A Double Inequality for Divided Differences and Some Identities of Psi and Polygamma Functions

A DOUBLE INEQUALITY FOR DIVIDED DIFFERENCES AND SOME IDENTITIES OF PSI AND POLYGAMMA FUNCTIONS

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Abstract. In this short note, from the logarithmically completely monotonic property of the function $(x+c)^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)}$, a double inequality for the divided differences and some identities of the psi and polygamma functions are presented.

1. Introduction

Recall [1, 8] that a positive function f is called logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies $(-1)^k[\ln f(x)]^{(k)} \ge 0$ for all $k \in \mathbb{N}$ on I. For more detailed information, please refer to [1, 2, 3, 4, 7, 10, 11] and the related references therein.

It is well known that the classical Euler's gamma function $\Gamma(x)$ plays a central role in the theory of special functions and has much extensive applications in many branches, for example, statistics, physics, engineering, and other mathematical sciences. The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$, is called psi or digamma function, and $\psi^{(i)}(x)$ for $i \in \mathbb{N}$ are known as the polygamma or multigamma functions.

For real numbers α and β with $\alpha \ne \beta$, $(\alpha,\beta) \ne (0,1)$ and $(\alpha,\beta) \ne (1,0)$ and for $t \in \mathbb{R}$, let

$$q_{\alpha,\beta}(t)=\begin{cases}\dfrac{e^{-\alpha t}-e^{-\beta t}}{1-e^{-t}}, & t \ne 0,\\ \beta-\alpha, & t=0.\end{cases} \tag{1}$$

From the necessary and sufficient conditions such that the function $q_{\alpha,\beta}(t)$ is monotonic, which were established in [5, 6], the following logarithmically complete monotonicity was obtained.

Lemma 1 ([9]). *Let a, b and c be real numbers and $\rho=\min\{a,b,c\}$. Then the function*

$$H_{a,b,c}(x)=(x+c)^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} \tag{2}$$

is logarithmically completely monotonic in $(-\rho,\infty)$ if and only if

$$\begin{aligned}(a,b,c) \in D_1(a,b,c) &\triangleq \{(a,b,c):(b-a)(1-a-b+2c)\ge 0\}\\ &\cap \{(a,b,c):(b-a)(|a-b|-a-b+2c)\ge 0\}\end{aligned}$$

2000 *Mathematics Subject Classification.* Primary 33B15, 26A48, 26A51; Secondary 26D07.

Key words and phrases. inequality, divided difference, identity, psi function, polygamma function, logarithmically completely monotonic function.

This paper was typeset using $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX.

$$\setminus \{(a,b,c) : a = c+1 = b+1\} \setminus \{(a,b,c) : b = c+1 = a+1\}, \quad (3)$$

so is $H_{b,a,c}(x)$ *in* $(-\rho,\infty)$ *if and only if*

$$\begin{aligned}(a,b,c) \in D_2(a,b,c) \triangleq \{(a,b,c) : (b-a)(1-a-b+2c) \le 0\} \\ \cap \{(a,b,c) : (b-a)(|a-b|-a-b+2c) \le 0\} \\ \setminus \{(a,b,c) : b = c+1 = a+1\} \setminus \{(a,b,c) : a = c+1 = b+1\}. \quad (4)\end{aligned}$$

The first aim of this short note is to deduce a double inequality for the divided differences of the polygamma functions from Lemma 1 as follows.

Theorem 1. *Let* $b > a \ge 0$ *and* $k \in \mathbb{N}$. *Then the double inequality*

$$\frac{(k-1)!}{(x+\alpha)^k} \le \frac{(-1)^{k-1}\big[\psi^{(k-1)}(x+b) - \psi^{(k-1)}(x+a)\big]}{b-a} \le \frac{(k-1)!}{(x+\beta)^k} \quad (5)$$

for $x \in (0,\infty)$ *holds if and only if* $\alpha \ge \max\big\{a, \frac{a+b-1}{2}\big\}$ *and* $0 \le \beta \le \min\big\{a, \frac{a+b-1}{2}\big\}$.

The second aim of this short note is to show some identities of the psi and polygamma functions by using Theorem 1.

Theorem 2. *Let* $k \in \mathbb{N}$. *then the following identities of polygamma functions are valid:*

$$\psi^{(k-1)}\left(\left(\frac{\sqrt{5}+1}{2}\right)^2\right) - \psi^{(k-1)}\left(\frac{\sqrt{5}+1}{2}\right) = (-1)^{k-1}(k-1)!\left(\frac{\sqrt{5}-1}{2}\right)^k, \quad (6)$$

$$\begin{aligned}\psi^{(k-1)}\left(\left(\frac{\sqrt[3]{9-\sqrt{69}}+\sqrt[3]{9+\sqrt{69}}}{\sqrt[3]{18}}\right)^3\right) - \psi^{(k-1)}\left(\frac{\sqrt[3]{9-\sqrt{69}}+\sqrt[3]{9+\sqrt{69}}}{\sqrt[3]{18}}\right) \\ = (-1)^{k-1}(k-1)!\left(\frac{\sqrt[3]{18}}{\sqrt[3]{9-\sqrt{69}}+\sqrt[3]{9+\sqrt{69}}}\right)^k, \quad (7)\end{aligned}$$

