A Functional Associated with Two Bounded Linear Operators in Hilbert Spaces and Related Inequalities

A FUNCTIONAL ASSOCIATED WITH TWO BOUNDED LINEAR OPERATORS IN HILBERT SPACES AND RELATED INEQUALITIES

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Abstract. In this paper several inequalities for the functional $\mu(A,B) := \sup_{\|x\|=1}\{\|Ax\|\,\|Bx\|\}$ under various assumptions for the operators involved, including operators satisfying the uniform (α,β)-property and operators for which the transform $C_{\alpha,\beta}(\cdot,\cdot)$ is accretive, are given.

1. Introduction

Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [9, p. 1]:

$$W(T) = \{\langle Tx,x\rangle,\ x\in H,\ \|x\|=1\}.$$

The *numerical radius* $w(T)$ of an operator T on H is given by [9, p. 8]:

$$w(T) = \sup\{|\lambda|, \lambda\in W(T)\} = \sup\{|\langle Tx,x\rangle|, \|x\|=1\}. \tag{1.1}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T: H\to H$. This norm is equivalent to the operator norm. In fact, the following more precise result holds [9, p. 9]:

$$w(T)\le\|T\|\le 2w(T), \tag{1.2}$$

for any $T\in B(H)$

For other results on numerical radii, see [10], Chapter 11. For some recent and interesting results concerning inequalities for the numerical radius, see [11] and [12].

If A,B are two bounded linear operators on the Hilbert space $(H,\langle\cdot,\cdot\rangle)$, then

$$w(AB)\le 4w(A)\,w(B). \tag{1.3}$$

In the case that $AB=BA$, then

$$w(AB)\le 2w(A)\,w(B). \tag{1.4}$$

The following results are also well known [9, p. 38]:

If A is a unitary operator that commutes with another operator B, then

$$w(AB)\le w(B). \tag{1.5}$$

If A is an isometry and $AB=BA$, then (1.5) also holds true.

We say that A and B *double commute* if $AB=BA$ and $AB^*=B^*A$. If the operators A and B double commute, then [9, p. 38]

$$w(AB)\le w(B)\,\|A\|. \tag{1.6}$$

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As a consequence of the above, we have [9, p. 39]:

Let A be a normal operator commuting with B, then

$$w(AB) \leq w(A) w(B). \tag{1.7}$$

For other results and historical comments on the above see [9, p. 39–41].

For two bounded linear operators A, B in the Hilbert space $(H, \langle \cdot, \cdot \rangle)$ we define the functional

$$\mu(A, B) := \sup_{\|x\|=1} \{\|Ax\| \|Bx\|\} (\geq 0). \tag{1.8}$$

It is obvious that μ is symmetric and sub-additive in each variable, $\mu(A, A) = \|A\|^2$, $\mu(A, I) = \|A\|$, where I is the identity operator, $\mu(\alpha A, \beta B) = |\alpha\beta| \mu(A, B)$ and $\mu(A, B) \leq \|A\| \|B\|$. We also have the following inequalities

$$\mu(A, B) \geq w(B^* A) \tag{1.9}$$

and

$$\mu(A, B) \|A\| \|B\| \geq \mu(AB, BA). \tag{1.10}$$

The inequality (1.9) follows by the Schwarz inequality $\|Ax\| \|Bx\| \geq |\langle Ax, Bx \rangle|$, $x \in H$, while (1.10) can be obtained by multiplying the inequalities $\|ABx\| \leq \|A\| \|Bx\|$ and $\|BAx\| \leq \|B\| \|Ax\|$.

From (1.9) we also get

$$\|A\|^2 \geq \mu(A, A^*) \geq w\left(A^2\right) \tag{1.11}$$

for any A.

Motivated by the above results we establish in this paper several inequalities for the functional $\mu(\cdot, \cdot)$ under various assumptions for the operators involved, including operators satisfying the uniform (α, β) −property and operators for which the transform $C_{\alpha,\beta}(\cdot, \cdot)$ is accretive.

2. General Inequalities

The following result concerning some general power operator inequalities may be stated:

Theorem 1. *For any $A, B \in B(H)$ and $r \geq 1$ we have the inequality*

$$\mu^r(A, B) \leq \frac{1}{2} \|(A^* A)^r + (B^* B)^r\|. \tag{2.1}$$

The constant $\frac{1}{2}$ is best possible.

Proof. Utilising the arithmetic mean - geometric mean inequality and the convexity of the function $f(t) = t^r$ for $r \geq 1$ we have successively

$$\begin{aligned} \|Ax\| \|Bx\| &\leq \frac{1}{2} [\langle A^* Ax, x \rangle + \langle B^* Bx, x \rangle] \\ &\leq \left[\frac{\langle A^* Ax, x \rangle^r + \langle B^* Bx, x \rangle^r}{2}\right]^{\frac{1}{r}}, \end{aligned} \tag{2.2}$$

for any $x \in H$.

