Subadditive and Superadditive Properties of Polygamma Functions

SUBADDITIVE AND SUPERADDITIVE PROPERTIES OF POLYGAMMA FUNCTIONS

FENG QI AND BAI-NI GUO

ABSTRACT. In this short paper, it is proved that the function $|\psi^{(i)}(e^x)|$ for $i \in \mathbb{N}$ is subadditive in $(\ln\theta_0, \infty)$ and superadditive in $(-\infty, \ln\theta_0)$, where $\theta_0 \in (0,1)$ is the unique root of equation $2|\psi^{(i)}(\theta)| = |\psi^{(i)}(\theta^2)|$.

1. Introduction

Recall [2, 3, 4] that a function f defined on an interval I is said to be subadditive on I if $f(x+y) \le f(x) + f(y)$ holds for all $x, y \in I$ such that $x+y \in I$. If $f(x+y) \ge f(x) + f(y)$, then f is called superadditive on I.

The subadditive and superadditive functions play an important role in the theory of differential equations, in the study of semi-groups, in number theory, and also in the theory of convex bodies. A lot of literature for the subadditive and superadditive functions can be found in [2, 3] and the references therein.

It is well known that the classical Euler's gamma function Γ can be defined for $x > 0$ as $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,\mathrm{d}t$. The digamma or psi function ψ is defined as the logarithmic derivative of Γ and $\psi^{(i)}$ for $i \in \mathbb{N}$ are called polygamma functions.

In [3], a subadditive property of the gamma function Γ was proved: The function $[\Gamma(x)]^\alpha$ is subadditive in $(0,\infty)$ if and only if $\frac{\ln 2}{\ln \Delta} \le \alpha \le 0$, where $\Delta = \min_{x \ge 0} \frac{\Gamma(2x)}{\Gamma(x)}$.

In [4], a subadditive property of the psi function was obtained: The function $\psi(a + e^x)$ is subadditive in $(-\infty, \infty)$ if and only if $a \ge c_0$, where c_0 is the unique positive zero of $\psi(x)$.

In this short paper, we would like to discuss the subadditive and superadditive properties of the polygamma functions $\psi^{(i)}(x)$ for $i \in \mathbb{N}$.

Our main result is the following Theorem 1.

Theorem 1. *The function $|\psi^{(i)}(e^x)|$ for $i \in \mathbb{N}$ is superadditive in $(-\infty, \ln\theta_0)$ or subadditive in $(\ln\theta_0, \infty)$, where $\theta_0 \in (0,1)$ is the unique root of equation $2|\psi^{(i)}(\theta)| = |\psi^{(i)}(\theta^2)|$.*

2. Proof of Theorem 1

In [5], the monotonicity of the function $x^\alpha |\psi^{(i)}(x+\beta)|$ was researched thoroughly, which is a generalization of the corresponding results in [1, 4], as follows:

(1) The function $x^\alpha |\psi^{(i)}(x)|$ in $(0,\infty)$ is strictly increasing if and only if $\alpha \ge i+1$ and strictly decreasing if and only if $0 \le \alpha \le i$.

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(2) For $\beta \geq \frac{1}{2}$, the function $x^{\alpha}\left|\psi^{(i)}(x+\beta)\right|$ is strictly increasing in $[0,\infty)$ if and only if $\alpha \geq i$.

(3) Let $\delta : (0,\infty) \to \left(0,\frac{1}{2}\right)$ be defined by

$$\delta(t) = \frac{e^t(t-1)+1}{(e^t-1)^2} \tag{1}$$

for $t \in (0,\infty)$ and $\delta^{-1} : \left(0,\frac{1}{2}\right) \to (0,\infty)$ stand for the inverse function of δ. If $0 < \beta < \frac{1}{2}$ and

$$\alpha \geq i+1-\left[\frac{e^{\delta^{-1}(\beta)}}{e^{\delta^{-1}(\beta)}-1}+\beta-1\right]\delta^{-1}(\beta), \tag{2}$$

then the function $x^{\alpha}\left|\psi^{(i)}(x+\beta)\right|$ is strictly increasing in $(0,\infty)$.

It is noted that

$$0 < \left[\frac{e^{\delta^{-1}(\beta)}}{e^{\delta^{-1}(\beta)}-1}+\beta-1\right]\delta^{-1}(\beta) < 1 \tag{3}$$

for $\beta \in (0,1)$, since $\lim_{\beta\to0^+}[\beta\delta^{-1}(\beta)] = 0$.

Let

$$f_\beta(x,y) = \left|\psi^{(i)}(\beta+x)\right| + \left|\psi^{(i)}(\beta+y)\right| - \left|\psi^{(i)}(\beta+xy)\right| \tag{4}$$

for $x > 0$ and $y > 0$, where $\beta \geq 0$ and $i \in \mathbb{N}$. In order to show Theorem 1, it is sufficient to prove the positivity or negativity of the function $f_\beta(x,y)$. Directly calculating yields

$$\begin{aligned}\frac{\partial f_\beta(x,y)}{\partial x} &= y\left|\psi^{(i+1)}(\beta+xy)\right| - \left|\psi^{(i+1)}(\beta+x)\right| \\ &= \frac{1}{x}\left[xy\left|\psi^{(i+1)}(\beta+xy)\right| - x\left|\psi^{(i+1)}(\beta+x)\right|\right].\end{aligned} \tag{5}$$

From the monotonicity of the function $x^{\alpha}\left|\psi^{(i)}(x+\beta)\right|$ in [5] mentioned above, it follows easily that $\frac{\partial f_0(x,y)}{\partial x} \gtreqless 0$ if and only if $y \lesseqgtr 1$. This means that the function $f_0(x,y)$ is strictly increasing for $y < 1$ and strictly decreasing for $y > 1$ in $x \in (0,\infty)$. Since $\lim_{x\to\infty} f_0(x,y) = \left|\psi^{(i)}(y)\right| > 0$, then for $y > 1$ the function $f_0(x,y)$ is positive in $x \in (0,\infty)$.

For $y < 1$, by the increasingly monotonicity of $f_0(x,y)$, it is deduced that

(1) if $x > 1$, then $f_0(1,y) = \left|\psi^{(i)}(1)\right| < f_0(x,y) < \left|\psi^{(i)}(y)\right|$,
(2) if $x < 1$, then $f_0(x,y) < f_0(1,y) = \left|\psi^{(i)}(1)\right|$,
(3) if $y < x < 1$, then $f_0(y,y) < f_0(x,y)$,
(4) if $x < y < 1$, then $f_0(x,x) < f_0(x,y)$.

This implies that

$$f_0(\theta,\theta) = 2\left|\psi^{(i)}(\theta)\right| - \left|\psi^{(i)}(\theta^2)\right| < f_0(x,y) \tag{6}$$

for $y < 1$, where $\theta < 1$ with $\theta < x$ and $\theta < y$.

Since $f_0(\theta,\theta)$ is strictly increasing in $\theta \in (0,1)$ such that $f_0(1,1) = \left|\psi^{(i)}(1)\right| > 0$ and $\lim_{\theta\to0^+} f_0(\theta,\theta) = -\infty$, then the function $f_0(\theta,\theta)$ has a unique zero $\theta_0 \in (0,1)$ and $f_0(\theta,\theta) > 0$ for $1 > \theta > \theta_0$.

In conclusion, the function $f_0(x,y)$ is positive for all $x,y > \theta_0$ or negative for $0 < x,y < \theta_0$. The proof of Theorem 1 is complete.

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