Sharp Function Estimates for Vector-Valued Multilinear Operator of Multiplier Operator

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Abstract: In this paper, we establish a sharp function estimate for the vector-valued multilinear operator of the multiplier. As the application, we obtain the weighted $L^p(1 < p < \infty)$ norm inequality for the multilinear operator.

Keywords: Vector-valued multilinear operator; Multiplier operator; Sharp estimate; BMO.

MR Subject Classification: 42B20, 42B25.

1. Introduction and Results

Let $b \in BMO(R^n)$ and T be the Calderón-Zygmund singular integral operator. The commutator $[b, T]$ generated by b and T is defined by $[b, T](f)(x) = b(x)T(f)(x) - T(bf)(x)$. As the development of the Calderón-Zygmund singular integral operators, their commutators and multilinear operators have been well studied(see [1-7][15-18]). In [10], Hu and Yang proved a variant sharp function estimate for the multilinear singular integral operators. In [18], C.Pérez and R.Trujillo-Gonzalez obtained a sharp weighted estimates for the singular integral operators and their commutators. In [23], You proved that the commutator $[b, T]$ is bounded in $L^p(R^n)$ when T is a multiplier operator and $b \in \dot{\Lambda}_\beta(R^n)$. In [24][25], Zhang studied the $(L^p, \dot{F}_p^{\beta,\infty})-$ boundedness of the commutator of the multipliers. In this paper, we will introduce the vector-valued multilinear operator associated to the multiplier operator and study the sharp function inequality of the vector-valued multilinear operator. By using the sharp inequality, we obtain the weighted L^p- norm inequality for the vector-valued multilinear operator.

First, let us introduce some notations. In this paper, Q will denote a cube of R^n with sides

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parallel to the axes. For a cube Q and a locally integrable function b, let $b_Q = |Q|^{-1}\int_Q b(x)dx$. The sharp function of b is defined by

$$b^{\#}(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |b(y) - b_Q|dy.$$

It is well-known that (see [9])

$$b^{\#}(x) \approx \sup_{Q\ni x} \inf_{c\in \mathbf{C}} \frac{1}{|Q|}\int_Q |b(y) - c|dy$$

and

$$||b - b_{2^k Q}||_{BMO} \le Ck||b||_{BMO} \text{ for } k \ge 1.$$

We say that b belongs to $BMO(R^n)$, if $b^{\#}$ belongs to $L^\infty(R^n)$ and define

$$\|b\|_{BMO} = \|b^{\#}\|_{L^\infty}.$$

Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|dy,$$

we write that $M_p(f) = (M(f^p))^{1/p}$ for $0 < p < \infty$.

We denote the Muckenhoupt weights by A_1 (see [9]), that is

$$A_1 = \{w : M(w)(x) \le Cw(x), a.e.\}.$$

A bounded measurable function k defined on $R^n\backslash\{0\}$ is called a multiplier. The multiplier operator T associated with k is defined by

$$T(f)(x) = k(x)\hat{f}(x), \text{ for } f \in S(R^n),$$

where $\hat{f}$ denotes the Fourier transform of f and $S(R^n)$ is the Schwartz test function class. Now, we recall the definition of the class $M(s,l)$. Denote by $|x| \sim t$ the fact that the value of x lies in the annulus $\{x \in R^n : at < |x| < bt\}$, where $0 < a \le 1 < b < \infty$ are values specified in each instance.

Definition 1. ([11]) Let $l \ge 0$ be a real number and $1 \le s \le 2$. we say that the multiplier k satisfies the condition $M(s,l)$, if

$$\left(\int_{|\xi|\sim R} |D^\alpha k(\xi)|^s d\xi\right)^{\frac{1}{s}} < CR^{n/s-|\alpha|}$$

for all $R > 0$ and multi-indices α with $|\alpha| \leq l$, when l is a positive integer, and, in addition, if

$$\left(\int_{|\xi|\sim R} |D^\alpha k(\xi) - D^\alpha k(\xi - z)|^s d\xi\right)^{\frac{1}{s}} \leq C(\frac{|z|}{R})^\gamma R^{\frac{n}{s}-|\alpha|}$$

for all $|z| < R/2$ and all multi-indices α with $|\alpha| = [l]$, the integer part of l,i.e., $[l]$ is the greatest integer less than or equal to l, and $l = [l] + \gamma$ when l is not an integer.