$$\begin{aligned}\psi^{(k-1)}\left(\frac{1}{8}\left(\sqrt{a-b+\frac{2}{\sqrt{b-a}}}+\sqrt{b-a}\right)^4\right) \\ - \psi^{(k-1)}\left(\frac{1}{2}\sqrt{a-b+\frac{2}{\sqrt{b-a}}}+\frac{\sqrt{b-a}}{2}\right) \\ = (-1)^{k-1}2^k(k-1)!\left(\sqrt{a-b+\frac{2}{\sqrt{b-a}}}+\sqrt{b-a}\right)^{-k}, \quad (8)\end{aligned}$$

where $a = 4\sqrt[3]{\frac{2}{3(9+\sqrt{849})}}$ *and* $b = \sqrt[3]{\frac{9+\sqrt{849}}{18}}$.

For $v > 1$ *and* $\alpha > 1$, *let* $v_0 > 1$ *denote the real root of equation* $v^\alpha - v - 1 = 0$, *then*

$$v_0^k\big[\psi^{(k-1)}(v_0^\alpha) - \psi^{(k-1)}(v_0)\big] = (-1)^{k-1}(k-1)!. \quad (9)$$

For $0 < v < 1$ *and* $\alpha < 0$, *let* $v_0 < 1$ *be the real root of equation* $v^\alpha - v - 1 = 0$, *then identity* (9) *is also valid.*

2. Proofs of theorems

Proof of Theorem 1. From the logarithmically complete monotonicity of the function $H_{a,b,c}(x)$ in Lemma 1, it follows that

$$\begin{aligned} 0 &\le (-1)^k[\ln H_{a,b,c}(x)]^{(k)} \\ &= (-1)^k\left[\psi^{(k-1)}(x+a)-\psi^{(k-1)}(x+b)+\frac{(-1)^{k-1}(b-a)(k-1)!}{(x+c)^k}\right] \end{aligned} \tag{10}$$

for $(a,b,c)\in D_1(a,b,c)$, then the left hand side inequality in (5) is deduced straightforwardly by standard arguments.

The right hand side inequality in (5) can be deduced from $(-1)^k[\ln H_{b,a,c}(x)]^{(k)}\ge 0$ for $(a,b,c)\in D_2(a,b,c)$. □

Proof of Theorem 2. Inequality (5) in Theorem 1 can be rearranged as

$$\begin{aligned} \frac{(k-1)!}{[\max\{v,(u+v-1)/2\}]^k} &\le \frac{(-1)^{k-1}\left[\psi^{(k-1)}(u)-\psi^{(k-1)}(v)\right]}{u-v} \\ &= \frac{(-1)^{k-1}}{u-v}\int_v^u \psi^{(k)}(t)\,\mathrm{d}\,t \le \frac{(k-1)!}{[\min\{v,(u+v-1)/2\}]^k} \end{aligned} \tag{11}$$

for $u>v>0$.

Substituting $u=v^2$ for $v>1$ in (11) yields

$$\begin{aligned} \frac{(k-1)!(v^2-v)}{[\max\{v,(v^2+v-1)/2\}]^k} &\le (-1)^{k-1}\left[\psi^{(k-1)}(v^2)-\psi^{(k-1)}(v)\right] \\ &\le \frac{(k-1)!(v^2-v)}{[\min\{v,(v^2+v-1)/2\}]^k}. \end{aligned} \tag{12}$$

Since equation $v^2-v-1=0$ has a unique root $\frac{\sqrt5+1}{2}$ greater than 1, then, if $1<v\le\frac{\sqrt5+1}{2}$,

$$\begin{aligned} (k-1)!\left(\frac{1}{v^{k-2}}-\frac{1}{v^{k-1}}\right) &\le (-1)^{k-1}\left[\psi^{(k-1)}(v^2)-\psi^{(k-1)}(v)\right] \\ &\le \frac{(k-1)!2^k v(v-1)}{(v^2+v-1)^k}; \end{aligned} \tag{13}$$

if $v\ge\frac{\sqrt5+1}{2}$, above inequality reverses. Taking $v\to\frac{\sqrt5+1}{2}$ in (12) or (13) yields identity (6).

