Reverse Triangle Inequality for Hilbert C-Modules*

REVERSE TRIANGLE INEQUALITY FOR HILBERT C^*-MODULES

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Abstract. We prove several versions of reverse triangle inequality in Hilbert C^*-modules based on some works of S. S. Dragomir. In particular, we show that if $e_1, \cdots, e_m$ are vectors in a Hilbert module $\mathfrak{X}$ over a C^*-algebra $\mathfrak{A}$ such that $\langle e_i, e_j\rangle = 0$ $(1 \le i \neq j \le m)$ and $\|e_i\| = 1$ $(1 \le i \le m)$, and also $r_k, \rho_k \in \mathbb{R}$ $(1 \le k \le m)$ and $x_1, \cdots, x_n \in \mathfrak{X}$ satisfy

$$0 \le r_k^2\|x_j\| \le Re\langle r_k e_k, x_j\rangle\,, \quad 0 \le \rho_k^2\|x_j\| \le Im\langle \rho_k e_k, x_j\rangle\,,$$

then

$$\left[\sum_{k=1}^{m}(r_k^2+\rho_k^2)\right]^{\frac{1}{2}}\sum_{j=1}^{n}\|x_j\| \le \left\|\sum_{j=1}^{n}x_j\right\|\,,$$

and the equality holds if and only if

$$\sum_{j=1}^{n}x_j = \sum_{j=1}^{n}\|x_j\|\sum_{k=1}^{m}(r_k+i\rho_k)e_k\,.$$

1. Introduction and preliminaries

The triangle inequality is one of the most fundamental inequalities in mathematics. Several mathematician have been investigated its generalizations and reverses.
Petrovitch [15] in 1917, proved that for complex number $z_1, \cdots, z_n$, we have

$$|\sum_{j=1}^{n}z_j| \ge \cos\theta\sum_{j=1}^{n}|z_j|\,,$$

where $0 < \theta < \frac{\pi}{2}$ and $\alpha - \theta < \arg z_j < \alpha + \theta$ $(1 \le j \le n)$ for a real number α.
This inequality can be found also in Karamata's book [8]. The first generalization of the triangle inequality in Hilbert space was given by Diaz and Matcalf [4]. They proved that for $x_1, \cdots, x_n$ in a Hilbert space H, if e be a unit vector H such that $0 \le r \le \frac{\mathrm{Re}\langle x_j, e\rangle}{\|x_j\|}$ for some $r \in \mathbb{R}$ and each $1 \le j \le n$, then

$$r\sum_{j=1}^{n}\|x_j\| \le \|\sum_{j=1}^{n}x_j\|\,,$$

2000 *Mathematics Subject Classification.* Primary 46L08; Secondary 15A39, 26D15, 46L05, 51M16.
Key words and phrases. Triangle inequality, Reverse inequality, Hilbert C^*-module.

and the equality holds if and only if $\sum_{j=1}^{n} x_j = r \sum_{j=1}^{n} \|x_j\| e$.

Recently, a number of mathematicians have represented several refinements of the reverse triangle inequality in Hilbert spaces and normed spaces. See [1, 2, 3, 6, 7, 9, 12, 14].

Our aim is to give some generalizations of results in Hilbert spaces to the framework of Hilbert C^*-modules. For this purpose, first we recall some fundamental definitions in the theory of Hilbert modules. We also use the elementary C^*-algebra theory, in particular we utilize this property that if $a \leq b$ then $a^{1/2} \leq b^{1/2}$, where a, b are positive elements of a C^*-algebra $\mathfrak{A}$. We also repeatedly apply the following known relation:

$$\frac{1}{2}(aa^* + a^*a) = (\mathrm{Re}a)^2 + (\mathrm{Im}a)^2\,, \tag{$\diamond$}$$

where a is an arbitrary element of $\mathfrak{A}$. For details on C^*-algebra theory we referred the readers to [13].

