Generalizations of a Theorem of I. Schur

GENERALIZATIONS OF A THEOREM OF I. SCHUR

FENG QI, WEI LI, AND BAI-NI GUO

Abstract. In the article, the monotonicities of two functions $\left(1+\frac{1}{x}\right)^{x+\alpha}$ and $\left(1+\frac{\alpha}{x}\right)^{x+\beta}$ and their corresponding sequences $\left(1+\frac{1}{n}\right)^{n+\alpha}$ and $\left(1+\frac{\alpha}{n}\right)^{n+\beta}$ are presented, an equivalent relations between the monotonicities of either these two functions or these two sequences are verified, an inclusion that a logarithmically absolutely monotonic function is also absolutely monotonic is revealed, and the logarithmically complete monotonicity and the logarithmically absolute monotonicity of the function $\left(1+\frac{\alpha}{x}\right)^{x+\beta}$ are proved, where α and β are given real parameters. As by-products, some new inequalities for the naturally logarithmic function $\ln t$ are obtained.

1. Introduction

In standard textbooks of calculus or advanced mathematical analysis, to prove that the limits $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ and $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ exist, it is sufficient to verify the sequence $\left(1+\frac{1}{n}\right)^n$ and the function $\left(1+\frac{1}{x}\right)^x$ are bounded and increasing respectively. These can be done traditionally by Newton's binomial expansion, by the arithmetic-geometric-harmonic means inequalities ([8] and [9, pp. 223–226]), by Bernoulli's inequality [10], by Young's inequality [7], by mathematical induction [9], and so on. See also [22]. As is well-known, the number e is contained in the interval $\left(1+\frac{1}{n}\right)^n<e<\left(1+\frac{1}{n}\right)^{n+1}$, where the sequence $\left(1+\frac{1}{n}\right)^{n+1}$ is decreasing ([11, pp. 357–371] and [13, pp. 266–268]).

A theorem of I. Schur [14, pp. 30, 186] states that the sequence $\left(1+\frac{1}{n}\right)^{n+\alpha}$ is decreasing if and only if $\alpha\ge\frac{1}{2}$. In [6] it was verified that the sequence $\left(1+\frac{1}{n}\right)^{n+\alpha}$ is increasing if and only if $\alpha\le\frac{2\ln3-3\ln2}{2\ln2-\ln3}$. In [9, 10, 11, 13] and [15, Vol. I, Part I, Chapter 4, p. 38] it was proved that the sequence $\left(1+\frac{1}{n}\right)^n\left(1+\frac{\beta}{n}\right)$ is decreasing if and only if $\beta\ge\frac{1}{2}$; the sequence $\left(1+\frac{\gamma}{n}\right)^{n+1}$ decreases for $0<\gamma\le2$ and increases for $\gamma>2$ and $n\ge\frac{\gamma}{\gamma-2}+1$; the sequence $\left(1+\frac{\theta}{n}\right)^n$ increases for $\theta>0$. It is easy to see that the function $\left(1+\frac{\alpha}{x}\right)^x$ is increasing with respect to $x>\max\{0,-\alpha\}$ for $\alpha\ne0$. In the proof in [13, 3.6.3 on p. 267], it was presented that for fixed $x>0$ the function $\left(1+\frac{x}{p}\right)^p$ is increasing with $p\in(0,\infty)$ and the function $\left(1+\frac{x}{p}\right)^{p+x/2}$ is decreasing with $p\in(0,\infty)$. Some related generalizations of I. Schur's theorem

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have been studied in [4, 5, 17, 20, 28]. For some new recent developments on this topic, please look up in [12, pp. 86–88 and pp. 291–292].

Now it is an entirely natural way to pose the following problem: How about the monotonicities of the functions $\left(1+\frac{1}{x}\right)^{x+\alpha}$ and $\left(1+\frac{\alpha}{x}\right)^{x+\beta}$ and the corresponding sequences $\left(1+\frac{1}{n}\right)^{n+\alpha}$ and $\left(1+\frac{\alpha}{n}\right)^{n+\beta}$ for all $\alpha\ne 0$ and $\beta\in\mathbb{R}$, respectively?

In this article, using analytic method, we will prove the following theorems which answer the problem posed above.

Theorem 1. *For $x>0$, the function $f_\alpha(x)=\left(1+\frac{1}{x}\right)^{x+\alpha}$ increases if and only if $\alpha\le 0$ and decreases if and only if $\alpha\ge\frac{1}{2}$.*

For $x<-1$, the function $f_\alpha(x)$ decreases if and only if $\alpha\ge 1$ and increases if and only if $\alpha\le\frac{1}{2}$.

The necessary and sufficient conditions such that the sequence $a_n=\left(1+\frac{1}{n}\right)^{n+\alpha}$ decreases or increases are $\alpha\ge\frac{1}{2}$ or $\alpha\le\frac{2\ln 3-3\ln 2}{2\ln 2-\ln 3}$ respectively.