Denote $D(R^n) = \{\phi \in S(R^n) : \text{supp}(\phi) \text{ is compact}\}$ and $\hat{D}_0(R^n) = \{\phi \in S(R^n) : \hat{\phi} \in D(R^n)$ and $\hat{\phi}$ vanishes in a neighbourhood of the origin$\}$. The following boundedness property of T on $L^p(R^n)$ is proved by Strömberg and Torkinsky (see [11-14]).

Lemma 1.([11]) Let $k \in M(s,l), 1 \leq s \leq 2$, and $l > \frac{n}{s}$. Then the associated mapping T, defined a priori for $f \in \hat{D}_0(R^n)$, $T(f)(x) = (f * K)(x)$, extends to a bounded mapping from $L^p(R^n)$ into itself for $1 < p < \infty$ and $K(x) = \check{k}(x)$.

Definition 2.([11]) For a real number $\tilde{l} \geq 0$ and $1 \leq \tilde{s} < \infty$, we say that K verifies the condition $\tilde{M}(\tilde{s},\tilde{l})$, and write $K \in \tilde{M}(\tilde{s},\tilde{l})$, if

$$\left(\int_{|x|\sim R} |D^{\tilde{\alpha}} K(x)|^{\tilde{s}} dx\right)^{\frac{1}{\tilde{s}}} \leq CR^{n/\tilde{s}-n-|\tilde{\alpha}|}, \quad R > 0$$

for all multi-indices $|\tilde{\alpha}| \leq \tilde{l}$ and, in addition, if

$$\left(\int_{|x|\sim R} |D^{\tilde{\alpha}} K(x) - D^{\tilde{\alpha}} K(x-z)|^{\tilde{s}} dx\right)^{\frac{1}{\tilde{s}}} \leq C(\frac{|z|}{R})^v R^{\frac{n}{\tilde{s}}-n-u}, \qquad if \ \ 0 < v < 1,$$

$$\left(\int_{|x|\sim R} |D^{\tilde{\alpha}} K(x) - D^{\tilde{\alpha}} K(x-z)|^{\tilde{s}} dx\right)^{\frac{1}{\tilde{s}}} \leq C(\frac{|z|}{R})(log\frac{R}{|z|})R^{\frac{n}{\tilde{s}}-n-u}, \qquad if \ \ v = 1,$$

for all $|z| < \frac{R}{2}, R > 0$, and all multi-indices $\tilde{\alpha}$ with$|\tilde{\alpha}| = u$, where u denotes the largest integer strictly less than $\tilde{l}$ with $\tilde{l} = u + v$.

Lemma 2. ([11]) Suppose $k \in M(s,l)$, $1 \leq s \leq 2$. Given $1 \leq \tilde{s} < \infty$, let $r \geq 1$ be such that $\frac{1}{r} = \max\{\frac{1}{s}, 1 - \frac{1}{\tilde{s}}\}$. Then $K \in \tilde{M}(\tilde{s},\tilde{l})$, where $\tilde{l} = l - \frac{n}{r}$.

Lemma 3. Let $1 \leq s \leq 2$, suppose that l is a positive real number with $l > n/r$, $1/r = \max\{1/s, 1-1/\tilde{s}\}$, and $k \in M(s,l)$. Then there is a positive constant a, such that

$$\left(\int_{B_k} |K(x-z) - K(x_Q - z)|^{\tilde{s}} dz\right)^{1/\tilde{s}} \leq C2^{-ka}(2^k h)^{-n/\tilde{s}'}.$$

Proof. We split our proof into two cases:

Case 1. $1 \leq s \leq 2$ and $0 < l - n/s \leq 1$. We choose a real number $1 < \tilde{s} < \infty$ such that $s \leq \tilde{s}$, and set $\tilde{l} = l - \frac{n}{s} > 0$. Since $k \in M(s,l)$, then by Lemma 3, there is $K \in \tilde{M}(\tilde{s},\tilde{l})$.

When $\tilde{l} = l - \frac{n}{s} < 1$, noting that l is a positive real number and $l > \frac{n}{s}$. Applying the condition $K \in \tilde{M}(\tilde{s}, \tilde{l})$ for $v = l - \frac{n}{s}$ and $u = 0$, one has

$$\left(\int_{B_k} |K(x-z) - K(x_Q - z)|^{\tilde{s}} dz\right)^{\frac{1}{\tilde{s}}} \leq C2^{-k(l-\frac{n}{s})}(2^k h)^{-\frac{n}{\tilde{s}'}},$$

let $a = l - \frac{n}{s}$,

$$\left(\int_{B_k} |K(x-z) - K(x_Q - z)|^{\tilde{s}} dz\right)^{\frac{1}{\tilde{s}}} \leq C2^{-ka}(2^k h)^{-\frac{n}{\tilde{s}'}}.$$