Suppose that $\mathfrak{A}$ is a C^*-algebra and $\mathfrak{X}$ is a linear space which is an algebraic right $\mathfrak{A}$-module. The space $\mathfrak{X}$ is called a pre-Hilbert $\mathfrak{A}$-module (or an inner product $\mathfrak{A}$-module) if there exists an $\mathfrak{A}$-valued inner product $\langle .,. \rangle : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{A}$ with the following properties:
(i) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$
(ii) $\langle x, \lambda y + z \rangle = \lambda \langle x, y \rangle + \langle x, z \rangle$
(iii) $\langle x, ya \rangle = \langle x, y \rangle a$
(iv) $\langle x, y \rangle^* = \langle y, x \rangle$
for all $x, y, z \in \mathfrak{X},\ a \in \mathfrak{A},\ \lambda \in \mathbb{C}$. By (ii) and (iv), $\langle .,. \rangle$ is conjogate linear in the first variable. Using the Cauchy–Schwartz inequality $\langle y, x \rangle \langle x, y \rangle \leq \|\langle x, x \rangle\| \langle y, y \rangle$ [10, Page 5], it follows that $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$ is a norm on $\mathfrak{X}$ making it into a right normed module. The pre-Hilbert module $\mathfrak{X}$ is called a Hilbert $\mathfrak{A}$-module if it is complete with respect to this norm. Notice that the inner structure of a C^*-algebra is essentially more complicated than complex numbers. For instance, the notations such as orthogonality and theorems such as Riesz' representation in the complex Hilbert space theory cannot simply be generalized or transferred to the theory of Hilbert C^*-modules.

One may define an "$\mathfrak{A}$-valued norm" $|.|$ by $|x| = \langle x, x \rangle^{1/2}$. Clearly, $\|\,|x|\,\| = \|x\|$ for each $x \in \mathfrak{X}$. It is known that $|.|$ does not satisfy the triangle inequality in general. See [10, 11] for more information on Hilbert C^*-modules.

2. Main results

Utilizing some C^*-algebraic techniques we present our first result as a generalization of [6, Theorem 2.3].

Theorem 2.1. *Let $\mathfrak{A}$ be a C^*-algebra with unit 1, let $\mathfrak{X}$ be a Hilbert $\mathfrak{A}$-module and let $x_1, \cdots, x_n \in \mathfrak{X}$. If there exist real numbers $k_1, k_2 \geq 0$ with*

$$0 \leq k_1\|x_j\| \leq Re\langle e, x_j\rangle\,, \quad 0 \leq k_2\|x_j\| \leq Im\langle e, x_j\rangle\,,$$

for some $e \in \mathfrak{X}$ with $|e| \leq 1$ and all $1 \leq j \leq n$, then

$$(k_1^2 + k_2^2)^{\frac{1}{2}} \sum_{j=1}^{n} \|x_j\| \leq \left\| \sum_{j=1}^{n} x_j \right\| . \tag{2.1}$$

Proof. Applying the Cauchy–Schwarz inequality, we get

$$|\langle e, \sum_{j=1}^{n} x_j\rangle|^2 \leq \|e\|^2 \left| \sum_{j=1}^{n} x_j \right|^2 \leq \left\| \sum_{j=1}^{n} x_j \right\|^2 ,$$

and

$$|\langle \sum_{j=1}^{n} x_j, e\rangle|^2 \leq \left\| \sum_{j=1}^{n} x_j \right\|^2 |e|^2 \leq \left\| \sum_{j=1}^{n} x_j \right\|^2 ,$$

whence

$$\begin{aligned}
\left\| \sum_{j=1}^{n} x_j \right\|^2 &\geq \tfrac{1}{2}\Big(|\langle e, \textstyle\sum_{j=1}^{n} x_j\rangle|^2 + |\langle \textstyle\sum_{j=1}^{n} x_j, e\rangle|^2 \Big) \\
&= \tfrac{1}{2}\Big(\langle e, \textstyle\sum_{j=1}^{n} x_j\rangle^* \langle e, \textstyle\sum_{j=1}^{n} x_j\rangle + \langle \textstyle\sum_{j=1}^{n} x_j, e\rangle^* \langle \textstyle\sum_{j=1}^{n} x_j, e\rangle \Big) \\
&= (\mathrm{Re}\langle e, \textstyle\sum_{j=1}^{n} x_j\rangle)^2 + (\mathrm{Im}\langle e, \textstyle\sum_{j=1}^{n} x_j\rangle)^2 \qquad \big(\text{by } (\diamond)\big) \\
&= (\mathrm{Re} \textstyle\sum_{j=1}^{n} \langle e, x_j\rangle)^2 + (\mathrm{Im} \textstyle\sum_{j=1}^{n} \langle e, x_j\rangle)^2 \\
&\geq k_1^2 (\textstyle\sum_{j=1}^{n} \|x_j\|)^2 + k_2^2 (\textstyle\sum_{j=1}^{n} \|x_j\|)^2 \\
&= (k_1^2 + k_2^2)(\textstyle\sum_{j=1}^{n} \|x_j\|)^2 .
\end{aligned}$$

□

Using the same argument as in the proof of Theorem 2.1 one can obtain the following result, where k_1, k_2 are hermitian elements of $\mathfrak{A}$.