Theorem 2. *Let $b_n=\left(1+\frac{\alpha}{n}\right)^{n+\beta}$ for $\alpha>-1$ and $\alpha\ne 0$ and $F_{\alpha,\beta}(x)=\left(1+\frac{\alpha}{x}\right)^{x+\beta}$ for $\alpha\ne 0$ and either $x>\max\{0,-\alpha\}$ or $x<\min\{0,-\alpha\}$.*

1. *For $x>\max\{0,-\alpha\}$, the function $F_{\alpha,\beta}(x)$ increases if and only if either $\alpha>0\ge\beta$, or $\alpha<0\le\beta$, or $\alpha\le 2\beta<0$; the function $F_{\alpha,\beta}(x)$ decreases if and only if either $2\beta\ge\alpha>0$ or $\beta\le\alpha<0$.*
2. *For $x<\min\{0,-\alpha\}$, the function $F_{\alpha,\beta}(x)$ increases if and only if either $\alpha>0\ge\beta$, or $0<2\beta\le\alpha$, or $\alpha<0\le\beta$; the function $F_{\alpha,\beta}(x)$ decreases if and only if either $0>\alpha\ge 2\beta$ or $0<\alpha<\beta$.*
3. *The sequence b_n increases if and only if either $\beta\le\frac{\ln(1+\alpha)-2\ln(1+\alpha/2)}{\ln(1+\alpha/2)-\ln(1+\alpha)}$ and $\alpha>0$ or $-1<\alpha<0$ and $\alpha\le 2\beta$. The sequence b_n decreases if and only if either $-1<\alpha<\beta\le\frac{\ln(1+\alpha)-2\ln(1+\alpha/2)}{\ln(1+\alpha/2)-\ln(1+\alpha)}<0$ or $0<\alpha\le 2\beta$.*

Theorem 3. *Theorem 1 and Theorem 2 are equivalent to each other.*

Remark 1. As by-products of proofs of Theorem 1 and Theorem 2, some inequalities for the naturally logarithmic function $\ln t$ are obtained as follows.

$$\ln t\ge\frac{t+3}{t+1}\ln\frac{1+t}{2},\qquad t>0; \tag{1}$$

$$\ln t\ge\frac{1+t}{t}\ln\frac{1+t}{2},\qquad t\in(0,1); \tag{2}$$

$$\frac{2t}{2+t}\le\ln(1+t)\le\frac{t(2+t)}{2(1+t)},\qquad t>0; \tag{3}$$

$$\ln(1+t)>\frac{t(1-t)}{1+t},\qquad t\in(-1,1). \tag{4}$$

When $-1<t<0$, inequality (3) is reversed.

These inequalities from (1) to (4) play important roles in theory of gamma functions. The left hand side of inequality (3), which is the same as the left hand side of [13, 3.6.19] essentially, improves a related problem of the 11th William Lowell Putnam Mathematical Competition. The right hand side of inequality (3) is weaker than the right hand side of [13, 3.6.18 and 3.6.19]. For further information, please refer to [11, pp. 367–368] or [13].

Remark 2. Note that some errors of mathematical expression in [4, 5, 17, 28] are corrected by Theorem 2.

Recall [1, 2, 16, 18, 19, 21, 23, 25, 27] that a function f is said to be completely monotonic on an interval I if f has derivative of all orders on I such that $(-1)^n f^{(n)}(x) \ge 0$ for $x \in I$ and $n \ge 0$. The set of the completely monotonic functions on I is denoted by $\mathcal{C}[I]$. In [21], it was coined explicitly that a positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies $(-1)^k[\ln f(x)]^{(k)} \ge 0$ for $k \in \mathbb{N}$ on I. This is a similarity of completely monotonic function [18, 24, 25, 27]. The set of the logarithmically completely monotonic functions on I is denoted by $\mathcal{C}_{\mathcal{L}}[I]$. For more information about the logarithmically completely monotonic functions, please refer to [2, 16, 23, 25] and the references therein.

A function f is said to be absolutely monotonic on an interval I if it has derivatives of all orders and $f^{(k)}(t) \ge 0$ for $t \in I$ and $k \in \mathbb{N}$. The set of the absolutely monotonic functions on I is denoted by $\mathcal{A}[I]$. See [16, 18, 24, 25, 27] and the references therein. As a similarity of this terminology and an analogy of logarithmically completely monotonic function, now a new notion is introduced below.

Definition 1. A positive function f is said to be logarithmically absolutely monotonic on an interval I if it has derivatives of all orders and $[\ln f(t)]^{(k)} \ge 0$ for $t \in I$ and $k \in \mathbb{N}$.

The set of the logarithmically absolutely monotonic functions on I is denoted by $\mathcal{A}_{\mathcal{L}}[I]$. It is well known that a logarithmic convex function is also convex. As a generalization of this conclusion and an analogy of the inclusion $\mathcal{C}_{\mathcal{L}}[I] \subset \mathcal{C}[I]$ in [2, 19, 21], the following Theorem 4 is established.

Theorem 4. *A logarithmically absolutely monotonic function on an interval I is also absolutely monotonic on I, equivalently, $\mathcal{A}_{\mathcal{L}}[I] \subset \mathcal{A}[I]$.*

Theorem 4 hints us that in order to show some functions, especially the power-exponential functions or the exponential functions, are absolutely monotonic, maybe it is much simpler or easier to prove the stronger statement that they are logarithmically absolutely monotonic.

In [23, Theorem 1.2] and [29], it was proved that $F_{\alpha,\beta}(x) \in \mathcal{C}_{\mathcal{L}}[(0,\infty)]$ for $\alpha > 0$ and $\beta \in \mathbb{R}$ if and only if $2\beta \ge \alpha > 0$. From $\mathcal{C}_{\mathcal{L}}[I] \subset \mathcal{C}[I]$ it is deduced that the function $(1+\frac{\alpha}{x})^{x+\beta} - e^{\alpha} \in \mathcal{C}[(0,\infty)]$ if and only if $0 < \alpha \le 2\beta$, which is a conclusion obtained in [1].