When $\tilde{l} = l - \frac{n}{s} = 1$, we choose $0 < \xi < 1$, such that $t^{1-\xi}\log(1/t) \leq C$ for $0 < t < 1/2$. Noting that $K \in \tilde{M}(\tilde{s}, \tilde{l})$, by Definition 2, for $u = 0, v = 1$,

$$\begin{aligned} &\left(\int_{B_k} |K(x-z) - K(x_Q - z)|^{\tilde{s}} dz\right)^{\frac{1}{\tilde{s}}} \\ \leq\ & C\left(\frac{|y - x_Q|}{2^k h}\right)^{\xi}\left(\frac{|y - x_Q|}{2^k h}\right)^{1-\xi}\left(\log\frac{2^k h}{|y - x_Q|}\right)(2^k h)^{n/\tilde{s}-n} \\ \leq\ & C2^{-k\xi}(2^k h)^{-n/\tilde{s}'}, \end{aligned}$$

let $a = \xi$, then

$$\left(\int_{B_k} |K(x-z) - K(x_Q - z)|^{\tilde{s}} dz\right)^{\frac{1}{\tilde{s}}} \leq C2^{-ka}(2^k h)^{-n/\tilde{s}'}.$$

Case 2. $1 \leq s \leq 2$ and $l - n/s > 1$. Set $d = [l - n/s]$, if $l - n/s > 1$ is not an integer, and $d = l - n/s - 1$ if $l - n/s > 1$ is an integer. Choose $l_1 = l - d$; then $0 < l_1 - n/s \leq 1$ and $0 < l_1 < l$. So, from $k \in M(s, l)$ we know $k \in M(s, l_1)$. Set $\tilde{l} = l_1 - n/s$; by Lemma 3, $K \in \tilde{M}(\tilde{s}, \tilde{l})$. Repeating the proof of **Case 1**, except for replacing l by l_1, we can obtain the same result under the assumption $l - n/s > 1$. We omit the details here.

Certainly when $0 < \tilde{s}' < s$, which is the same as the above.

Now we can define the vector-valued multilinear operator associated to the multiplier operator T. Let m_j be the positive integers($j = 1, \cdots, l$), $m_1 + \cdots + m_l = m$ and b_j be the functions on R^n($j = 1, \cdots, l$). Set, for $1 \leq j \leq m$,

$$R_{m_j+1}(b_j; x, y) = b_j(x) - \sum_{|\alpha| \leq m_j} \frac{1}{\alpha!} D^{\alpha} b_j(y)(x-y)^{\alpha}.$$

By Lemma 1, $T(f)(x) = (K * f)(x)$ for $K(x) = \check{k}(x)$. Given the functions f_i defined on R^n, $i = 1, 2, \ldots$, for $1 < r < \infty$, the the vector-valued multilinear operator associated to T is defined by

$$|T_b(f)(x)|_r = \left(\sum_{i=1}^{\infty} (T_b(f_i)(x))^r\right)^{1/r},$$

where

$$T_b(f_i)(x) = \int_{R^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(b_j; x, y)}{|x-y|^m} K(x-y) f_i(y) dy.$$

Set

$$|T(f)(x)|_r = \left(\sum_{i=1}^{\infty} |T(f_i)(x)|^r \right)^{1/r} \quad \text{and} \quad |f(x)|_r = \left(\sum_{i=1}^{\infty} |f_i(x)|^r \right)^{1/r}.$$

Note that when $m = 0$, $|T_b|_r$ is just the vector-valued multilinear commutator of T and b_j(see [18]). While when $m > 0$, $|T_b|_r$ is non-trivial generalizations of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [2-5][7]). Hu and Yang (see [10]) proved a variant sharp estimate for the multilinear singular integral operators. In [17], Pérez and Trujillo-Gonzalez prove a sharp estimate for the multilinear commutator when $b_j \in Osc_{expL^{r_j}}(R^n)$. The main purpose of this paper is to prove a sharp function inequality for the vector-valued multilinear multiplier operator when $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$. As the application, we obtain the $L^p (p > 1)$ norm inequality for the vector-valued multilinear operator.

We shall prove the following theorems.

