On Some Inequalities in Normed Algebras

ON SOME INEQUALITIES IN NORMED ALGEBRAS

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Abstract. Some inequalities in normed algebras that provides lower and upper bounds for the norm of $\sum_{j=1}^{n} a_j x_j$ are obtained. Applications for estimating the quantities $\left\| \left\| x^{-1} \right\| x \pm \left\| y^{-1} \right\| y \right\|$ and $\left\| \left\| y^{-1} \right\| x \pm \left\| x^{-1} \right\| y \right\|$ for invertible elements x, y in unital normed algebras are also given.

1. Introduction

In [1], in order to provide a generalisation of a norm inequality for n vectors in a normed linear space obtained by Pečarić and Rajić in [2], the author obtained the following result:

$$\text{(1.1)} \quad \max_{k\in\{1,\dots,n\}} \left\{ |\alpha_k| \left\| \sum_{j=1}^{n} x_j \right\| - \sum_{j=1}^{n} |\alpha_j - \alpha_k| \|x_j\| \right\}$$
$$\leq \left\| \sum_{j=1}^{n} \alpha_j x_j \right\| \leq \min_{k\in\{1,\dots,n\}} \left\{ |\alpha_k| \left\| \sum_{j=1}^{n} x_j \right\| + \sum_{j=1}^{n} |\alpha_j - \alpha_k| \|x_j\| \right\},$$

where x_j, $j \in \{1, \dots, n\}$ are vectors in the normed linear space $(X, \|\cdot\|)$ over $\mathbb{K}$ while α_j, $j \in \{1, \dots, n\}$ are scalars in $\mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$.

For $\alpha_k = \frac{1}{\|x_k\|}$, with $x_k \neq 0$, $k \in \{1, \dots, n\}$ the above inequality produces the following result established by Pečarić and Rajić in [2]:

$$\text{(1.2)} \quad \max_{k\in\{1,\dots,n\}} \left\{ \frac{1}{\|x_k\|} \left[\left\| \sum_{j=1}^{n} x_j \right\| - \sum_{j=1}^{n} \left| \|x_j\| - \|x_k\| \right| \right] \right\}$$
$$\leq \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \leq \min_{k\in\{1,\dots,n\}} \left\{ \frac{1}{\|x_k\|} \left[\left\| \sum_{j=1}^{n} x_j \right\| + \sum_{j=1}^{n} \left| \|x_j\| - \|x_k\| \right| \right] \right\},$$

which implies the following refinement and reverse of the generalised triangle inequality due to M. Kato et al. [3]:

$$\text{(1.3)} \quad \min_{k\in\{1,\dots,n\}} \{\|x_k\|\} \left[n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right]$$

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$$\leq \sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n} x_j\right\| \leq \max_{k\in\{1,\dots,n\}}\{\|x_k\|\}\left[n - \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right].$$

The other natural choice, $\alpha_k = \|x_k\|$, $k \in \{1,\dots,n\}$ in (1.1) produces the result

$$\max_{k\in\{1,\dots,n\}}\left\{\|x_k\|\left\|\sum_{j=1}^{n} x_j\right\| - \sum_{j=1}^{n}\left|\|x_j\| - \|x_k\|\right|\|x_j\|\right\} \tag{1.4}$$
$$\leq \left\|\sum_{j=1}^{n}\|x_j\| x_j\right\| \leq \min_{k\in\{1,\dots,n\}}\left\{\|x_k\|\left\|\sum_{j=1}^{n} x_j\right\| + \sum_{j=1}^{n}\left|\|x_j\| - \|x_k\|\right|\|x_j\|\right\},$$

which in its turn implies another refinement and reverse of the generalised triangle inequality:

$$(0\leq)\frac{\sum_{j=1}^{n}\|x_j\|^2 - \left\|\sum_{j=1}^{n}\|x_j\| x_j\right\|}{\max_{k\in\{1,\dots,n\}}\{\|x_k\|\}} \tag{1.5}$$
$$\leq \sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n} x_j\right\| \leq \frac{\sum_{j=1}^{n}\|x_j\|^2 - \left\|\sum_{j=1}^{n}\|x_j\| x_j\right\|}{\min_{k\in\{1,\dots,n\}}\{\|x_k\|\}},$$

provided $x_k \neq 0$, $k \in \{1,\dots,n\}$.

