CUBATURE RULES FROM A GENERALIZED TAYLOR PERSPECTIVE

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by

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Abstract

The accuracy and efficiency of computing multiple integrals is a very important problem that arises in many scientific, financial and engineering applications. The research conducted in this thesis is designed to build on past work and develop and analyze new numerical methods to evaluate double integrals efficiently. The *fundamental aim* is to develop and assess techniques for (numerically) evaluating double integrals with high accuracy.

The general approach presented in this thesis involves the development of new multivariate approximations from a generalaised Taylor perspective in terms of Appell type polynomials and to study their use in multi-dimensional integration. The expectation is that the new methods will provide polynomial and polynomial-like approximations that can be used for application in a straight forward manner with better accuracy. That is, we aim to devise and investigate new multiple integration formulae and as well as provide information on *a priori* error bounds.

A further major contribution of the work builds on the research conducted in the field of Grüss type inequalities and leads to a new approximation of the one and two dimensional finite Fourier transform. The approximations are in terms of the complex exponential mean and estimate of the error of approximation for different classes of functions of bounded variation defined on finite intervals.

It is believed that this work will also have an impact in the area of numerical multidimensional integral evaluation for other integral operators.

Declaration

I, George Hanna, hereby declare that the PhD thesis entitled [Cubature Rules from a Generalised Taylor Perspective] is my own work and that, to the best of my knowledge and belief. It is no more than 100,000 words in length including quotes and exclusive of tables, figures, appendices, bibliography, references and footnotes. This thesis contains no material that has been submitted previously, in whole or in part, for the award of any other academic degree or diploma. Except where otherwise indicated, this thesis is my own work.

Signature Date

Acknowledgment

A journey is easier when you travel together. Depending on each other is certainly more valuable than independence. This thesis is the result of four and half years of work whereby I have been accompanied and supported by many people. It is a pleasant aspect that I have now the opportunity to express my gratitude to all of them.

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Dedication

To the memory of my father " TODARY ".
To the memory of my mother " MARY ".

كم كنت اتمنى
ان احقق هذا الحلم
فى وجودك !!!!

Last but not least, I praise **GOD** for leading me to pursue this dream, answering all my prayers and fulfilling all my needs during this study

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Papers published during the author's candidature

From the material in this thesis there are, at the time of submission, some papers which have been published in refereed publications or have been submitted for publication.

- Part of the work from Chapter 2 has been published in the *Tamkang Journal of Mathematics* **33**(4), (2002), pp. 319-333.
- Part of the work from Chapter 2 has been published in the *Journal of Concrete and Applicable Mathematics*, **3**(4), (2005), pp. 389-404.
- Part of the work from Chapter 2 has been published as a chapter in the book: S. S. Dragomir and T. M. Rassias (Eds), *Ostrowski Type Inequalities and Application in Numerical Integration*, Kluwer Academic Publishers, pp. 331-372.
- Part of Chapter 3 has been published in the *Tamsui Oxford Journal of Mathematical Sciences*, **18**(1), (2002), pp. 1-16.
- Part of Chapter 4 has been published in the *Journal of the Korea Soc. for Industrial Math.*, **9**(2), (2005), pp. 1-16.
- Part of Chapter 5 has been published in the *Bull. Korean Math. Soc.*, **42**(4), (2005), pp. 725-738.
- Part of Chapter 6 has been published in the *Journal of the Korea Soc. for Industrial Math.*, **8**(1), (2004), pp. 31-40.
- Part of Chapter 7 has been published in the *Proc. 4th Int. Conf. on Modeling and Simulation*, Victoria University, (2002), pp. 375-380.
- Part of Chapter 7 is in the preprint *RGMIA Res. Rep. Coll*, **7**(2), (2004), Article 5 and has been submitted for publication.

In addition, the following papers have also been published during the author's candidature and which are not directly related to the thesis,

- G. Hanna and S. S. Dragomir, Some Ostrowski Type Inequalities for Double Integrals of Functions whose Partial Derivatives Satisfy Certain Convexity Properties. (Published in Y. J. Cho, J. K. Kim and S. S. Dragomir (Eds), (2003), *Inequality Theory and Applications*, **Volume 2**, New York, Nova Science Publishers, Inc, pp. 113-126).
- G. Hanna, J. Roumeliotis and A. Kucera. Collocation and Fredholm Integral Equations of the First Kind. (Published in the *Journal of Inequalities in Pure and Applied Mathematics* , (2005), **Volume 6**, Issue 5, Article 131).

Chapter 1

Introduction

1.1 Background

Integration is an important, indeed fundamental part of countless problems of interest. In practice, integrals are not known exactly and require numerical treatment. This procedure is known as *numerical integration* or *quadrature*. Much effort over the past 150 years has been expended into the search for efficient integration routines and an analysis of their accuracy. The bulk of this work considered the treatment of single dimensional integrals. The problem of two or higher dimensions is much more difficult. Notwithstanding the contributions of a number of authors over the last 30 years, there is still little knowledge of multi-dimensional integration in comparison with the univariate quadrature with much scope for further development.

The accurate and efficient evaluation of multiple integrals is a very important problem that arises in many scientific, financial and engineering applications. The principal idea behind this research is to build on past work and develop and analyze numerical methods to evaluate double integrals efficiently. The *fundamental aim* is to develop and assess techniques for numerically evaluating multidimensional integrals.

The general method presented in this thesis involves the development of new multivariate approximations, known as Taylor like approximations and to study their use in integration. It is expected that the new methods will provide polynomial and polynomial-like approximations that can be used for application in a straight forward manner with better accuracy . That is, the intention behind this research is to devise and investigate new multiple integration formulae and as well as provide *a priori* error information. A similar method has been used with much success for one dimensional problems. It is believed that this work will also have an impact in the area of numerical multidimensional integral evaluation for other integral operators.

