Four Logarithmically Completely Monotonic Functions Involving Gamma Function and Originating from Problems of Traffic Flow

FOUR LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS INVOLVING GAMMA FUNCTION AND ORIGINATING FROM PROBLEMS OF TRAFFIC FLOW

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Abstract. In this paper, two classes of functions, involving a parameter and the Euler gamma function, and two functions, involving the Euler gamma function, are verified to be logarithmically completely monotonic in $\left(-\frac{1}{2},\infty\right)$ or $(0,\infty)$ and an inequality involving the Euler gamma function, due to J. Wendel, is refined partially.

1. Introduction

The Kershaw's inequality in [9] states that the double inequality

$$\left(x+\frac{s}{2}\right)^{1-s}<\frac{\Gamma(x+1)}{\Gamma(x+s)}<\left(x-\frac{1}{2}+\sqrt{s+\frac{1}{4}}\right)^{1-s} \tag{1}$$

holds for $0<s<1$ and $x\ge1$, where Γ denotes the classical Euler gamma function and $\psi=\frac{\Gamma'}{\Gamma}$, the logarithmic derivative of Γ. If taking $s=\frac{1}{2}$ in (1), then

$$\sqrt{x+\frac{1}{4}}<\frac{\Gamma(x+1)}{\Gamma(x+1/2)}<\sqrt{x+\frac{\sqrt{3}-1}{2}}\,. \tag{2}$$

Let s and t be nonnegative numbers and $\alpha=\min\{s,t\}$. For $x\in(-\alpha,\infty)$, define

$$z_{s,t}(x)=\begin{cases}\left[\dfrac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}-x, & s\ne t,\\ e^{\psi(x+s)}-x, & s=t.\end{cases} \tag{3}$$

In order to establish the best bounds in Kershaw's inequality (1), among other things, the papers [4, 6, 15, 19] established the following monotonicity and convexity property of $z_{s,t}(x)$: The function $z_{s,t}(x)$ is either convex and decreasing for $|t-s|<1$ or concave and increasing for $|t-s|>1$. This result was further generalized in the papers [12, 13].

In [5, p. 123] and [10], while ones studied certain problems of traffic flow, the following double inequality was obtained for $n\in\mathbb{N}$:

$$2\Gamma\left(n+\frac{1}{2}\right)\le\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)\le2^n\Gamma\left(n+\frac{1}{2}\right), \tag{4}$$

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which can be rearranged for $n > 1$ as

$$1 \le \left[\frac{\Gamma(1/2)\Gamma(n+1)}{2\Gamma(n+1/2)}\right]^{1/(n-1)} \le 2. \tag{5}$$

In [23], by using the following double inequality due to J. Wendel in [27]:

$$\left(\frac{x}{x+a}\right)^{1-a} \le \frac{\Gamma(x+a)}{x^a\Gamma(x)} \le 1 \tag{6}$$

for $0 < a < 1$ and $x > 0$, inequality (4) was extended and refined as

$$\sqrt{x} \le \frac{\Gamma(x+1)}{\Gamma(x+1/2)} \le \sqrt{x+\frac{1}{2}} \tag{7}$$

for $x > 0$.

It is clear that the double inequality (7) is weaker than (2).

Recall [14, 24, 28] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and $0 \le (-1)^n f^{(n)}(x) < \infty$ for $x \in I$ and $n \ge 0$. The set of the completely monotonic functions on I is denoted by $\mathcal{C}[I]$. The well known Bernstein's Theorem [28, p. 161] states that $f \in \mathcal{C}[(0,\infty)]$ if and only if $f(x) = \int_0^\infty e^{-xs}\,\mathrm{d}\mu(s)$, where μ is a nonnegative measure on $[0,\infty)$ such that the integral converges for all $x > 0$. This expresses that $f \in \mathcal{C}[(0,\infty)]$ if and only if f is a Laplace transform of the measure μ.