Theorem 2.2. *If the vectors $x_1, \cdots, x_n \in \mathfrak{X}$ satisfy the conditions*

$$0 \leq k_1^2\|x_j\|^2 \leq (Re\langle e, x_j\rangle)^2\,, \quad 0 \leq k_2^2\|x_j\|^2 \leq (Im\langle e, x_j\rangle)^2\,,$$

for some hermitian elements k_1, k_2 in $\mathfrak{A}$, some $e \in \mathfrak{X}$ with $|e| \leq 1$ and all $1 \leq j \leq n$ then the inequality 2.1 holds.

One may observe an integral version of inequality (2.1) as follows:

Corollary 2.3. *Let $\mathfrak{X}$ be a Hilbert module over a C^*-algebra $\mathfrak{A}$ with unit 1 and let $f : [a,b] \to \mathfrak{X}$ be strongly measurable such that the Lebesgue integral $\int_a^b \|f(t)\|dt$ exist and be finite. If there exist self-adjoint elements a_1, a_2 in $\mathfrak{A}$ with*

$$a_1^2\|f(t)\|^2 \leq Re\langle f(t), e\rangle^2\,, \quad a_2^2\|f(t)\|^2 \leq Im\langle f(t), e\rangle^2 \quad (\text{ a.e. } t \in [a,b])\,,$$

where $e \in \mathfrak{X}$ with $|e| \leq 1$, then

$$(a_1^2 + a_2^2)^{\frac{1}{2}} \int_a^b \|f(t)\|dt \leq \|\int_a^b f(t)dt\|\,.$$

Now we prove a useful lemma which is frequently applied in the next theorems.

Lemma 2.4. *Let $\mathfrak{X}$ be a Hilbert $\mathfrak{A}$-module and let $x, y \in \mathfrak{X}$. If $|\langle x, y\rangle| = \|x\|\|y\|$, then*

$$y = \frac{x\langle x, y\rangle}{\|x\|^2}\,.$$

Proof. For $x, y \in \mathfrak{X}$ we have

$$\begin{aligned} 0 \leq \left|y - \tfrac{x\langle x,y\rangle}{\|x\|^2}\right|^2 &= \langle y - \tfrac{x\langle x,y\rangle}{\|x\|^2}, y - \tfrac{x\langle x,y\rangle}{\|x\|^2}\rangle \\ &= \langle y, y\rangle - \tfrac{1}{\|x\|^2}\langle y, x\rangle\langle x, y\rangle + \tfrac{1}{\|x\|^4}\langle y, x\rangle\langle x, x\rangle\langle x, y\rangle - \tfrac{1}{\|x\|^2}\langle y, x\rangle\langle x, y\rangle \\ &\leq |y|^2 - \tfrac{1}{\|x\|^2}|\langle x, y\rangle|^2 = |y|^2 - \tfrac{1}{\|x\|^2}\|x\|^2\|y\|^2 \\ &= |y|^2 - \|y\|^2 \leq 0\,, \end{aligned}$$

whence $\left|y - \frac{x\langle x,y\rangle}{\|x\|^2}\right| = 0$. Hence $y = \frac{x\langle x,y\rangle}{\|x\|^2}$. □

Using the Cauchy-Schwarz inequality, we have the following theorem for Hilbert modules, which is similar to [1, Theorem 2.5].