This research aims to extend the work for multidimensional integration and hence its impact on real world problems. In particular, the methodology to be created involves developing a general Taylor-like expansion for multivariate functions and representing the remainder in an integral form, which will allow a better estimation using the Theory of Integral Inequalities. This will provide new tools for the numerical evaluation of double integrals via Bernoulli and Euler polynomials, the properties of which are well documented in the literature.

The research will also give numerical approximations that can be used in the numerical analysis of partial differential equations, or integral equations for two independent variables, and provide new tools for the approximation of integral operators expressed in terms of double integrals (for example, Fourier transform in two dimensional optics or Hankel transforms, etc.).

In addition, this research focuses on the symbolic computation of Appell polynomials using the computer algebra system “Maple" (Char *et al.* (1991)). The work will be tested against a comparable procedure for different examples of Appell polynomials and indeed comparison with the more common multidimensional integration techniques will be made. The procedures will be implemented and some software developed.

Numerical integration of univariate integrals has a long history. Classical rules such as the trapezoidal (Lyness and Genz (1980)) and Simpson rule (Kohler (1991)) calculate the integral exactly for polynomials of degree 1 and 3 respectively. High order rules have been developed to give exact results for arbitrary order polynomials (known as Newton-Cotes integration).

Assessment of the error of such approximations is based on Taylor's series. For example, the error in the integration $\int_a^b f(x)dx$ for Simpson's rule is bounded by

$$\frac{(b-a)^5}{2880}\left\|f^{(4)}\right\|_\infty$$

where $\left\|f^{(4)}\right\|_\infty$ is the maximum value of the 4^{th} derivative of the function.
In 1975, G. N. Milovanović generalised the inequality in Theorem 2.1 due to Ostrowski (1938) to the case where f is a function of several variables.

Recently, the Research Group in Mathematical Inequalities and Applications (RGMIA) within the School Computer Science and Mathematics in Victoria University, has carried out a considerable amount of work in the application of the Modern Theory of Inequalities to obtain *a priori* bounds of a variety of Newton-Cotes rules (Anastassiou and Dragomir (2001), Cerone *et al.* (1999a), Cerone *et al.* (2000), Cerone (2001), Dragomir and Wang (1998a), Roumeliotis *et al.* (1999)) for which the classical rules of mid-point, trapezoidal and Simpson's are special cases. Error bounds in terms of a variety of norms (a term used to describe a measure of the behavior of the function) were provided (see Barnett and Dragomir (2001), Cerone *et al.* (1999a), Dragomir (2001a), Dragomir and Wang (1998a)).

In another important development, Matić *et al.* (1999) derived an estimation using a perturbed generalized one-dimensional Taylor's formula. Using his theorem any integral $\int_a^b f(t)dt$ can be expressed as follows

$$\int_a^b f(t)dt = A_n(f;a,x) + R_n(f;a,x)$$

where the approximation to the integral $A_n(f;a,x)$ can be evaluated, and the error $R_n(f;a,x)$ is a one dimensional integral of a product Appell polynomial (Appell 1880) and the $(n+1)^{th}$ derivative of the function to be integrated. The importance of this result is again the ability to determine *a-priori* error bounds which are also be sharper than the classical bounds in some cases.

For multiple integrals the pioneering work was done by Stroud (1971). More recently, Cools and his group (Cools *et al.* (1997)) (Numerical Integration, Nonlinear Equation and Software – NINES), have developed CUBPACK++ which is specifically designed for double integrals over a variety of regions (cubature). Their work can be seen as an extension of Stroud's work (Stroud (1971)) (see also Cools (1999)). The NINES group presented both theoretical and practical aspects of multidimensional integration, a comprehensive bibliography, and provided cubature rules for different shaped regions. Assessment of the bounds is done via a Taylor-like expansion due to Sard (Sard (1963))(see also Stroud (1971, p. 138)). However these bounds cannot be determined prior to estimating the integral. There are, of course, other methods that have been developed to estimate multiple integrals (see Hanna *et al.* (2000), Hanna *et al.* (2002b) and Sloan and Lyness (1989)). The Monte Carlo Method (MCM) is one of the most popular methods used. The basic idea in MCM is to replace an analytic problem with a probabilistic problem of the same solution, and then investigate the latter problem by statistical simulation. These are useful for functions whose convergence is slow, for integral domains that are irregular, or for larger dimensions.

Other methods have been stated for decreasing the error in the MCM. All such approximations are called Quasi-Monte Carlo Methods. Many different Quasi-Monte Carlo Methods were developed by Haber (see Haber (1967), Haber (1970)). An extensive theory of number-theoretic-methods (NTM) is given by Korbove (Korobov (1963)). Recently, new references for NTM have been given by Fang and Wang (1994), Fang and Zhang (1999).

Some other numerical methods and techniques have been used for multidimensional integration. For example, adaptive quadrature (Rice (1973)) is a powerful automatic procedure for increasing the accuracy of numerical approximation to an integral by increasing the number of samples of the integrand. It should be noted that:

(i) When an adaptive algorithm is used, the nodes at which the integrand is evaluated cannot be determined beforehand. Therefore, adaptive techniques are inappropriate for tabulated integrands. An even more important consequence is that *a priori* error results are not available. This contrasts with the current research which aims to provide such *a priori* bounds.

(ii) Often adaptive strategies for multiple dimensions are simply iterated decomposition of single dimensional integrals. The work here seeks to evaluate and provide error bounds for multiple dimensional integrals without resorting to decompositions since this will help in sharpening the error bound.

To date there has been little work in developing *a priori* bounds for multiple integrals. It is expected that by combining the approach used by RGMIA for single integration and generalizing the theorem given by Matić *et al.* (2001), that general progress in this direction can be made.