It is well known that, if P is a positive operator, then for any $r \geq 1$ and $x \in H$ with $\|x\| = 1$ we have the inequality (see for instance [13])

$$\langle Px, x \rangle^r \leq \langle P^r x, x \rangle. \tag{2.3}$$

Applying this inequality to the positive operators A^*A and B^*B we deduce that

$$\left[\frac{\langle A^*Ax,x\rangle^r+\langle B^*Bx,x\rangle^r}{2}\right]^{\frac{1}{r}}\leq\left\langle\frac{\left[(A^*A)^r+(B^*B)^r\right]x}{2},x\right\rangle^{\frac{1}{r}}, \tag{2.4}$$

for any $x\in H$ with $\|x\|=1$.

Now, on making use of the inequalities (2.2) and (2.4) we get

$$\|Ax\|\,\|Bx\|\leq\left\langle\frac{\left[(A^*A)^r+(B^*B)^r\right]x}{2},x\right\rangle^{\frac{1}{r}}, \tag{2.5}$$

for any $x\in H$ with $\|x\|=1$. Taking the supremum over $x\in H$ with $\|x\|=1$ we obtain the desired result (2.1).

For $r=1$ and $B=A$ we get in both sides of (2.1) the same quantity $\|A\|^2$ which shows that the constant $\frac{1}{2}$ is best possible in general in the inequality (2.1). ∎

Corollary 1. *For any $A\in B(H)$ and $r\geq1$ we have the inequality*

$$\mu^r(A,A^*)\leq\frac{1}{2}\left\|(A^*A)^r+(AA^*)^r\right\| \tag{2.6}$$

and the inequality

$$\|A\|^r\leq\frac{1}{2}\left\|(A^*A)^r+I\right\|, \tag{2.7}$$

respectively.

The following similar result for powers of operators can be stated as well:

Theorem 2. *For any $A,B\in B(H)$, any $\alpha\in(0,1)$ and $r\geq1$ we have the inequality*

$$\mu^{2r}(A,B)\leq\left\|\alpha\cdot(A^*A)^{r/\alpha}+(1-\alpha)\cdot(B^*B)^{r/(1-\alpha)}\right\|. \tag{2.8}$$

The inequality is sharp.

Proof. Observe that, for any $\alpha\in(0,1)$ we have

$$\begin{aligned}\|Ax\|^2\|Bx\|^2&=\langle(A^*A)x,x\rangle\langle(B^*B)x,x\rangle\\&=\left\langle\left[(A^*A)^{1/\alpha}\right]^\alpha x,x\right\rangle\left\langle\left[(B^*B)^{1/(1-\alpha)}\right]^{1-\alpha}x,x\right\rangle,\end{aligned} \tag{2.9}$$

where $x\in H$.

It is well known that (see for instance [13]), if P is a positive operator and $q\in(0,1)$, then

$$\langle P^qx,x\rangle\leq\langle Px,x\rangle^q. \tag{2.10}$$

Applying this property to the positive operators $(A^*A)^{1/\alpha}$ and $(B^*B)^{1/(1-\alpha)}$, where $\alpha\in(0,1)$, we have

$$\begin{aligned}&\left\langle\left[(A^*A)^{1/\alpha}\right]^\alpha x,x\right\rangle\left\langle\left[(B^*B)^{1/(1-\alpha)}\right]^{1-\alpha}x,x\right\rangle\\&\qquad\leq\left\langle(A^*A)^{1/\alpha}x,x\right\rangle^\alpha\left\langle(B^*B)^{1/(1-\alpha)}x,x\right\rangle^{1-\alpha},\end{aligned} \tag{2.11}$$

for any $x\in H$ with $\|x\|=1$.

Now, on utilising the weighted arithmetic mean-geometric mean inequality, i.e.,

$$a^{\alpha}b^{1-\alpha}\leq\alpha a+(1-\alpha)b,\quad\text{where }\alpha\in(0,1)\text{ and }a,b\geq0;$$

we get

$$\left\langle (A^*A)^{1/\alpha}x,x\right\rangle^{\alpha}\left\langle (B^*B)^{1/(1-\alpha)}x,x\right\rangle^{1-\alpha}\leq\alpha\cdot\left\langle (A^*A)^{1/\alpha}x,x\right\rangle+(1-\alpha)\cdot\left\langle (B^*B)^{1/(1-\alpha)}x,x\right\rangle, \tag{2.12}$$

for any $x\in H$ with $\|x\|=1$.

Moreover, by the elementary inequality

$$\alpha a+(1-\alpha)b\leq(\alpha a^r+(1-\alpha)b^r)^{1/r},\quad\text{where }\alpha\in(0,1)\text{ and }a,b\geq0;$$

we have successively

$$\begin{aligned}&\alpha\cdot\left\langle (A^*A)^{1/\alpha}x,x\right\rangle+(1-\alpha)\cdot\left\langle (B^*B)^{1/(1-\alpha)}x,x\right\rangle\\&\leq\left[\alpha\cdot\left\langle (A^*A)^{1/\alpha}x,x\right\rangle^{r}+(1-\alpha)\cdot\left\langle (B^*B)^{1/(1-\alpha)}x,x\right\rangle^{r}\right]^{\frac{1}{r}}\\&\leq\left[\alpha\cdot\left\langle (A^*A)^{r/\alpha}x,x\right\rangle+(1-\alpha)\cdot\left\langle (B^*B)^{r/(1-\alpha)}x,x\right\rangle\right]^{\frac{1}{r}},\end{aligned} \tag{2.13}$$

for any $x\in H$ with $\|x\|=1$, where for the last inequality we have used the property (2.3) for the positive operators $(A^*A)^{1/\alpha}$ and $(B^*B)^{1/(1-\alpha)}$.

