On a Result of Hardy and Ramanujan

ON A RESULT OF HARDY AND RAMANUJAN

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Abstract. In this paper, we introduce some explicit approximations for the summation $\sum_{k\leq n}\Omega(k)$, where $\Omega(k)$ is the total number of prime factors of k.

1. Introduction

Letting $\Omega(k)$ be the total number of prime factors of k, a result of Hardy and Ramanujan [7] asserts that

$$\sum_{k\leq n}\Omega(k)=n\log\log n+M'n+O\Big(\frac{n}{\log n}\Big),$$

where

$$M'=\gamma+\sum_{p}\Big(\log\Big(1-p^{-1}\Big)+(p-1)^{-1}\Big)\approx 1.0346538818.$$

The aim of this paper is finding explicit version of this result. We proceed by letting $n!=\prod_{p\leq n}p^{v_p(n!)}$, standard factorization of $n!$ into primes. It is known that $v_p(n!)=\sum_{k=1}^{m}\lfloor\frac{n}{p^k}\rfloor$, with $m=m_{n,p}=\lfloor\frac{\log n}{\log p}\rfloor$ and $\lfloor x\rfloor$ is the largest integer less than or equal to x (see for example [10]). We introduce some explicit (and neat) approximations for the summation $\Upsilon(n)=\sum_{p\leq n}v_p(n!)$. Then, considering

$$\sum_{k\leq n}\Omega(k)=\Omega(n!)=\Upsilon(n),$$

we obtain the main result as follows.

Main Theorem. *For every $n\geq 3$ we have*

$$\left|\sum_{k\leq n}\Omega(k)-(n-1)\log\log(n-1)\right|<23(n-1).$$

Note that one can modify above result to the following one:

$$\left|\sum_{k\leq n}\Omega(k)-n\log\log n\right|<23n,$$

which is an explicit version of the result of Hardy and Ramanujan.

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2. Proof of the Main Theorem

Starting point of proof, is considering the inequality

$$\frac{n-p}{p-1}-\frac{\log n}{\log p}<v_p(n!)\leq\frac{n-1}{p-1}, \tag{2.1}$$

(see [8] for a proof). Taking the summation $\sum_{p\leq n}$ from parts of this inequality, we will require to approximate summations of the form $\sum_{p\leq n}f(p)$ with $f(p)=\frac{1}{\log p}$ and $f(p)=\frac{1}{p-1}$ (and more generally, for a given function $f\in C^1(\mathbb{R}^+)$). To do this, we use reduction of a Riemann-Stieljes integral to a finite sum [3]; let α be a step function defined on $[a,b]$ with jump α_k at x_k, where $a\leq x_1<x_2<\cdots<x_n\leq b$. Let f be defined on $[a,b]$ in such a way that not both f and α are discontinuous from the right or from the left t each x_k. Then $\int_a^b fd\alpha$ exists and we have

$$\int_a^b f(x)d\alpha(x)=\sum_{k=1}^n f(x_k)\alpha_k.$$

Thus, integrating by parts yields

$$\sum_{k=1}^n f(x_k)\alpha_k=\int_a^b f(x)d\alpha(x)=\int_a^b \alpha(x)\frac{d}{dx}(-f(x))dx+f(b)\alpha(b)-f(a)\alpha(a).$$

Also if for the sequence a_k we let $f(x)=a_k$ when $k-1<x\leq k$ with $f(0)=0$, then

$$\sum_{k=1}^n a_k=\sum_{k=1}^n f(k)=\int_0^n f(x)d\lfloor x\rfloor.$$

This allow us to get some ways for evaluating the summation $\sum_{p\leq n}f(p)$; two of them are:

- Using $\vartheta(x)=\sum\limits_{p\leq x}\log p$, which ends to the approximation

  $$\sum_{p\leq n}f(p)=\int_{2^-}^n\frac{f(x)}{\log x}d\vartheta(x)=\frac{f(n)\vartheta(n)}{\log n}+\int_2^n\vartheta(x)\frac{d}{dx}\left(\frac{-f(x)}{\log x}\right)dx,$$

  and it is known that for $x>1$, we have $200\log^2 x|\vartheta(x)-x|<793x$, and $\log^4 x|\vartheta(x)-x|<1717433x$ (see [6] for more details).
- Using $\pi(x)=\#\mathbb{P}\cap[2,x]$, which ends to the approximation

  $$\sum_{p\leq n}f(p)=f(x)\pi(x)+\int_2^n\pi(x)\frac{d}{dx}\big(-f(x)\big)dx,$$

  and we have some explicit bounds for $\pi(x)$ (again see [6] for lots of them). In this paper we will use the following neat one:

$$\pi(x)\leq\frac{x}{\log x}\Big(1+\frac{1.2762}{\log x}\Big)\qquad (x>1). \tag{2.2}$$