In [2], the authors have shown that the case $n = 2$ in (1.2) produces the *Maligranda-Mercer inequality*:

$$\frac{\|x-y\| - \left|\|x\| - \|y\|\right|}{\min\{\|x\|,\|y\|\}} \leq \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \leq \frac{\|x-y\| + \left|\|x\| - \|y\|\right|}{\max\{\|x\|,\|y\|\}}, \tag{1.6}$$

for any $x, y \in X\setminus\{0\}$.

We notice that Maligranda proved the right inequality in [5] while Mercer proved the left inequality in [4].

We have shown in [1] that the following dual result for two vectors is also valid:

$$(0\leq)\frac{\|x-y\|}{\min\{\|x\|,\|y\|\}} - \frac{\left|\|x\| - \|y\|\right|}{\max\{\|x\|,\|y\|\}} \tag{1.7}$$
$$\leq \left\|\frac{x}{\|y\|} - \frac{y}{\|x\|}\right\| \leq \frac{\|x-y\|}{\max\{\|x\|,\|y\|\}} + \frac{\left|\|x\| - \|y\|\right|}{\min\{\|x\|,\|y\|\}},$$

for any $x, y \in X\setminus\{0\}$.

Motivated by the above results, the aim of the present paper is to establish lower and upper bounds for the norm of $\sum_{j=1}^{n} a_j x_j$, where $a_j, x_j, j \in \{1,\dots,n\}$ are elements in a normed algebra $(A, \|\cdot\|)$ over the real or complex number field $\mathbb{K}$. In the case where $(A, \|\cdot\|)$ is a unital algebra and x, y are invertible, lower and upper bounds for the quantities

$$\left\|\left\|x^{-1}\right\| x \pm \left\|y^{-1}\right\| y\right\| \quad \text{and} \quad \left\|\left\|y^{-1}\right\| x \pm \left\|x^{-1}\right\| y\right\|$$

are provided as well.

2. Inequalities for n Pairs of Elements

Let $(A, \|\cdot\|)$ be a normed algebra over the real or complex number field $\mathbb{K}$.

Theorem 1. *If* $(a_j, x_j) \in A^2$, $j \in \{1, \dots, n\}$, *then*

$$
\begin{aligned}
(2.1)\qquad & \max_{k\in\{1,\dots,n\}} \left\{ \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| - \sum_{j=1}^{n} \|a_j - a_k\| \, \|x_j\| \right\} \\
& \le \max_{k\in\{1,\dots,n\}} \left\{ \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| - \sum_{j=1}^{n} \|(a_j - a_k)\, x_j\| \right\} \\
& \le \left\| \sum_{j=1}^{n} a_j x_j \right\| \\
& \le \min_{k\in\{1,\dots,n\}} \left\{ \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| + \sum_{j=1}^{n} \|(a_j - a_k)\, x_j\| \right\} \\
& \le \min_{k\in\{1,\dots,n\}} \left\{ \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| + \sum_{j=1}^{n} \|a_j - a_k\| \, \|x_j\| \right\}.
\end{aligned}
$$

Proof. Observe that for any $k \in \{1, \dots, n\}$ we have

$$\sum_{j=1}^{n} a_j x_j = a_k \left(\sum_{j=1}^{n} x_j \right) + \sum_{j=1}^{n} (a_j - a_k)\, x_j.$$

Taking the norm and utilising the triangle inequality and the normed algebra properties, we have

$$
\begin{aligned}
\left\| \sum_{j=1}^{n} a_j x_j \right\| & \le \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| + \left\| \sum_{j=1}^{n} (a_j - a_k)\, x_j \right\| \\
& \le \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| + \sum_{j=1}^{n} \|(a_j - a_k)\, x_j\| \\
& \le \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| + \sum_{j=1}^{n} \|a_j - a_k\| \, \|x_j\|,
\end{aligned}
$$

for any $k \in \{1, \dots, n\}$, which implies the second part in (2.1).