Now, on making use of the identity (2.9) and the inequalities (2.11)-(2.13) we get

$$\|Ax\|^2\|Bx\|^2\leq\left[\left\langle\left[\alpha\cdot(A^*A)^{r/\alpha}+(1-\alpha)\cdot(B^*B)^{r/(1-\alpha)}\right]x,x\right\rangle\right]^{\frac{1}{r}},$$

for any $x\in H$ with $\|x\|=1$. Taking the supremum over $x\in H$ with $\|x\|=1$ we deduce the desired result (2.8).

Notice that the inequality is sharp since for $r=1$ and $B=A$ we get in both sides of (2.8) the same quantity $\|A\|^4$. ∎

Corollary 2. *For any $A\in B(H)$, any $\alpha\in(0,1)$ and $r\geq1$, we have the inequalities*

$$\mu^{2r}(A,A^*)\leq\left\|\alpha\cdot(A^*A)^{r/\alpha}+(1-\alpha)\cdot(AA^*)^{r/(1-\alpha)}\right\|,$$

$$\|A\|^{2r}\leq\left\|\alpha\cdot(A^*A)^{r/\alpha}+(1-\alpha)\cdot I\right\|$$

and

$$\|A\|^{4r}\leq\left\|\alpha\cdot(A^*A)^{r/\alpha}+(1-\alpha)\cdot(A^*A)^{r/(1-\alpha)}\right\|,$$

respectively.

The following reverse of the inequality (1.9) maybe stated as well:

Theorem 3. *For any $A,B\in B(H)$ we have the inequality*

$$(0\leq)\,\mu(A,B)-w(B^*A)\leq\frac{1}{2}\|A-B\|^2 \tag{2.14}$$

and the inequality

$$\mu\left(\frac{A+B}{2},\frac{A-B}{2}\right)\leq\frac{1}{2}w(B^*A)+\frac{1}{2}\|A-B\|^2, \tag{2.15}$$

respectively.

Proof. We have

$$\begin{aligned} \|Ax - Bx\|^2 &= \|Ax\|^2 + \|Bx\|^2 - 2\operatorname{Re}\langle B^*Ax, x\rangle \\ &\geq 2\|Ax\|\|Bx\| - 2|\langle B^*Ax, x\rangle|, \end{aligned} \tag{2.16}$$

for any $x \in H, \|x\| = 1$, which gives the inequality

$$\|Ax\|\|Bx\| \leq |\langle B^*Ax, x\rangle| + \frac{1}{2}\|Ax - Bx\|^2,$$

for any $x \in H, \|x\| = 1$.

Taking the supremum over $\|x\| = 1$ we deduce the desired result (2.14).

By the parallelogram identity in the Hilbert space H we also have

$$\begin{aligned} \|Ax\|^2 + \|Bx\|^2 &= \frac{1}{2}\left(\|Ax + Bx\|^2 + \|Ax - Bx\|^2\right) \\ &\geq \|Ax + Bx\|\|Ax - Bx\|, \end{aligned}$$

for any $x \in H$.

Combining this inequality with the first part of (2.16) we get

$$\|Ax + Bx\|\|Ax - Bx\| \leq \|Ax - Bx\|^2 + 2|\langle B^*Ax, x\rangle|,$$

for any $x \in H$. Taking the supremum in this inequality over $\|x\| = 1$ we deduce the desired result (2.15). ∎

Corollary 3. *Let $A \in B(H)$. If $\operatorname{Re}(A) := \frac{A+A^*}{2}$ and $\operatorname{Im}(A) := \frac{A-A^*}{2i}$ are the real and imaginary parts of A, then we have the inequality*

$$(0 \leq)\, \mu(A, A^*) - w\left(A^2\right) \leq 2 \cdot \|\operatorname{Im}(A)\|^2$$

and

$$\mu(\operatorname{Re}(A), \operatorname{Im}(A)) \leq \frac{1}{2}w\left(A^2\right) + 2 \cdot \|\operatorname{Im}(A)\|^2,$$

respectively.

Moreover, we have

$$(0 \leq)\, \mu(\operatorname{Re}(A), \operatorname{Im}(A)) - w(\operatorname{Re}(A)\operatorname{Im}(A)) \leq \frac{1}{2}\|A\|^2.$$

Corollary 4. *For any $A \in B(H)$ and $\lambda \in \mathbb{C}$ with $\lambda \neq 0$ we have the inequality (see also* [6]*)*

$$(0 \leq)\, \|A\| - w(A) \leq \frac{1}{2|\lambda|}\|A - \lambda I\|^2. \tag{2.17}$$

For a bounded linear operator T consider the quantity $\ell(T) := \inf_{\|x\|=1}\|Tx\|$. We can state the following result as well.

