A Note on l^2 Norms of Weighted Mean Matrices

A NOTE ON l^2 NORMS OF WEIGHTED MEAN MATRICES

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Abstract. We give a proof of Cartlidge's result on the l^p operator norms of weighted mean matrices for $p = 2$ on interpreting the norms as eigenvalues of certain matrices.

1. Introduction

Suppose throughout that $p \neq 0, \frac{1}{p} + \frac{1}{q} = 1$. Let l^p be the Banach space of all complex sequences $\mathbf{a} = (a_n)_{n \geq 1}$ with norm

$$||\mathbf{a}|| := (\sum_{n=1}^{\infty} |a_n|^p)^{1/p} < \infty.$$

The celebrated Hardy's inequality ([7, Theorem 326]) asserts that for $p > 1$,

$$\sum_{n=1}^{\infty} \Big| \frac{1}{n} \sum_{k=1}^{n} a_k \Big|^p \leq (\frac{p}{p-1})^p \sum_{k=1}^{\infty} |a_k|^p. \tag{1.1}$$

Hardy's inequality can be regarded as a special case of the following inequality:

$$\sum_{j=1}^{\infty} \Big| \sum_{k=1}^{\infty} c_{j,k} a_k \Big|^p \leq U \sum_{k=1}^{\infty} |a_k|^p,$$

in which $C = (c_{j,k})$ and the parameter p are assumed fixed ($p > 1$), and the estimate is to hold for all complex sequences $\mathbf{a}$. The l^p operator norm of C is then defined as the p-th root of the smallest value of the constant U:

$$||C||_{p,p} = U^{\frac{1}{p}}.$$

Hardy's inequality thus asserts that the Cesáro matrix operator C, given by $c_{j,k} = 1/j, k \leq j$ and 0 otherwise, is bounded on l^p and has norm $\leq p/(p-1)$. (The norm is in fact $p/(p-1)$.)

We say a matrix A is a summability matrix if its entries satisfy: $a_{j,k} \geq 0$, $a_{j,k} = 0$ for $k > j$ and $\sum_{k=1}^{j} a_{j,k} = 1$. We say a summability matrix A is a weighted mean matrix if its entries satisfy:

$$a_{j,k} = \lambda_k / \Lambda_j, \ 1 \leq k \leq j; \Lambda_j = \sum_{i=1}^{j} \lambda_i, \lambda_i \geq 0, \lambda_1 > 0. \tag{1.2}$$

Hardy's inequality (1.1) now motivates one to determine the l^p operator norm of an arbitrary summability matrix A. In an unpublished dissertation [4], Cartlidge studied weighted mean matrices as operators on l^p and obtained the following result (see also [2, p. 416, Theorem C]).

Theorem 1.1. *Let $1 < p < \infty$ be fixed. Let A be a weighted mean matrix given by* (1.2). *If*

$$L = \sup_n (\frac{\Lambda_{n+1}}{\lambda_{n+1}} - \frac{\Lambda_n}{\lambda_n}) < p \ , \tag{1.3}$$

then $||A||_{p,p} \leq p/(p-L)$.

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We note here there are several published proofs of Cartlidge's result. Borwein [3] proved a far more general result than Theorem 1.1 on the l^p norms of generalized Hausdorff matrices. Rhoades [11, Theorem 1] obtained a slightly general result than Theorem 1.1, using a modification of the proof of Cartlidge. Recently, the author [6] also gave a simple proof of Theorem 1.1.

It is our goal in this note to give another proof of Theorem 1.1 for the case $p = 2$, following an approach of Wang and Yuan in [13], which interprets the left-hand side of (1.1) when $p = 2$ as a quadratic form so that Hardy's inequality follows from estimations of the eigenvalues of the corresponding matrix associated to the quadratic form. We will show in the next section that the same idea also works for the case of weighted mean matrices.