Observing that

$$\sum_{j=1}^{n} a_j x_j = a_k \left(\sum_{j=1}^{n} x_j \right) - \sum_{j=1}^{n} (a_k - a_j)\, x_j$$

and utilising the continuity of the norm, we have

$$\begin{aligned}\left\|\sum_{j=1}^{n}a_jx_j\right\| &\geq \left\|a_k\left(\sum_{j=1}^{n}x_j\right)-\sum_{j=1}^{n}(a_k-a_j)\,x_j\right\| \\ &\geq \left\|a_k\left(\sum_{j=1}^{n}x_j\right)\right\|-\left\|\sum_{j=1}^{n}(a_k-a_j)\,x_j\right\| \\ &\geq \left\|a_k\left(\sum_{j=1}^{n}x_j\right)\right\|-\sum_{j=1}^{n}\left\|(a_k-a_j)\,x_j\right\| \\ &\geq \left\|a_k\left(\sum_{j=1}^{n}x_j\right)\right\|-\sum_{j=1}^{n}\left\|a_k-a_j\right\|\left\|x_j\right\|\end{aligned}$$

for any $k\in\{1,\dots,n\}$, which implies the first part in (2.1). ∎

Remark 1. *If there exists $r>0$ so that $\|a_j-a_k\|\leq r\|a_k\|$ for any $j,k\in\{1,\dots,n\}$, then, by the second part of (2.1), we have*

$$\left\|\sum_{j=1}^{n}a_jx_j\right\|\leq\min_{k\in\{1,\dots,n\}}\{\|a_k\|\}\left[\left\|\sum_{j=1}^{n}x_j\right\|+r\sum_{j=1}^{n}\|x_j\|\right]. \tag{2.2}$$

Corollary 1. *If $x_j\in A$, $j\in\{1,\dots,n\}$, then*

$$\begin{aligned}&\max_{k\in\{1,\dots,n\}}\left\{\left\|x_k\left(\sum_{j=1}^{n}x_j\right)\right\|-\sum_{j=1}^{n}\|x_j-x_k\|\|x_j\|\right\} \\ &\leq \max_{k\in\{1,\dots,n\}}\left\{\left\|x_k\left(\sum_{j=1}^{n}x_j\right)\right\|-\sum_{j=1}^{n}\|(x_j-x_k)\,x_j\|\right\}\leq\left\|\sum_{j=1}^{n}x_j^2\right\| \\ &\leq \min_{k\in\{1,\dots,n\}}\left\{\left\|x_k\left(\sum_{j=1}^{n}x_j\right)\right\|+\sum_{j=1}^{n}\|(x_j-x_k)\,x_j\|\right\} \\ &\leq \min_{k\in\{1,\dots,n\}}\left\{\left\|x_k\left(\sum_{j=1}^{n}x_j\right)\right\|+\sum_{j=1}^{n}\|x_j-x_k\|\|x_j\|\right\}.\end{aligned} \tag{2.3}$$

Corollary 2. *Assume that A is a unital normed algebra. If $x_j\in A$ are invertible for any $j\in\{1,\dots,n\}$, then*

$$\begin{aligned}&\min_{k\in\{1,\dots,n\}}\left\|x_k^{-1}\right\|\left[\sum_{j=1}^{n}\|x_j\|-\left\|\sum_{j=1}^{n}x_j\right\|\right] \\ &\leq \sum_{j=1}^{n}\left\|x_j^{-1}\right\|\|x_j\|-\left\|\sum_{j=1}^{n}\left\|x_j^{-1}\right\|x_j\right\| \\ &\leq \max_{k\in\{1,\dots,n\}}\left\|x_k^{-1}\right\|\left[\sum_{j=1}^{n}\|x_j\|-\left\|\sum_{j=1}^{n}x_j\right\|\right].\end{aligned} \tag{2.4}$$