Theorem 1. Let $1 < r < \infty$ and $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \cdots, l$. Then there exists a constant $C > 0$ such that for every $f \in C_0^\infty(R^n)$, $1 < s < \infty$ and $\tilde{x} \in R^n$,

$$(|T_b(f)|_r)_s^{\#}(\tilde{x}) \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} b_j||_{BMO} \right) M_s(|f|_r)(\tilde{x}).$$

Theorem 2. Let $1 < r < \infty$ and $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \cdots, l$. Then $|T_b|_r$ is bounded on $L^p(w)$ for any $w \in A_1$ and $1 < p < \infty$, that is

$$|||T_b(f)|_r||_{L^p(w)} \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} b_j||_{BMO} \right) |||f|_r||_{L^p(w)}.$$

2. Proofs of Theorems

To prove the theorems, we need the following lemmas.

Lemma 4.([4]) Let b be a function on R^n and $D^\alpha b \in L^q(R^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then

$$|R_m(b; x, y)| \le C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha b(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 5.([8][11]) Let T be the multiplier operator. Then, for every $f \in L^p(R^n), 1 < p < \infty$,

$$|||T(f)|_r||_{L^p} \le C|||f|_r||_{L^p}.$$

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|}\int_Q ||T^A(f)(x)|_r - C_0|dx \le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)M_s(|f|_r)(x).$$

Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{b}_j(x) = b_j(x) - \sum_{|\alpha|=m_j}\frac{1}{\alpha!}(D^\alpha b_j)_{\tilde{Q}}x^\alpha$, then $R_{m_j+1}(b_j; x, y) = R_{m_j+1}(\tilde{b}_j; x, y)$ and $D^\alpha\tilde{b}_j = D^\alpha b_j - (D^\alpha b_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We split $f = g + h = \{g_i\} + \{h_i\}$ for $g_i = f_i\chi_{\tilde{Q}}$ and $h_i = f_i\chi_{R^n\setminus\tilde{Q}}$. Write

$$\begin{aligned}
T_b(f_i)(x) \quad = \quad & \int_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, y)}{|x-y|^m}K(x-y)g_i(y)dy \\
& - \sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m}D^{\alpha_1}\tilde{b}_1(y)K(x-y)g_i(y)dy \\
& - \sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m}D^{\alpha_2}\tilde{b}_2(y)K(x-y)g_i(y)dy \\
& + \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m}K(x-y)g_i(y)dy \\
& + \int_{R^n}\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j; x, y)}{|x-y|^m}K(x-y)h_i(y)dy \\
& = T\left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, \cdot)}{|x-\cdot|^m}g_i\right)(x) \\
& -T\left(\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\frac{R_{m_2}(\tilde{b}_2; x, \cdot)(x-\cdot)^{\alpha_1}D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m}g_i\right)(x) \\
& -T\left(\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\frac{R_{m_1}(\tilde{b}_1; x, \cdot)(x-\cdot)^{\alpha_2}D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}g_i\right)(x) \\
& +T\left(\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\frac{(x-\cdot)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}g_i\right)(x) + T_{\tilde{b}}(h_i)(x),
\end{aligned}$$

then, by Minkowski' inequality,

$$\frac{1}{|Q|}\int_Q \left||T_b(f)(x)|_r - |T_{\tilde{b}}(h)(x_0)|_r\right| dx$$

$$
\begin{aligned}
&\leq \frac{1}{|Q|}\int_Q \left(\sum_{i=1}^{\infty}|T_b(f_i)(x)-T_{\tilde{b}}(h_i)(x_0)|^r\right)^{1/r}dx \\
&\leq \frac{1}{|Q|}\int_Q \left(\sum_{i=1}^{\infty}\left|T\left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}g_i\right)(x)\right|^r\right)^{1/r}dx \\
&+\frac{1}{|Q|}\int_Q \left(\sum_{i=1}^{\infty}\left|T\left(\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1}D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m}g_i\right)(x)\right|^r\right)^{1/r}dx \\
&+\frac{1}{|Q|}\int_Q \left(\sum_{i=1}^{\infty}\left|T\left(\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2}D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}g_i\right)(x)\right|^r\right)^{1/r}dx \\
&+\frac{1}{|Q|}\int_Q \left(\sum_{i=1}^{\infty}\left|T\left(\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\frac{(x-\cdot)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}g_i\right)(x)\right|^r\right)^{1/r}dx \\
&+\frac{1}{|Q|}\int_Q \left(\sum_{i=1}^{\infty}|T_{\tilde{b}}(h_i)(x)-T_{\tilde{b}}(h_i)(x_0)|^r\right)^{1/r}dx \\
&:= I_1+I_2+I_3+I_4+I_5.
\end{aligned}
$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5, respectively. First, for $x \in Q$ and $y \in \tilde{Q}$, by Lemma 4, we get