Recall [2, 7, 14, 16, 17, 20, 21] also that a positive function f is called logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies $0 \le (-1)^k[\ln f(x)]^{(k)} < \infty$ for all $k \in \mathbb{N}$ on I. The set of the logarithmically completely monotonic functions on I is denoted by $\mathcal{L}[I]$. In [3, Theorem 1.1] and [7, 21] it is pointed out that the logarithmically completely monotonic functions on $(0,\infty)$ can be characterized as the infinitely divisible completely monotonic functions studied by Horn in [8, Theorem 4.4].

It was proved in [3, 14, 18, 20, 21, 24] that $\mathcal{L}[I] \subset \mathcal{C}[I]$, but not conversely. Stimulated by the papers [17, 20], among other things, it was further revealed in [3] that $\mathcal{S}\setminus\{0\} \subset \mathcal{L}[(0,\infty)] \subset \mathcal{C}[(0,\infty)]$, where $\mathcal{S}$ denotes the set of Stieltjes transforms.

Let $\gamma = 0.57721566\cdots$ be Euler-Mascheroni's constant. For $x \in \left(-\frac{1}{2},\infty\right)$, let

$$g(x) = \begin{cases} \left[\dfrac{\Gamma(1/2)\Gamma(x+1)}{2\Gamma(x+1/2)}\right]^{1/(x-1)}, & x \ne 1, \\ \exp\left[1-\gamma-\psi\left(\dfrac{3}{2}\right)\right], & x = 1. \end{cases} \tag{8}$$

The first aim of this paper is to show the logarithmically complete monotonicity of $g(x)$. The first main result of this paper is as follows.

Theorem 1. *Let $g(x)$ be the function defined by* (8)*. Then $g(x) \in \mathcal{L}\left[\left(-\frac{1}{2},\infty\right)\right]$ with $\lim_{x\to-\frac{1}{2}^+} g(x) = \infty$ and $\lim_{x\to\infty} g(x) = 1$.*

Remark 1. From the decreasingly monotonic property of $g(x)$ and $\lim_{x\to\infty} g(x) = 1$ in Theorem 1 and $g(1) = \exp\left[1-\gamma-\psi\left(\frac{3}{2}\right)\right]$, it is obtained that

$$2\Gamma\left(x+\frac{1}{2}\right) \le \Gamma\left(\frac{1}{2}\right)\Gamma(x+1) \le 2\Gamma\left(x+\frac{1}{2}\right)\exp\left\{(x-1)\left[1-\gamma-\psi\left(\frac{3}{2}\right)\right]\right\} \tag{9}$$

for $x \in [1, \infty)$. From $g(0) = 2$, $\lim_{x\to\infty} g(x) = 1$ and the decreasingly monotonicity of $g(x)$, it is also revealed that

$$2\Gamma\left(x + \frac{1}{2}\right) \le \Gamma\left(\frac{1}{2}\right)\Gamma(x+1) \le 2^x\Gamma\left(x + \frac{1}{2}\right) \tag{10}$$

for $x \in (0, \infty)$.

Inequalities (9) and (10) extend (4) and (5) and the right hand side inequality of (9) refines the right hand side inequality of (4) and (5). Therefore, it can be said that Theorem 1 generalizes, extends, and refines the double inequalities (4) and (5).

By the way, numerical calculation shows

$$\sqrt{1 + \frac{\sqrt{3}-1}{2}} = 1.168\cdots > \frac{2}{\Gamma(1/2)} \exp\left\{(1-1)\left[1 - \gamma - \psi\left(\frac{3}{2}\right)\right]\right\} = 1.128\cdots$$

and

$$\sqrt{2 + \frac{\sqrt{3}-1}{2}} = 1.538\cdots < \frac{2}{\Gamma(1/2)} \exp\left\{(2-1)\left[1 - \gamma - \psi\left(\frac{3}{2}\right)\right]\right\} = 1.660\cdots,$$