Proof. If $1 \in A$ is the unity, then on choosing $a_k = \left\| x_k^{-1} \right\| \cdot 1$ in (2.1) we get

$$\max_{k \in \{1,\dots,n\}} \left\{ \left\| x_k^{-1} \right\| \left\| \sum_{j=1}^n x_j \right\| - \sum_{j=1}^n \left| \left\| x_j^{-1} \right\| - \left\| x_k^{-1} \right\| \right| \left\| x_j \right\| \right\} \tag{2.5}$$
$$\leq \left\| \sum_{j=1}^n \left\| x_j^{-1} \right\| x_j \right\|$$
$$\leq \min_{k \in \{1,\dots,n\}} \left\{ \left\| x_k^{-1} \right\| \left\| \sum_{j=1}^n x_j \right\| + \sum_{j=1}^n \left| \left\| x_j^{-1} \right\| - \left\| x_k^{-1} \right\| \right| \left\| x_j \right\| \right\}.$$

Now, assume that $\min_{k \in \{1,\dots,n\}} \left\{ \left\| x_k^{-1} \right\| \right\} = \left\| x_{k_0}^{-1} \right\|$. Then

$$\left\| x_{k_0}^{-1} \right\| \left\| \sum_{j=1}^n x_j \right\| + \sum_{j=1}^n \left| \left\| x_j^{-1} \right\| - \left\| x_{k_0}^{-1} \right\| \right| \left\| x_j \right\|$$
$$= - \left\| x_{k_0}^{-1} \right\| \left(\sum_{j=1}^n \left\| x_j \right\| - \left\| \sum_{j=1}^n x_j \right\| \right) + \sum_{j=1}^n \left\| x_j^{-1} \right\| \left\| x_j \right\|.$$

Utilising the second inequality in (2.5), we deduce

$$\left\| x_{k_0}^{-1} \right\| \left(\sum_{j=1}^n \left\| x_j \right\| - \left\| \sum_{j=1}^n x_j \right\| \right) \leq \sum_{j=1}^n \left\| x_j^{-1} \right\| \left\| x_j \right\| - \left\| \sum_{j=1}^n \left\| x_j^{-1} \right\| x_j \right\|$$

and the first inequality in (2.4) is proved.

The second part of (2.4) can be proved in a similar manner, however, the details are omitted. ∎

Remark 2. *An equivalent form of (2.4) is:*

$$\frac{\sum_{j=1}^n \left\| x_j^{-1} \right\| \left\| x_j \right\| - \left\| \sum_{j=1}^n \left\| x_j^{-1} \right\| x_j \right\|}{\max_{k \in \{1,\dots,n\}} \left\| x_k^{-1} \right\|} \tag{2.6}$$
$$\leq \sum_{j=1}^n \left\| x_j \right\| - \left\| \sum_{j=1}^n x_j \right\| \leq \frac{\sum_{j=1}^n \left\| x_j^{-1} \right\| \left\| x_j \right\| - \left\| \sum_{j=1}^n \left\| x_j^{-1} \right\| x_j \right\|}{\min_{k \in \{1,\dots,n\}} \left\| x_k^{-1} \right\|},$$

which provides both a refinement and a reverse inequality for the generalised triangle inequality.

3. Inequalities for Two Pairs of Elements

The following particular case of Theorem 1 is of interest for applications.

Lemma 1. *If $(a,b), (x,y) \in A^2$, then*

$$\max \left\{ \left\| a(x \pm y) \right\| - \left\| (b-a) y \right\|, \left\| b(x \pm y) \right\| - \left\| (b-a) x \right\| \right\} \tag{3.1}$$
$$\leq \left\| ax \pm by \right\| \leq \min \left\{ \left\| a(x \pm y) \right\| + \left\| (b-a) y \right\|, \left\| b(x \pm y) \right\| + \left\| (b-a) x \right\| \right\}$$

or, equivalently,

$$\begin{aligned}(3.2)\qquad &\frac{1}{2}\left\{\|a(x\pm y)\|+\|b(x\pm y)\|-[\|(b-a)y\|+\|(b-a)x\|]\right\}\\ &\quad+\frac{1}{2}\left|\|a(x\pm y)\|-\|b(x\pm y)\|+\|(b-a)y\|-\|(b-a)x\|\right|\\ &\leq\|ax\pm by\|\\ &\leq\frac{1}{2}\left\{\|a(x\pm y)\|+\|b(x\pm y)\|+[\|(b-a)y\|+\|(b-a)x\|]\right\}\\ &\quad-\frac{1}{2}\left|\|a(x\pm y)\|+\|b(x\pm y)\|-\|(b-a)y\|-\|(b-a)x\|\right|.\end{aligned}$$

Proof. The inequality (3.1) follows from Theorem 1 for $n=2$, $a_1=a$, $a_2=b$, $x_1=x$ and $x_2=\pm y$.

