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A Note on the Volume of Sections of $B^n p$

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A NOTE ON THE VOLUME OF SECTIONS OF B_p^n

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ABSTRACT. Let B_p^n denote the unit ball in ℓ_p^n with $p \geq 1$. We prove that $\mathrm{Vol}_{n-1}(H \cap B_p^n) \geq \left(\mathrm{Vol}_n(B_p^n)\right)^{(n-1)/n}$ for any $(n-1)$-dimensional subspace H of $\mathbb{R}^n$. This is a consequence of bounding the isotropy constant of B_p^n above by $1/\sqrt{12}$ and we show that one can replace $1/\sqrt{12}$ by a possibly smaller number for $n \geq 2$.

1. INTRODUCTION

A symmetric convex body K in $\mathbb{R}^n$ is said to be in isotropic position if there is a constant (the isotropy constant) L_K such that

$$\int_K x_i x_j dx = L_K^2 \delta_{ij} \left(\mathrm{Vol}_n(K)\right)^{(n+2)/n}, \quad (1 \leq i, j \leq n),$$

where δ_{ij} is the Kronecker symbol. A well-known conjecture is that there exists a universal constant $c > 0$ such that $L_K < c$ for all convex centrally symmetric bodies in all dimensions. The best estimate known to date is due to Bourgain [9] that

$$L_K < c n^{\frac{1}{4}} \ln n.$$

In addition, the conjecture was verified for large classes of bodies (see [16], [7], [13], [14]) and it is equivalent to the famous hyperplane conjecture, which states that there is a universal constant $c > 0$ such that, for any convex centrally symmetric body $K \subset \mathbb{R}^n$, there is an $(n-1)$-dimensional subspace H for which

$$(1.1) \qquad\qquad \mathrm{Vol}_{n-1}(H \cap K) \geq c \cdot \left(\mathrm{Vol}_n(K)\right)^{(n-1)/n}.$$

Now let K be the unit ball B_p^n in ℓ_p^n with $p \geq 1$, that is,

$$B_p^n = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^p \leq 1 \right\}, \quad 1 \leq p < +\infty,$$

$$B_\infty^n = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \max_{1 \leq i \leq n} |x_i| \leq 1 \right\}.$$

In this case Meyer and Pajor [15] proved (1.1) (in fact for any $(n-1)$-dimensional subspace H) with $c = 1$ for $p = 1$ and $p \geq 2$. Later Schmuckenschläger [18] gave a proof for the case $1 < p < 2$ with $c = 1$ but the proof of the inequality he proposed was not correct and this was fixed by Bastero, Galve, Peña and Romance in [8]. The approach of Schmuckenschläger and Bastero et al. is based on an estimation of $L_{B_p^n}$, for which there is an explicit expression involving the gamma function $\Gamma(x)$. It is the goal of this paper to extend their results to all $p \geq 1$ via this approach and also to do it in a way that involves less direct computations.

2. Gamma and Polygamma Functions

The digamma (or psi) function $\psi(x)$ for $x > 0$ is defined as the logarithmic derivative of $\Gamma(x)$ and the derivatives of $\psi(x)$ are known as polygamma functions. We note here that $\psi'(x)$ is completely monotonic on $(0, +\infty)$. (A function $f(x)$ is said to be completely monotonic on (a, b) if it has derivatives of all orders and $(-1)^n f^{(n)}(x) \geq 0, x \in (a, b), n = 0, 1, 2, \ldots$).

We now collect here a few facts about the gamma and polygamma functions, these can be found, for example, in [1, (7.1)], [2, (1.1)-(1.5), (3.39)].

Lemma 2.1. *For $x > 0$ we have*

$$(2.1) \qquad \psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt,$$

$$(2.2) \qquad (-1)^{n+1}\psi^{(n)}(x) = \int_0^\infty e^{-xt} \frac{t^n}{1 - e^{-t}} dt = n! \sum_{k=0}^\infty \frac{1}{(x+k)^{n+1}}, \quad n \geq 1,$$

$$(2.3) \qquad \psi^{(n)}(x+1) = \psi^{(n)}(x) + (-1)^n \frac{n!}{x^{n+1}}, \quad n \geq 0,$$

$$(2.4) \qquad \ln\Gamma(x) = (x - \frac{1}{2})\ln x - x + \frac{1}{2}\ln(2\pi) + O\left(\frac{1}{x}\right), \quad x \to +\infty,$$

$$(2.5) \qquad \psi(x) = \ln x - \frac{1}{2x} - \frac{1}{12x^2} + O\left(\frac{1}{x^3}\right), \quad x \to +\infty,$$

$$(2.6) \qquad (-1)^{n+1}\psi^{(n)}(x) = \frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + O\left(\frac{1}{x^{n+2}}\right), \quad n \geq 1, \quad x \to +\infty,$$

$$(2.7) \qquad \psi'(x) < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3},$$

where $\gamma = 0.57721\ldots$ denotes Euler's constant.