$$
R_m(\tilde{b}_j;x,y) \leq C|x-y|^m \sum_{|\alpha_j|=m}||D^{\alpha_j}b_j||_{BMO},
$$

thus, by the Hölder's inequality and L^s-boundedness of $|T|_r$(Lemma 5), we obtain

$$
\begin{aligned}
I_1 &\leq C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)\frac{1}{|Q|}\int_Q |T(g)(x)|_r dx \\
&\leq C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)\left(\frac{1}{|Q|}\int_{R^n}|T(g)(x)|_r^s dx\right)^{1/s} \\
&\leq C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)\left(\frac{1}{|Q|}\int_{R^n}|g(x)|_r^s dx\right)^{1/s} \\
&\leq C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)|_r^s dx\right)^{1/s} \\
&\leq C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)M_s(|f|_r)(\tilde{x}).
\end{aligned}
$$

For I_2, denoting $s = pq$ for $1 < p < \infty$, $q > 1$ and $1/q + 1/q' = 1$, we have, by Lemma 5,

$$
I_2 \leq C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}b_2||_{BMO}\sum_{|\alpha_1|=m_1}\frac{1}{|Q|}\int_Q |T(D^{\alpha_1}\tilde{b}_1 g)(x)|_r dx
$$

$$
\begin{aligned}
&\leq C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2} b_2||_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1} \tilde{b}_1 g)(x)|_r^p dx \right)^{1/p} \\
&\leq C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2} b_2||_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|} \int_{R^n} |D^{\alpha_1} \tilde{b}_1(x)||g(x)|_r^p dx \right)^{1/p} \\
&\leq C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2} b_2||_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |D^{\alpha_1} b_1(x) - (D^{\alpha} b_j)_{\tilde{Q}}|^{pq'} dx \right)^{1/pq'} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |f(x)|_r^{pq} dx \right)^{1/pq} \\
&\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^{\alpha} b_j||_{BMO} \right) M_s(|f|_r)(\tilde{x}).
\end{aligned}
$$

For I_3, similar to the proof of I_2, we get

$$
I_3 \leq C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^{\alpha} b_j||_{BMO} \right) M_s(|f|_r)(\tilde{x}).
$$

Similarly, for I_4, denoting $s = pq_3$ for $1 < p < \infty$, $q_1, q_2, q_3 > 1$ and $1/q_1 + 1/q_2 + 1/q_3 = 1$, we obtain

$$
\begin{aligned}
I_4 &\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{|Q|} \int_Q |T(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 g)(x)|_r dx \\
&\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 g)(x)|_r^p dx \right)^{1/p} \\
&\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{|Q|} \int_{R^n} |D^{\alpha_1} \tilde{b}_1(x) D^{\alpha_2} \tilde{b}_2(x)||g(x)|_r^p dx \right)^{1/p} \\
&\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \prod_{j=1}^{2} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |D^{\alpha_j} \tilde{b}_j(x)|^{pq_j} dx \right)^{1/pq_j} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |f(x)|_r^{pq_3} dx \right)^{1/pq_3} \\
&\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^{\alpha} b_j||_{BMO} \right) M_s(|f|_r)(\tilde{x}).
\end{aligned}
$$

For I_5, we write

$$
\begin{aligned}
&T_{\tilde{b}}(h_i)(x) - T_{\tilde{b}}(h_i)(x_0) = \int_{R^n} (K(x-y) - K(x_0-y)) \frac{1}{|x-y|^m} \prod_{j=1}^{2} R_{m_j}(\tilde{b}_j; x, y) h_i(y) dy \\
&+ \int_{R^n} \left(\frac{1}{|x-y|^m} - \frac{1}{|x_0-y|^m} \right) K(x_0-y) \prod_{j=1}^{2} R_{m_j}(\tilde{b}_j; x, y) h_i(y) dy \\
&+ \int_{R^n} \left(R_{m_1}(\tilde{b}_1; x, y) - R_{m_1}(\tilde{b}_1; x_0, y) \right) \frac{R_{m_2}(\tilde{b}_2; x, y)}{|x_0-y|^m} K(x_0-y) h_i(y) dy \\
&+ \int_{R^n} \left(R_{m_2}(\tilde{b}_2; x, y) - R_{m_2}(\tilde{b}_2; x_0, y) \right) \frac{R_{m_1}(\tilde{b}_1; x_0, y)}{|x_0-y|^m} K(x_0-y) h_i(y) dy
\end{aligned}
$$