hence, the right hand side inequality of (2) and the following inequality

$$\frac{\Gamma(x+1)}{\Gamma(x+1/2)} \le \frac{2}{\Gamma(1/2)} \exp\left\{(x-1)\left[1 - \gamma - \psi\left(\frac{3}{2}\right)\right]\right\} \tag{11}$$

for $x \in [1, \infty)$, which is deduced from the right hand side inequality of (9), are not included each other. Similarly, it is easy to show that inequality (11) and the right hand side inequality in (7) are also not included each other in $x \in [1, \infty)$.

For $x > 0$ and $a > 0$, let

$$h_a(x) = \frac{(x+a)^{1-a}\Gamma(x+a)}{x\Gamma(x)}. \tag{12}$$

The second aim of this paper is to prove the logarithmically complete monotonicity of $h_a(x)$. The second main result of this paper is as follows.

Theorem 2. *Let $h_a(x)$ be the function defined by* (12)*. Then*

(1) $\lim_{x\to 0+} h_a(x) = \frac{\Gamma(a+1)}{a^a}$ *and* $\lim_{x\to\infty} h_a(x) = 1$ *for any* $a > 0$*,*
(2) $h_a(x) \in \mathcal{L}[(0, \infty)]$ *if* $0 < a < 1$*,*
(3) $[h_a(x)]^{-1} \in \mathcal{L}[(0, \infty)]$ *if* $a > 1$*.*

Let $\gamma = 0.57721566\cdots$ be the Euler-Mascheroni's constant. For $x \in (0, \infty)$, define

$$p(x) = \begin{cases} \left[\dfrac{x^x}{\Gamma(x+1)}\right]^{1/(1-x)}, & x \ne 1, \\ e^{-\gamma}, & x = 1. \end{cases} \tag{13}$$

The third aim of this paper is to present the logarithmically complete monotonicity of $p(x)$. The third main result of this paper is as follows.

Theorem 3. *Let $p(x)$ be the function defined by* (13)*. Then* $p(x) \in \mathcal{L}[(0, \infty)]$ *with* $\lim_{x\to 0+} p(x) = 1$ *and* $\lim_{x\to\infty} p(x) = \frac{1}{e}$*.*

For $x \in (0, \infty)$ and $a \in (0, \infty)$, let

$$f_a(x) = \frac{\Gamma(x+a)}{x^a \Gamma(x)}. \tag{14}$$

The fourth aim of this paper is to consider the logarithmically complete monotonicity of $f_a(x)$. The fourth main result of this paper is as follows.

Theorem 4. *Let $f_a(x)$ be the function defined by* (14)*. Then*

(1) $\lim_{x\to\infty} f_a(x) = 1$ *for any* $a \in (0, \infty)$,
(2) $f_a(x) \in \mathcal{L}[(0, \infty)]$ *and* $\lim_{x\to 0+} f_a(x) = \infty$ *if* $a > 1$,
(3) $[f_a(x)]^{-1} \in \mathcal{L}[(0, \infty)]$ *and* $\lim_{x\to 0+} f_a(x) = 0$ *if* $0 < a < 1$.

As a straightforward consequence of Theorem 2 and Theorem 4, the following refinement of inequality (6) is established.

Theorem 5. *Let $x \in (0, \infty)$. If $0 < a < 1$, then*

$$\left(\frac{x}{x+a}\right)^{1-a} < \frac{\Gamma(x+a)}{x^a\Gamma(x)} < \begin{cases} \dfrac{\Gamma(a+1)}{a^a}\left(\dfrac{x}{x+a}\right)^{1-a} \le 1, & 0 < x \le \dfrac{ap(a)}{1-p(a)}, \\ 1, & \dfrac{ap(a)}{1-p(a)} < x < \infty, \end{cases} \tag{15}$$

where $p(x)$ is defined by (13)*. If $a > 1$, the reversed inequality of* (15) *holds.*

Remark 2. The graph of the function $\frac{ap(a)}{1-p(a)}$, pictured by Mathematica 5.2, shows that it is an increasing function from $(0, \infty)$ to $(0, \infty)$.