1.2 Aims and Outcomes

The principal aim of this thesis is to develop techniques and, in particular, assess these with regards to numerically evaluating double integrals. The discretisation will be predetermined to produce an estimate within given tolerance limits. The performance of the current methodology will be evaluated with respect to two dimensions, for specificity. The expected outcomes are:

- the development of a general Taylor-like expansion for functions of two variables in terms of Appell type polynomials;
- the representation of the remainder in a double integral form which will allow a better estimation using the Theory of Integral Inequalities (including Grüss type inequalities);
- the provision of new tools for the numerical evaluation of double integrals via Bernoulli and Euler polynomials;
- achievement of a sharper analysis of the error bounds;
- numerical approximation that can be used in the numerical analysis of partial differential equations or integral equations for two independent variables;
- provision of new tools for approximation of integral operators expressed in terms of double integrals (for example Fourier transform in two dimensional optics or Hankel transforms, etc.).

It is believed that the current work will have a significant impact in the area of numerical multidimensional integral evaluation.

The project achieves generalization for two arbitrary polynomials in two variables of Appell type (Appell (1880)) employing the Taylor-like formula of Sard (1963). This will play a fundamental role in obtaining kernel theorems and error estimates for the remainders in cubature formulae. Applications to integral operators, with a broad scope for applications in Physics, Engineering, and other practical domains will be an outcome of the current investigation.

1.3 Outline of the Thesis

A review of the one Ostrowski type inequalities are investigated, and some recent results relating to it are given in Chapter 2.

In Chapter 3, the utilization of the theorem obtained by Sard (1963) to develop an inequality for Taylor's expansion of two variables defined on a rectangular plane will be considered. Also, a development of a Grüss type inequality for double integrals where Korkine's identity is applied. Moreover, utilizing the result obtained to develop a perturbed version of the Taylor expansion. An application for this expansion and some related numerical results are demonstrated.

Chapter 4 aims to extend the work of Chapter 3 to explore a new Taylor's expansion which is comprised of the product of two polynomials, each of which satisfies the Appell condition (Appell (1880)). Also, new multiple integration formulae which provide *a priori* error information are devised and investigated.

In Chapter 5, we consider a reverse of the Cauchy-Bunyakovsky-Schwarz integral inequality for complex-valued functions. A pre-Grüss type inequality is obtained when one of the factors is known and some bounds for the second factor are provided. Numerical and graphical experiments of the obtained results are given for some functions with different behaviours.

In Chapter 6, some approximations of the finite Fourier transform in terms of the exponential mean and estimate the error of approximation for different classes of mappings of bounded variation defined on finite intervals for functions of one variable are established. Also, some numerical and graphical results are shown.

The focus on approximating the two dimensional finite Fourier transform to obtain some integral inequalities for the estimation is taken up in Chapter 7. Finally, a method is developed which provides for the possibility to approximate the integral of the product of functions in terms of the product of integrals is developed.

Chapter 2

History of the Ostrowski Inequality

A review of the one and two dimensional Ostrowski type inequalities are investigated, and some recent results are given in this chapter. Applications of the cubature formulas are produced and some related numerical results are demonstrated.

The chapter is arranged in the following manner. In Section 2.2, a short definition of the Peano Kernel is given. In Section 2.3, a review of the Ostrowski type inequalities using different types of norms is undertaken. Also, the three point technique of the Ostrowski inequalities in terms of L_p-norms $(1 \leq p \leq \infty)$, where at most the first derivatives are involved in the bound, are demonstrated. Some generalisations of Ostrowski type inequalities in one dimension for $n-$times differentiable functions are illustrated.

In Section 2.4, results attained by utilizing the techniques used in the previous section to obtain two dimensional Ostrowski inequalities in different types of norms, as well as, the two dimensional three points are given. Also, some generalisations of Ostrowski type inequalities in two dimensions for $n-$times differentiable functions are shown. The results involve integral inequalities with bounds in terms of the n^{th} derivative of the integrand. These are then employed to approximate double integrals using one dimension integrals and functions evaluated at the interior points.

In Section 2.5, applications of some of the cubature formulas which are produced in the previous section are illustrated numerically and some related plots are demonstrated.

Finally, Section 2.6 focuses on two dimensional integral inequalities. It shows weighted first and second order double integral inequalities, where the focus is on minimising the bound for different weights and weight null-spaces.

2.1 Introduction

Many of the techniques used for developing multiple integral inequalities are based on analogous one dimensional results. With this in mind, this section will focus on one dimensional integral inequalities and we review some recent results.

The classical Ostrowski integral inequality in one dimension stipulates an error bound in approximating a function evaluated at an interior point *x* by the average of the function f over an interval (see for example, Mitrinović *et al.* (1994, p. 468)). That is,

THEOREM 2.1. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I°, $(I^\circ$ *is the interior of* $I)$ *and let* $a, b \in I^\circ$ *with* $a < b$. *If* $f' : (a, b) \to \mathbb{R}$ *is bounded on* (a, b), *that is,*

$$||f'||_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty,$$

then we have the inequality:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \left\| f' \right\|_\infty \tag{2.1}$$

for all $x \in [a, b]$. *The constant* $\frac{1}{4}$ *is sharp in the sense that it cannot be replaced by a smaller quantity.*

2.2 Peano Kernel

From an estimation or error analysis point of view, we observe that a method like the Peano kernel formula for quadrature rule errors is more general and can be applied in other cases besides interpolation. Further, it can be used for error bounds as well as for study of the behavior of the error itself. Consider all the functions $f \in C^{n+1}[a, b]$, then the error $E[f]$ can be represented by the formula $E[f] = \int_a^b f^{(n+1)}(t)K(t)dt$ where $K(t)$

is the Peano kernel for the error and is defined by

$$K(t) = \frac{1}{n!} E[g(x;t)], \tag{2.2}$$

$$g(x;t) = (x-t)_+^n = \begin{cases} (x-t)^n & if \quad x \geq t, \\ 0 \quad if \quad x \leq t. \end{cases}$$

where t is just a parameter in the g function and the E operates only with respect to the x variable. The fruitful thing about the Peano kernel, is that it can be used to determine the error in integration rules explicitly, as well as being applied for the case when the function has only a low order of differentiability. See Engelbrecht *et al.* (2003) for full definition about the Peano Kernel,

For full definition of the Peano Kernel,

2.3 One Dimensional Integral Inequalities

It is natural to obtain the corresponding bounds in term of the p-norms with $p \in [1, \infty)$. This was explicitly done by Dragomir and Wang (1997) and Dragomir and Wang (1998b). These results are stated below.