Now it is natural to pose a problem: How about the logarithmically complete (absolute) monotonicity of the function $F_{\alpha,\beta}(x)$ for all real numbers $\alpha \ne 0$ and β in the interval $(-\infty, \min\{0,-\alpha\})$ or $(\max\{0,-\alpha\},\infty)$? The following Theorem 5 answers this problem.

Theorem 5. *For $\alpha < 0$, $F_{\alpha,\beta}(x) \in \mathcal{C}_{\mathcal{L}}[(-\alpha,\infty)]$ if and only if $\beta \le \alpha$ and $[F_{\alpha,\beta}(x)]^{-1} \in \mathcal{C}_{\mathcal{L}}[(-\alpha,\infty)]$ if and only if $2\beta \ge \alpha$.*

For $\alpha > 0$, $F_{\alpha,\beta}(x) \in \mathcal{C}_{\mathcal{L}}[(0,\infty)]$ if and only if $2\beta \ge \alpha$ and $[F_{\alpha,\beta}(x)]^{-1} \in \mathcal{C}_{\mathcal{L}}[(0,\infty)]$ if and only if $\beta \le 0$.

For $\alpha < 0$, $F_{\alpha,\beta}(x) \in \mathcal{A}_{\mathcal{L}}[(-\infty,0)]$ *if and only if* $\beta \ge 0$ *and* $[F_{\alpha,\beta}(x)]^{-1} \in \mathcal{A}_{\mathcal{L}}[(-\infty,0)]$ *if and only if* $2\beta \le \alpha$.

For $\alpha > 0$, $F_{\alpha,\beta}(x) \in \mathcal{A}_{\mathcal{L}}[(-\infty,-\alpha)]$ *if and only if* $2\beta \le \alpha$ *and* $[F_{\alpha,\beta}(x)]^{-1} \in \mathcal{A}_{\mathcal{L}}[(-\infty,-\alpha)]$ *if and only if* $\beta \ge \alpha$.

As an immediate consequence of combining Theorem 5 with $\mathcal{C}_{\mathcal{L}}[I] \subset \mathcal{C}[I]$, the following complete monotonicity relating to the function $F_{\alpha,\beta}(x)$, which extends the corresponding result obtained in [1, 23, 29] and mentioned above, can be obtained easily.

Theorem 6. *For* $\alpha > 0$, $(1+\frac{\alpha}{x})^{x+\beta} - e^{\alpha} \in \mathcal{C}[(0,\infty)]$ *if and only if* $\alpha \le 2\beta$ *and* $(1+\frac{\alpha}{x})^{-(x+\beta)} - e^{-\alpha} \in \mathcal{C}[(0,\infty)]$ *if and only if* $\beta \le 0$. *For* $\alpha < 0$, $(1+\frac{\alpha}{x})^{x+\beta} - e^{\alpha} \in \mathcal{C}[(-\alpha,\infty)]$ *if and only if* $\beta \le \alpha$ *and* $\frac{1}{F_{\alpha,\beta}(x)} \in \mathcal{C}[(-\alpha,\infty)]$ *if and only if* $2\beta \ge \alpha$.

2. Proofs of theorems and inequalities

2.1. **Proof of Theorem 1.** Direct calculation gives

$$[\ln f_\alpha(x)]' = \ln\Bigl(1+\frac{1}{x}\Bigr) - \frac{x+\alpha}{x(x+1)} \quad \text{and} \quad [\ln f_\alpha(x)]'' = \frac{(2\alpha-1)x+\alpha}{x^2(x+1)^2}.$$

For $x > 0$, it is easy to see that $[\ln f_\alpha(x)]'' > 0$ and $[\ln f_\alpha(x)]'$ increases if and only if $\alpha \ge \frac{1}{2}$. Since $\lim_{x\to\infty}[\ln f_\alpha(x)]' = 0$ for any $\alpha \in \mathbb{R}$, thus $[\ln f_\alpha(x)]' < 0$ for $\alpha \ge \frac{1}{2}$ (This implies the right hand side of inequality (3)), which means $f_\alpha'(x) < 0$ and $f_\alpha(x)$ decreases. This implies also that the sequence a_n is decreasing for $\alpha \ge \frac{1}{2}$.

For $x > 0$, it is clear that $[\ln f_\alpha(x)]'' < 0$ and $[\ln f_\alpha(x)]'$ decreases if and only if $\alpha \le 0$. Then $[\ln f_\alpha(x)]' > 0$, $f_\alpha'(x) > 0$ and $f_\alpha(x)$ increases for $\alpha \le 0$. This implies that the sequence $\bigl(1+\frac{1}{n}\bigr)^{n+\alpha}$ is increasing for $\alpha \le 0$.