2. Proofs of theorems

It is well-known that, for $x > 0$ and $\omega > 0$,

$$\frac{1}{x^\omega} = \frac{1}{\Gamma(\omega)} \int_0^\infty t^{\omega-1} e^{-xt}\, \mathrm{d}t, \tag{16}$$

and that, for $k \in \mathbb{N}$ and $x > 0$,

$$\psi(x) = \ln x + \int_0^\infty \left(\frac{1}{u} - \frac{1}{1-e^{-u}}\right) e^{-xu}\, \mathrm{d}u, \tag{17}$$

$$\psi^{(k)}(x) = (-1)^{k+1} \int_0^\infty \frac{t^k}{1-e^{-t}} e^{-xt}\, \mathrm{d}t. \tag{18}$$

Moreover, as $x \to \infty$, the following asymptotic formula holds:

$$x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 + \frac{(a-b)(a+b-1)}{2x} + O\left(\frac{1}{x^2}\right), \tag{19}$$

where a and b are two constants.

It is remaked that formulas (16), (17), (18) and (19) can be found in [1, p. 257 and p. 259] and [11, 14, 16, 17, 19, 20, 21, 25, 26].

Proof of Theorem 1. Taking logarithm of $g(x)$ leads to

$$\ln g(x) = \frac{\ln\Gamma(x+1) + \ln\Gamma(1/2) - \ln\Gamma(x+1/2) - \ln 2}{x-1}$$

$$
\begin{aligned}
&= \frac{\ln\Gamma(x+1)-\ln 2}{x-1} - \frac{\ln\Gamma(x+1/2)-\ln\Gamma(1/2)}{x-1} \\
&= \frac{\ln\Gamma(x+1)-\ln\Gamma(1+1)}{x-1} - \frac{\ln\Gamma(x+1/2)-\ln\Gamma(1+1/2)}{x-1} \\
&= \frac{1}{x-1}\int_1^x \psi(u+1)\,\mathrm{d}u - \frac{1}{x-1}\int_1^x \psi(u+1/2)\,\mathrm{d}u \\
&= \frac{1}{x-1}\int_1^x [\psi(u+1)-\psi(u+1/2)]\,\mathrm{d}u \\
&= \frac{1}{x-1}\int_1^x\int_{1/2}^1 \psi'(u+t)\,\mathrm{d}t\,\mathrm{d}u \\
&= \int_{1/2}^1\left[\frac{1}{x-1}\int_1^x \psi'(u+t)\,\mathrm{d}u\right]\mathrm{d}t \\
&= \int_{1/2}^1\int_0^1 \psi'((x-1)u+t+1)\,\mathrm{d}u\,\mathrm{d}t
\end{aligned}
$$

and, for $k\in\mathbb{N}$,

$$
(-1)^k[\ln g(x)]^{(k)} = \int_{1/2}^1\int_0^1 u^k\big[(-1)^k\psi^{(k+1)}((x-1)u+t+1)\big]\,\mathrm{d}u\,\mathrm{d}t \ge 0
$$

by considering formula (18). This means $g(x)\in\mathcal{L}\big[\big(-\frac12,\infty\big)\big]$.

By L'Hospital's rule and formula (17), it is deduced that

$$
\begin{aligned}
\lim_{x\to\infty}\ln g(x) &= \lim_{x\to\infty}\left[\psi(x+1)-\psi\left(x+\frac12\right)\right] \\
&= \lim_{x\to\infty}\ln\frac{x+1}{x+1/2} + \lim_{x\to\infty}\int_0^\infty\left(\frac1u-\frac{1}{1-e^{-u}}\right)\big(e^{-u}-e^{-u/2}\big)e^{-xu}\,\mathrm{d}u = 0,
\end{aligned}
$$

which can be restated as $\lim_{x\to\infty} g(x)=1$.

