A Remark on the Sum of Reciprocal of Imaginary Parts of Zeta Zeros

A REMARK ON THE SUM OF RECIPROCAL OF IMAGINARY PARTS OF ZETA ZEROS

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Abstract. In this note, we give some explicit upper and lower bounds for the summation $\sum_{0<\gamma\leq T}\frac{1}{\gamma}$, where γ is the imaginary part of nontrivial zeros $\rho=\beta+i\gamma$ of $\zeta(s)$, that is a zero with $0\leq\beta\leq 1$. In a research report appeared in "*RGMIA Research Report Collection*, 9(2), Article 15, 2006", we have given some bounds, but here we give the following more clean ones:

$$\frac{1}{4\pi}\log^2 T-\frac{\log(2\pi)}{2\pi}\log T+\frac{15}{250}<\sum_{0<\gamma\leq T}\frac{1}{\gamma}<\frac{1}{4\pi}\log^2 T-\frac{\log(2\pi)}{2\pi}\log T+\frac{109}{250},$$

where the left hand side holds for $T\geq 2$ and the right hand side holds for $T\geq 2.222$.

1. Introduction

The Riemann zeta function is defined for $\Re(s)>1$ by $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ and extended by analytic continuation to the complex plan with one singularity at $s=1$; in fact a simple pole with residues 1. The functional equation for this function in symmetric form, is $\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$, where $\Gamma(s)=\int_0^{\infty}e^{-t}t^{s-1}dt$ is a meromorphic function of the complex variable s, with simple poles at $s=0,-1,-2,\cdots$ (see [3]). By this equation, trivial zeros of $\zeta(s)$ are $s=-2,-4,-6,\cdots$. Also, it implies symmetry of nontrivial zeros (other zeros $\rho=\beta+i\gamma$ which have the property $0\leq\beta\leq 1$) according to the line $\Re(s)=\frac{1}{2}$. The summation

$$\mathcal{A}(T)=\sum_{0<\gamma\leq T}\frac{1}{\gamma},$$

where γ is the imaginary part of nontrivial zeros appears in some explicit approximation of primes, and having some explicit approximations of it can be useful for careful computations. This is a summation over imaginary part of zeta zeros, and for approximating such summations we use Stieljes integral and integrating by parts; let $N(T)$ be the number of zeros ρ of $\zeta(s)$ with $0<\Im(\rho)\leq T$ and $0\leq\Re(\rho)\leq 1$. Then, supposing $1<U\leq V$ and

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$\Phi(t) \in C^1(U,V)$ to be non-negative, we have

$$\sum_{U<\gamma\leq V} \Phi(\gamma) = \int_U^V \Phi(t)dN(t) = -\int_U^V N(t)\Phi'(t)dt + N(V)\Phi(V) - N(U)\Phi(U). \tag{1.1}$$

About $N(T)$, Riemann [5] guessed that

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \tag{1.2}$$

This conjecture of Riemann proved by H. von Mangoldt more than 30 years later [1, 2]. An immediate corollary of above approximate formula, which is known as Riemann-van Mangoldt formula is $\mathcal{A}(T) = O(\log^2 T)$, which follows by partial summation from Riemann-van Mangoldt formula [2]. In 1941, Rosser [6] introduced the following approximation of $N(T)$:

$$|N(T) - F(T)| \leq R(T) \qquad (T \geq 2), \tag{1.3}$$

where $F(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8}$, and $R(T) = \frac{137}{1000}\log T + \frac{433}{1000}\log\log T + \frac{397}{250}$. This approximation allows us to make some explicit approximation of $\mathcal{A}(T)$.

2. Approximation of $\mathcal{A}(T)$

2.1. Approximate Estimation of $\mathcal{A}(T)$. As we set above, γ_1 is the imaginary part of first nontrivial zero of the Riemann zeta function in the upper half plane and computations [4] give us $\gamma_1 = 14.13472514\cdots$. On using (1.1) with $\Phi(\gamma) = \frac{1}{\gamma}$, $0 < U < \gamma_1$ and $V = T$, we obtain

$$\mathcal{A}(T) = \int_U^T \frac{dN(t)}{t} = \int_U^T \frac{N(t)}{t^2}dt + \frac{N(T)}{T}. \tag{2.1}$$

Substituting $N(T)$ from (1.2), we obtain

$$\mathcal{A}(T) = \frac{1}{2\pi}\int_U^T \frac{\log\left(\frac{t}{2\pi}\right)}{t}dt - \frac{1}{2\pi}\int_U^T \frac{dt}{t} + \frac{1}{2\pi}\log\frac{T}{2\pi} - \frac{1}{2\pi} + O\left(\int_U^T \frac{\log(t)}{t^2}dt\right) + O\Big(\frac{\log T}{T}\Big).$$