For $x > 0$, when $0 < \alpha < \frac{1}{2}$, the function $[\ln f_\alpha(x)]''$ has a unique zero point $x_0 = \frac{\alpha}{1-2\alpha} > 0$ which is a supremum point of $[\ln f_\alpha(x)]'$, this supremum equals $[\ln f_\alpha(x_0)]' = \ln\bigl(\frac{1}{\alpha}-1\bigr) + 2(2\alpha-1) > 0$ (This implies the left hand side of inequality (3)). Since $\lim_{x\to0^+}[\ln f_\alpha(x)]' = -\infty$ for $\alpha > 0$ and $\lim_{x\to\infty}[\ln f_\alpha(x)]' = 0$ for any $\alpha \in \mathbb{R}$, it is yielded that the functions $[\ln f_\alpha(x)]'$ and $f_\alpha'(x)$ have only one zero point $x_1 > 0$, which is a unique infimum point of $f_\alpha(x)$ on $(0,\infty)$. Consequently, the sufficient and necessary condition of the sequence a_n being increasing is $f_\alpha(1) \le f_\alpha(2)$ which is equivalent to $\alpha \le \frac{2\ln3-3\ln2}{2\ln2-\ln3}$.

For $x < -1$, the function $[\ln f_\alpha(x)]'' > 0$ and $[\ln f_\alpha(x)]'$ is increasing if and only if $\alpha \le \frac{1}{2}$. From $\lim_{x\to-\infty}[\ln f_\alpha(x)]' = 0$ it is deduced that $[\ln f_\alpha(x)]' > 0$ and $f_\alpha'(x) > 0$ in $(-\infty,-1)$. Consequently, the function $f_\alpha(x)$ is increasing in $(-\infty,-1)$ if $\alpha \le \frac{1}{2}$.

For $x < -1$, the function $[\ln f_\alpha(x)]'' < 0$ and $[\ln f_\alpha(x)]'$ is decreasing if and only if $\alpha \ge 1$. From $\lim_{x\to-\infty}[\ln f_\alpha(x)]' = 0$ it follows that $[\ln f_\alpha(x)]' < 0$ and $f_\alpha'(x) < 0$ in $(-\infty,-1)$. Accordingly, the function $f_\alpha(x)$ decreases in $(-\infty,-1)$ if $\alpha \ge 1$.

For $x < -1$ and $\frac{1}{2} < \alpha < 1$, the function $[\ln f_\alpha(x)]''$ has a unique zero point $x_0 = \frac{\alpha}{1-2\alpha} < -1$ which is a minimum point of $[\ln f_\alpha(x)]'$. Since $\lim_{x\to(-1)^-}[\ln f_\alpha(x)]' = \infty$ and $\lim_{x\to-\infty}[\ln f_\alpha(x)]' = 0$, then the functions $[\ln f_\alpha(x)]'$ and $f_\alpha'(x)$ have only one zero point $x_1 > 0$, which is a unique infimum point of $f_\alpha(x)$ on $(0,\infty)$. This completes the proof of Theorem 1.

2.2. **Proofs of Inequalities.** Let $G(t) = (2+t)\ln(1+t) - (4+t)\ln\bigl(1+\frac{t}{2}\bigr)$ for $t > -1$. Then

$$G'(t) = \ln(1+t) - \ln\Bigl(1+\frac{t}{2}\Bigr) + \frac{1}{1+t} - \frac{2}{2+t} \quad \text{and} \quad G''(t) = \frac{t(3+2t)}{(1+t)^2(2+t)^2}.$$

It is clear that $G''(t)$ has a unique zero $t = 0$ for $t > -1$ and $G'(t)$ takes the infimum $G'(0) = 0$, thus $G'(t) > 0$ (This implies the inequality (4)) and $G(t)$ is increasing. From $G(0) = 0$, it follows that $G(t)\begin{cases} > 0, & t > 0, \\ < 0, & t < 0. \end{cases}$ From this, we conclude the inequality (1) and

$$\frac{\ln(1+t) - 2\ln\bigl(1+\frac{t}{2}\bigr)}{\ln\bigl(1+\frac{t}{2}\bigr) - \ln(1+t)} < \frac{t}{2} \tag{5}$$

for $t > -1$ and $t \neq 0$.

The inequality (2) follows from standard arguments.

2.3. **Proof of monotonicity of $\bigl(1+\frac{\alpha}{x}\bigr)^{x+\beta}$.** Calculating directly yields

$$\ln F_{\alpha,\beta}(x) = (x+\beta)\ln\Bigl(1+\frac{\alpha}{x}\Bigr), \tag{6}$$

$$[\ln F_{\alpha,\beta}(x)]' = \ln\Bigl(1+\frac{\alpha}{x}\Bigr) - \frac{\alpha(x+\beta)}{x(x+\alpha)}, \tag{7}$$

$$[\ln F_{\alpha,\beta}(x)]'' = \frac{\alpha[(2\beta-\alpha)x + \alpha\beta]}{x^2(x+\alpha)^2}. \tag{8}$$

2.3.1. *The case of* $x > \max\{0, -\alpha\}$. It is not difficult to see that the function $F_{\alpha,0}(x)$ is increasing for $x > \max\{0, -\alpha\}$.