Many interesting inequalities arise from the study of the asymptotic behavior of the polygamma functions. For example, one sees from (2.6) that $x^n(-1)^{n+1}\psi^{(n)}(x)$ is asymptotically $(n-1)!$, hence it's natural to ask how it approaches this constant. For $n = 1$, a result of Ronning [17] asserts that $x\psi'(x)$ is strictly decreasing. We note here this is also equivalent to a result of Alzer [2, Lemma 2.4], which asserts that $\psi(e^x)$ is strictly concave on $(-\infty, +\infty)$. The cases $n > 1$ have been studied in [11] and [3]. One can certainly ask a more general question on the behavior of $f_{a,n}(x) = x^n(-1)^{n+1}\psi^{(n)}(x+a)$ for any non-negative number a. When $a = 1$ and $n = 1$, this was investigated by Anderson and Qiu [5] and later proved to be strictly increasing for $x > -1$ by Elbert and Laforgia [12]. Borwein et al. showed that [10, Lemma 2.1] $f_{1,1}(x)$ is even completely monotonic on $(0, +\infty)$. Alzer and Ruehr [4] showed that $f_{a,1}(x)$ is strictly increasing for $a \geq 1/2$. We now summarize these results in the following lemma.

Lemma 2.2. *For fixed $n \geq 1, a \geq 0$, the function $f_{a,n}(x) = x^n(-1)^{n+1}\psi^{(n)}(x+a)$ is increasing on $[0, +\infty)$ if and only if $a \geq 1/2$. Also, $f_{0,n}(x)$ is decreasing on $(0, +\infty)$.*

Proof. From (2.6) we see that

$$
\begin{aligned}
\frac{f'_{a,n}(x)}{x^{n-1}} &= n(-1)^{n+1}\psi^{(n)}(x+a) - x(-1)^{n+2}\psi^{(n+1)}(x+a) \\
&= \frac{n!(a - 1/2)}{(x+a)^{n+1}} + O\left(\frac{1}{(x+a)^{n+2}}\right), \quad x \to +\infty.
\end{aligned}
$$

It then follows that it is necessary to have $a \geq 1/2$ for $f_{a,n}(x)$ to be increasing on $[0, +\infty)$.

Assume now $a \geq 1/2$, we use the integral representation in (2.2) for $(-1)^{n+1}\psi^{(n)}(x)$ to deduce that

$$f_{a,n}(x) = \int_0^\infty e^{-(x+a)t} \frac{(xt)^n}{1 - e^{-t}} dt.$$

It follows from this that $f'_{a,n}(0) = 0$. For $x > 0$, we make a change of variable $xt = s$ in the above integral to get

$$f_{a,n}(x) = \int_0^\infty e^{-s} s^{n-1} \frac{re^{-ar}}{1 - e^{-r}} ds,$$

where $r = s/x$. We then obtain for $x > 0$,

$$f'_{a,n}(x) = \int_0^\infty e^{-s} s^n \frac{e^{-(a+1)r}\Big((ar - 1)(e^r - 1) + r \Big)}{\Big(x(1 - e^{-r}) \Big)^2} ds,$$

One then checks easily that $(ar - 1)(e^r - 1) + r \geq 0$ for $r \geq 0, a \geq 1/2$ and this implies $f'_{a,n}(x) \geq 0$ for $x > 0, a \geq 1/2$. Similarly, one shows that $f'_{0,n}(x) \leq 0$ for $x > 0$ and this completes the proof. $\square$

Before we proceed to prove our main result in the next section, we state more auxiliary results here.

Lemma 2.3. *Let $0 \leq x \leq 1/4$ and $y \geq 0$. The function*

$$u(x,y) = \psi(1 + xy) - \psi\Big(1 + (y + 2)x\Big) + \frac{(y + 2)x}{2}\psi'(1 + xy) - \frac{xy}{2}\psi'\Big(1 + (y + 2)x\Big)$$

is non-positive. Moreover, $u(1/2, y) < 0$ for $y \geq 0$ and $u(1, y) \leq 0$ for $y \geq 1$.