Computing integrals and error terms, and then letting $U \to \gamma_1^-$, we get the following approximation

$$\mathcal{A}(T) = \frac{1}{4\pi}\log^2 T - \frac{\log(2\pi)}{2\pi}\log T + O(1).$$

2.2. **Explicit Estimation of $\mathcal{A}(T)$.** Considering (1.3) and using (2.1) with $2 \leq U < \gamma_1$, for every $T \geq 2$ implies

$$\int_U^T \frac{F(t)}{t^2}dt - \int_U^T \frac{R(t)}{t^2}dt + \frac{F(T)-R(T)}{T} \leq \mathcal{A}(T) \leq \int_U^T \frac{F(t)}{t^2}dt + \int_U^T \frac{R(t)}{t^2}dt + \frac{F(T)+R(T)}{T}.$$

A simple calculation, yields $\frac{F(t)}{t^2} = \frac{d}{dt}\{\frac{1}{4\pi}\log^2 t - \frac{1+\log(2\pi)}{2\pi}\log t + \frac{\log^2(2\pi)-2\log(2\pi)}{4\pi} - \frac{7}{8t}\}$, and setting $\mathfrak{E}(t) = \int_1^\infty \frac{ds}{st^s}$, we also have $\frac{R(t)}{t^2} = \frac{d}{dt}\{-\frac{433}{1000}\frac{\log\log t}{t} - \frac{137}{1000}\frac{\log t}{t} - \frac{69}{40t} - \frac{433}{1000}\mathfrak{E}(t)\}$. The integral of $\mathfrak{E}(t)$ converges for $t > 1$; in fact $\mathfrak{E}(t) \sim \frac{1}{t\log t}$ when $t \to \infty$. Using the fact $\frac{d}{dt}\mathfrak{E}(t) = -\frac{1}{t^2\log t}$, we get $\frac{1}{t\log t} - \frac{1}{t\log^2 t} < \mathfrak{E}(t) < \frac{1}{t\log t} - \frac{31}{95t\log^2 t}$ for $t \geq 2$. Therefore, after letting $U \to \gamma_1^-$, we get the following explicit upper bound

$$\mathcal{A}(T) < \frac{1}{4\pi}\log^2 T - \frac{\log(2\pi)}{2\pi}\log T + \mathfrak{c}_{\mathfrak{au}} - \frac{137\log^2 T + 433\log T - 433}{1000T\log^2 T} \qquad (T \geq 2),$$

where $\mathfrak{c}_{\mathfrak{au}} = 0.43596427\cdots < \frac{109}{250}$ and for $T \geq 2.222$ we have $-\frac{137\log^2 T+433\log T-433}{1000T\log^2 T} < 0$. Thus, we obtain $\mathcal{A}(T) < \frac{1}{4\pi}\log^2 T - \frac{\log(2\pi)}{2\pi}\log T + \frac{109}{250}$ for $T \geq 2.222$. Similarly, we get

$$\begin{aligned}\mathcal{A}(T) \quad &> \quad \frac{1}{4\pi}\log^2 T - \frac{\log(2\pi)}{2\pi}\log T + \mathfrak{c}_{\mathfrak{al}} \\ &+ \quad \frac{274\log^3 T + 866(\log\log T)\log^2 T + 3313\log^2 T + 433\log T - 433}{1000T\log^2 T} \qquad (T \geq 2),\end{aligned}$$

where $\mathfrak{c}_{\mathfrak{al}} = 0.06058187\cdots > \frac{3}{50}$ and for $T \geq 2$ the last term in the above inequality is positive. So, we obtain $\mathcal{A}(T) > \frac{1}{4\pi}\log^2 T - \frac{\log(2\pi)}{2\pi}\log T + \frac{3}{50}$ for $T \geq 2$. Therefore we have proved the following result:

Proposition 2.1. *Letting $\mathcal{A}(T) = \sum_{0<\gamma\leq T}\frac{1}{\gamma}$ with γ is imaginary part of zeta zeros, we have*

$$\frac{15}{250} < \mathcal{A}(T) - \left\{\frac{1}{4\pi}\log^2 T - \frac{\log(2\pi)}{2\pi}\log T\right\} < \frac{109}{250}, \tag{2.2}$$

where the left hand side holds for $T \geq 2$ and the right hand side holds for $T \geq 2.222$.

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