2. Proof of Theorem 1.1 for $p = 2$

We may assume $a_n > 0$ without loss of generality and it suffices to prove the theorem for any finite summation from $n = 1$ to N with $N \geq 1$. We also note that it follows from our assumption on L that $\lambda_n > 0$. Now consider

$$\sum_{n=1}^{N}\Big(\sum_{i=1}^{n}\frac{\lambda_i}{\Lambda_n}a_i\Big)^2=\sum_{n=1}^{N}\Big(\sum_{i,j=1}^{n}\frac{\lambda_i\lambda_j}{\Lambda_n^2}a_ia_j\Big)=\sum_{n=1}^{N}\alpha_{i,j}a_ia_j,\quad \alpha_{i,j}=\sum_{k\geq \max(i,j)}^{N}\frac{\lambda_i\lambda_j}{\Lambda_k^2}.$$

We view the above as a quadratic form and define the associated matrix A to be

$$A=\Big(\alpha_{i,j}\Big)_{1\leq i,j\leq N}.$$

We note that the matrix A here is certainly positive definite, being equal to B^tB with B a lower-triangular matrix,

$$B=\Big(b_{i,j}\Big)_{1\leq i,j\leq N},\quad b_{i,j}=\lambda_j/\Lambda_i,\ 1\leq j\leq i;\quad b_{i,j}=0, j>i.$$

In order to establish our assertion, it suffices to show that the maximum eigenvalue of A is less than $4/(2-L)^2$ or the minimum eigenvalue of its inverse A^{-1} is greater than $(2-L)^2/4$ which is equivalent to proving that the matrix $A^{-1}-\lambda I_N$ is positive definite, where $\lambda=(2-L)^2/4$ and I_N is the $N\times N$ identity matrix.

It is easy to check that the entries of B^{-1} are given by

$$\big(B^{-1}\big)_{i,i}=\frac{\Lambda_i}{\lambda_i},\quad \big(B^{-1}\big)_{i+1,i}=-\frac{\Lambda_i}{\lambda_{i+1}},\quad \big(B^{-1}\big)_{i,j}=0\ \text{ otherwise.}$$

It follows from this that A^{-1} is a symmetric tridiagonal matrix with its entries given by

$$\big(A^{-1}\big)_{i,i}=\frac{\Lambda_i^2+\Lambda_{i-1}^2}{\lambda_i^2},\quad \big(A^{-1}\big)_{i,i+1}=\big(A^{-1}\big)_{i+1,i}=-\frac{\Lambda_i^2}{\lambda_i\lambda_{i+1}},\quad \big(A^{-1}\big)_{i,j}=0\ \text{ otherwise.}$$

Here $1\leq i\leq N$ and we define $\Lambda_0=0$. We note here when $\lambda_i=1$, the expression of A^{-1} is given explicitly in [13] while a generalization is given in [12], from which one can easily deduce our case here.

It follows from the expression for A^{-1} and induction that if we let Δ_k, $1\leq k\leq N$ denote the k-th principal minor determinant of the matrix $A^{-1}-\lambda I_N$ for any λ, then for $1\leq k\leq N-1$,

$$\Delta_{k+1}=\Big(\frac{1}{\lambda_{k+1}^2}(\Lambda_k^2+\Lambda_{k+1}^2)-\lambda\Big)\Delta_k-\frac{\Lambda_k^4}{\lambda_k^2\lambda_{k+1}^2}\Delta_{k-1}.$$

Here we define $\Delta_0=1$ and note that $\Delta_1=1-\lambda$. To simplify the above relation, we define for $1\leq k\leq N$,

$$x_k=\Delta_k/\Delta_{k-1},$$

so that for $1 \leq k \leq N-1$,

$$x_{k+1} = \frac{1}{\lambda_{k+1}^2}(\Lambda_k^2 + \Lambda_{k+1}^2) - \lambda - \frac{\Lambda_k^4}{\lambda_k^2 \lambda_{k+1}^2} \frac{1}{x_k}. \tag{2.1}$$

We further define for $1 \leq k \leq N$,

$$P_k = \frac{\lambda_k^2}{\Lambda_k^2} x_k,$$

so that for $1 \leq k \leq N-1$,

$$P_{k+1} = 1 + \frac{\Lambda_k^2}{\Lambda_{k+1}^2}(1 - \frac{1}{P_k}) - \frac{\lambda_{k+1}^2}{\Lambda_{k+1}^2}\lambda.$$