Theorem 4. *For any $A, B \in B(H)$ with $A \neq B$ and such that $\ell(B) \geq \|A - B\|$ we have*

$$(0 \leq)\, \mu^2(A, B) - w^2(B^*A) \leq \|A\|^2\|A - B\|^2. \tag{2.18}$$

Proof. Denote $r := \|A - B\| > 0$. Then for any $x \in H$ with $\|x\| = 1$ we have $\|Bx\| \geq r$ and by the first part of (2.16) we can write that

$$\|Ax\|^2 + \left(\sqrt{\|Bx\|^2 - r^2}\right)^2 \leq 2|\langle B^*Ax, x\rangle|, \tag{2.19}$$

for any $x \in H$ with $\|x\| = 1$.

On the other hand we have

$$\|Ax\|^2 + \left(\sqrt{\|Bx\|^2 - r^2}\right)^2 \geq 2 \cdot \|Ax\| \sqrt{\|Bx\|^2 - r^2}, \tag{2.20}$$

for any $x \in H$ with $\|x\| = 1$.

Combining (2.19) with (2.20) we deduce

$$\|Ax\| \sqrt{\|Bx\|^2 - r^2} \leq |\langle B^*Ax, x\rangle|$$

which is clearly equivalent with

$$\|Ax\|^2 \|Bx\|^2 \leq |\langle B^*Ax, x\rangle|^2 + \|Ax\|^2 \|A - B\|^2, \tag{2.21}$$

for any $x \in H$ with $\|x\| = 1$. Taking the supremum in (2.21) over $x \in H$ with $\|x\| = 1$, we deduce the desired inequality (2.18). ∎

Corollary 5. *For any $A \in B(H)$ a non self adjoint operator and such that $\ell(A^*) \geq 2 \cdot \|\mathrm{Im}(A)\|$ we have*

$$(0 \leq)\, \mu^2(A, A^*) - w^2\left(A^2\right) \leq 4 \cdot \|A\|^2 \|\mathrm{Im}(A)\|^2. \tag{2.22}$$

Corollary 6. *For any $A \in B(H)$ and $\lambda \in \mathbb{C}$ with $\lambda \neq 0$ and $|\lambda| \geq \|A - \lambda I\|$ we have the inequality (see also* [6]*)*

$$(0 \leq)\, \|A\|^2 - w^2(A) \leq \frac{1}{|\lambda|^2} \cdot \|A\|^2 \|A - \lambda I\|^2$$

or, equivalently,

$$(0 \leq) \sqrt{1 - \frac{\|A - \lambda I\|^2}{|\lambda|^2}} \leq \frac{w(A)}{\|A\|} (\leq 1).$$

3. Inequalities for Operators Satisfying the Uniform (α, β)-property

The following result that may be of interest in itself, holds:

Lemma 1. *Let $T \in B(H)$ and $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \beta$. The following statements are equivalent:*

(i) *We have*

$$\mathrm{Re}\,\langle \beta y - Tx, Tx - \alpha y\rangle \geq 0, \tag{3.1}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$;

(ii) *We have*

$$\left\| Tx - \frac{\alpha + \beta}{2} \cdot y \right\| \leq \frac{1}{2} |\alpha - \beta|, \tag{3.2}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Proof. Follows by the following identity

$$\mathrm{Re}\,\langle \beta y - Tx, Tx - \alpha y\rangle = \frac{1}{4} |\alpha - \beta|^2 - \left\| Tx - \frac{\alpha + \beta}{2} \cdot y \right\|^2,$$

that holds for any $x, y \in H$ with $\|x\| = \|y\| = 1$. ∎

Remark 1. *For any operator $T \in B(H)$ if we choose $\alpha = a\|T\|(1+2i)$ and $\beta = a\|T\|(1-2i)$ with $a \geq 1$, then*

$$\frac{\alpha+\beta}{2} = a\|T\| \quad \text{and} \quad \frac{|\alpha-\beta|}{2} = 2a\|T\|$$

showing that

$$\begin{aligned}\left\|Tx - \frac{\alpha+\beta}{2}\cdot y\right\| &\leq \|Tx\| + \left|\frac{\alpha+\beta}{2}\right| \leq \|T\| + a\|T\| \\ &\leq 2a\|T\| = \frac{1}{2}\cdot|\alpha-\beta|,\end{aligned}$$

that holds for any $x, y \in H$ with $\|x\| = \|y\| = 1$, i.e., T satisfies the condition (3.1) with the scalars α and β given above.