THEOREM 2.2. *Let f as be in Theorem 2.1 and let $f' \in L_p[a,b], (p > 1, \frac{1}{p} + \frac{1}{q} = 1)$, then the following inequality holds*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \frac{1}{b-a} \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}} \|f'\|_p, \tag{2.3}$$

where

$$\|f'\|_p := \left(\int_a^b |f'(t)|^p\, dt \right)^{\frac{1}{p}},$$

is the $L_p[a,b]$-norm.

and

THEOREM 2.3. *Let f be defined as in (2.1). Further, let $f' \in L_1[a,b]$. The following inequality holds*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left| \frac{1}{2} + \frac{x - \frac{a+b}{2}}{b-a} \right| \|f'\|_1 \tag{2.4}$$

for all $x \in [a,b]$ *and* $\|f'\|_1 := \int_a^b |f'(t)|\, dt$. *The constant* $\frac{1}{2}$ *is the best possible.*

Notice that the above inequalities (2.3) and (2.4) can be obtained in an equivalent form from Fink (1992) by choosing $n = 1$ and performing the corresponding calculations.

The above three theorems can be proved by utilizing the Peano kernel $K(.,.) : [a,b]^2 \to \mathbb{R}$,

$$K(x,t) := \begin{cases} t-a & \text{if } t \in [a,x] \\ t-b & \text{if } t \in (x,b], \end{cases} \tag{2.5}$$

and the Montgomery identity (see for example, Mitrinović *et al.* (1994, Chapter XVII, P. 565)):

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + \frac{1}{b-a}\int_a^b K(x,t)f'(t)dt, \quad x \in [a,b]. \tag{2.6}$$

Since Ostrowski first produced his inequality in 1937, Anastassiou (1995) established an optimal upper bound on the deviation of n-time differentiable function from its average. He gave a different proof to Theorem 2.1 than Ostrowski's original proof (see Ostrowski (1938)). Also, he obtained more general Ostrowski type inequalities as follows.

THEOREM 2.4. *Let* $f \in C^{n+1}([a,b]), n \in \mathbb{N}$ *and* $x \in [a,b]$ *be fixed, such that* $f^{(k)}(x) = 0, \quad k = 1, \cdots, n$. *Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{(x-a)^{n+2} + (b-x)^{n+2}}{b-a} \right] \frac{\|f^{(n+1)}\|_\infty}{(n+2)!}. \tag{2.7}$$

Corollary 2.4.1. *Let* $f \in C^{n+1}([a,b]), n \in \mathbb{N}$ *be such that* $f^{(k)}((a+b)/2) = 0, \quad all \quad k \in \{1, \cdots, n\}$. *Then*

$$\left| f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \frac{(b-a)^{n+1}}{2^{n+1}} \times \frac{\|f^{(n+1)}\|_\infty}{(n+2)!}. \tag{2.8}$$

Further, Milovanović and Pečarić (1976) generalised the order of the derivative in (2.1) to an arbitrary n by considering n-times differentiable mappings as shown in the following theorem.

THEOREM 2.5. *Let* $f(x)$ *be an* $n(\ge 1)$ *times differentiable function such that* $f^{(n)} \in L_\infty[a,b]$ *for* $x \in (a,b)$. *Then, for every* $x \in [a,b]$

$$\left| \frac{1}{n}\left(f(x) + \sum_{k=1}^{n-1} F_k \right) - \frac{1}{b-a}\int_a^b f(y)dy \right| \le \left[\frac{(x-a)^{n+1} + (b-x)^{n+1}}{b-a} \right] \frac{\|f^{(n)}\|_\infty}{n(n+1)!}, \tag{2.9}$$

where F_k is defined by

$$F_k = F_k(f; n; x, a, b) = \frac{n-k}{k!} \frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a}. \tag{2.10}$$

Equation (2.9) was proved by employing Taylor's formula

$$f(x) = f(y) + \sum_{k=1}^{n-1} \frac{1}{k!} f^{(k)}(y)(x-y)^k + \frac{1}{n!} f^{(n)}(\xi)(x-y)^n \tag{2.11}$$

and integration by parts, (see Mitrinović *et al.* (1994) for the complete proof).

Remark 2.5.1. *Substituting $n = 1$ in (2.9) produces (2.1).*

Fink (1992) used the integral remainder form of a Taylor series to generalize the Milovanović and Pečarić (1976) result (Theorem 2.5) to include functions in L_p spaces.