Direct calculation yields

$$\lim_{x\to\infty}[\ln F_{\alpha,\beta}(x)]'' = 0, \qquad \lim_{x\to\infty}[\ln F_{\alpha,\beta}(x)]' = 0, \tag{9}$$

$$\lim_{x\to 0^+}[\ln F_{\alpha,\beta}(x)]' = -\operatorname{sgn}\beta\cdot\infty \qquad \text{if } \alpha > 0, \tag{10}$$

$$\lim_{x\to(-\alpha)^+}[\ln F_{\alpha,\beta}(x)]' = -\infty \qquad \text{if } \beta = \alpha < 0, \tag{11}$$

$$\lim_{x\to(-\alpha)^+}[\ln F_{\alpha,\beta}(x)]' = \operatorname{sgn}(\beta-\alpha)\cdot\infty \qquad \text{if } \alpha < 0,\ \beta \neq 0 \text{ and } \beta \neq \alpha. \tag{12}$$

From (8), it follows that $[\ln F_{\alpha,\beta}(x)]'' > 0$ and $[\ln F_{\alpha,\beta}(x)]'$ is increasing for $\alpha = 2\beta > 0$. Further, from (9), it follows also that $[\ln F_{\alpha,\beta}(x)]' < 0$ (From this, we can obtain the right hand side of inequality (3)) and $\ln F_{\alpha,\beta}(x)$ decreases. Therefore $F_{\alpha,\beta}(x)$ decreases for $\alpha = 2\beta > 0$. By the same argument, it can be deduced that if $\alpha = 2\beta < 0$ the function $F_{\alpha,\beta}(x)$ increases.

From (8), if $\alpha \neq 2\beta$, the function $[\ln F_{\alpha,\beta}(x)]''$ may have one zero point $x_0 = \frac{\alpha\beta}{\alpha-2\beta}$ at most.

If $x_0 \leq \max\{0, -\alpha\}$, then the function $[\ln F_{\alpha,\beta}(x)]''$ has no zero point. This means that if $\alpha > 0 > \beta$, or $0 < \alpha < 2\beta$, or $\alpha < 0 < \beta$, or $\alpha < 2\beta < 0$, or $\beta \leq \alpha < 0$, the function $[\ln F_{\alpha,\beta}(x)]''$ keep the same sign and $[\ln F_{\alpha,\beta}(x)]'$ is monotonic. Furthermore, utilizing (10), (11) and (12), it is concluded that when either $\alpha > 0 > \beta$, or $\alpha < 0 < \beta$ or $\alpha < 2\beta < 0$, the function $[\ln F_{\alpha,\beta}(x)]' > 0$, and then $\ln F_{\alpha,\beta}(x)$ and $F_{\alpha,\beta}(x)$ increases; when either $2\beta > \alpha > 0$ or $\beta \le \alpha < 0$, the function $[\ln F_{\alpha,\beta}(x)]' < 0$, then $\ln F_{\alpha,\beta}(x)$ and $F_{\alpha,\beta}(x)$ decreases.

If $x_0 > \max\{0, -\alpha\}$, the function $[\ln F_{\alpha,\beta}(x)]''$ has a unique zero x_0. If $\alpha(2\beta - \alpha) > 0$, the function $[\ln F_{\alpha,\beta}(x)]'$ has a unique minimum attained at x_0; if $\alpha(2\beta - \alpha) < 0$, the function $[\ln F_{\alpha,\beta}(x)]'$ has a unique maximum attained at x_0. This implies that for $\alpha < \beta < \frac{\alpha}{2} < 0$ the function $[\ln F_{\alpha,\beta}(x)]'$ has a unique zero point which is a maximum point of $\ln F_{\alpha,\beta}(x)$ and $F_{\alpha,\beta}(x)$ and that for $0 < 2\beta < \alpha$ the function $[\ln F_{\alpha,\beta}(x)]'$ has a unique zero point which is a minimum point of $\ln F_{\alpha,\beta}(x)$ and $F_{\alpha,\beta}(x)$.

2.3.2. *The case of $x < \min\{0, -\alpha\}$.* It is not difficult to see that the function $F_{\alpha,0}(x)$ increases for $x < \min\{0, -\alpha\}$.

Straightforward computation leads to

$$\lim_{x\to-\infty}[\ln F_{\alpha,\beta}(x)]'' = 0, \qquad \lim_{x\to-\infty}[\ln F_{\alpha,\beta}(x)]' = 0, \tag{13}$$

$$\lim_{x\to(-\alpha)^-}[\ln F_{\alpha,\beta}(x)]' = -\operatorname{sgn}(\beta-\alpha)\cdot\infty \qquad \text{if } \alpha > 0, \tag{14}$$

$$\lim_{x\to0^-}[\ln F_{\alpha,\beta}(x)]' = -\infty \qquad \text{if } \beta = \alpha < 0, \tag{15}$$

$$\lim_{x\to0^-}[\ln F_{\alpha,\beta}(x)]' = \operatorname{sgn}\beta\cdot\infty \qquad \text{if } \alpha < 0 \text{ and } \beta \ne \alpha. \tag{16}$$

By (8), if $\alpha = 2\beta > 0$, then the function $[\ln F_{\alpha,\beta}(x)]'' > 0$ and $[\ln F_{\alpha,\beta}(x)]'$ is increasing. Considering (13) gives $[\ln F_{\alpha,\beta}(x)]' > 0$, and then $\ln F_{\alpha,\beta}(x)$ and $F_{\alpha,\beta}(x)$ are increasing for $\alpha = 2\beta > 0$. Similarly, if $\alpha = 2\beta < 0$, the function $F_{\alpha,\beta}(x)$ is decreasing.

Observing (8), when $\alpha \ne 2\beta$, the function $[\ln F_{\alpha,\beta}(x)]''$ may have at most one zero point $x_0 = \frac{\alpha\beta}{\alpha-2\beta}$.