Utilising the properties of real numbers,

$$\min\{\alpha,\beta\}=\frac{1}{2}[\alpha+\beta-|\alpha-\beta|],\quad\max\{\alpha,\beta\}=\frac{1}{2}[\alpha+\beta+|\alpha-\beta|];\quad\alpha,\beta\in\mathbb{R};$$

the inequality (3.1) is clearly equivalent with (3.2). ∎

The following result contains some upper bounds for $\|ax\pm by\|$ that are perhaps more useful for applications.

Theorem 2. *If* $(a,b),(x,y)\in A^2$, *then*

$$\begin{aligned}(3.3)\qquad\|ax\pm by\|&\leq\min\{\|a(x\pm y)\|,\|b(x\pm y)\|\}+\|b-a\|\max\{\|x\|,\|y\|\}\\ &\leq\|x\pm y\|\min\{\|a\|,\|b\|\}+\|b-a\|\max\{\|x\|,\|y\|\}\end{aligned}$$

and

$$\begin{aligned}(3.4)\qquad\|ax\pm by\|&\leq\|x\pm y\|\max\{\|a\|,\|b\|\}+\min\{\|(b-a)x\|,\|(b-a)y\|\}\\ &\leq\|x\pm y\|\max\{\|a\|,\|b\|\}+\|b-a\|\min\{\|x\|,\|y\|\}.\end{aligned}$$

Proof. Observe that $\|(b-a)x\|\leq\|b-a\|\|x\|$ and $\|(b-a)y\|\leq\|b-a\|\|y\|$, and then

$$(3.5)\qquad\|(b-a)x\|,\|(b-a)y\|\leq\|b-a\|\max\{\|x\|,\|y\|\},$$

which implies that

$$\begin{aligned}&\min\{\|a(x\pm y)\|+\|(b-a)y\|,\|b(x\pm y)\|+\|(b-a)x\|\}\\ &\leq\min\{\|a(x\pm y)\|,\|b(x\pm y)\|\}+\|b-a\|\max\{\|x\|,\|y\|\}\\ &\leq\|x\pm y\|\min\{\|a\|,\|b\|\}+\|b-a\|\max\{\|x\|,\|y\|\}.\end{aligned}$$

Utilising the second inequality in (3.1), we deduce (3.3).

Also, since $\|a(x\pm y)\|\leq\|a\|\|x\pm y\|$ and $\|b(x\pm y)\|\leq\|b\|\|x\pm y\|$, hence

$$\|a(x\pm y)\|,\|b(x\pm y)\|\leq\|x\pm y\|\max\{\|a\|,\|b\|\},$$

which implies that

$$\begin{aligned}&\min\{\|a(x\pm y)\|+\|(b-a)y\|,\|b(x\pm y)\|+\|(b-a)x\|\}\\ &\leq\|x\pm y\|\max\{\|a\|,\|b\|\}+\min\{\|(b-a)x\|,\|(b-a)y\|\}\\ &\leq\|x\pm y\|\max\{\|a\|,\|b\|\}+\|b-a\|\min\{\|x\|,\|y\|\},\end{aligned}$$

and the inequality (3.4) is also proved. ∎

The following corollary may be more useful for applications.