$$
\begin{aligned}
& - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \left[\frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} K(x-y) - \frac{R_{m_2}(\tilde{b}_2; x_0, y)(x_0-y)^{\alpha_1}}{|x_0-y|^m} K(x_0-y) \right] \\
& \times D^{\alpha_1}\tilde{b}_1(y) h_i(y) dy \\
& - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \left[\frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} K(x-y) - \frac{R_{m_1}(\tilde{b}_1; x_0, y)(x_0-y)^{\alpha_2}}{|x_0-y|^m} K(x_0-y) \right] \\
& \times D^{\alpha_2}\tilde{b}_2(y) h_i(y) dy \\
& + \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} K(x-y) - \frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m} K(x_0-y) \right] \\
& \times D^{\alpha_1}\tilde{b}_1(y) D^{\alpha_2}\tilde{b}_2(y) h_i(y) dy \\
= \;& I_5^{(1)} + I_5^{(2)} + I_5^{(3)} + I_5^{(4)} + I_5^{(5)} + I_5^{(6)} + I_5^{(7)}.
\end{aligned}
$$

By Lemma 4 and the following inequality(see [20])

$$
|b_{Q_1} - b_{Q_2}| \le C \log(|Q_2|/|Q_1|) ||b||_{BMO} \text{ for } Q_1 \subset Q_2,
$$

we know that, for $x \in Q$ and $y \in 2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}$,

$$
\begin{aligned}
|R_m(\tilde{b}; x, y)| &\le C|x-y|^m \sum_{|\alpha|=m} (||D^\alpha b||_{BMO} + |(D^\alpha b)_{\tilde{Q}(x,y)} - (D^\alpha b)_{\tilde{Q}}|) \\
&\le Ck|x-y|^m \sum_{|\alpha|=m} ||D^\alpha b||_{BMO}.
\end{aligned}
$$

Note that $|x-y| \sim |x_0-y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, we have

$$
|I_5^{(1)}| \le C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO} \right) \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} k^2 |K(x-y) - K(x_0-y)||f_i(y)|dy,
$$

thus, by the Minkowski' inequality and Lemma 3, we obtain

$$
\begin{aligned}
\left(\sum_{i=1}^\infty ||I_5^{(1)}||^r \right)^{1/r} &\le C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO} \right) \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} k^2 |K(x-y) - K(x_0-y)||f(y)|_r dy \\
&\le C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO} \right) \sum_{k=0}^\infty k^2 \left(\int_{2^{k+1}\tilde{Q}} |f(y)|_r^s dy \right)^{1/s} \\
&\quad \times \left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} |K(x-y) - K(x_0-y)|^{s'} dy \right)^{1/s'} \\
&\le C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO} \right) \sum_{k=1}^\infty k^2 2^{-ka} \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |f(y)|_r^s dy \right)^{1/s}
\end{aligned}
$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha b_j||_{BMO}\right)\sum_{k=1}^{\infty}k^2 2^{-ka}M_s(|f|_r)(\tilde{x})$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha b_j||_{BMO}\right)M_s(|f|_r)(\tilde{x}).$$

For $I_5^{(2)}$, by the Minkowski' inequality and Lemma 2, we obtain

$$\begin{aligned}\left(\sum_{i=1}^{\infty}||I_5^{(2)}||^r\right)^{1/r} &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha b_j||_{BMO}\right)\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k^2\frac{|x-x_0|}{|x_0-y|}|K(x_0-y)||f(y)|_r dy\\ &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha b_j||_{BMO}\right)\sum_{k=0}^{\infty}k^2 2^{-k}\left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|K(x_0-y)|^{s'}dy\right)^{1/s'}\\ &\quad\times\left(\int_{2^{k+1}\tilde{Q}}|f(y)|_r^s dy\right)^{1/s}\\ &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha b_j||_{BMO}\right)\sum_{k=0}^{\infty}k^2 2^{-k}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|_r^s dy\right)^{1/s}\\ &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha b_j||_{BMO}\right)M_s(|f|_r)(\tilde{x}).\end{aligned}$$