Definition 1. *For given $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \beta$ and $y \in H$ with $\|y\| = 1$, we say that the operator $T \in B(H)$ has the (α, β, y)-property if either (3.1) or, equivalently, (3.2) holds true for any $x \in H$ with $\|x\| = 1$. Moreover, if T has the (α, β, y)-property for any $y \in H$ with $\|y\| = 1$, then we say that this operator has the uniform (α, β)-property.*

Remark 2. *The above Remark 1 shows that any bounded linear operator has the uniform (α, β)-property for infinitely many (α, β) appropriately chosen. For a given operator satisfying an (α, β)-property, it is an open problem to find the possibly nonzero lower bound for the quantity $|\alpha - \beta|$.*

The following results may be stated:

Theorem 5. *Let $A, B \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ with $\alpha \neq \beta$ and $\gamma \neq \delta$. For $y \in H$ with $\|y\| = 1$ assume that A^* has the (α, β, y)-property while B^* has the (γ, δ, y)-property, then*

$$\left|\|Ay\|\|By\| - \|BA^*\|\right| \leq \frac{1}{4}|\beta-\alpha||\gamma-\delta|. \tag{3.3}$$

Moreover, if A^* has the uniform (α, β)-property and B^* has the uniform (γ, δ)-property, then

$$|\mu(A, B) - \|BA^*\|| \leq \frac{1}{4}|\beta-\alpha||\gamma-\delta|. \tag{3.4}$$

Proof. Since A^* has the (α, β, y)-property while B^* has the (γ, δ, y)-property, then on making use of Lemma 1 we have that

$$\left\|A^*x - \frac{\alpha+\beta}{2}\cdot y\right\| \leq \frac{1}{2}|\beta-\alpha|$$

and

$$\left\|B^*z - \frac{\gamma+\delta}{2}\cdot y\right\| \leq \frac{1}{2}|\gamma-\delta|$$

for any $x, z \in H$, with $\|x\| = \|z\| = 1$.

Now, we make use of the following Grüss type inequality for vectors in inner product spaces obtained by the author in [1] (see also [2] or [7, p. 43]):

Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $u, v, e \in H$, $\|e\| = 1$, and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ such that

$$\operatorname{Re}\langle\beta e - u, u - \alpha e\rangle \geq 0, \qquad \operatorname{Re}\langle\delta e - v, v - \gamma e\rangle \geq 0 \tag{3.5}$$

or, equivalently,

$$\left\| u - \frac{\alpha+\beta}{2} e \right\| \leq \frac{1}{2} |\beta - \alpha|, \qquad \left\| v - \frac{\gamma+\delta}{2} e \right\| \leq \frac{1}{2} |\delta - \gamma|, \tag{3.6}$$

then

$$|\langle u, v \rangle - \langle u, e \rangle \langle e, v \rangle| \leq \frac{1}{4} |\beta - \alpha| |\delta - \gamma|. \tag{3.7}$$

Applying (3.7) for $u = A^* x$, $v = B^* z$ and $e = y$ we deduce

$$|\langle BA^* x, z \rangle - \langle x, Ay \rangle \langle z, By \rangle| \leq \frac{1}{4} |\beta - \alpha| |\delta - \gamma|, \tag{3.8}$$

for any $x, z \in H$, $\|x\| = \|z\| = 1$, which is an inequality of interest in itself.

Observing that

$$||\langle BA^* x, z \rangle| - |\langle x, Ay \rangle \langle z, By \rangle|| \leq |\langle BA^* x, z \rangle - \langle x, Ay \rangle \langle z, By \rangle|,$$

then by (3.7) we deduce the inequality

$$||\langle BA^* x, z \rangle| - |\langle x, Ay \rangle \langle z, By \rangle|| \leq \frac{1}{4} |\beta - \alpha| |\delta - \gamma|,$$

for any $x, z \in H$, $\|x\| = \|z\| = 1$. This is equivalent with the following two inequalities

$$|\langle BA^* x, z \rangle| \leq |\langle x, Ay \rangle \langle z, By \rangle| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma| \tag{3.9}$$

and

$$|\langle x, Ay \rangle \langle z, By \rangle| \leq |\langle BA^* x, z \rangle| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma|, \tag{3.10}$$

for any $x, z \in H, \|x\| = \|z\| = 1$.

Taking the supremum over $x, z \in H$, $\|x\| = \|z\| = 1$ in (3.9) and (3.10) we get the inequalities

$$\|BA^*\| \leq \|Ay\| \|By\| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma| \tag{3.11}$$

and

$$\|Ay\| \|By\| \leq \|BA^*\| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma|, \tag{3.12}$$

which are clearly equivalent with (3.3).

Now, if A has the uniform (α, β)-property and B has the uniform (γ, δ)-property, then the inequalities (3.11) and (3.12) hold for any $y \in H$ with $\|y\| = 1$. Taking the supremum over $y \in H$ with $\|y\| = 1$ in these inequalities we deduce

$$\|BA^*\| \leq \mu(A, B) + \frac{1}{4} |\beta - \alpha| |\delta - \gamma|$$

and

$$\mu(A, B) \leq \|BA^*\| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma|$$

which are equivalent with (3.4). ∎

Corollary 7. *Let $A \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ with $\alpha \neq \beta$ and $\gamma \neq \delta$. For $y \in H$ with $\|y\| = 1$ assume that A has the (α, β, y)-property while A^* has the (γ, δ, y)-property, then*

$$\left|\|A^* y\| \|Ay\| - \left\|A^2\right\|\right| \leq \frac{1}{4} |\beta - \alpha| |\gamma - \delta| .$$