Theorem 2.5. *Let $e_1, \cdots, e_m$ be a family of vectors in a Hilbert C^*-module $\mathfrak{X}$ such that $\langle e_i, e_j\rangle = 0$ $(1 \leq i \neq j \leq m)$ and $\|e_i\| = 1$ $(1 \leq i \leq m)$. Suppose that $r_k, \rho_k \in \mathbb{R}$ $(1 \leq k \leq m)$ and that the vectors $x_1, \cdots, x_n \in \mathfrak{X}$ satisfy*

$$0 \leq r_k^2\|x_j\| \leq Re\langle r_k e_k, x_j\rangle\,, \quad 0 \leq \rho_k^2\|x_j\| \leq Im\langle \rho_k e_k, x_j\rangle\,,$$

Then

$$\left[\sum_{k=1}^m (r_k^2 + \rho_k^2)\right]^{\frac{1}{2}} \sum_{j=1}^n \|x_j\| \leq \left\|\sum_{j=1}^n x_j\right\|\,, \tag{2.2}$$

and the equality holds if and only if

$$\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} \|x_j\| \sum_{k=1}^{m} (r_k + i\rho_k) e_k \,. \tag{2.3}$$

Proof. There is nothing to prove if $\sum_{k=1}^{m}(r_k^2+\rho_k^2)=0$. Assume that $\sum_{k=1}^{m}(r_k^2+\rho_k^2)\neq 0$. From the hypothesis we have

$$\begin{aligned}
\big(\textstyle\sum_{k=1}^{m}(r_k^2+\rho_k^2)\big)^2\big(\sum_{j=1}^{n}\|x_j\|\big)^2 &\le \big(\mathrm{Re}\langle\textstyle\sum_{k=1}^{m} r_k e_k, \sum_{j=1}^{n} x_j\rangle + \mathrm{Im}\langle\sum_{k=1}^{m}\rho_k e_k, \sum_{j=1}^{n} x_j\rangle\big)^2 \\
&= \big(\mathrm{Re}\langle\textstyle\sum_{j=1}^{n} x_j, \sum_{k=1}^{m}(r_k+i\rho_k)e_k\rangle\big)^2 \\
&\qquad \Big(\text{by } \mathrm{Im}(a)=\mathrm{Re}(ia^*), \mathrm{Re}(a^*)=\mathrm{Re}(a) \quad (a\in\mathfrak{A})\Big) \\
&\le \tfrac{1}{2}|\langle\textstyle\sum_{j=1}^{n} x_j, \sum_{k=1}^{m}(r_k+i\rho_k)e_k\rangle|^2 \\
&\quad + |\langle\textstyle\sum_{k=1}^{m}(r_k+i\rho_k)e_k, \sum_{j=1}^{n} x_j\rangle|^2 \qquad \big(\text{by } (\diamond)\big) \\
&\le \tfrac{1}{2}\|\textstyle\sum_{j=1}^{n} x_j\|^2|\sum_{k=1}^{m}(r_k+i\rho_k)e_k|^2 \\
&\quad + \|\textstyle\sum_{k=1}^{m}(r_k+i\rho_k)e_k\|^2|\sum_{j=1}^{n} x_j|^2 \\
&\le \|\textstyle\sum_{j=1}^{n} x_j\|^2\,\|\sum_{k=1}^{m}(r_k+i\rho_k)e_k\|^2 \\
&\qquad \big(\text{by } |a|\le\|a\| \quad (a\in\mathfrak{A})\big) \\
&= \|\textstyle\sum_{j=1}^{n} x_j\|^2\,\|\langle\sum_{k=1}^{m}(r_k+i\rho_k)e_k, \sum_{k=1}^{m}(r_k+i\rho_k)e_k\rangle\| \\
&= \|\textstyle\sum_{j=1}^{n} x_j\|^2\sum_{k=1}^{m}|r_k+i\rho_k|^2\|e_k\|^2 \\
&= \|\textstyle\sum_{j=1}^{n} x_j\|^2\sum_{k=1}^{m}(r_k^2+\rho_k^2)\,.
\end{aligned}$$

Hence

$$\left[\sum_{k=1}^{m}(r_k^2+\rho_k^2)\right](\sum_{j=1}^{n}\|x_j\|)^2 \le \left\|\sum_{j=1}^{n} x_j\right\|^2 .$$

By taking square roots the desired result follows.