THEOREM 2.6. *Let $f^{(n-1)}$ be absolutely continuous on $[a, b]$ with $f^{(n)} \in L_p[a, b]$ then*

$$\left| \frac{1}{n}\left(f(x) + \sum_{k=1}^{n-1} F_k \right) - \frac{1}{b-a} \int_a^b f(y)dy \right| \leq K(n, p, x) \|f^{(n)}\|_p \tag{2.12}$$

where

$$K(n, p, x) = \frac{1}{n!} \frac{[(x-a)^{n+\frac{1}{q}} + (b-x)^{n+\frac{1}{q}}]^{1/q}}{b-a} B((n-1)q+1, q+1)^{1/q},$$

for $1 < p \leq \infty,\ \frac{1}{p} + \frac{1}{q} = 1$

and

$$K(n, 1, x) = \frac{(n-1)^{n-1}}{n^n n!} \frac{\max\{(x-a)^n, (b-x)^n\}}{b-a}$$

with $B(x, y)$ representing the beta function of Euler, that is

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad x, y > 0.$$

Remark 2.6.1. *It is easily observed that for $n = 1$, the result is as in Theorem 2.2.*

Cerone *et al.* (2000), proved the following perturbed inequality of Ostrowski type for mappings which are twice differentiable:

THEOREM 2.7. *Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable mapping on (a, b) and $f'' \in L_p(a, b)(p > 1)$. Then, we have the following inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \left(x - \frac{a+b}{2}\right) f'(x) \right| \tag{2.13}$$
$$\leq \frac{1}{2(b-a)(2q+1)^{\frac{2}{q}}} \left[(x-a)^{2q+1} + (b-x)^{2q+1}\right]^{\frac{1}{q}} \|f''\|_p$$
$$\leq \frac{(b-a)^{1+\frac{1}{q}} \|f''\|_p}{2(2q+1)^{\frac{1}{q}}}$$

for all $x \in [a, b]$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$.

Dragomir and Sofo (2000) obtained the following inequality in the case where the second derivative belongs to the L_∞ norm.

THEOREM 2.8. *Let $f : [a, b] \to \mathbb{R}$ be a mapping whose first derivative is absolutely continuous on $[a, b]$ and assume that the second derivative $f'' \in L_\infty[a, b]$. Then, the following inequality holds:*

$$\left| \int_a^b f(t)\,dt - \frac{1}{2}\left[f(x) + \frac{f(a)+f(b)}{2}\right](b-a) + \frac{(b-a)}{2}\left(x - \frac{a+b}{2}\right) f'(x) \right| \tag{2.14}$$
$$\leq \left(\frac{1}{3}\left|x - \frac{a+b}{2}\right|^3 + \frac{(b-a)^3}{48}\right) \|f''\|_\infty$$

for all $x \in [a, b]$.

Cerone *et al.* (1999b), established a generalization of the Ostrowski inequality for n-times differentiable mappings which naturally generalizes the result from (2.1), as given in the following theorem:

THEOREM 2.9. *Let $f : [a, b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$ and $f^{(n)} \in L_\infty[a, b]$. Then for all $x \in [a, b]$, we have the inequality:*

$$\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x) \right|$$
$$\leq \frac{[(x-a)^{n+1} + (b-x)^{n+1}]}{(n+1)!} \|f^{(n)}\|_\infty \leq \frac{(b-a)^{n+1}}{(n+1)!} \|f^{(n)}\|_\infty \tag{2.15}$$

where

$$||f^{(n)}||_\infty := \sup_{t \in [a,b]} |f^{(n)}(t)| < \infty.$$

The theorem is proved utilizing mathematical induction and using the Peano kernel $K(.,.) : [a,b]^2 \to \mathbb{R}$,

$$K(x,t) := \begin{cases} \frac{(t-a)^n}{n!}, & \text{if } t \in [a,x], \\ \frac{(t-b)^n}{n!}, & \text{if } t \in (x,b]. \end{cases} \tag{2.16}$$

The kernel (2.16) is similar in sense to that of (2.5). It vanishes at the boundary points and is discontinuous at the interior point, thus producing a rule that provides sampling at the interior point and not at the end points. Since (2.16) is a polynomial of order n, an integral inequality in the n^{th} derivative will result in (2.15). We can compare this to (2.1) which has a bound in the first derivative due to the linear Peano kernel.

Another extension was proposed and explored by Cerone (2001) wherein the constants 'a' and 'b' in the kernel (2.5) were replaced by linear parametric functions- the zeroes and discontinuity of the kernel were themselves functions whose positions were allowed to change.
The kernel is

$$K(x,t) := \begin{cases} t-\alpha(x), & \text{if } t \in [a,x], \\ t-\beta(x), & \text{if } t \in (x,b], \end{cases} \tag{2.17}$$

where

$$\alpha(x) = \gamma x + (1-\gamma)a \qquad \text{and} \qquad \beta(x) = \gamma x + (1-\gamma)b \tag{2.18}$$

$\gamma \in [0,1]$ and $x \in [a,b]$. Hence the sampling occurs at three points, the boundary 'a' and 'b' and the point x. The sampling is controlled by the parameter γ, (see also, Cerone and Dragomir (2003a), Cerone and Dragomir (2003b), Cerone and Dragomir (2003c)). This is shown in the next theorem.

THEOREM 2.10. *Let f and f' be as in Theorem 2.1. Further, let $\alpha : [a,b] \to \mathbb{R}$ and $\beta : [a,b] \to \mathbb{R}$ with $a \le \alpha(x) \le x \le \beta(x) \le b$. Then, for all $x \in [a,b]$, we have the inequality*

$$\begin{aligned} &\left| \int_a^b f(t)dt - [(\beta(x)-\alpha(x))f(x) + (b-\beta(x))f(b) + (\alpha(x)-a)f(a)] \right| \\ &\qquad \le \left\{ \frac{1}{2}\left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{b-a}{2}\right)^2 \right] \right. \\ &\qquad\quad \left. + \left(\alpha(x) - \frac{a+x}{2}\right)^2 + \left(\beta(x) - \frac{b+x}{2}\right)^2 \right\} ||f'||_\infty . \end{aligned} \tag{2.19}$$

Proof. Let $K(.,.) : [a,b]^2 \to \mathbb{R}$, where $K(x,t)$ is the kernel (2.17) and consider the integral

$$\int_a^b K(x,t)f'(t)dt.$$

Integrating by parts over the given intervals using (2.17) and simplifing produces an identity from which, taking the modulus and using well known properties of the modulus and integral, the results follows. □

Inspection of the bound in (2.19) reveals that α and β should be linear functions for the bound to be minimized. Thus the motivation is to prescribe a linear parameterization in (2.18). Utilizing equation (2.18), we get the following theorem.