Proof. We have

$$\begin{aligned}
&\frac{(y + 2)x}{2}\psi'(1 + xy) - \frac{xy}{2}\psi'\Big(1 + (y + 2)x\Big) \\
=\ &\frac{xy + 1/2}{2}\psi'(1 + xy) - \frac{x(y + 2) + 1/2}{2}\psi'\Big(1 + (y + 2)x\Big) \\
&+ (x - \frac{1}{4})\psi'(1 + xy) + (x + \frac{1}{4})\psi'\Big(1 + (y + 2)x\Big) \\
\leq\ &(x - \frac{1}{4})\psi'(1 + xy) + (x + \frac{1}{4})\psi'\Big(1 + (y + 2)x\Big),
\end{aligned}$$

where the inequality above follows from the case $n = 1, a = 1/2$ of Lemma 2.2. Also by Cauchy's mean value theorem, we obtain

$$(2.8) \qquad \psi(1 + xy) - \psi\Big(1 + (y + 2)x\Big) < -2x\psi'\Big(1 + (y + 2)x\Big).$$

These estimations yield

$$u(x,y) \leq (x - \frac{1}{4})\left(\psi'(1 + xy) - \psi'\Big(1 + (y + 2)x\Big) \right) \leq 0,$$

for $0 \leq x \leq 1/4$ and $y \geq 0$.

In the case $x = 1/2$, we obtain by setting $z = y/2$ that

$$\begin{aligned}
u(1/2, y) &= \psi(1 + z) - \psi(1 + z + 1) + \frac{z + 1}{2}\psi'(1 + z) - \frac{z}{2}\psi'(1 + z + 1) \\
&= -\frac{1}{1 + z} + \frac{1}{2}\psi'(1 + z) + \frac{z}{2(1 + z)^2},
\end{aligned}$$

where we have used (2.3) for $n = 0, 1$ above. We now use the bound (2.7) for $\psi'(x)$ to get for $z \geq 0$:

$$2u(1/2, y) < -\frac{1}{2(1+z)^2} + \frac{1}{6(1+z)^3} < 0.$$

Lastly, we use (2.3) to express $u(1, y)$ as

$$u(1, y) = -\frac{1}{y+1} - \frac{1}{y+2} + \psi'(1+y) + \frac{y}{2(y+1)^2} + \frac{y}{2(y+2)^2}.$$

We further apply (2.7) to get

$$
\begin{aligned}
u(1, y) &\leq \frac{1}{2(y+1)} - \frac{1}{2(y+2)} - \frac{1}{(y+2)^2} + \frac{1}{6(y+1)^3} \\
&= \frac{(y+2)^2 - 3y(y+1)^2}{6(y+1)^3(y+2)^2} \leq 0,
\end{aligned}
$$

where the last inequality follows since $3y \geq y + 2$ and $(y+1)^2 \geq y + 2$ for $y \geq 1$ and this completes the proof. $\qquad\square$

Lemma 2.4. *Let $1/2 \leq x \leq 1$ and $y \geq 0$. The function*

$$v(x, y) = \psi(1 + xy) - \psi\left(1 + (y+2)x\right) + (y+2)x\psi'(1+xy) - (y+2)x\psi'\left(1 + (y+2)x\right)$$

is non-negative.

Proof. Let $R = [1/2, 1] \times [0, +\infty)$ and we need to show $v(x, y) \geq 0$ for $(x, y) \in R$. Let $(x_0, y_0) \in R$ be the point in which the absolute minimum of $v(x, y)$ is reached and assume first that (x_0, y_0) is an interior point of R, then we obtain

$$\frac{\partial v}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) = 0.$$

Calculations yield

$$
\begin{aligned}
\frac{1}{x}\frac{\partial v}{\partial y} &= 2\psi'(1+xy) - 2\psi'\left(1 + (y+2)x\right) + x(y+2)\psi''(1+xy) - x(y+2)\psi''\left(1 + (y+2)x\right), \\
\frac{\partial v}{\partial x} &= 2(y+1)\psi'(1+xy) - 2(y+2)\psi'\left(1 + (y+2)x\right) + xy(y+2)\psi''(1+xy) \\
&\quad -x(y+2)^2\psi'\left(1 + (y+2)x\right).
\end{aligned}
$$