It is easy to see that $\lim_{x\to-\frac12+}\ln g(x)=\infty$ and $\lim_{x\to-\frac12+} g(x)=\infty$. The proof of Theorem 1 is complete. □

Proof of Theorem 2. Using the differences equation $\Gamma(x+1)=x\Gamma(x)$ and taking limit directly gives

$$
\lim_{x\to0+} h_a(x) = \lim_{x\to0}\frac{\Gamma(x+a+1)}{(x+a)^a\Gamma(x+1)} = \frac{\Gamma(1+a)}{a^a}.
$$

Using the asymptotic expansion (19) yields

$$
h_a(x) = \frac{\Gamma(x+a+1)}{(x+a)^a\Gamma(x+1)} = \left(1+\frac ax\right)^{-a}\left[1+\frac{a(a+1)}{2x}+O\left(\frac{1}{x^2}\right)\right]\to1
$$

as $x\to\infty$, which means $\lim_{x\to\infty}h_a(x)=1$.

Taking logarithm of $h_a(x)$, differentiating with respect to x successively and utilizing formulas (16) and (18) leads to

$$
\ln h_a(x) = (1-a)\ln(x+a)+\ln\Gamma(x+a)-\ln\Gamma(x+1)
$$

and, for $n\in\mathbb{N}$ and $n>1$,

$$
[\ln h_a(x)]^{(n)} = (1-a)\frac{(-1)^{n-1}(n-1)!}{(x+a)^n} + \psi^{(n-1)}(x+a)-\psi^{(n-1)}(x+1)
$$

$$
\begin{aligned}
&= (-1)^n\left[(a-1)\frac{(n-1)!}{(x+a)^n} + (-1)^n\psi^{(n-1)}(x+a) - (-1)^n\psi^{(n-1)}(x+1)\right]\\
&= (-1)^n\left\{\int_0^\infty (a-1)e^{-(x+a)t}t^{n-1}\,\mathrm{d}t + \int_0^\infty\left[e^{-(x+a)t} - e^{-(x+1)t}\right]\frac{t^{n-1}}{1-e^{-t}}\,\mathrm{d}t\right\}\\
&= (-1)^n\left\{\int_0^\infty \frac{e^{-(x+a)t}t^{n-1}}{1-e^{-t}}\left[(a-1)\big(1-e^{-t}\big) + 1 - e^{(a-1)t}\right]\mathrm{d}t\right\}\\
&\triangleq (-1)^n\int_0^\infty \frac{e^{-(x+a)t}t^{n-1}}{1-e^{-t}} r(t)\,\mathrm{d}t.
\end{aligned}
$$

It is clear that $r'(t) = (a-1)e^{-t}\big(1-e^{at}\big)$ and $r(0) = 0$. Therefore, the function $r(t)$ is non-positive for $a > 1$ and $r(t) \ge 0$ for $0 < a < 1$. As a result, $(-1)^n[\ln h_a(x)]^{(n)} \le 0$ for $a > 1$ and $(-1)^n[\ln h_a(x)]^{(n)} \ge 0$ for $0 < a < 1$.

By formula (17), it is easy to see that

$$
\begin{aligned}
[\ln h_a(x)]' &= \frac{1-a}{x+a} + \psi(x+a) - \psi(x+1)\\
&= \frac{1-a}{x+a} + \ln\frac{x+a}{x+1} + \int_0^\infty\left(\frac{1}{t} - \frac{1}{1-e^{-t}}\right)\big(e^{-at} - e^{-t}\big)e^{-xt}\,\mathrm{d}t\\
&\to 0
\end{aligned}
$$

as $x \to \infty$. Since $[\ln h_a(x)]'$ is decreasing for $a > 1$ and increasing for $0 < a < 1$, then $[\ln h_a(x)]' \ge 0$ for $a > 1$ and $[\ln h_a(x)]' \le 0$ for $0 < a < 1$.