Lastly, we define for $1 \leq k \leq N$,

$$\theta_k = \frac{\Lambda_k}{\lambda_k}(1 - P_k),$$

so that for $1 \leq k \leq N-1$,

$$\theta_{k+1} = \frac{\lambda_{k+1}}{\Lambda_{k+1}}\lambda + \frac{\Lambda_k}{\Lambda_{k+1}} \frac{\lambda_k \theta_k / \lambda_{k+1}}{1 - \lambda_k \theta_k / \Lambda_k}.$$

From now on we let $\lambda = (2-L)^2/4$ and note from our discussions above that

$$\Delta_1 = x_1 = P_1 = 1 - \lambda, \quad \theta_1 = \lambda.$$

Our goal is to show that $A^{-1} - \lambda I_N$ is positive definite and it suffices to show that $\Delta_k > 0$, $1 \leq k \leq N$ or equivalently, $\theta_k < \Lambda_k/\lambda_k$ for $1 \leq k \leq N$. We now prove this by establishing the following

Lemma 2.1. *With the assumption of Theorem 1.1 and $\lambda = (2-L)^2/4$, we have for $1 \leq k \leq N$,*

$$0 < \theta_k \leq \frac{2-L}{2} - \frac{L(2-L)}{4} \frac{\lambda_k}{\Lambda_k}. \tag{2.2}$$

Proof. We prove (2.2) by induction on k. The case $k = 1$ follows from $\theta_1 = \lambda = (2-L)^2/4$. Now we assume both inequalities of (2.2) hold for θ_k with $k \leq N-1$ and we note here the case $n = 1$ of (1.3) implies $L > 0$. As we assume $L < 2$, it follows from this and our assumption on θ_k that

$$\theta_k \leq \frac{2-L}{2} - \frac{L(2-L)}{4} \frac{\lambda_k}{\Lambda_k} \leq \frac{2-L}{2} < 1 \leq \frac{\Lambda_k}{\lambda_k}.$$

This immediately implies that the left-hand side inequality of (2.2) holds for θ_{k+1}. We now move on to show that the right-hand side inequality of (2.2) holds for θ_{k+1}. For this, we denote $a = (2-L)/2$, $b = L(2-L)/4$ and note that

$$\begin{aligned} \theta_{k+1} &= \frac{\lambda_{k+1}}{\Lambda_{k+1}}\lambda + \frac{\Lambda_k}{\Lambda_{k+1}} \frac{\lambda_k \theta_k / \lambda_{k+1}}{1 - \lambda_k \theta_k / \Lambda_k} \\ &\leq \frac{\lambda_{k+1}}{\Lambda_{k+1}}\lambda + \frac{\Lambda_k}{\Lambda_{k+1}} \frac{\lambda_k / \lambda_{k+1}(a - b\lambda_k/\Lambda_k)}{1 - \lambda_k/\Lambda_k(a - b\lambda_k/\Lambda_k)}. \end{aligned} \tag{2.3}$$

We write

$$\lambda_k/\lambda_{k+1}(a - b\lambda_k/\Lambda_k) = a\Big(1 - \lambda_k/\Lambda_k\Big(a - b\lambda_k/\Lambda_k\Big)\Big) + S,$$

where

$$S = a^2 \frac{\lambda_k}{\Lambda_k} - ab\frac{\lambda_k^2}{\Lambda_k^2} - b\frac{\lambda_k^2}{\Lambda_k \lambda_{k+1}} - a\frac{\lambda_{k+1} - \lambda_k}{\lambda_{k+1}}.$$

Using this notation in (2.3), we have

$$\theta_{k+1} \leq a - b\frac{\lambda_{k+1}}{\Lambda_{k+1}} + (\lambda + b - a)\frac{\lambda_{k+1}}{\Lambda_{k+1}} + \frac{\Lambda_k}{\Lambda_{k+1}} \frac{S}{1 - \lambda_k/\Lambda_k(a - b\lambda_k/\Lambda_k)}.$$

It is easy to see that $\lambda + b - a = 0$ and as we have already pointed out above that

$$1 - \lambda_k/\Lambda_k(a - b\lambda_k/\Lambda_k) > 0.$$

Hence in order to prove the right-hand side inequality of (2.3) holds for θ_{k+1}, it suffices to show $S \leq 0$, or equivalently,

$$a^2 - ab\frac{\lambda_k}{\Lambda_k} - b\frac{\lambda_k}{\lambda_{k+1}} - a\Lambda_k\Big(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}}\Big) \leq 0. \tag{2.4}$$