It is easy to see that equation $v^3-v-1=0$ has a unique real root

$$\sqrt[3]{\frac12-\frac16\sqrt{\frac{23}{3}}}+\sqrt[3]{\frac12+\frac16\sqrt{\frac{23}{3}}}=\frac{\sqrt[3]{9-\sqrt{69}}+\sqrt[3]{9+\sqrt{69}}}{\sqrt[3]{2}\,\sqrt[3]{9}}=1.324\cdots. \tag{14}$$

Substituting $u=v^3$ for $v>1$ in (11) yields

$$\begin{aligned} \frac{(k-1)!(v^3-v)}{[\max\{v,(v^3+v-1)/2\}]^k} &\le (-1)^{k-1}\left[\psi^{(k-1)}(v^3)-\psi^{(k-1)}(v)\right] \\ &\le \frac{(k-1)!(v^3-v)}{[\min\{v,(v^3+v-1)/2\}]^k}. \end{aligned} \tag{15}$$

If $1 < v \le \frac{\sqrt[3]{9-\sqrt{69}}+\sqrt[3]{9+\sqrt{69}}}{\sqrt[3]{2}\sqrt[3]{9}}$,

$$
\begin{aligned}
(k-1)!\biggl(\frac{1}{v^{k-3}}-\frac{1}{v^{k-1}}\biggr) &\le (-1)^{k-1}\bigl[\psi^{(k-1)}(v^3)-\psi^{(k-1)}(v)\bigr]\\
&\le \frac{(k-1)!2^kv(v^2-1)}{(v^3+v-1)^k}; \quad (16)
\end{aligned}
$$

if $v \ge \frac{\sqrt[3]{9-\sqrt{69}}+\sqrt[3]{9+\sqrt{69}}}{\sqrt[3]{2}\sqrt[3]{9}}$, above inequality reverses. Identity (7) follows from taking $v \to \frac{\sqrt[3]{9-\sqrt{69}}+\sqrt[3]{9+\sqrt{69}}}{\sqrt[3]{2}\sqrt[3]{9}}$ in (15) or (16).

It is not difficult to see that the quartic equation $v^4 - v - 1 = 0$ has a unique real root

$$
\begin{aligned}
&\frac12\sqrt{4\sqrt[3]{\frac{2}{3\bigl(9+\sqrt{849}\,\bigr)}}-\sqrt[3]{\frac{9+\sqrt{849}}{18}}+\frac{2}{\sqrt{\sqrt[3]{\frac{9+\sqrt{849}}{18}}-4\sqrt[3]{\frac{2}{3(9+\sqrt{849}\,)}}}}}\\
&\quad+\frac12\sqrt{\sqrt[3]{\frac{9+\sqrt{849}}{18}}-4\sqrt[3]{\frac{2}{3\bigl(9+\sqrt{849}\,\bigr)}}} = 1.220\cdots \quad (17)
\end{aligned}
$$

Replacing u by v^4 for $v > 1$ in (11) gives

$$
\begin{aligned}
\frac{(k-1)!(v^4-v)}{[\max\{v,(v^4+v-1)/2\}]^k} &\le (-1)^{k-1}\bigl[\psi^{(k-1)}(v^4)-\psi^{(k-1)}(v)\bigr]\\
&\le \frac{(k-1)!(v^4-v)}{[\min\{v,(v^4+v-1)/2\}]^k}. \quad (18)
\end{aligned}
$$

If $1 < v \le \frac12\sqrt{a-b+\frac{2}{\sqrt{b-a}}}+\frac12\sqrt{b-a}$, then

$$
\begin{aligned}
(k-1)!\biggl(\frac{1}{v^{k-4}}-\frac{1}{v^{k-1}}\biggr) &\le (-1)^{k-1}\bigl[\psi^{(k-1)}(v^4)-\psi^{(k-1)}(v)\bigr]\\
&\le \frac{(k-1)!2^kv(v^3-1)}{(v^4+v-1)^k}; \quad (19)
\end{aligned}
$$

if $v \ge \frac12\sqrt{a-b+\frac{2}{\sqrt{b-a}}}+\frac12\sqrt{b-a}$, above inequality reverses. Identity (8) follows from taking $v \to \frac12\sqrt{a-b+\frac{2}{\sqrt{b-a}}}+\frac12\sqrt{b-a}$ in (18) or (19).

For $v > 1$ and $\alpha > 1$, since the function $f_\alpha(v) = v^\alpha - v - 1$ satisfying

$$\lim_{v\to1^+} f_\alpha(v) = -1 \quad\text{and}\quad \lim_{v\to\infty} f_\alpha(v) = \infty, \tag{20}$$

the equation $v^\alpha - v - 1 = 0$ must have at least one root v_0 greater than 1. Letting $u = v^\alpha > v > 1$ and taking limit $v \to v_0$ in (11) leads to

$$\psi^{(k-1)}(v_0^4)-\psi^{(k-1)}(v_0) = \frac{(-1)^{k-1}(k-1)!}{v_0^k}. \tag{21}$$

Identity (9) is proved for $v > 1$ and $\alpha > 1$.

For $0 < v < 1$ and $\alpha < 0$, since the function $f_\alpha(v) = v^\alpha - v - 1$ satisfying

$$\lim_{v\to1^+} f_\alpha(v) = -1 \quad\text{and}\quad \lim_{v\to0^+} f_\alpha(v) = \infty, \tag{22}$$

the equation $v^\alpha - v - 1 = 0$ must have at least one root v_0 less than 1. Letting $u = v^\alpha > 1 > v$ and taking limit $v \to v_0$ in (11) leads to (21). Hence, identity (9) is proved for $0 < v < 1$ and $\alpha < 0$. □

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