For $I_5^{(3)}$, by the formula (see [5]):

$$R_{m_j}(\tilde{b};x,y)-R_{m_j}(\tilde{b};x_0,y)=\sum_{|\beta|<m_j}\frac{1}{\beta!}R_{m_j-|\beta|}(D^\beta\tilde{b};x,x_0)(x-y)^\beta$$

and Lemma 4, we have

$$|R_{m_j}(\tilde{b};x,y)-R_{m_j}(\tilde{b};x_0,y)|\leq C\sum_{|\beta|<m_j}\sum_{|\alpha|=m_j}|x-x_0|^{m_j-|\beta|}|x-y|^{|\beta|}||D^\alpha b||_{BMO},$$

thus, by the Minkowski' inequality and Lemma 2, we obtain

$$\begin{aligned}\left(\sum_{i=1}^{\infty}||I_5^{(3)}||^r\right)^{1/r} &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha b_j||_{BMO}\right)\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k^2\frac{|x-x_0|}{|x_0-y|}|K(x_0-y)||f(y)|_r dy\\ &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha b_j||_{BMO}\right)\sum_{k=0}^{\infty}k^2 2^{-k}\left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|K(x_0-y)|^{s'}dy\right)^{1/s'}\\ &\quad\times\left(\int_{2^{k+1}\tilde{Q}}|f(y)|_r^s dy\right)^{1/s}\\ &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha b_j||_{BMO}\right)\sum_{k=0}^{\infty}k^2 2^{-k}(2^k d)^{-n/s}(2^k d)^{n/s}M_s(|f|_r)(\tilde{x})\end{aligned}$$

$$\leq \quad C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}b_j||_{BMO}\right)M_s(|f|_r)(\tilde{x}).$$

Similarly,

$$\left(\sum_{i=1}^{\infty}||I_5^{(4)}||^r\right)^{1/r}\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)M_s(|f|_r)(\tilde{x}).$$

For $I_5^{(5)}$, similar to the proofs of $I_5^{(1)}$, $I_5^{(2)}$, $I_5^{(3)}$ and I_4, we get, for $1<s_1,s_2<\infty$ with $1/s_1+1/s_2+1/s=1$,

$$\begin{aligned}
&\left(\sum_{i=1}^{\infty}||I_5^{(5)}||^r\right)^{1/r}\\
\leq\;& C\sum_{|\alpha_1|=m_1}\int_{(\tilde{Q})^c}\frac{|(x-y)^{\alpha_1}|}{|x-y|^m}|R_{m_2}(\tilde{b}_2;x,y)-R_{m_2}(\tilde{b}_2;x_0,y)||K(x-y)D^{\alpha_1}\tilde{b}_1(y)||f(y)|_r dy\\
&+C\sum_{|\alpha_1|=m_1}\int_{(\tilde{Q})^c}\left|\frac{(x-y)^{\alpha_1}}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1}}{|x_0-y|^m}\right||R_{m_2}(\tilde{b}_2;x_0,y)K(x-y)D^{\alpha_1}\tilde{b}_1(y)||f(y)|_r dy\\
&+C\sum_{|\alpha_1|=m_1}\int_{(\tilde{Q})^c}\frac{|R_{m_2}(\tilde{b}_2;x_0,y)||(x_0-y)^{\alpha_1}|}{|x_0-y|^m}|K(x-y)-K(x_0-y)||D^{\alpha_1}\tilde{b}_1(y)||f(y)|_r dy\\
\leq\;& C\sum_{|\alpha|=m_2}||D^{\alpha}b_2||_{BMO}\sum_{|\alpha_1|=m_1}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k\frac{|x-x_0|}{|x_0-y|}|K(x-y)||D^{\alpha_1}\tilde{b}_1(y)||f(y)|_r dy\\
&+C\sum_{|\alpha|=m_2}||D^{\alpha}b_2||_{BMO}\sum_{|\alpha_1|=m_1}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k|K(x-y)-K(x_0-y)||D^{\alpha_1}\tilde{b}_1(y)||f(y)|_r dy\\
\leq\;& C\sum_{|\alpha|=m_2}||D^{\alpha}b_2||_{BMO}\sum_{|\alpha_1|=m_1}\sum_{k=0}^{\infty}k2^{-k}\left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|K(x-y)|^{s_1}dy\right)^{1/s_1}\\
&\times\left(\int_{2^{k+1}\tilde{Q}}|D^{\alpha_1}\tilde{b}_1(y)|^{s_2}dy\right)^{1/s_2}\left(\int_{2^{k+1}\tilde{Q}}|f(y)|_r^s dy\right)^{1/s}\\
&+C\sum_{|\alpha|=m_2}||D^{\alpha}b_2||_{BMO}\sum_{|\alpha_1|=m_1}\sum_{k=0}^{\infty}k\left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|K(x-y)-K(x_0-y)|^{s_1}dy\right)^{1/s_1}\\
&\times\left(\int_{2^{k+1}\tilde{Q}}|D^{\alpha_1}\tilde{b}_1(y)|^{s_2}dy\right)^{1/s_2}\left(\int_{2^{k+1}\tilde{Q}}|f(y)|_r^s dy\right)^{1/s}\\
\leq\;& C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}b_j||_{BMO}\right)\sum_{k=1}^{\infty}k^2(2^{-k}+2^{-ka})\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|_r^s dy\right)^{1/s}\\
\leq\;& C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}b_j||_{BMO}\right)M_s(|f|_r)(\tilde{x}).
\end{aligned}$$