We note here

$$\Lambda_k\Big(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}}\Big) = \frac{\Lambda_k}{\lambda_k} - \frac{\Lambda_{k+1}}{\lambda_{k+1}} + 1. \tag{2.5}$$

Note also that

$$a^2 - b - (1-L)a = 0.$$

Hence inequality (2.4) will follow from

$$b\frac{\lambda_{k+1} - \lambda_k}{\lambda_{k+1}} - ab\frac{\lambda_k}{\Lambda_k} - \Big(L - \Big(\frac{\Lambda_{k+1}}{\lambda_{k+1}} - \frac{\Lambda_k}{\lambda_k}\Big)\Big)a \leq 0.$$

We recast the above inequality as

$$b\Lambda_k\Big(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}}\Big) - ab - \Big(L - \Big(\frac{\Lambda_{k+1}}{\lambda_{k+1}} - \frac{\Lambda_k}{\lambda_k}\Big)\Big)a\frac{\Lambda_k}{\lambda_k} \leq 0.$$

Using (2.5) again, we can further rewrite the above inequality as

$$b - ab - aL\frac{\Lambda_k}{\lambda_k} + \Big(\frac{\Lambda_{k+1}}{\lambda_{k+1}} - \frac{\Lambda_k}{\lambda_k}\Big)\Big(a\frac{\Lambda_k}{\lambda_k} - b\Big) \leq 0.$$

Note that

$$a\frac{\Lambda_k}{\lambda_k} - b \geq a - b = \lambda > 0,$$

and by (1.3),

$$\frac{\Lambda_{k+1}}{\lambda_{k+1}} - \frac{\Lambda_k}{\lambda_k} \leq L.$$

It follows that

$$\begin{aligned} & b - ab - aL\frac{\Lambda_k}{\lambda_k} + \Big(\frac{\Lambda_{k+1}}{\lambda_{k+1}} - \frac{\Lambda_k}{\lambda_k}\Big)\Big(a\frac{\Lambda_k}{\lambda_k} - b\Big) \\ \leq \quad & b - ab - aL\frac{\Lambda_k}{\lambda_k} + L\Big(a\frac{\Lambda_k}{\lambda_k} - b\Big) = b - ab - bL = -bL/2 < 0. \end{aligned}$$

Thus inequality (2.4) holds, which implies the right-hand side inequality of (2.2) holds for θ_{k+1} and this completes the proof of the lemma. □

As we have discussed in the proof of Lemma 2.1 that the bounds in (2.2) imply that $\theta_k < \Lambda_k/\lambda_k$ for $1 \leq k \leq N$ and this completes our proof of Theorem 1.1 for $p = 2$.

3. Further Discussions

Our proof of Theorem 1.1 for $p = 2$ in the previous section in fact establishes that the minimum eigenvalue of its inverse A^{-1} defined there is greater than $(2-L)^2/4$. Using the expression $A^{-1} = B^{-1}(B^{-1})^t$, we see that what we have shown in the previous section implies that for any integer $N \geq 1$ and any real sequence $\mathbf{a} = (a_n)_{1 \leq n \leq N}$,

$$\sum_{n=1}^{N-1} \Big(\frac{\Lambda_n}{\lambda_n} a_n - \frac{\Lambda_n}{\lambda_{n+1}} a_{n+1}\Big)^2 + \frac{\Lambda_N^2}{\lambda_N^2} a_N^2 \geq \frac{(2-L)^2}{4} \sum_{n=1}^{N} a_n^2. \tag{3.1}$$

Conversely, the above inequalities also imply Theorem 1.1 for $p = 2$. It is therefore interesting to study the above inequalities of their own. As analogues, we point out the following discrete inequalities of Wirtinger's type studied by Fan, Taussky and Todd [5, Theorem 8]:

$$a_1^2 + \sum_{n=1}^{N-1} (a_n - a_{n+1})^2 + a_N^2 \geq 2\Big(1 - \cos\frac{\pi}{N+1}\Big) \sum_{n=1}^{N} a_n^2. \tag{3.2}$$

Converses of the above inequalities were found by Milovanović and Milovanović [9]:

$$a_1^2 + \sum_{n=1}^{N-1} (a_n - a_{n+1})^2 + a_N^2 \leq 2\Big(1 + \cos\frac{\pi}{N+1}\Big) \sum_{n=1}^{N} a_n^2. \tag{3.3}$$

Simple proofs of inequalities (3.2) and (3.3) were given by Redheffer [10] and Alzer [1], respectively.