Clearly we have equality in 2.2 if condition 2.3 holds. To see the converse, first note that if equality holds in 2.2, then all inequalities in the above relations should be equality. Therefore

$$r_k^2\|x_j\| = Re\langle r_k e_k, x_j\rangle\,, \quad \rho_k^2\|x_j\| = Im\langle \rho_k e_k, x_j\rangle\,,$$

$$\mathrm{Re}\langle\sum_{j=1}^{n} x_j, \sum_{k=1}^{m}(r_k+i\rho_k)e_k\rangle = \langle\sum_{j=1}^{n} x_j, \sum_{k=1}^{m}(r_k+i\rho_k)e_k\rangle\,,$$

and also

$$|\langle\sum_{k=1}^{m}(r_k+i\rho_k)e_k, \sum_{j=1}^{n} x_j\rangle| = \|\sum_{j=1}^{n} x_j\|\|\sum_{k=1}^{m}(r_k+i\rho_k)e_k\|\,.$$

From Lemma 2.4 and these equalities we have

$$\begin{aligned}\sum_{j=1}^n x_j &= \frac{\sum_{k=1}^m (r_k+i\rho_k)e_k}{\|\sum_{k=1}^m (r_k+i\rho_k)e_k\|^2}\langle \sum_{k=1}^m (r_k+i\rho_k)e_k, \sum_{j=1}^n x_j\rangle \\ &= \frac{\sum_{k=1}^m (r_k+i\rho_k)e_k}{\sum_{k=1}^m (r_k^2+\rho_k^2)}\mathrm{Re}\langle \sum_{k=1}^m (r_k+i\rho_k)e_k, \sum_{j=1}^n x_j\rangle \\ &= \frac{\sum_{k=1}^m (r_k+i\rho_k)e_k}{\sum_{k=1}^m (r_k^2+\rho_k^2)}\sum_{k=1}^m\sum_{j=1}^n (r_k^2\|x_j\| + \rho_k^2\|x_j\|) \\ &= \sum_{j=1}^n \|x_j\| \sum_{k=1}^m (r_k+i\rho_k)e_k\,,\end{aligned}$$

which is the desired result. □

From now on we assume that $\mathfrak{X}$ is a right Hilbert module over a C^*-algebra $\mathfrak{A}$, which is an algebraic left A-module subject to

$$\langle x, ay\rangle = a\langle x, y\rangle \quad (x, y \in \mathfrak{X}, a \in \mathfrak{A})\,. \tag{$\dagger$}$$

For example if $\mathfrak{A}$ is a C*-algebra and $\mathfrak{I}$ be a commutative right ideal of $\mathfrak{A}$, then $\mathfrak{I}$ is a right Hilbert module over $\mathfrak{A}$ and

$$\langle x, ay\rangle = x^*(ay) = ax^*y = a\langle x, y\rangle \quad (x, y \in \mathfrak{I},\ a \in \mathfrak{A})\,.$$

The next theorem is a refinement of [6, Theorem 2.1]. To prove it we need the following lemma.

Lemma 2.6. *Let $\mathfrak{X}$ be a Hilbert $\mathfrak{A}$-module and $e_1, \cdots, e_n \in \mathfrak{X}$ be a family of vectors such that $\langle e_i, e_j\rangle = 0$ $(i \neq j)$ and $\|e_i\| = 1$. If $x \in \mathfrak{X}$, then*

$$|x|^2 \geq \sum_{k=1}^n |\langle e_k, x\rangle|^2 \quad \textit{and} \quad |x|^2 \geq \sum_{k=1}^n |\langle x, e_k\rangle|^2\,.$$

Proof. The first result follows from the following inequality:

$$\begin{aligned}0 \leq |x - \sum_{k=1}^n e_k\langle e_k, x\rangle|^2 &= \langle x - \sum_{k=1}^n e_k\langle e_k, x\rangle, x - \sum_{j=1}^n e_j\langle e_j, x\rangle\rangle \\ &= \langle x, x\rangle + \sum_{k=1}^n\sum_{j=1}^n \langle e_k, x\rangle^*\langle e_k, e_j\rangle\langle e_j, x\rangle - 2\sum_{k=1}^n |\langle e_k, x\rangle|^2 \\ &= \langle x, x\rangle + \sum_{k=1}^n \langle e_k, x\rangle^*\langle e_k, e_k\rangle\langle e_k, x\rangle - 2\sum_{k=1}^n |\langle e_k, x\rangle|^2 \\ &\leq |x|^2 + \sum_{k=1}^n \langle e_k, x\rangle^*\langle e_k, x\rangle - 2\sum_{k=1}^n |\langle e_k, x\rangle|^2 \\ &= |x|^2 - \sum_{k=1}^n |\langle e_k, x\rangle|^2\,.\end{aligned}$$

By considering $|x - \sum_{k=1}^n \langle e_k, x\rangle e_k|^2$, similarly, we have the second one. □

Now we are able to prove the next theorem without using the Cauchy–Schwarz inequality.