Both of these methods are applicable for the summation $\sum_{p\leq n}\frac{1}{p-1}$, while first method on the summation $\sum_{p\leq n}\frac{1}{\log p}$ ends to some integrals hard for approximating. Here, based on some known approximations for both of these summations, which are obtained on the second method, we give some neat bounds for them.

Proposition 2.1. *For every $n \geq 3$, we have*

$$\log\log(n-1) - 14 < \sum_{p \leq n} \frac{1}{p-1} < \log\log(n-1) + 23.$$

Proof. It is known [9] that

$$\sum_{p \leq n} \frac{1}{p-1} > \log\log n + a + \frac{n}{(n-1)\log n} - \frac{1717433n}{(n-1)\log^5 n} \qquad (n \geq 2),$$

where $a \approx -11.86870152$. But, for $n \geq 3564183$ we have

$$\log\log n + a + \frac{n}{(n-1)\log n} - \frac{1717433n}{(n-1)\log^5 n} > \log\log(n-1) - 14.$$

Thus, for $n \geq 3564183$ we obtain

$$\sum_{p \leq n} \frac{1}{p-1} > \log\log(n-1) - 14,$$

which holds true for $2 \leq n \leq 3564182$, too.

Also, we have

$$\sum_{p \leq n} \frac{1}{p-1} < \log\log(n-1) + b + \frac{n}{(n-1)\log n} + \frac{1717433n}{(n-1)\log^5 n} \qquad (n \geq 2),$$

where $b \approx 21.18095291$. In the other hand, for $n \geq 7126157$ we have

$$b + \frac{n}{(n-1)\log n} + \frac{1717433n}{(n-1)\log^5 n} < 23.$$

So, for $n \geq 7126157$ we obtain

$$\sum_{p \leq n} \frac{1}{p-1} < \log\log(n-1) + 23,$$

which holds true by computation for $3 \leq n \leq 7126156$, too. This completes the proof. □

Proposition 2.2. *For every $n \geq 2$, we have*

$$\left| \sum_{p \leq n} \frac{1}{\log p} - \left\{ \frac{n}{\log^2 n} + \frac{2n}{\log^3 n} + \frac{6n}{\log^4 n} \right\} \right| < 271382 \frac{n}{\log^5 n}.$$

Proof. In a similar process [9], we have

$$\frac{n}{\log^2 n} + \frac{2n}{\log^3 n} + \frac{6n}{\log^4 n} + \frac{1607n}{100\log^5 n} - \frac{1717433n}{\log^6 n} + a < \sum_{p \leq n} \frac{1}{\log p} \qquad (n \geq 564),$$

where $a \approx -16.42613005$. Also, we have

$$\sum_{p \leq n} \frac{1}{\log p} < \frac{n}{\log^2 n} + \frac{2n}{\log^3 n} + \frac{6n}{\log^4 n} + \frac{54281n}{800\log^5 n} + \frac{1717433n}{\log^6 n} + b \qquad (n \geq 2),$$

where $b \approx 30.52238614$. Computation gives

$$\frac{-271382n}{\log^5 n} < \frac{1607n}{100\log^5 n} - \frac{1717433n}{\log^6 n} + a \qquad (n \geq 564).$$

Also

$$\frac{54281n}{800\log^5 n} + \frac{1717433n}{\log^6 n} + b < \frac{271382n}{\log^5 n} \qquad (n \geq 569).$$

Therefore, we obtain the following inequality:

$$\left|\sum_{p\le n}\frac{1}{\log p}-\left\{\frac{n}{\log^2 n}+\frac{2n}{\log^3 n}+\frac{6n}{\log^4 n}\right\}\right|<271382\frac{n}{\log^5 n}\qquad (n\ge 569).$$

A computer program verifies above inequality for $2\le n\le 568$, too. The proof is complete. □