Moreover, if A has the uniform (α, β)-property and A^* has the uniform (γ, δ)-property, then

$$\left|\mu(A, A^*) - \left\|A^2\right\|\right| \leq \frac{1}{4} |\beta - \alpha| |\gamma - \delta| .$$

The following results may be stated as well:

Theorem 6. *Let $A, B \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ with $\alpha + \beta \neq 0$ and $\gamma + \delta \neq 0$. For $y \in H$ with $\|y\| = 1$ assume that A^* has the (α, β, y)-property while B^* has the (γ, δ, y)-property, then*

$$\begin{aligned} (3.13) \qquad & |\|Ay\| \|By\| - \|BA^*\|| \\ & \leq \frac{1}{4} \cdot \frac{|\beta - \alpha| |\delta - \gamma|}{\sqrt{|\beta + \alpha| |\delta + \gamma|}} \sqrt{(\|A\| + \|Ay\|)(\|B\| + \|By\|)}. \end{aligned}$$

Moreover, if A^* has the uniform (α, β)-property and B^* has the uniform (γ, δ)-property, then

$$(3.14) \qquad |\mu(A, B) - \|BA^*\|| \leq \frac{1}{2} \cdot \frac{|\beta - \alpha| |\delta - \gamma|}{\sqrt{|\beta + \alpha| |\delta + \gamma|}} \sqrt{\|A\| \|B\|}.$$

Proof. We make use of the following inequality obtained by the author in [5] (see also [7, p. 65]):

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $u, v, e \in H$, $\|e\| = 1$, and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ with $\alpha + \beta \neq 0$ and $\gamma + \delta \neq 0$ and such that

$$\operatorname{Re} \langle \beta e - u, u - \alpha e \rangle \geq 0, \qquad \operatorname{Re} \langle \delta e - v, v - \gamma e \rangle \geq 0$$

or, equivalently,

$$\left\| u - \frac{\alpha + \beta}{2} e \right\| \leq \frac{1}{2} |\beta - \alpha| , \left\| v - \frac{\gamma + \delta}{2} e \right\| \leq \frac{1}{2} |\delta - \gamma| ,$$

then

$$\begin{aligned} (3.15) \qquad & |\langle u, v \rangle - \langle u, e \rangle \langle e, v \rangle| \\ & \leq \frac{1}{4} \cdot \frac{|\beta - \alpha| |\delta - \gamma|}{\sqrt{|\beta + \alpha| |\delta + \gamma|}} \sqrt{(\|u\| + |\langle u, e \rangle|)((\|v\| + |\langle v, e \rangle|)}. \end{aligned}$$

Applying (3.15) for $u = A^* x$, $v = B^* z$ and $e = y$ we deduce

$$\begin{aligned} & |\langle BA^* x, z \rangle - \langle x, Ay \rangle \langle z, By \rangle| \\ & \leq \frac{1}{4} \cdot \frac{|\beta - \alpha| |\delta - \gamma|}{\sqrt{|\beta + \alpha| |\delta + \gamma|}} \sqrt{(\|A^* x\| + |\langle x, Ay \rangle|)((\|B^* z\| + |\langle z, By \rangle|)}, \end{aligned}$$

for any $x, y, z \in H$, $\|x\| = \|y\| = \|z\| = 1$.

Now, on making use of a similar argument to the one from the proof of Theorem 5, we deduce the desired results (3.13) and (3.14). The details are omitted. ∎

Corollary 8. *Let $A \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ with $\alpha \neq \beta$ and $\gamma \neq \delta$. For $y \in H$ with $\|y\| = 1$ assume that A has (α, β, y)-property while A^* has the (γ, δ, y)-property, then*

$$\left| \|A^* y\| \|Ay\| - \|A^2\| \right| \leq \frac{1}{4} \cdot \frac{|\beta - \alpha| |\delta - \gamma|}{\sqrt{|\beta + \alpha| |\delta + \gamma|}} \sqrt{(\|A\| + \|A^* y\|)(\|A\| + \|Ay\|)}.$$

Moreover, if A has the uniform (α, β)-property and A^* has the uniform (γ, δ)-property, then

$$\left| \mu(A, A^*) - \|A^2\| \right| \leq \frac{1}{2} \cdot \frac{|\beta - \alpha| |\delta - \gamma|}{\sqrt{|\beta + \alpha| |\delta + \gamma|}} \|A\|.$$

4. The Transform $C_{\alpha,\beta}(\cdot, \cdot)$ and Other Inequalities

For two given operators $T, U \in B(H)$ and two given scalars $\alpha, \beta \in \mathbb{C}$ consider the transform

$$C_{\alpha,\beta}(T, U) = (T^* - \bar{\alpha} U^*)(\beta U - T).$$

This transform generalizes the transform $C_{\alpha,\beta}(T) := (T^* - \bar{\alpha} I)(\beta I - T) = C_{\alpha,\beta}(T, I)$, where I is the identity operator, which has been introduced in [8] in order to provide some generalizations of the well known Kantorovich inequality for operators in Hilbert spaces.

We recall that a bounded linear operator T on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ is called *accretive* if $\operatorname{Re} \langle Ty, y \rangle \geq 0$ for any $y \in H$.