If $x_0 \ge \min\{0, -\alpha\}$, then $[\ln F_{\alpha,\beta}(x)]''$ has no zero point. This implies that if either $0 > \alpha > 2\beta$, or $\alpha > 0 > \beta$, or $0 < 2\beta < \alpha$, or $0 < \alpha < \beta$, or $\alpha < 0 < \beta$ then the function $[\ln F_{\alpha,\beta}(x)]''$ does not change its sign and $[\ln F_{\alpha,\beta}(x)]'$ is monotonic. Employing (14), (15) and (16) concludes that when either $\alpha > 0 > \beta$, or $0 < 2\beta < \alpha$, or $\alpha < 0 < \beta$ the function $[\ln F_{\alpha,\beta}(x)]' > 0$, and then $\ln F_{\alpha,\beta}(x)$ and $F_{\alpha,\beta}(x)$ increases and that when either $0 > \alpha > 2\beta$ or $0 < \alpha < \beta$ the function $[\ln F_{\alpha,\beta}(x)]' < 0$, and then $\ln F_{\alpha,\beta}(x)$ and $F_{\alpha,\beta}(x)$ decreases.

If $x_0 < \min\{0, -\alpha\}$, the function $[\ln F_{\alpha,\beta}(x)]''$ has a unique zero x_0. If $\alpha(2\beta - \alpha) > 0$, the function $[\ln F_{\alpha,\beta}(x)]'$ has a unique minimum attained at x_0; if $\alpha(2\beta - \alpha) < 0$, the function $[\ln F_{\alpha,\beta}(x)]'$ has a unique maximum attained at x_0. This implies that for $2\beta > \alpha > \beta > 0$ the function $[\ln F_{\alpha,\beta}(x)]'$ has a unique zero point which is a minimum point of $\ln F_{\alpha,\beta}(x)$ and $F_{\alpha,\beta}(x)$ and that for $0 > 2\beta > \alpha$ the function $[\ln F_{\alpha,\beta}(x)]'$ has a unique zero point which is a maximum point of $\ln F_{\alpha,\beta}(x)$ and $F_{\alpha,\beta}(x)$.

2.4. **Proof of monotonicity of $\left(1 + \frac{\alpha}{n}\right)^{n+\beta}$.** It has been proved above that the function $F_{\alpha,\beta}(x)$ has a unique maximum if $\alpha < \beta < \frac{\alpha}{2} < 0$ and that the function $F_{\alpha,\beta}(x)$ has a unique minimum if $0 < 2\beta < \alpha$. Consequently, if $F_{\alpha,\beta}(1) \le F_{\alpha,\beta}(2)$ for $\alpha > 2\beta > 0$ the sequence b_n increases; if $F_{\alpha,\beta}(1) \ge F_{\alpha,\beta}(2)$ for $2\beta < \alpha <$

$\beta < 0$ the sequence b_n decreases; otherwise, b_n is not monotonic. Namely, when $\alpha > 2\beta > 0$ and $\beta \le \frac{\ln(1+\alpha)-2\ln(1+\alpha/2)}{\ln(1+\alpha/2)-\ln(1+\alpha)}$, the sequence $b_n = \left(1+\frac{\alpha}{n}\right)^{n+\beta}$ increases; when $2\beta < \alpha < \beta < 0$ and $\beta \le \frac{\ln(1+\alpha)-2\ln(1+\alpha/2)}{\ln(1+\alpha/2)-\ln(1+\alpha)}$, the sequence b_n decreases. As a result, using inequality (1) or (5), the sufficient and necessary conditions of the sequence $\left(1+\frac{\alpha}{n}\right)^{n+\beta}$ being monotonic are concluded. The proof of Theorem 2 is complete.

2.5. **Proof of Theorem 3.** It is clear that Theorem 1 is a special case of Theorem 2.

In order to prove Theorem 3, it is sufficient to conclude Theorem 2 directly from Theorem 1. For this purpose, taking $\frac{\alpha}{x} = \frac{1}{y}$ for $\alpha > 0$ yields $F_{\alpha,\beta}(x) = [f_{\beta/\alpha}(y)]^{\alpha}$. This shows that the functions $F_{\alpha,\beta}(x)$ and $f_{\beta/\alpha}(y)$ have the same monotonicity as $\alpha > 0$. On the other hand, if $\alpha < 0$, setting $\frac{-\alpha}{x+\alpha} = \frac{1}{y}$ leads to $F_{\alpha,\beta}(x) = [f_{1-\beta/\alpha}(y)]^{\alpha}$. This tells us that the functions $F_{\alpha,\beta}(x)$ and $f_{1-\beta/\alpha}(y)$ have the opposite monotonicity as $\alpha < 0$. From Theorem 1, the monotonicities of $F_{\alpha,\beta}(x)$ can be deduced immediately.

By similar arguments, the equivalence between the necessary and sufficient conditions of the monotonicities of the sequences a_n and b_n can be obtained easily. The proof of Theorem 3 is complete.