Corollary 3. *If* $(a,b),(x,y)\in A^2$, *then*

$$\|ax\pm by\|\leq\|x\pm y\|\cdot\frac{\|a\|+\|b\|}{2}+\|b-a\|\cdot\frac{\|x\|+\|y\|}{2}. \tag{3.6}$$

Proof. Follows from Theorem 2 by adding the last inequality in (3.3) to the last inequality (3.4) and utilising the property that $\min\{\alpha,\beta\}+\max\{\alpha,\beta\}=\alpha+\beta$, $\alpha,\beta\in\mathbb{R}$. ∎

The following lower bounds for $\|ax\pm by\|$ can be stated as well:

Theorem 3. *For any* (a,b) *and* $(x,y)\in A^2$, *we have:*

$$\begin{aligned}&\max\{|\|ax\|-\|ay\||,|\|bx\|-\|by\||\}-\|b-a\|\max\{\|x\|,\|y\|\}\\&\leq\max\{\|a(x\pm y)\|,\|b(x\pm y)\|\}-\|b-a\|\max\{\|x\|,\|y\|\}\\&\leq\|ax\pm by\|\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&\min\{|\|ax\|-\|ay\||,|\|bx\|-\|by\||\}-\|b-a\|\min\{\|x\|,\|y\|\}\\&\leq\min\{|\|ax\|-\|ay\||,|\|bx\|-\|by\||\}-\min\{\|(b-a)x\|,\|(b-a)y\|\}\\&\leq\|ax\pm by\|.\end{aligned} \tag{3.8}$$

Proof. Observe that, by (3.5) we have that

$$\begin{aligned}&\max\{\|a(x\pm y)\|-\|(b-a)y\|,\|b(x\pm y)\|-\|(b-a)x\|\}\\&\geq\max\{\|ax\pm ay\|,\|bx\pm by\|\}-\|b-a\|\max\{\|x\|,\|y\|\}\\&\geq\max\{|\|ax\|-\|ay\||,|\|bx\|-\|by\||\}-\|b-a\|\max\{\|x\|,\|y\|\}\end{aligned}$$

and on utilising the first inequality in (3.1), the inequality (3.7) is proved.

Observe also that, since

$$\|a(x\pm y)\|,\|b(x\pm y)\|\geq\min\{|\|ax\|-\|ay\||,|\|bx\|-\|by\||\},$$

then

$$\begin{aligned}&\max\{\|a(x\pm y)\|-\|(b-a)y\|,\|b(x\pm y)\|-\|(b-a)x\|\}\\&\geq\min\{|\|ax\|-\|ay\||,|\|bx\|-\|by\||\}-\min\{\|(b-a)x\|,\|(b-a)y\|\}\\&\geq\min\{|\|ax\|-\|ay\||,|\|bx\|-\|by\||\}-\|b-a\|\min\{\|x\|,\|y\|\}.\end{aligned}$$

Then, by the first inequality in (3.1), we deduce (3.8). ∎

Corollary 4. *For any* $(a,b),(x,y)\in A^2$, *we have*

$$\frac{1}{2}\cdot[|\|ax\|-\|ay\||+|\|bx\|-\|by\||]-\|b-a\|\cdot\frac{\|x\|+\|y\|}{2}\leq\|ax\pm by\|. \tag{3.9}$$

The proof follows from Theorem 3 by adding (3.7) to (3.8). The details are omitted.

4. Applications for Two Invertible Elements

In this section we assume that A is a unital algebra with the unity 1. The following results provide some upper bounds for the quantity $\left\|\left\|x^{-1}\right\|x\pm\left\|y^{-1}\right\|y\right\|$, where x and y are invertible in A.

Proposition 1. *If $(x,y)\in A^2$ are invertible, then*

$$\left\| \left\| x^{-1}\right\| x\pm \left\| y^{-1}\right\| y\right\| \leq \left\| x\pm y\right\| \min\left\{ \left\| x^{-1}\right\| ,\left\| y^{-1}\right\| \right\} +\left| \left\| x^{-1}\right\| -\left\| y^{-1}\right\| \right| \max\left\{ \left\| x\right\| ,\left\| y\right\| \right\} \tag{4.1}$$

and

$$\left\| \left\| x^{-1}\right\| x\pm \left\| y^{-1}\right\| y\right\| \leq \left\| x\pm y\right\| \max\left\{ \left\| x^{-1}\right\| ,\left\| y^{-1}\right\| \right\} +\left| \left\| x^{-1}\right\| -\left\| y^{-1}\right\| \right| \min\left\{ \left\| x\right\| ,\left\| y\right\| \right\} . \tag{4.2}$$