We then deduce from the above that

$$\psi'(1 + x_0 y_0) + x_0(y_0 + 2)\psi''(1 + x_0 y_0) = 0.$$

Note from Lemma 2.2 for the case $n = 1, a = 0$ we also have

$$\psi'(1 + x_0 y_0) + (1 + x_0 y_0)\psi''(1 + x_0 y_0) \leq 0 = \psi'(1 + x_0 y_0) + x_0(y_0 + 2)\psi''(1 + x_0 y_0),$$

which implies

$$(1 - 2x_0)\psi''(1 + x_0 y_0) \leq 0,$$

a contradiction. Thus we conclude that (x_0, y_0) is a boundary point of R. Hence we need to check $v(x, y) \geq 0$ for the cases $x = 1/2, 1$ or $y = 0, y \to +\infty$. It follows from the asymptotic expressions (2.6) and (2.5) that

$$\lim_{y \to +\infty} v(x, y) = 0.$$

Now for $x = 1/2$, using the relation (2.3) for $n = 0, 1$ and by setting $z = y/2$, we obtain

$$
\begin{aligned}
v(1/2, 2z) &= v(1/2, y) = \psi(1+z) - \psi(1+z+1) + (1+z)\psi'(1+z) - (1+z)\psi'(1+z+1) \\
&= -\frac{1}{1+z} + (1+z)\frac{1}{(1+z)^2} = 0.
\end{aligned}
$$

Similarly, for $x = 1$, we have

$$v(1, y) = -\frac{1}{y+2} - \frac{1}{y+1} + (y+2)\Big(\frac{1}{(y+2)^2} + \frac{1}{(y+1)^2}\Big) > 0.$$

It remains to check the case $y = 0$ and we get

$$v(x, 0) = \psi(1) - \psi(1 + 2x) + 2x\psi'(1) - 2x\psi'(1 + 2x),$$

and that

$$
\begin{aligned}
\frac{1}{2}\frac{\partial v}{\partial x}(x, 0) &= \psi'(1) - 2\psi'(1 + 2x) - 2x\psi''(1 + 2x) \\
&= \psi'(1) - \psi'(1 + 2x) + \psi''(1 + 2x) - \Big(\psi'(1 + 2x) + (1 + 2x)\psi''(1 + 2x)\Big) \\
&\geq \psi'(1) - \psi'(1 + 2x) + \psi''(1 + 2x),
\end{aligned}
$$

where the last inequality follows from the case $n = 1, a = 0$ of Lemma 2.2. Now by Cauchy's mean value theorem, we have

$$\psi'(1) - \psi'(1 + 2x) \geq -2x\psi''(1 + 2x),$$

which implies that

$$\frac{1}{2}\frac{\partial v}{\partial x}(x, 0) \geq (1 - 2x)\psi''(1 + 2x) \geq 0.$$

Thus

$$v(x, 0) \geq v(1/2, 0) = 0,$$

and this completes the proof. □

Lemma 2.5. *Let $0 \leq x \leq 1$ and $y > 0$. For fixed x, the function*

$$f(x, y) = (1 + \frac{2}{y})\ln\Gamma(1 + xy) - \ln\Gamma\Big(1 + (y+2)x\Big)$$

is a decreasing function of y for $y > 0$ when $0 \leq x \leq 1/2$ and for $y \geq 2$ when $1/2 < x \leq 1$.

Proof. We define

$$g(x, y) := y^2\frac{\partial f}{\partial y} = -2\ln\Gamma(1 + xy) + y(y+2)x\psi(1 + xy) - xy^2\psi\Big(1 + (y+2)x\Big).$$

It suffices to show $g(x, y) \leq 0$ for $0 \leq x \leq 1/2, y \geq 0$ and $1/2 < x \leq 1, y \geq 2$. We show first that $g(x, y) \leq 0$ for $0 \leq x \leq 1/4$ and $y \geq 0$. Since $g(0, y) = g(x, 0) = 0$, we may assume $x, y > 0$ and note that

$$\frac{1}{2xy}\frac{\partial g}{\partial y} = u(x, y), \quad \frac{1}{y^2}\frac{\partial g}{\partial x} = v(x, y),$$

where $u(x, y), v(x, y)$ are as defined in Lemma 2.3 and Lemma 2.4, respectively. By Lemma 2.3, $u(x, y) \leq 0$ for $0 \leq x \leq 1/4, \ y \geq 0$ and it follows that $g(x, y) \leq g(x, 0) = 0$ for $0 \leq x \leq 1/4, \ y \geq 0$.