Summing up, for any positive integer $k \in \mathbb{N}$, if $a > 1$ then $(-1)^k[\ln h_a(x)]^{(k)} \le 0$, if $0 < a < 1$ then $(-1)^k[\ln h_a(x)]^{(k)} \ge 0$. The proof of Theorem 2 is complete. □

Proof of Theorem 3. From the differences equation $\Gamma(x+1) = x\Gamma(x)$, it follows easily that

$$
\psi(x+1) - \psi(x) = \frac{1}{x} \tag{20}
$$

for $x > 0$. Taking logarithm of $p(x)$ and utilizing (20) gives

$$
\begin{aligned}
\ln p(x) &= \frac{x\ln x - \ln\Gamma(x+1)}{1-x}\\
&= \frac{\ln\Gamma(x+1) - \ln\Gamma(1+1)}{x-1} - \frac{x\ln x - 1\ln 1}{x-1}\\
&= \frac{1}{x-1}\int_1^x \psi(u+1)\,\mathrm{d}u - \frac{1}{x-1}\int_1^x (1+\ln u)\,\mathrm{d}u\\
&= \frac{1}{x-1}\int_1^x [\psi(u+1) - \ln u]\,\mathrm{d}u - 1\\
&= \frac{1}{x-1}\int_1^x \left[\psi(u) - \ln u + \frac{1}{u}\right]\mathrm{d}u - 1\\
&\triangleq \frac{1}{x-1}\int_1^x \Psi(u)\,\mathrm{d}u - 1\\
&= \int_0^1 \Psi((x-1)u+1)\,\mathrm{d}u - 1.
\end{aligned}
$$

Formulas (16) and (17) imply that

$$
\Psi(x) = \psi(x) - \ln x + \frac{1}{x} = \int_0^\infty\left[\frac{e^u - 1 - u}{u(e^u - 1)}\right]e^{-xu}\,\mathrm{d}u.
$$

Since $e^u - 1 - u > 0$ for $u \in (0,\infty)$, then $\Psi(x) \in \mathcal{C}[(0,\infty)]$. Therefore, for $k \in \mathbb{N}$,

$$(-1)^k[\ln p(x)]^{(k)} = \int_0^1 u^k\big[(-1)^k\Psi^{(k)}((x-1)u+1)\big]\,\mathrm{d}u \ge 0.$$

This means $p(x) \in \mathcal{L}[(0,\infty)]$.

The L'Hospital's rule and formulas (17) and (20) yield

$$\begin{aligned}\lim_{x\to\infty}\ln p(x) &= \lim_{x\to\infty}\frac{x\ln x - \ln\Gamma(x+1)}{1-x} = \lim_{x\to\infty}[\psi(x+1)-(1+\ln x)]\\ &= \lim_{x\to\infty}\left[\psi(x)-\ln x+\frac{1}{x}\right]-1 = \lim_{x\to\infty}[\psi(x)-\ln x]+\lim_{x\to\infty}\frac{1}{x}-1=-1.\end{aligned}$$

Thus, it follows easily that $\lim_{x\to\infty}p(x)=\frac{1}{e}$.

It is clear that $\lim_{x\to0+}\ln p(x)=0$. Hence, the limit $\lim_{x\to0+}p(x)=1$ follows. The proof of Theorem 3 is complete. □

Proof of Theorem 4. Applying (19) reveals

$$f_a(x) = \frac{\Gamma(x+a)}{x^a\Gamma(x)} = 1+\frac{a(a-1)}{2x}+O\left(\frac{1}{x^2}\right)\to 1$$

as $x\to\infty$ for $a\in(0,\infty)$.