We now explore the ideas in [10] and [1] to see whether they can give another proof of (3.2) or not. For any integer $n \geq 1$, we consider for $a_n \geq 0$ the function

$$f(a_n) = \Big(\frac{\Lambda_n}{\lambda_n} a_n - \frac{\Lambda_n}{\lambda_{n+1}} a_{n+1}\Big)^2 - \mu_n a_n^2,$$

where $a_{n+1} \geq 0, 0 < \mu_n < \Lambda_n^2/\lambda_n^2$ are being fixed and we shall think of μ_n as a parameter whose value is going to be specified later. It is easy to check that

$$f(a_n) \geq f\Big(\frac{\Lambda_n^2/(\lambda_n\lambda_{n+1})}{\Lambda_n^2/\lambda_n^2 - \mu_n} a_{n+1}\Big),$$

or explicitly,

$$\Big(\frac{\Lambda_n}{\lambda_n} a_n - \frac{\Lambda_n}{\lambda_{n+1}} a_{n+1}\Big)^2 - \mu_n a_n^2 \geq -\frac{\Lambda_n^2/\lambda_{n+1}^2}{\Lambda_n^2/\lambda_n^2 - \mu_n} \mu_n a_{n+1}^2.$$

Summing the above inequality for $n = 1, \ldots, N-1$ yields:

$$\begin{aligned} &\sum_{n=1}^{N-1} \Big(\frac{\Lambda_n}{\lambda_n} a_n - \frac{\Lambda_n}{\lambda_{n+1}} a_{n+1}\Big)^2 + \frac{\Lambda_N^2}{\lambda_N^2} a_N^2 \\ \geq\ & \mu_1 a_1^2 + \sum_{n=1}^{N-2} \Big(\mu_{n+1} - \frac{\Lambda_n^2/\lambda_{n+1}^2}{\Lambda_n^2/\lambda_n^2 - \mu_n}\mu_n\Big) a_{n+1}^2 + \Big(\frac{\Lambda_N^2}{\lambda_N^2} - \frac{\Lambda_{N-1}^2/\lambda_N^2}{\Lambda_{N-1}^2/\lambda_{N-1}^2 - \mu_{N-1}}\mu_{N-1}\Big) a_N^2. \end{aligned} \tag{3.4}$$

We want to show for any integer $n \geq 1$, one can choose μ_n such that the following inequality holds for $1 \leq n \leq N-1$:

$$\mu_{n+1} - \frac{\Lambda_n^2/\lambda_{n+1}^2}{\Lambda_n^2/\lambda_n^2 - \mu_n}\mu_n \geq \frac{(2-L)^2}{4}. \tag{3.5}$$

For this purpose, we set

$$\eta_n = \Lambda_n^2/\lambda_n^2 - \mu_n$$

so that we can rewrite inequality (3.5) as

$$\frac{1}{\lambda_{n+1}^2}(\Lambda_n^2+\Lambda_{n+1}^2)-\eta_{n+1}-\frac{\Lambda_n^4}{\lambda_n^2\lambda_{n+1}^2}\frac{1}{\eta_n}\geq\frac{(2-L)^2}{4}.$$

Note that the above inequality follows from (2.1) if we set $\lambda=(2-L)^2/4$ and $x_n=\eta_n, 1\leq n\leq N$ there. Note also that $x_N>0$ so that $\Lambda_N^2/\lambda_N^2>\mu_N$ and we have

$$\frac{\Lambda_N^2}{\lambda_N^2}-\frac{\Lambda_{N-1}^2/\lambda_N^2}{\Lambda_{N-1}^2/\lambda_{N-1}^2-\mu_{N-1}}\mu_{N-1}>\mu_N-\frac{\Lambda_{N-1}^2/\lambda_N^2}{\Lambda_{N-1}^2/\lambda_{N-1}^2-\mu_{N-1}}\mu_{N-1}=\frac{(2-L)^2}{4}. \tag{3.6}$$

Moreover, we have

$$\mu_1=\Lambda_1^2/\lambda_1^2-\eta_1=1-x_1=\frac{(2-L)^2}{4}.$$

The above inequality combined with inequalities (3.4)-(3.6) now implies (3.1).