Theorem 2.7. *Let $e_1, \cdots, e_m \in \mathfrak{X}$ be a family of vectors with $\langle e_i, e_j\rangle = 0$ $(1 \le i \ne j \le m)$ and $\|e_i\| = 1$ $(1 \le i \le m)$. If the vectors $x_1, \cdots, x_n \in \mathfrak{X}$ satisfy the conditions*

$$0 \le r_k\|x_j\| \le Re\langle e_k, x_j\rangle\,, \quad 0 \le \rho_k\|x_j\| \le Im\langle e_k, x_j\rangle \quad (1 \le j \le n, 1 \le k \le m)\,, \tag{2.4}$$

where $r_k, \rho_k \in [0, \infty)$ $(1 \le k \le m)$, then

$$\left[\sum_{k=1}^{m}(r_k^2 + \rho_k^2)\right]^{\frac{1}{2}} \sum_{j=1}^{n} \|x_j\| \le \left|\sum_{j=1}^{n} x_j\right|\,. \tag{2.5}$$

Proof. Applying the previous lemma for $x = \sum_{j=1}^n x_j$, we obtain

$$\begin{aligned}
\left|\sum_{j=1}^n x_j\right|^2 &\ge \tfrac{1}{2}\left(\sum_{k=1}^m |\langle e_k, \sum_{j=1}^n x_j\rangle|^2 + \sum_{k=1}^m |\langle \sum_{j=1}^n x_j, e_k\rangle|^2\right)\\
&= \sum_{k=1}^m \tfrac{1}{2}\left(\langle e_k, \sum_{j=1}^n x_j\rangle^* \langle e_k, \sum_{j=1}^n x_j\rangle + \langle \sum_{j=1}^n x_j, e_k\rangle^* \langle \sum_{j=1}^n x_j, e_k\rangle\right)\\
&= \sum_{k=1}^m (\mathrm{Re}\langle e_k, \sum_{j=1}^n x_j\rangle)^2 + (\mathrm{Im}\langle e_k, \sum_{j=1}^n x_j\rangle)^2 \qquad \big(\text{by } (\diamond)\big)\\
&= \sum_{k=1}^m (\mathrm{Re}\sum_{j=1}^n \langle e_k, x_j\rangle)^2 + (\mathrm{Im}\sum_{j=1}^n \langle e_k, x_j\rangle)^2\\
&\ge \sum_{k=1}^m (r_k^2(\sum_{j=1}^n \|x_j\|)^2 + \rho_k^2(\sum_{j=1}^n \|x_j\|)^2) \qquad \big(\text{by } (2.4)\big)\\
&= \sum_{k=1}^m (r_k^2 + \rho_k^2)(\sum_{j=1}^n \|x_j\|)^2\,.
\end{aligned}$$

□

Proposition 2.8. *In Theorem 2.7, if $\langle e_k, e_k\rangle = 1$, then the equality holds in (2.5) if and only if*

$$\sum_{j=1}^n x_j = (\sum_{j=1}^n \|x_j\|) \sum_{k=1}^m (r_k + i\rho_k) e_k\,. \tag{2.6}$$

Proof. If (2.6) holds, then inequality in (2.5) turns trivially into equality.