Utilizing the following identity

$$\begin{aligned} \operatorname{Re} \langle C_{\alpha,\beta}(T, U) x, x \rangle &= \operatorname{Re} \langle C_{\beta,\alpha}(T, U) x, x \rangle \\ &= \frac{1}{4} |\beta - \alpha|^2 \|Ux\|^2 - \left\| Tx - \frac{\alpha + \beta}{2} \cdot Ux \right\|^2, \end{aligned} \tag{4.1}$$

that holds for any scalars α, β and any vector $x \in H$, we can give a simple characterization result that is useful in the following:

Lemma 2. *For $\alpha, \beta \in \mathbb{C}$ and $T, U \in B(H)$ the following statements are equivalent:*

(i) *The transform $C_{\alpha,\beta}(T, U)$ (or, equivalently, $C_{\beta,\alpha}(T, U)$) is accretive;*

(ii) *We have the norm inequality*

$$\left\| Tx - \frac{\alpha + \beta}{2} \cdot Ux \right\| \leq \frac{1}{2} |\beta - \alpha| \|Ux\|, \tag{4.2}$$

for any $x \in H$.

As a consequence of the above lemma we can state

Corollary 9. *Let $\alpha, \beta \in \mathbb{C}$ and $T, U \in B(H)$. If $C_{\alpha,\beta}(T, U)$ is accretive, then*

$$\left\| T - \frac{\alpha + \beta}{2} \cdot U \right\| \leq \frac{1}{2} |\beta - \alpha| \|U\|. \tag{4.3}$$

Remark 3. *In order to give examples of operators $T, U \in B(H)$ and numbers $\alpha, \beta \in \mathbb{C}$ such that the transform $C_{\alpha,\beta}(T, U)$ is accretive, it suffices to select two bounded linear operator S and V and the complex numbers z, w $(w \neq 0)$ with the property that $\|Sx - zVx\| \leq |w| \|Vx\|$ for any $x \in H$, and, by choosing $T = S$, $U = V$, $\alpha = \frac{1}{2}(z + w)$ and $\beta = \frac{1}{2}(z - w)$ we observe that T and U satisfy (4.2), i.e., $C_{\alpha,\beta}(T, U)$ is accretive.*

We are able now to give the following result concerning other reverse inequalities for the case when the involved operators satisfy the accretivity property described above.

Theorem 7. *Let $\alpha, \beta \in \mathbb{C}$ and $A, B \in B(H)$. If $C_{\alpha,\beta}(A,B)$ is accretive, then*

$$(0 \leq)\, \mu^2(A,B) - w^2(B^*A) \leq \frac{1}{4} \cdot |\beta - \alpha|^2 \|B\|^4 . \tag{4.4}$$

Moreover, if $\alpha + \beta \neq 0$, then

$$(0 \leq)\, \mu(A,B) - w(B^*A) \leq \frac{1}{4} \cdot \frac{|\beta - \alpha|^2}{|\beta + \alpha|} \|B\|^2 . \tag{4.5}$$

In addition, if $\operatorname{Re}\left(\alpha\bar{\beta}\right) > 0$, then also

$$(1 \leq)\, \frac{\mu(A,B)}{w(B^*A)} \leq \frac{1}{2} \cdot \frac{|\beta + \alpha|}{\sqrt{\operatorname{Re}\left(\alpha\bar{\beta}\right)}} \tag{4.6}$$

and

$$(0 \leq)\, \mu^2(A,B) - w^2(B^*A) \leq \left(|\beta + \alpha| - 2 \cdot \sqrt{\operatorname{Re}\left(\alpha\bar{\beta}\right)}\right) w(B^*A) \|B\|^2 , \tag{4.7}$$

respectively.

Proof. By Lemma 2, since $C_{\alpha,\beta}(A,B)$ is accretive, then

$$\left\| Ax - \frac{\alpha + \beta}{2} \cdot Bx \right\| \leq \frac{1}{2} |\beta - \alpha| \|Bx\| , \tag{4.8}$$

for any $x \in H$.

We utilize the following reverse of the Schwarz inequality in inner product spaces obtained by the author in [3] (see also [7, p. 4]):

If $\gamma, \Gamma \in \mathbb{K}$ ($\mathbb{K} = \mathbb{C}, \mathbb{R}$) and $u, v \in H$ are such that

$$\operatorname{Re}\langle \Gamma v - u, u - \gamma v \rangle \geq 0 \tag{4.9}$$

or, equivalently,

$$\left\| u - \frac{\gamma + \Gamma}{2} \cdot v \right\| \leq \frac{1}{2} |\Gamma - \gamma| \|v\| , \tag{4.10}$$

then

$$0 \leq \|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2 \|v\|^4 . \tag{4.11}$$

Now, on making use of (4.11) for $u = Ax$, $v = Bx$, $x \in H$, $\|x\| = 1$ and $\gamma = \alpha, \Gamma = \beta$ we can write the inequality

$$\|Ax\|^2 \|Bx\|^2 \leq |\langle B^*Ax, x \rangle|^2 + \frac{1}{4} |\beta - \alpha|^2 \|Bx\|^4 ,$$

for any $x \in H$, $\|x\| = 1$. Taking the supremum over $\|x\| = 1$ in this inequality produces the desired result (4.4).