Proof. Follows by Theorem 2 on choosing $a=\left\| x^{-1}\right\| \cdot 1$ and $b=\left\| y^{-1}\right\| \cdot 1$. ∎

Corollary 5. *With the above assumption for x and y, we have*

$$\left\| \left\| x^{-1}\right\| x\pm \left\| y^{-1}\right\| y\right\| \leq \left\| x\pm y\right\| \cdot \frac{\left\| x^{-1}\right\| +\left\| y^{-1}\right\| }{2}+\left| \left\| x^{-1}\right\| -\left\| y^{-1}\right\| \right| \cdot \frac{\left\| x\right\| +\left\| y\right\| }{2}. \tag{4.3}$$

Lower bounds for $\left\| \left\| x^{-1}\right\| x\pm \left\| y^{-1}\right\| y\right\|$ are provided below:

Proposition 2. *If $(x,y)\in A^2$ are invertible, then*

$$\left\| x\pm y\right\| \max\left\{ \left\| x^{-1}\right\| ,\left\| y^{-1}\right\| \right\} -\left| \left\| x^{-1}\right\| -\left\| y^{-1}\right\| \right| \max\left\{ \left\| x\right\| ,\left\| y\right\| \right\} \leq \left\| \left\| x^{-1}\right\| x\pm \left\| y^{-1}\right\| y\right\| \tag{4.4}$$

and

$$\left\| x\pm y\right\| \min\left\{ \left\| x^{-1}\right\| ,\left\| y^{-1}\right\| \right\} -\left| \left\| x^{-1}\right\| -\left\| y^{-1}\right\| \right| \min\left\{ \left\| x\right\| ,\left\| y\right\| \right\} \leq \left\| \left\| x^{-1}\right\| x\pm \left\| y^{-1}\right\| y\right\| . \tag{4.5}$$

Proof. The first inequality in (4.4) follows from the second inequality in (3.7) on choosing $a=\left\| x^{-1}\right\| \cdot 1$ and $b=\left\| y^{-1}\right\| \cdot 1$.

We know from the proof of Theorem 3 that

$$\max\left\{ \left\| a\left(x\pm y\right) \right\| -\left\| \left(b-a\right) y\right\| ,\left\| b\left(x\pm y\right) \right\| -\left\| \left(b-a\right) x\right\| \right\} \leq \left\| ax\pm by\right\| . \tag{4.6}$$

If in this inequality we choose $a=\left\| x^{-1}\right\| \cdot 1$ and $b=\left\| y^{-1}\right\| \cdot 1$, then we get

$$\begin{aligned} &\left\| \left\| x^{-1}\right\| x\pm \left\| y^{-1}\right\| y\right\| \\ &\geq \max\left\{ \left\| x^{-1}\right\| \left\| x\pm y\right\| -\left| \left\| x^{-1}\right\| -\left\| y^{-1}\right\| \right| \left\| y\right\| ,\left\| y^{-1}\right\| \left\| x\pm y\right\| -\left| \left\| x^{-1}\right\| -\left\| y^{-1}\right\| \right| \left\| x\right\| \right\} \\ &\geq \left\| x\pm y\right\| \min\left\{ \left\| x^{-1}\right\| ,\left\| y^{-1}\right\| \right\} -\left| \left\| x^{-1}\right\| -\left\| y^{-1}\right\| \right| \min\left\{ \left\| x\right\| ,\left\| y\right\| \right\} \end{aligned}$$

and the inequality (4.5) is obtained. ∎

Corollary 6. *If $(x,y)\in A^2$ are invertible, then*

$$\left\| x\pm y\right\| \cdot \frac{\left\| x^{-1}\right\| +\left\| y^{-1}\right\| }{2}-\left| \left\| x^{-1}\right\| -\left\| y^{-1}\right\| \right| \cdot \frac{\left\| x\right\| +\left\| y\right\| }{2} \leq \left\| \left\| x^{-1}\right\| x\pm \left\| y^{-1}\right\| y\right\| . \tag{4.7}$$

Remark 3. *We observe that the inequalities (4.3) and (4.7) are in fact equivalent with:*

$$\left| \left\| \left\| x^{-1} \right\| x \pm \left\| y^{-1} \right\| y \right\| - \left\| x \pm y \right\| \cdot \frac{\left\| x^{-1} \right\| + \left\| y^{-1} \right\|}{2} \right| \leq \left| \left\| x^{-1} \right\| - \left\| y^{-1} \right\| \right| \cdot \frac{\|x\| + \|y\|}{2}. \tag{4.8}$$

Now we consider the dual expansion $\left\| \left\| y^{-1} \right\| x \pm \left\| x^{-1} \right\| y \right\|$, for which the following upper bounds can be stated.