THEOREM 2.11. *Let the conditions of Theorem 2.10 hold, then*

$$\left|\int_a^b f(t)dt - (b-a)\left\{(1-\gamma)f(x) + \gamma\left[\left(\frac{x-a}{b-a}\right)f(a) + \left(\frac{b-x}{b-a}\right)f(b)\right]\right\}\right| \leq 2\left[\frac{1}{4} + \left(\gamma - \frac{1}{2}\right)^2\right]\left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2\right]||f'||_\infty. \quad (2.20)$$

Remark 2.11.1. $\gamma = 0$ *in (2.20) reproduces Ostrowski's inequality equation (2.1) whose bound is sharpest where* $x = \frac{a+b}{2}$*, giving the mid-point inequality.*

Remark 2.11.2. $\gamma = 1$ *produces the generalized trapezoidal inequality for which again the best bound occurs when* $x = \frac{a+b}{2}$ *giving the standard trapezoidal-type inequality.*

Remark 2.11.3. $\gamma = \frac{1}{3}$ *gives a Simpson-type rule for which the value* $x = \frac{a+b}{2}$*, gives the optimal bound when only the assumption of a bounded first derivative is used.*

Further, the stated three-point rule when $f' \in L_p[a,b]$, is as shown below.

THEOREM 2.12. *Let* $f : [a,b] \in \mathbb{R}$ *be a differentiable mapping on (a,b) and* $f' \in L_p(a,b)$ *where* $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*. Then the following inequality holds for all* $x \in$

$[a,b],\ \alpha(x)\in[a,x]$ *and* $\beta(x)\in(x,b]$,

$$\begin{aligned}
&\left|\int_a^b f(t)dt-[(\beta(x)-\alpha(x))f(x)+(b-\beta(x))f(b)+(\alpha(x)-a)f(a)]\right|\\
&\le[(\alpha(x)-a)^{q+1}+(x-\alpha(x))^{q+1}+(\beta(x)-x)^{q+1}+(b-\beta(x))^{q+1}]^{\frac{1}{q}}(q+1)^{\frac{1}{q}}||f'||_p\\
&\qquad\le\left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}}||f'||_p\\
&\qquad\qquad\le(b-a)\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}}||f'||_p. \qquad (2.21)
\end{aligned}$$

Inequalities for higher order derivative norms are not the only extensions to Theorem 2.1. Introducing more branches of the Peano kernel; that is extending the number of discontinuities, will produce an integral inequality with many sampling points. This avenue has been explored by Dragomir with bounds involving the first derivative and by A. Sofo (see Dragomir and Rassias (2002), Chapter 2) involving the n^{th} derivative. Sofo used the Peano kernel

$$K_{n,k}(t):=\begin{cases}\frac{(t-\alpha_1)^n}{n!}, & t\in[a,x_1)\\ \frac{(t-\alpha_2)^n}{n!}, & t\in[x_1,x_2)\\ \vdots & \\ \frac{(t-\alpha_{k-1})^n}{n!}, & t\in[x_{k-2},x_{k-1})\\ \frac{(t-\alpha_k)^n}{n!}, & t\in[x_{k-1},b].\end{cases} \qquad (2.22)$$

To begin, it is immediately evident that $K_{n,k}(t)$ is of order n, thus the integral inequality will be bounded by a measure of $f^{(n)}$. In addition, (2.22) has discontinuities at $x_1,x_2,\cdots,x_{k-1}$ and does not vanish at the boundary, thus we would expect sampling at the points
$\{a,x_1,x_2,\cdots,x_{k-1},b\}$. The integral inequality furnished for this kernel is

$$\begin{aligned}
&\left|\int_a^b f(t)dt+\sum_{j=1}^{n}\frac{(-1)^j}{j!}\left[\sum_{i=0}^{k}\{(x_i-\alpha_i)^j-(x_i-\alpha_{i+1})^j\}f^{(j-1)}(x_i)\right]\right| \qquad (2.23)\\
&\le\frac{||f^{(n)}||_\infty}{(n+1)!}\sum_{i=0}^{k-1}\{(\alpha_{i+1}-x_i)^{n+1}+(x_{i+1}-\alpha_{i+1})^{n+1}\}\\
&\le\frac{||f^{(n)}||_\infty}{(n+1)!}\sum_{i=0}^{k-1}h_i^{n+1}\\
&\le\frac{||f^{(n)}||_\infty}{(n+1)!}(b-a)\nu^n(h)\quad if\quad f^{(n)}\in L_\infty[a,b],
\end{aligned}$$

where $h_i := x_{i+1} - x_i$ and $\nu(h) :=$ max $\{h_i \,|i = 0, \cdots, k-1\}$.

In recent years a number of articles have been written about generalizations of Ostrowski's inequality (see Anastassiou (1997), Cerone *et al.* (1999a), Hanna *et al.* (2000), Matić *et al.* (2000), Matić and Pečarić (2001), Pachpatte (2002b), Cerone and Dragomir (2004), Dragomir (2004), Pachpatte (2004), Ujević (2004a), Ujević (2004b) and Ujević (2005)). See also, Cheng (2001), Dragomir and Gomm (2003), Pachpatte (2002a) and Ujević (2003a).