Now, by utilizing the result from [5] (see also [7, p. 29]), namely:

If $\gamma, \Gamma \in \mathbb{K}$ with $\gamma + \Gamma \neq 0$ and $u, v \in H$ are such that either (4.9) or, equivalently, (4.9) holds true, then

$$0 \leq \|u\| \|v\| - |\langle u, v \rangle| \leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \|v\|^2 . \tag{4.12}$$

Now, on making use of (4.12) for $u = Ax$, $v = Bx$, $x \in H$, $\|x\| = 1$ and $\gamma = \alpha, \Gamma = \beta$ and using the same procedure outlined above, we deduce the second inequality (4.5).

The inequality (4.6) follows from the result presented below obtained in [4] (see also [7, p. 21]):

If $\gamma, \Gamma \in \mathbb{K}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$ and $u, v \in H$ are such that either (4.9) or, equivalently, (4.9) holds true, then

$$\|u\| \|v\| \leq \frac{1}{2} \cdot \frac{|\Gamma + \gamma|}{\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}} |\langle u, v\rangle| , \tag{4.13}$$

by choosing $u = Ax$, $v = Bx$, $x \in H$, $\|x\| = 1$ and $\gamma = \alpha, \Gamma = \beta$ and taking the supremum over $\|x\| = 1$.

Finally, on making use of the inequality (see [6])

$$\|u\|^2 \|v\|^2 - |\langle u, v\rangle|^2 \leq \left(|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right) |\langle u, v\rangle| \|v\|^2 \tag{4.14}$$

that is valid provided $\gamma, \Gamma \in \mathbb{K}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$ and $u, v \in H$ are such that either (4.9) or, equivalently, (4.9) holds true, we obtain the last inequality (4.7). The details are omitted. ∎

Remark 4. *Let $M > m > 0$ and $A, B \in B(H)$. If $C_{m,M}(A, B)$ is accretive, then*

$$(0 \leq)\, \mu^2(A, B) - w^2(B^*A) \leq \frac{1}{4} \cdot (M - m)^2 \|B\|^4 ,$$

$$(0 \leq)\, \mu(A, B) - w(B^*A) \leq \frac{1}{4} \cdot \frac{(M - m)^2}{m + M} \|B\|^2 ,$$

$$(1 \leq)\, \frac{\mu(A, B)}{w(B^*A)} \leq \frac{1}{2} \cdot \frac{m + M}{\sqrt{mM}}$$

and

$$(0 \leq)\, \mu^2(A, B) - w^2(B^*A) \leq \left(\sqrt{M} - \sqrt{m}\right)^2 w(B^*A) \|B\|^2 ,$$

respectively.

Corollary 10. *Let $\alpha, \beta \in \mathbb{C}$ and $A \in B(H)$. If $C_{\alpha,\beta}(A, A^*)$ is accretive, then*

$$(0 \leq)\, \mu^2(A, A^*) - w^2\left(A^2\right) \leq \frac{1}{4} \cdot |\beta - \alpha|^2 \|A\|^4 .$$

Moreover, if $\alpha + \beta \neq 0$, then

$$(0 \leq)\, \mu(A, A^*) - w\left(A^2\right) \leq \frac{1}{4} \cdot \frac{|\beta - \alpha|^2}{|\beta + \alpha|} \|A\|^2 .$$

In addition, if $\operatorname{Re}\left(\alpha\bar{\beta}\right) > 0$, then also

$$(1 \leq)\, \frac{\mu(A, A^*)}{w(A^2)} \leq \frac{1}{2} \cdot \frac{|\beta + \alpha|}{\sqrt{\operatorname{Re}\left(\alpha\bar{\beta}\right)}}$$

and

$$(0 \leq)\, \mu^2(A, A^*) - w^2\left(A^2\right) \leq \left(|\beta + \alpha| - 2 \cdot \sqrt{\operatorname{Re}\left(\alpha\bar{\beta}\right)}\right) w\left(A^2\right) \|A\|^2 ,$$

respectively.

Remark 5. *In a similar manner, if $N > n > 0$, $A \in B(H)$ and $C_{n,N}(A, A^*)$ is accretive, then*

$$(0 \leq)\, \mu^2 (A, A^*) - w^2 \left(A^2\right) \leq \frac{1}{4} \cdot (N-n)^2 \|A\|^4 ,$$

$$(0 \leq)\, \mu (A, A^*) - w \left(A^2\right) \leq \frac{1}{4} \cdot \frac{(N-n)^2}{n+N} \|A\|^2 ,$$

$$(1 \leq)\, \frac{\mu (A, A^*)}{w (A^2)} \leq \frac{1}{2} \cdot \frac{n+N}{\sqrt{nN}}$$

and

$$(0 \leq)\, \mu^2 (A, A^*) - w^2 \left(A^2\right) \leq \left(\sqrt{N} - \sqrt{n}\right)^2 w \left(A^2\right) \|A\|^2 ,$$

respectively.

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