Now let $D = [1/4, 1/2] \times [0, +\infty)$. To show $g(x, y) \leq 0$ for $(x, y) \in D$, we let $(x_0, y_0) \in D$ be the point in which the absolute maximum of $g(x, y)$ is reached and assume first that (x_0, y_0) is an interior point of D, then we obtain

$$\frac{\partial g}{\partial x}(x_0, y_0) = \frac{\partial g}{\partial y}(x_0, y_0) = 0.$$

From our expressions for $u(x, y)$ and $v(x, y)$, one deduces that

$$(y_0 + 2)x_0\psi'(1 + x_0y_0) = (y_0 + 4)x_0\psi'\Big(1 + (y_0 + 2)x_0\Big),$$

which further implies that

$$\frac{1}{2x_0y_0}\frac{\partial g}{\partial y}(x_0, y_0) = \psi(1 + x_0y_0) - \psi\Big(1 + (y_0 + 2)x_0\Big) + 2x_0\psi'\Big(1 + (y_0 + 2)x_0\Big) = 0,$$

which is certainly impossible in view of (2.8). Thus we conclude that (x_0, y_0) is a boundary point of D. Hence we need to check $g(x, y) \leq 0$ for the cases $x_0 = 1/4, 1/2$ or $y = 0, y \to +\infty$. The cases $g(x, 0) = 0$ and $g(1/4, y) \leq 0$ follow from our discussion on the situation $x \leq 1/4, y \geq 0$ above and for the case $y \to +\infty$, using the asymptotic expression (2.4) and (2.5), we deduce via simple calculations that as $y \to +\infty$,

$$g(x, y) = -\ln y + O(1) < 0.$$

It thus remains to check the case $x = 1/2$. In this case it follows from Lemma 2.3 that $u(1/2, y) < 0$ so that $g(1/2, y) \leq g(1/2, 0) = 0$.

Lastly, we need to show that $g(x, y) \leq 0$ for $1/2 < x \leq 1$ and $y \geq 1$. We note by Lemma 2.4 that in this case $g(x, y) \leq g(1, y)$ and also by Lemma 2.3 that $g(1, y)$ is a decreasing function of y. Hence it suffices to check that $g(1, 2) \leq 0$. In this case one checks easily by using the well-known fact $\Gamma(n + 1) = n!$, relation (2.3) and the observation that $\psi(1) = -\gamma$ from (2.1) that

$$\frac{g(1, 2)}{2} = 3 - \ln 2 - 2\gamma - 1/2 - 2/3 < 0,$$

and this completes the proof. $\qquad\square$

3. Volume of sections of B_p^n

We now apply Lemma 2.5 to estimate the volume of sections of B_p^n.

Theorem 3.1. *Let $n \in \mathbb{N}$, $n \geq 2$, $p \geq 1$ and let H be any $(n-1)$-dimensional subspace in $\mathbb{R}^n$. Then*

$$(3.1) \qquad \frac{\mathrm{Vol}_{n-1}(H \cap B_p^n)}{\left(\mathrm{Vol}_n(B_p^n)\right)^{(n-1)/n}} \geq \sqrt{\frac{\Gamma(1 + \frac{4}{p})\Gamma(1 + \frac{1}{p})^3}{\Gamma(1 + \frac{2}{p})^2\Gamma(1 + \frac{3}{p})}} \geq 1.$$

Proof. Let H be a hyperplane in $\mathbb{R}^n$. A well-known result (see [6, (11)]) ensures that

$$\mathrm{Vol}_{n-1}(H \cap B_p^n) L_{B_p^n} \geq \frac{1}{\sqrt{12}} \left(\mathrm{Vol}_n(B_p^n)\right)^{(n-1)/n}$$

where $L_{B_p^n}$ is (see [18], [8])

$$L_{B_p^n}^2 = \frac{\Gamma(1 + \frac{3}{p})\Gamma(1 + \frac{n}{p})^{1+2/n}}{12\Gamma(1 + \frac{n+2}{p})\Gamma(1 + \frac{1}{p})^3}.$$

Now it follows from Lemma 2.5 with $x = 1/p, y = n$ that for $n \geq 2$,

$$L_{B_p^n}^2 \leq L_{B_p^2}^2,$$

from which one deduces the first inequality of (3.1). The second inequality of (3.1) now follows from Lemma 2.5 for the case $p \geq 2$ and [8, Proposition 1.2] for the case $1 \leq p < 2$. $\qquad\square$

We remark here Theorem 3.1 recovers [18, Proposition 3.1] for the case $1 < p < 2$.

Acknowledgement

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