2.6. **Proof of Theorem 4.** The Faá di Bruno's formula [3, 26] gives an explicit formula for the n-th derivative of the composition $g(h(t))$: If $g(t)$ and $h(t)$ are functions for which all the necessary derivatives are defined, then

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}[g(h(x))] = \sum_{\substack{1\leqslant i\leqslant n, i_k\geqslant 0\\ \sum_{k=1}^n i_k = i\\ \sum_{k=1}^n k i_k = n}} \frac{n!}{\prod_{k=1}^n i_k!} g^{(i)}(h(x)) \prod_{k=1}^n \left[\frac{h^{(k)}(x)}{k!}\right]^{i_k}. \tag{17}$$

Applying (17) to $g(x) = e^x$ and $h(x) = \ln f(x)$ leads to

$$f^{(n)}(x) = \left[e^{\ln f(x)}\right]^{(n)} = n! f(x) \sum_{\substack{1\leqslant i\leqslant n, i_k\geqslant 0\\ \sum_{k=1}^n i_k = i\\ \sum_{k=1}^n k i_k = n}} \prod_{k=1}^n \frac{\left\{[\ln f(x)]^{(k)}\right\}^{i_k}}{[i_k!(k!)^{i_k}]} \tag{18}$$

for $n \in \mathbb{N}$. Theorem 4 follows easily.

2.7. **Proof of Theorem 5.** For $\alpha < 0$ and $x > -\alpha$, in virtue of formula

$$\frac{1}{x^r} = \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-xt}\,\mathrm{d}t \tag{19}$$

for $x > 0$ and $r > 0$, the equation (8) can be rewritten as

$$\begin{aligned}[\ln F_{\alpha,\beta}(x)]'' &= \frac{1}{x+\alpha} - \frac{1}{x} - \frac{\beta-\alpha}{(x+\alpha)^2} + \frac{\beta}{x^2}\\ &\triangleq \int_0^\infty [\beta - q_\alpha(t)] t \left(e^{\alpha t} - 1\right) e^{-(x+\alpha)t}\,\mathrm{d}t,\end{aligned} \tag{20}$$

where

$$q_\alpha(t)=\frac{e^{\alpha t}-\alpha t-1}{t\big(e^{\alpha t}-1\big)}=\frac{\alpha(e^u-u-1)}{u(e^u-1)}\triangleq\alpha q(u) \tag{21}$$

for $t>0$ and $u=\alpha t<0$. Since $q(u)$ is decreasing in $(-\infty,0)$ with $\lim_{u\to0^-}q(u)=\frac12$ and $\lim_{u\to-\infty}q(u)=1$, then when $\beta\le\alpha$ the function $(-1)^i[\ln F_{\alpha,\beta}(x)]^{(i+2)}\ge0$ and when $2\beta\ge\alpha$ the function $(-1)^i[\ln F_{\alpha,\beta}(x)]^{(i+2)}\le0$ in $(-\alpha,\infty)$ for $i\ge0$. Since $[\ln F_{\alpha,\beta}(x)]'$ increases for $\beta\le\alpha$ and decreases for $2\beta\ge\alpha$, considering the second limit in (9) shows that $[\ln F_{\alpha,\beta}(x)]'\le0$ for $\beta\le\alpha$ and $[\ln F_{\alpha,\beta}(x)]'\ge0$ for $2\beta\ge\alpha$. In conclusion, $(-1)^k[\ln F_{\alpha,\beta}(x)]^{(k)}\ge0$ for $\beta\le\alpha$ and $(-1)^k[\ln F_{\alpha,\beta}(x)]^{(k)}\le0$ for $2\beta\ge\alpha$ and $k\in\mathbb{N}$. This means that $F_{\alpha,\beta}(x)\in\mathcal{C_L}[(-\alpha,\infty)]$ for $\beta\le\alpha<0$ and $\frac{1}{F_{\alpha,\beta}(x)}\in\mathcal{C_L}[(-\alpha,\infty)]$ for both $2\beta\ge\alpha$ and $\alpha<0$. Conversely, if $F_{\alpha,\beta}(x)\in\mathcal{C_L}[(-\alpha,\infty)]$ for $\alpha<0$, then $[\ln F_{\alpha,\beta}(x)]'\le0$ which can be rearranged as

$$\beta\le x\left[\left(1+\frac{x}{\alpha}\right)\ln\left(1+\frac{\alpha}{x}\right)-1\right]\triangleq\theta_\alpha(x) \tag{22}$$

and $\lim_{x\to(-\alpha)^+}\theta_\alpha(x)=\alpha$, thus $\beta\le\alpha$. If $\frac{1}{F_{\alpha,\beta}(x)}\in\mathcal{C_L}[(-\alpha,\infty)]$ for $\alpha<0$, then $[\ln F_{\alpha,\beta}(x)]'\ge0$ which can be rearranged as $\beta\ge\theta_\alpha(x)\to\frac\alpha2$ as $x\to\infty$, hence $2\beta\ge\alpha$ holds.

If $\alpha>0$, the formulas (20) and (21) are valid for $x>0$ and $u>0$. Since $q(u)$ is decreasing in $(0,\infty)$ with $\lim_{u\to0^+}q(u)=\frac12$ and $\lim_{u\to\infty}q(u)=0$, by the same argument as above, it follows easily that $F_{\alpha,\beta}(x)\in\mathcal{C_L}[(0,\infty)]$ for $2\beta\ge\alpha$ and $\frac{1}{F_{\alpha,\beta}(x)}\in\mathcal{C_L}[(0,\infty)]$ for $\beta\le0$. Conversely, if $F_{\alpha,\beta}(x)\in\mathcal{C_L}[(0,\infty)]$ for $\alpha>0$, then $[\ln F_{\alpha,\beta}(x)]'\le0$ which can be rewritten as $\beta\ge\theta_\alpha(x)\to\frac\alpha2$ as x tends to ∞; if $\frac{1}{F_{\alpha,\beta}(x)}\in\mathcal{C_L}[(0,\infty)]$ for $\alpha>0$, then $[\ln F_{\alpha,\beta}(x)]'\ge0$ which can be rewritten as $\beta\le\theta_\alpha(x)\to0$ as $x\to0$.