2.4 Two Dimensional Integral Inequalities

Employing the Peano kernel and combining the work of Barnett and Dragomir (2001) and Hanna *et al.* (2000) produced an Ostrowski type inequality in two dimensions using the three point rule involving the $L_p,\ p \in [1, \infty)$, norms in terms of the first derivatives of the function. That is given in the following theorem:

THEOREM 2.13. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a differentiable mapping on $[a_1, b_1] \times [a_2, b_2]$ and let $f''_{t_1,t_2} = \frac{\partial^2 f}{\partial t_1 \partial t_2}$ be bounded on $(a_1, b_1) \times (a_2, b_2)$. That is,*

$$\left\| f''_{t_1,t_2} \right\|_\infty := \sup_{(x_1,x_2)\in(a_1,b_1)\times(a_2,b_2)} \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right| < \infty.$$

Furthermore, let $x_i \in (a_i, b_i)$ and introduce the parameterization α_i, β_i defined by

$$\alpha_i = (1-\gamma_i)\, a_i + \gamma_i x_i, \qquad \textit{and} \quad \beta_i = (1-\gamma_i)\, b_i + \gamma_i x_i,$$

where $\gamma_i \in [0,1]$, *for* $i = 1, 2$. *Then the following inequality holds*

$$|G(x_1,t_1,x_2,t_2)| \le \begin{cases} \frac{\left\|f''_{t_1,t_2}\right\|_\infty}{4}\left(1+(2\gamma_1-1)^2\right)\left[\left(\frac{b_1-a_1}{2}\right)^2+\left(x_1-\frac{a_1+b_1}{2}\right)^2\right] \\ \qquad \times\left(1+(2\gamma_2-1)^2\right)\left[\left(\frac{b_2-a_2}{2}\right)^2+\left(x_2-\frac{a_2+b_2}{2}\right)^2\right], \\ \frac{\left\|f''_{t_1,t_2}\right\|_p}{(q+1)^{\frac{2}{q}}}\left[\gamma_1^{q+1}+(1-\gamma_1)^{q+1}\right]^{\frac{1}{q}}\left[(x_1-a_1)^{q+1}+(b_1-x_1)^{q+1}\right]^{\frac{1}{q}} \\ \qquad \times\left[\gamma_2^{q+1}+(1-\gamma_2)^{q+1}\right]^{\frac{1}{q}}\left[(x_2-a_2)^{q+1}+(b_2-x_2)^{q+1}\right]^{\frac{1}{q}} \\ \qquad\qquad \textit{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \left\|f''_{t_1,t_2}\right\|_1 M_1M_2 \end{cases} \tag{2.24}$$

given that

$$G(x_1,t_1,x_2,t_2) = \sum_{k=1}^{3}\sum_{j=1}^{3} C_{k1}C_{j2}f_{jk} - \sum_{j=1}^{3}\left(C_{j1}I_{j2}+C_{j2}I_{j1}\right) + \int_{a_2}^{b_2}\int_{a_1}^{b_1} f\left(t_1,t_2\right)\,dt_1dt_2 \tag{2.25}$$

$$(f_{jk}) = \begin{pmatrix} f\left(a_1,a_2\right) & f\left(x_1,a_2\right) & f\left(b_1,a_2\right) \\ f\left(a_1,x_2\right) & f\left(x_1,x_2\right) & f\left(b_1,x_2\right) \\ f\left(a_1,b_2\right) & f\left(x_1,b_2\right) & f\left(b_1,b_2\right) \end{pmatrix}, \tag{2.26}$$

$$(C_{jk}) = \begin{pmatrix} \gamma_1(x_1-a_1) & \gamma_2(x_2-a_2) \\ (1-\gamma_1)\left(b_1-a_1\right) & (1-\gamma_2)\left(b_2-a_2\right) \\ \gamma_1\left(b_1-x_1\right) & \gamma_2(b_2-a_2) \end{pmatrix}, \tag{2.27}$$

$$(I_{jk}) = \begin{pmatrix} \int_{a_1}^{b_1} f(t_1,a_2)\,dt_1 & \int_{a_2}^{b_2} f(a_1,t_2)\,dt_2 \\ \int_{a_1}^{b_1} f(t_1,x_2)\,dt_1 & \int_{a_1}^{b_1} f(x_1,t_2)\,dt_2 \\ \int_{a_1}^{b_1} f(t_1,b_2)\,dt_1 & \int_{a_1}^{b_1} f(b_1,t_2)\,dt_2 \end{pmatrix}, \tag{2.28}$$

and

$$M_i = \frac{(b_i-a_i)}{4}\left[1+|2\gamma_i-1|\right] + 2\left|(x_i-\frac{a_i+b_i}{2})(1+|2\gamma_i-1|)\right|. \tag{2.29}$$

For the complete proof, see Hanna *et al.* (2000).

In addition, Pachpatte has obtained some inequalities involving functions of several independent variables and their first order partial derivatives as well as those of Ostrowski type in n independent variables (see Pachpatte (2001), Pachpatte (2002b)).

Further, in Pachpatte (2004) the author obtained some generalizations of the Ostrowski type inequality and also a new weighted integral and discrete inequalities of the Grüss type involving functions of several independent variables.

In the mean time, Hanna *et al.* (2002a) have obtained some generalizations of an Ostrowski type inequality in two dimensions for $n-$time differentiable mappings. The result is an integral inequality with bounded n^{th} derivatives. This is employed to approximate double integrals using integrals and function evaluations at the boundary and interior points.