Proposition 3. *If* (x, y) *are invertible in* A, *then*

$$\left\| \left\| y^{-1} \right\| x \pm \left\| x^{-1} \right\| y \right\| \leq \left\| x \pm y \right\| \min\left\{ \left\| x^{-1} \right\|, \left\| y^{-1} \right\| \right\} + \left| \left\| x^{-1} \right\| - \left\| y^{-1} \right\| \right| \max\left\{ \|x\|, \|y\| \right\} \tag{4.9}$$

and

$$\left\| \left\| y^{-1} \right\| x \pm \left\| x^{-1} \right\| y \right\| \leq \left\| x \pm y \right\| \max\left\{ \left\| x^{-1} \right\|, \left\| y^{-1} \right\| \right\} + \left| \left\| x^{-1} \right\| - \left\| y^{-1} \right\| \right| \min\left\{ \|x\|, \|y\| \right\}. \tag{4.10}$$

In particular,

$$\left\| \left\| y^{-1} \right\| x \pm \left\| x^{-1} \right\| y \right\| \leq \left\| x \pm y \right\| \cdot \frac{\left\| x^{-1} \right\| + \left\| y^{-1} \right\|}{2} + \left| \left\| x^{-1} \right\| - \left\| y^{-1} \right\| \right| \cdot \frac{\|x\| + \|y\|}{2}. \tag{4.11}$$

The proof follows from Theorem 2 on choosing $a = \left\| y^{-1} \right\| \cdot 1$ and $b = \left\| x^{-1} \right\| \cdot 1$.

The lower bounds for the quantity $\left\| \left\| y^{-1} \right\| x \pm \left\| x^{-1} \right\| y \right\|$ are incorporated in:

Proposition 4. *If* (x, y) *are invertible in* A, *then*

$$\left\| x \pm y \right\| \max\left\{ \left\| x^{-1} \right\|, \left\| y^{-1} \right\| \right\} - \left| \left\| x^{-1} \right\| - \left\| y^{-1} \right\| \right| \max\left\{ \|x\|, \|y\| \right\} \leq \left\| \left\| y^{-1} \right\| x \pm \left\| x^{-1} \right\| y \right\| \tag{4.12}$$

and

$$\left\| x \pm y \right\| \min\left\{ \left\| x^{-1} \right\|, \left\| y^{-1} \right\| \right\} - \left| \left\| x^{-1} \right\| - \left\| y^{-1} \right\| \right| \min\left\{ \|x\|, \|y\| \right\} \leq \left\| \left\| y^{-1} \right\| x \pm \left\| x^{-1} \right\| y \right\|. \tag{4.13}$$

In particular,

$$\left\| x \pm y \right\| \cdot \frac{\left\| x^{-1} \right\| + \left\| y^{-1} \right\|}{2} - \left| \left\| x^{-1} \right\| - \left\| y^{-1} \right\| \right| \cdot \frac{\|x\| + \|y\|}{2} \leq \left\| \left\| y^{-1} \right\| x \pm \left\| x^{-1} \right\| y \right\|. \tag{4.14}$$

Remark 4. *We observe that the inequalities (4.11) and (4.14) are equivalent with*

$$\left| \left\| \left\| y^{-1} \right\| x \pm \left\| x^{-1} \right\| y \right\| - \left\| x \pm y \right\| \cdot \frac{\left\| x^{-1} \right\| + \left\| y^{-1} \right\|}{2} \right| \leq \left| \left\| x^{-1} \right\| - \left\| y^{-1} \right\| \right| \cdot \frac{\|x\| + \|y\|}{2}. \tag{4.15}$$

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