From $f_a(x)=\frac{x^{1-a}\Gamma(x+a)}{\Gamma(x+1)}$, it follows that $\lim_{x\to0+}f_a(x)=\begin{cases}\infty, & a>1,\\ 0, & 0<a<1.\end{cases}$

Taking logarithm of $f_a(x)$ and differentiating yields

$$\ln f_a(x) = \ln\Gamma(x+a)-a\ln x-\ln\Gamma(x)$$

and, by (16) and (18) for $n>1$,

$$\begin{aligned}(-1)^n[\ln f_a(x)]^{(n)} &= (-1)^n\left[\psi^{(n-1)}(x+a)-\psi^{(n-1)}(x)-a\frac{(-1)^{n-1}(n-1)!}{x^n}\right]\\ &= \int_0^\infty\big(e^{-(x+a)t}-e^{-xt}\big)\frac{t^{n-1}}{1-e^{-t}}\,\mathrm{d}t+\int_0^\infty ae^{-xt}t^{n-1}\,\mathrm{d}t\\ &= \int_0^\infty\frac{t^{n-1}}{1-e^{-t}}\big[e^{-at}-1+a\big(1-e^{-t}\big)\big]e^{-xt}\,\mathrm{d}t\\ &\triangleq \int_0^\infty\frac{t^{n-1}}{1-e^{-t}}s(t)e^{-xt}\,\mathrm{d}t.\end{aligned}$$

It is clear that $s(0)=0$ and $s'(t)=a\big[1-e^{(1-a)t}\big]e^{-t}$. Thus, standard argument gives $s(t)\begin{cases}\ge 0, & a>1,\\ \le 0, & 0<a<1.\end{cases}$ This implies $(-1)^n[\ln f_a(x)]^{(n)}\begin{cases}\ge 0, & a>1,\\ \le 0, & 0<a<1.\end{cases}$

Since

$$\begin{aligned}[\ln f_a(x)]' &= \psi(x+a)-\psi(x)-\frac{a}{x}\\ &= \int_0^\infty\left(\frac{1}{t}-\frac{1}{1-e^{-t}}\right)\big(e^{-at}-1\big)e^{-xt}\,\mathrm{d}t+\ln\left(1+\frac{a}{x}\right)-\frac{a}{x}\to 0\end{aligned}$$

as $x\to\infty$ and the function $[\ln f_a(x)]'$ is increasing for $a>1$ and decreasing for $0<a<1$, then $[\ln f_a(x)]'\begin{cases}\le 0, & a>1,\\ \ge 0, & 0<a<1.\end{cases}$

In a word, for $k \in \mathbb{N}$, it follows that $(-1)^k[\ln f_a(x)]^{(k)} \begin{cases} \geq 0, & a > 1, \\ \leq 0, & 0 < a < 1. \end{cases}$ The proof of Theorem 4 is complete. □

Proof of Theorem 5. As a direct consequence of Theorem 2, a double inequality is obtained: For $0 < a < 1$ and $x > 0$,

$$\left(\frac{x}{x+a}\right)^{1-a} < \frac{\Gamma(x+a)}{x^a\Gamma(x)} < \frac{\Gamma(a+1)}{a^a}\left(\frac{x}{x+a}\right)^{1-a}. \tag{21}$$

For $a > 1$ and $x > 0$, the double inequality (21) is reversed.

As an easy consequence of Theorem 4, an inequality is deduced: For $0 < a < 1$, inequality

$$\frac{\Gamma(x+a)}{x^a\Gamma(x)} < 1 \tag{22}$$

is valid in $x \in (0, \infty)$. For $a > 1$, inequality (22) reverses.

It is a standard argument that $\frac{\Gamma(a+1)}{a^a}\left(\frac{x}{x+a}\right)^{1-a} \leq 1$ if and only if $0 < x \leq \frac{ap(a)}{1-p(a)}$ for $0 < a < 1$. The proof of Theorem 5 is complete. □

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