Next, assume that equality holds in (2.5). Then two inequalities in the proof of Theorem 2.7 should be equality. Hence

$$|\sum_{j=1}^n x_j|^2 = \sum_{k=1}^m |\langle e_k, \sum_{j=1}^n x_j\rangle|^2 \quad \text{and} \quad |\sum_{j=1}^n x_j|^2 = \sum_{k=1}^m |\langle \sum_{j=1}^n x_j, e_k\rangle|^2\,,$$

which is equivalent to

$$\sum_{j=1}^n x_j = \sum_{k=1}^m \sum_{j=1}^n e_k\langle e_k, x_j\rangle = \sum_{k=1}^m \sum_{j=1}^n \langle e_k, x_j\rangle e_k\,,$$

and also

$$r_k\|x_j\| = \mathrm{Re}\langle e_k, x_j\rangle\,, \quad \rho_k\|x_j\| = \mathrm{Im}\langle e_k, x_j\rangle\,.$$

So

$$\sum_{j=1}^{n} x_j = \sum_{k=1}^{m}\sum_{j=1}^{n} e_k\langle e_k, x_j\rangle = \sum_{k=1}^{m}\sum_{j=1}^{n} e_k(r_k + i\rho_k)\|x_j\| = (\sum_{j=1}^{n}\|x_j\|)\sum_{k=1}^{m}(r_k + i\rho_k)e_k\,.$$

□

There are some versions of additive reverse of triangle inequality. In [5], S. S. Dragomir established the following theorem:

Theorem 2.9. *Let* $\{e_k\}_{k=1}^m$ *be a family of orthonormal vectors in Hilbert space* H *and* $M_{jk} \geq 0$ $(1 \leq i \leq n, 1 \leq k \leq m)$ *such that*

$$\|x_j\| - Re\langle e_k, x_j\rangle \leq M_{jk}\,,$$

for each $1 \leq i \leq n$ *and* $1 \leq k \leq m$*. Then*

$$\sum_{j=1}^{n}\|x_j\| \leq \frac{1}{\sqrt{m}}\|\sum_{j=1}^{n} x_j\| + \frac{1}{m}\sum_{j=1}^{n}\sum_{k=1}^{m} M_{jk}\,;$$

and the equality holds if and only if

$$\sum_{j=1}^{n}\|x_j\| \geq \frac{1}{m}\sum_{j=1}^{n}\sum_{k=1}^{m} M_{jk}\,,$$

and

$$\sum_{j=1}^{n} x_j = \left(\sum_{j=1}^{n}\|x_j\| - \frac{1}{m}\sum_{j=1}^{n}\sum_{k=1}^{m} M_{jk}\right)\sum_{k=1}^{m} e_k\,.$$

Now we extend this theorem for Hilbert modules.

Theorem 2.10. *Let* $\{e_k\}_{k=1}^m$ *be a family of vectors in Hilbert* $\mathfrak{A}$*-module* $\mathfrak{X}$ *with* $|e_k| \leq 1$ $(1 \leq k \leq m)$ *and* $\langle e_i, e_j\rangle = 0$ $(1 \leq i \neq j \leq m)$ *and* $x_j \in \mathfrak{X}$ $(1 \leq i \leq n)$*. If for some scalars* $M_{jk} \geq 0$ $(1 \leq i \leq n,\ 1 \leq k \leq m)$*,*

$$\|x_j\| - Re\langle e_k, x_j\rangle \leq M_{jk} \qquad (1 \leq i \leq n, 1 \leq k \leq m)\,, \tag{2.7}$$

then

$$\sum_{j=1}^{n}\|x_j\| \leq \frac{1}{\sqrt{m}}\|\sum_{j=1}^{n} x_j\| + \frac{1}{m}\sum_{j=1}^{n}\sum_{k=1}^{m} M_{jk}\,. \tag{2.8}$$

Moreover, if $|e_k| = 1$ $(1 \leq k \leq m)$*, then the equality in (2.8) holds if and only if*

$$\sum_{j=1}^{n}\|x_j\| \geq \frac{1}{m}\sum_{j=1}^{n}\sum_{k=1}^{m} M_{jk}\,, \tag{2.9}$$

and

$$\sum_{j=1}^{n} x_j = \left(\sum_{j=1}^{n} \|x_j\| - \frac{1}{m}\sum_{j=1}^{n}\sum_{k=1}^{m} M_{jk}\right)\sum_{k=1}^{m} e_k\,. \tag{2.10}$$