Proof of the Main Theorem. Considering the right hand side of (2.1) and the Proposition 2.1, for every $n\ge 3$ we have

$$\Upsilon(n)\le (n-1)\sum_{p\le n}\frac{1}{p-1}<(n-1)\log\log(n-1)+23(n-1).$$

In the other hand, considering the left hand side of (2.1) and the Proposition 2.1, for every $n\ge 3$ we have

$$\Upsilon(n)>(n-1)\sum_{p\le n}\frac{1}{p-1}-\pi(n)-\log n\sum_{p\le n}\frac{1}{\log p}>(n-1)\log\log(n-1)-14(n-1)-\mathcal{R}(n),$$

where

$$\mathcal{R}(n)=\pi(n)+\log n\sum_{p\le n}\frac{1}{\log p},$$

and considering (2.2) and the Proposition 2.2, we have

$$\mathcal{R}(n)\le\frac{n}{\log n}\Big(1+\frac{1.2762}{\log n}\Big)+\frac{n}{\log n}+\frac{2n}{\log^2 n}+\frac{6n}{\log^3 n}+\frac{271382n}{\log^4 n}=\frac{2n}{\log n}+\frac{3.2762n}{\log^2 n}+\frac{6n}{\log^3 n}+\frac{271382n}{\log^4 n}.$$

But, for $n\ge 563206$ the right hand side of this relation is strictly less than $9(n-1)$. So, we obtain

$$\Upsilon(n)>(n-1)\log\log(n-1)-23(n-1),$$

for $n\ge 563206$, which holds true for $3\le n\le 563205$ by computations. This completes the proof. □

3. Remarks for Further Studies

3.1. **Explicit Approximation of the Function $\Omega(n)$.** Concerning the main theorem, considering $n!=\Gamma(n+1)$, one can reform above result as

$$|\Omega(\Gamma(n))-(n-2)\log\log(n-2)|<23(n-2),$$

and replacing n by $\Gamma^{-1}(n)$ (inverse of Gamma function), yields

$$\left|\Omega(n)-(\Gamma^{-1}(n)-2)\log\log(\Gamma^{-1}(n)-2)\right|<23\left(\Gamma^{-1}(n)-2\right).$$

This suggests an explicit approximation for the function $\Omega(n)$ in sense of inverse of Gamma function, which approximating Γ^{-1} one can make it in sense of elementary functions.

Remark 3.1. We don't know such approximations for Γ^{-1}. Professor Horst Alzer [1] recommended to read the papers [4] and [5] by Necdet Batir, who studied properties of the inverse gamma and polygamma functions. He believes that modifiying Batir's approach one can find properties of the inverse gamma function.

3.2. **An Extension of the Function $v_p(n!)$.** The function $v_p(n!)$ defined by

$$n! = \prod_{p \leq n} p^{v_p(n!)},$$

can be generalized for every positive integer $m \leq n$ instead prime $p \leq n$. Fix n and consider canonical decomposition

$$m = \prod_{p \leq n} p^{v_p(m)}.$$

Same to $v_p(n!)$, we define $v_m(n!)$ in which $m^{v_m(n!)} \| n!$. So,

$$m^{v_m(n!)} = \prod_{p \leq n} p^{v_p(m) v_m(n!)} \Big\| \prod_{p \leq n} p^{v_p(n!)}.$$

Therefore, we must have $v_p(m) v_m(n!) \leq v_p(n!)$ for every prime $p \leq n$; that is

$$v_m(n!) \leq \min_{\substack{p \leq n \\ v_p(m) \neq 0}} \left\{ \frac{v_p(n!)}{v_p(m)} \right\}.$$

This leads us to the following definition:

Definition. For positive integers m, n with $m \leq n$, we set

$$v_m(n!) = \left\lfloor \min_{\substack{p \leq n \\ v_p(m) \neq 0}} \left\{ \frac{v_p(n!)}{v_p(m)} \right\} \right\rfloor.$$

Note that in above definition, $v_p(N)$ for a positive integer N and prime p, is a well defined notation for the greatest power of p dividing N. Related by this generalization, the following question arise to mind:

Question. Find the function $\mathfrak{F}(n)$ such that

$$\sum_{m=1}^{n} v_m(n!) = \mathfrak{F}(n) \sum_{p \leq n} v_p(n!).$$

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