8}+e^{-c}\right)+c_{11},\end{aligned}$$

with

$$c_{10}=\frac{3\log 2-\log 147}{2\pi}+c_4\approx -0.3292229701,$$

and

$$c_{11}=\frac{c_6+(3\log 2)c_7}{2\pi}-\frac{4(c_8+c_9)}{\pi}+c_5\approx 5.277388010.$$

References

[1] Necdet Batir, Some New Inequalities for Gamma and Polygamma Functions, *Journal of Inequalities in Pure and Applied Mathematics (JIPAM)*, Volume 6, Issue 4, Article 103.

[2] R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey, D.E. Knuth, On the Lambert W function, *Adv. Comput. Math.* 5 (1996), no. 4, 329-359.

[3] Robert M. Corless, David J. Jeffrey, Donald E. Knuth, A sequence of series for the Lambert W function, *Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation (Kihei, HI)*, 197-204 (electronic), ACM, New York, 1997.

[4] H. Davenport, Multiplicative Number Theory (Second Edition), Springer-Verlag, 1980.

[5] M. Hassani, Approximation of the Lambert W Function, *RGMIA Research Report Collection*, **8**(4), Article 12, 2005.

[6] M. Hassani, Approximation of the Dilogarithm Function, *RGMIA Research Report Collection*, **8**(4), Article 18, 2005.

[7] Aleksandar Ivic, *The Riemann Zeta Function*, John Wiley & sons, 1985.

[8] N.N. Lebedev, *Special Functions and their Applications*, Translated and edited by Richard A. Silverman, Dover Publications, New York, 1972.

[9] http://www.dtc.umn.edu/˜odlyzko/zeta_tables/

[10] Bernhard Riemann, Ueber die Anzahl der Primzahlen unter einer gegebenen Grosse (On the Number of Prime Numbers less than a Given Quantity), *Monatsberichte der Berliner Akademie*, November 1859, Translated by David R. Wilkins.

[11] J. Barkley Rosser, Explicit bounds for some functions of prime numbers, *Amer. J. Math.*, Vol. 63, (1941) pp. 211-232.

[12] J. Barkley Rosser, the n-th prime is greater than $n\log n$, *Proc. London Math. Soc. (2)*, Vol. 45, (1939) pp. 21-44.

[13] J. Barkley Rosser & L. Schoenfeld, Approximate Formulas for Some Functions of Prime Numbers, *Illinois Journal Math.*, 6 (1962) pp. 64-94.

[14] J. Barkley Rosser & L. Schoenfeld, Sharper Bounds for the Chebyshev Functions $\theta(x)$ and $\psi(x)$, *Math. Of Computation*, Vol. 29, Number 129 (January 1975) pp. 243-269.

[15] Lowell Schoenfeld, Sharper Bounds for the Chebyshev Functions $\theta(x)$ and $\psi(x)$.II", *Math. Of Computation*, Vol. 30, Number 134 (April 1976) pp. 337-360.

[16] Eric W. Weisstein. "Dilogarithm." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/Dilogarithm.html

Institute for Advanced Studies in Basic Sciences, P.O. Box 45195-1159, Zanjan, Iran

E-mail address: mmhassany@srttu.edu