Similarly,

$$\left(\sum_{i=1}^{\infty}||I_5^{(6)}||^r\right)^{1/r} \leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)M_s(|f|_r)(\tilde{x}).$$

For $I_5^{(7)}$, similar to the proofs of $I_5^{(5)}$ and I_4, we get, for $1 < s_1, s_2, s_3 < \infty$ with $1/s_1 + 1/s_2 + 1/s_3 + 1/s = 1$,

$$\begin{aligned}
&\left(\sum_{i=1}^{\infty}||I_5^{(7)}||^r\right)^{1/r} \\
\leq\ & C\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\int_{(\tilde{Q})^c}\left|\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m}\right||K(x-y)D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)||f(y)|_r dy \\
&+C\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\int_{(\tilde{Q})^c}\frac{|(x_0-y)^{\alpha_1+\alpha_2}|}{|x_0-y|^m}|K(x-y)-K(x_0-y)||D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)||f(y)|_r dy \\
\leq\ & C\sum_{|\alpha_1|=m_1,\ |\alpha|=m_2}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0|}{|x_0-y|}|K(x-y)||D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)||f(y)|_r dy \\
&+C\sum_{|\alpha_1|=m_1,\ |\alpha|=m_2}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|K(x-y)-K(x_0-y)||D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)||f(y)|_r dy \\
\leq\ & C\sum_{|\alpha_1|=m_1,\ |\alpha|=m_2}\sum_{k=0}^{\infty}2^{-k}\left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|K(x-y)|^{s_1}dy\right)^{1/s_1}\left(\int_{2^{k+1}\tilde{Q}}|D^{\alpha_1}\tilde{b}_1(y)|^{s_2}dy\right)^{1/s_2} \\
&\times\left(\int_{2^{k+1}\tilde{Q}}|D^{\alpha_2}\tilde{b}_2(y)|^{s_3}dy\right)^{1/s_3}\left(\int_{2^{k+1}\tilde{Q}}|f(y)|_r^s dy\right)^{1/s} \\
&+C\sum_{|\alpha_1|=m_1,\ |\alpha|=m_2}\sum_{k=0}^{\infty}\left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|K(x-y)-K(x_0-y)|^{s_1}dy\right)^{1/s_1} \\
&\times\left(\int_{2^{k+1}\tilde{Q}}|D^{\alpha_1}\tilde{b}_1(y)|^{s_2}dy\right)^{1/s_2}\left(\int_{2^{k+1}\tilde{Q}}|D^{\alpha_2}\tilde{b}_2(y)|^{s_3}dy\right)^{1/s_3}\left(\int_{2^{k+1}\tilde{Q}}|f(y)|_r^s dy\right)^{1/s} \\
\leq\ & C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}b_j||_{BMO}\right)\sum_{k=1}^{\infty}k^2(2^{-k}+2^{-ka})\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|_r^s dy\right)^{1/s} \\
\leq\ & C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}b_j||_{BMO}\right)M_s(|f|_r)(\tilde{x}).
\end{aligned}$$

Thus

$$I_5 \leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}b_j||_{BMO}\right)M_s(|f|_r)(\tilde{x}).$$

This completes the proof of Theorem 1.

Proof of Theorem 2. We choose $1 < s < p$ in Theorem 1 and by [8], we get

$$||T_b(f)||_{L^p(w)} \leq ||M(T_b(f))||_{L^p(w)} \leq C||(T_b(f))^{\#}||_{L^p(w)}$$

$$\leq C \prod_{j=1}^{l} \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO} \right) ||M_s(|f|_r)||_{L^p(w)}$$
$$\leq C \prod_{j=1}^{l} \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO} \right) |||f|_r||_{L^p(w)}.$$

This finishes the proof.

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