For $\alpha<0$ and $x<0$, it is easy to obtain

$$\begin{aligned}[\ln F_{\alpha,\beta}(x)]''&=-\frac{1}{-(x+\alpha)}+\frac{1}{-x}-\frac{\beta-\alpha}{[-(x+\alpha)]^2}+\frac{\beta}{(-x)^2}\\&\triangleq\int_0^\infty[\beta+p_\alpha(t)]t\big(1-e^{\alpha t}\big)e^{xt}\,\mathrm{d}t,\end{aligned} \tag{23}$$

where

$$p_\alpha(t)=\frac{1+(\alpha t-1)e^{\alpha t}}{t(1-e^{\alpha t})}=\frac{\alpha[1+(u-1)e^u]}{u(1-e^u)}\triangleq\alpha p(u) \tag{24}$$

for $t>0$ and $u=\alpha t<0$ and $p(u)$ is decreasing in $(-\infty,0)$ with $\lim_{u\to-\infty}p(u)=0$ and $\lim_{u\to0^-}p(u)=-\frac12$. Accordingly, for $i\ge0$ and in $(-\infty,0)$, if $\beta-\frac\alpha2\le0$ then $[\ln F_{\alpha,\beta}(x)]^{(i+2)}\le0$, if $\beta\ge0$ then $[\ln F_{\alpha,\beta}(x)]^{(i+2)}\ge0$. By virtue of (13), it is deduced immediately that $[\ln F_{\alpha,\beta}(x)]^{(k)}\le0$ for $2\beta\le\alpha$ and $[\ln F_{\alpha,\beta}(x)]^{(k)}\ge0$ for $\beta\ge0$ and $k\in\mathbb{N}$ in $(-\infty,0)$. Conversely, if $F_{\alpha,\beta}(x)$ is logarithmically absolutely monotonic in $(-\infty,0)$, then $[\ln F_{\alpha,\beta}(x)]'\ge0$ which can be rewritten as $\beta\ge\theta_\alpha(x)$ for $x\in(-\infty,0)$. From $\lim_{x\to0^-}\theta_\alpha(x)=0$, it follows that $\beta\ge0$; if $\frac{1}{F_{\alpha,\beta}(x)}$ is logarithmically absolutely monotonic in $(-\infty,0)$, then $[\ln F_{\alpha,\beta}(x)]'\le0$ which can be rearranged as $\beta\le\theta_\alpha(x)$ for $x\in(-\infty,0)$. From $\lim_{x\to-\infty}\theta_\alpha(x)=\frac\alpha2$, it concludes that $2\beta\le\alpha$.

For $\alpha > 0$ and $x < -\alpha$, the formulas (23) and (24) hold for $x \in (-\infty, -\alpha)$ and $u > 0$. The function $p(u)$ is negative and decreasing in $(0, \infty)$ with $\lim_{u \to 0^+} p(u) = -\frac{1}{2}$ and $\lim_{u \to \infty} p(u) = -1$. Consequently, if $\beta - \frac{1}{2}\alpha \le 0$ then $[\ln F_{\alpha,\beta}(x)]^{(i+2)} \ge 0$ for $i \ge 0$ in $(-\infty, -\alpha)$, if $\beta - \alpha \ge 0$ then $[\ln F_{\alpha,\beta}(x)]^{(i+2)} \le 0$ for $i \ge 0$ in $(-\infty, -\alpha)$. In virtue of (13), it is concluded readily that $[\ln F_{\alpha,\beta}(x)]^{(k)} \ge 0$ for $2\beta \le \alpha$ and $[\ln F_{\alpha,\beta}(x)]^{(k)} \le 0$ for $\beta \ge \alpha$ and $k \in \mathbb{N}$ in $(-\infty, -\alpha)$. Conversely, if $F_{\alpha,\beta}(x)$ is logarithmically absolutely monotonic in $(-\infty, -\alpha)$, then $[\ln F_{\alpha,\beta}(x)]' \ge 0$ which can be rewritten as $\beta \le \theta_\alpha(x)$ for $x \in (-\infty, -\alpha)$. From the fact that $\lim_{x \to -\infty} \theta_\alpha(x) = \frac{\alpha}{2}$, it follows that $2\beta \le \alpha$; if $\frac{1}{F_{\alpha,\beta}(x)}$ is logarithmically absolutely monotonic in $(-\infty, -\alpha)$, then $[\ln F_{\alpha,\beta}(x)]' \le 0$ which can be rearranged as $\beta \ge \theta_\alpha(x)$ for $x \in (-\infty, -\alpha)$. From the fact that $\lim_{x \to (-\alpha)^-} \theta_\alpha(x) = \alpha$, it concludes that $\beta \ge \alpha$. The proof of Theorem 5 is complete.

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