We now give for the special case of $\lambda_n=1$ (corresponding to the original Hardy's inequality for $p=2$) an explicit expression for μ_n considered above. In this case we set

$$\mu_n=\frac{1}{2}n-\frac{1}{4}.$$

One checks easily that with the choice of μ_n, inequalities (3.5) and (3.6) are satisfied with $L=1$. As $\mu_1=1/4$, this implies (3.1) with $\lambda_n=1,\Lambda_n=n$ there.

To end this paper, we note the paper [8] contains several generalizations of inequalities of (3.2) and (3.3), one of them can be stated as:

Theorem 3.1. *For any real sequence $\mathbf{a}=(a_n)_{1\leq n\leq N}$, and two positive real numbers a,b,*

$$\Big(a^2+b^2-2ab\cos\frac{\pi}{N+1}\Big)\sum_{n=1}^N a_n^2\leq b^2a_1^2+\sum_{n=1}^{N-1}(aa_n-ba_{n+1})^2+a^2a_N^2\leq\Big(a^2+b^2+2ab\cos\frac{\pi}{N+1}\Big)\sum_{n=1}^N a_n^2. \tag{3.7}$$

The proof given in [8] to the above theorem is to regard

$$b^2a_1^2+\sum_{n=1}^{N-1}(aa_n-ba_{n+1})^2+a^2a_N^2$$

as a quadratic form with the associated matrix A being symmetric tridiagonal with its entries given by

$$\big(A\big)_{i,i}=a^2+b^2,\quad \big(A\big)_{i,i+1}=\big(A\big)_{i+1,i}=-ab,\quad \big(A\big)_{i,j}=0\ \text{ otherwise.}$$

The eigenvalues of A are shown in [8] to be $a^2+b^2+2ab\cos(\frac{k\pi}{N+1}), 1\leq k\leq N$, from which Theorem 3.1 follows easily.

We now give another proof of Theorem 3.1 following the methods in [10] and [1]. Consider the function:

$$f(a_n)=(aa_n-ba_{n+1})^2-\mu_na_n^2.$$

Here we regard a_{n+1},μ_n as being fixed. When $\mu_n>a^2$, it is easy to see that

$$f(a_n)\leq f(\frac{aba_{n+1}}{a^2-\mu_n})=\frac{b^2\mu_na_{n+1}^2}{\mu_n-a^2}.$$

Summing the above inequality for $n=1,\dots,N-1$ yields:

$$b^2a_1^2+\sum_{n=1}^{N-1}(aa_n-ba_{n+1})^2+a^2a_N^2\leq(b^2+\mu_1)a_1^2+\sum_{n=2}^{N-1}\Big(\frac{b^2\mu_{n-1}}{\mu_{n-1}-a^2}+\mu_n\Big)a_n^2+\Big(\frac{b^2\mu_{N-1}}{\mu_{N-1}-a^2}+a^2\Big)a_N^2.$$

On letting $\mu_n=a^2+ab\sin(n+1)t/\sin(nt), t=\pi/(N+1)$ and note that $\mu_n>a^2$ for $1\leq n\leq N-1$, it is easy to see that the right-hand side inequality of (3.7) follows from this.

Similarly, when $\mu_n < a^2$, we have

$$b^2a_1^2+\sum_{n=1}^{N-1}(aa_n-ba_{n+1})^2+a^2a_N^2 \geq (b^2+\mu_1)a_1^2+\sum_{n=2}^{N-1}\Big(\mu_n-\frac{b^2\mu_{n-1}}{a^2-\mu_{n-1}}\Big)a_n^2+\Big(a^2-\frac{b^2\mu_{N-1}}{a^2-\mu_{N-1}}\Big)a_N^2.$$

The left-hand side inequality of (3.7) now follows from this on taking $\mu_n = a^2 - ab\sin(n+1)t/\sin(nt), t = \pi/(N+1)$ here.

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