Proof. Taking the summation in (2.7) over i from 1 to n, we obtain

$$\sum_{j=1}^{n} \|x_j\| \le \mathrm{Re}\langle e_k, \sum_{j=1}^{n} x_j\rangle + \sum_{j=1}^{n} M_{jk}\,,$$

for each $k \in \{1, \cdots, m\}$. Summing these inequalities over k from 1 to m, we deduce

$$\sum_{j=1}^{n} \|x_j\| \le \frac{1}{m}\mathrm{Re}\langle\sum_{k=1}^{m} e_k, \sum_{j=1}^{n} x_j\rangle + \frac{1}{m}\sum_{k=1}^{m}\sum_{j=1}^{n} M_{jk}\,. \tag{2.11}$$

Using the Cauchy–Schwarz we obtain

$$\begin{aligned}\left(\mathrm{Re}\langle\textstyle\sum_{k=1}^{m} e_k, \sum_{j=1}^{n} x_j\rangle\right)^2 &\le \tfrac{1}{2}(|\langle\textstyle\sum_{k=1}^{m} e_k, \sum_{j=1}^{n} x_j\rangle|^2 + |\langle\sum_{k=1}^{m} e_k, \sum_{j=1}^{n} x_j\rangle^*|^2) \quad (\text{by } (\diamond))\\ &\le \tfrac{1}{2}(\|\textstyle\sum_{k=1}^{m} e_k\|^2|\sum_{j=1}^{n} x_j|^2 + |\sum_{k=1}^{m} e_k|^2\|\sum_{j=1}^{n} x_j\|^2)\\ &\le \|\textstyle\sum_{k=1}^{m} e_k\|^2\|\sum_{j=1}^{n} x_j\|^2\\ &\le m\|\textstyle\sum_{j=1}^{n} x_j\|^2\,,\end{aligned} \tag{2.12}$$

since

$$\|\sum_{k=1}^{m} e_k\|^2 = \|\langle\sum_{k=1}^{m} e_k, \sum_{k=1}^{m} e_k\rangle\| = \|\sum_{k=1}^{m}\sum_{l=1}^{m}\langle e_k, e_l\rangle\| = \|\sum_{k=1}^{m} |e_k|^2\| \le m\,.$$

Using (2.12) in (2.11), we deduce the desired inequality.

If (2.9) and (2.10) hold, then

$$\begin{aligned}\tfrac{1}{\sqrt{m}}\left\|\textstyle\sum_{j=1}^{n} x_j\right\| &= \tfrac{1}{\sqrt{m}}\left(\textstyle\sum_{j=1}^{n} \|x_j\| - \frac{1}{m}\sum_{j=1}^{n}\sum_{k=1}^{m} M_{jk}\right)\|\sum_{k=1}^{m} e_k\|\\ &= \textstyle\sum_{j=1}^{n} \|x_j\| - \frac{1}{m}\sum_{j=1}^{n}\sum_{k=1}^{m} M_{jk}\,,\end{aligned}$$

and then the equality in 2.8 holds true.

Conversely, if the equality holds in (2.8), then obviously (2.9) is valid and we have equalities all over in the above proof. This means that

$$\|x_j\| - \mathrm{Re}\langle e_k, x_j\rangle = M_{jk}\,,$$

$$\mathrm{Re}\langle\sum_{k=1}^{m} e_k, \sum_{j=1}^{n} x_j\rangle = \langle\sum_{k=1}^{m} e_k, \sum_{j=1}^{n} x_j\rangle\,,$$

and

$$|\langle\sum_{k=1}^{m} e_k, \sum_{j=1}^{n} x_j\rangle| = \|\sum_{k=1}^{m} e_k\|\|\sum_{j=1}^{n} x_j\|\,.$$

Now from Lemma 2.4 and these relations, we have

$$\begin{aligned}
\sum_{j=1}^n x_j &= \frac{\sum_{k=1}^m e_k}{\|\sum_{k=1}^m e_k\|^2}\langle \sum_{k=1}^m e_k, \sum_{j=1}^n x_j\rangle \\
&= \frac{\sum_{k=1}^m e_k}{m}\mathrm{Re}\langle \sum_{k=1}^m e_k, \sum_{j=1}^n x_j\rangle \\
&= \frac{\sum_{k=1}^m e_k}{m}\sum_{k=1}^m \sum_{j=1}^n (\|x_j\| - M_{jk}) \\
&= \left(\sum_{j=1}^n \|x_j\| - \frac{1}{m}\sum_{j=1}^n \sum_{k=1}^m M_{jk}\right)\sum_{k=1}^m e_k\,.
\end{aligned}$$

□

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