THEOREM 2.14. *Let* $f:[a,b]\times[c,d]\to\mathbb{R}$ *be continuous on* $[a,b]\times[c,d]$*, and assume that* $\frac{\partial^{n+m}f}{\partial t^n\partial s^m}$ *exist on* $(a,b)\times(c,d)$*. Further, consider* $K_n:[a,b]^2\to\mathbb{R}$*,* $S_m:[c,d]^2\to\mathbb{R}$ *given by*

$$K_n(x,t):=\begin{cases}\frac{(t-a)^n}{n!}, & t\in[a,x],\\ \frac{(t-b)^n}{n!}, & t\in(x,b],\end{cases}\qquad S_m(y,s):=\begin{cases}\frac{(s-c)^m}{m!}, & s\in[c,y],\\ \frac{(s-d)^m}{m!}, & s\in(y,d],\end{cases}\tag{2.30}$$

Then we have the inequality

$$\begin{aligned}&\left|\int_a^b\int_c^d f(t,s)\,ds\,dt-\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}X_k(x)\cdot Y_l(y)\frac{\partial^{l+k}f}{\partial t^k\partial s^l}(x,y)\right.\\&\left.-(-1)^m\sum_{k=0}^{n-1}X_k(x)\int_c^d S(y,s)\frac{\partial^{k+m}f}{\partial t^k\partial s^m}(x,s)\,ds-(-1)^n\sum_{l=0}^{m-1}Y_l(y)\int_a^b K(x,t)\frac{\partial^{n+l}f}{\partial t^n\partial s^l}(t,y)\,dt\right|\\&\le\begin{cases}\frac{1}{(n+1)!(m+1)!}\left[(x-a)^{n+1}+(b-x)^{n+1}\right]\times\left[(y-c)^{m+1}+(d-y)^{m+1}\right]\times\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_\infty,\\ \qquad\text{if } \frac{\partial^{n+m}f}{\partial t^n\partial s^m}\in L_\infty([a,b]\times[c,d]);\\ \frac{1}{n!m!}\left[\frac{(x-a)^{nq+1}+(b-x)^{nq+1}}{nq+1}\right]^{\frac{1}{q}}\times\left[\frac{(y-c)^{mq+1}+(d-y)^{mq+1}}{mq+1}\right]^{\frac{1}{q}}\times\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_p,\\ \qquad\text{if } \frac{\partial^{n+m}f}{\partial t^n\partial s^m}\in L_p([a,b]\times[c,d]),\ p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{4n!m!}\left[(x-a)^n+(b-x)^n+|(x-a)^n-(b-x)^n|\right]\\ \qquad\times\left[(y-c)^m+(d-y)^m+|(y-c)^m-(d-y)^m|\right]\times\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_1,\\ \qquad\text{if } \frac{\partial^{n+m}f}{\partial t^n\partial s^m}\in L_1([a,b]\times[c,d]).\end{cases}\end{aligned}\tag{2.31}$$

for all $(x,y)\in[a,b]\times[c,d]$*, where*

$$\begin{aligned}\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_\infty&=\sup_{(t,s)\in[a,b]\times[c,d]}\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\right|<\infty,\\ \left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_p&=\left(\int_c^d\int_a^b\left|\frac{\partial^{n+m}}{\partial t^n\partial s^m}f(t,s)\right|^p dtds\right)^{\frac{1}{p}}<\infty,\end{aligned}$$

and

$$X_k(x) = \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}, \qquad Y_l(y) = \frac{(d-y)^{l+1} + (-1)^l (y-c)^{l+1}}{(l+1)!}. \tag{2.32}$$

Keeping in mind that x and y are free parameters, then one can produce “mid-point" and “boundary-point" type results by choosing appropriate values for x and y. In addition’ choosing values for n and m will re-capture the earlier results of Hanna *et al.* (2000) and Dragomir *et al.* (2000).

An iterative approach is used in (Cerone (2003a)) to represent multidimensional integrals in terms of lower dimensional integrals and function evaluations. The procedure is quite general utilising one dimensional identities as the *seed* or *generator* to procure multidimensional identities. Bounds are obtained from the identities.
In the following theorem bounds for $\tau_n\left(\vec{a},\vec{x},\vec{b}\right)$ are obtained where

$$\begin{aligned} &\tau_n\left(\vec{a},\vec{x},\vec{b}\right) \qquad (2.33)\\ =\ & f(x_1,x_2,\ldots,x_n) - \sum_{i=1}^{n} \frac{1}{d_i}\int_{a_i}^{b_i} f(x_1,x_2,\ldots,x_{i-1},t_i,x_{i+1},\ldots,x_n)\,dt_i \\ &+ \sum_{i<j}^{n} \frac{1}{d_j d_i}\int_{a_j}^{b_j}\int_{a_i}^{b_i} f(x_1,\ldots,x_{i-1},t_i,x_{i+1},\ldots,x_{j-1},t_j,x_{j+1},\ldots,x_n)\,dt_i dt_j \\ &- \cdots\cdots\cdots - \frac{(-1)^n}{D_n}\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1} f(t_1,\ldots,t_n)\,dt_1\ldots dt_n \end{aligned}$$

and $\vec{z} = (z_1,z_2,\ldots,z_n)$.

THEOREM 2.15. *Let the conditions of Theorem 6.3 continue to hold. Then*

$$\left|\tau_n\left(\vec{a},\vec{x},\vec{b}\right)\right| \tag{2.34}$$

$$\leq \begin{cases} \prod_{i=1}^{n} P_i(1) \left\|\frac{\partial^n f}{\partial t_n \ldots \partial t_1}\right\|_\infty, & \frac{\partial^n f}{\partial t_n \ldots \partial t_1} \in L_\infty[I^n]; \\ \left(\prod_{i=1}^{n} P_i(q)\right)^{\frac{1}{q}} \left\|\frac{\partial^n f}{\partial t_n \ldots \partial t_1}\right\|_p, & \frac{\partial^n f}{\partial t_n \ldots \partial t_1} \in L_p[I^n], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \prod_{i=1}^{n} \theta_i \left\|\frac{\partial^n f}{\partial t_n \ldots \partial t_1}\right\|_1, & \frac{\partial^n f}{\partial t_n \ldots \partial t_1} \in L_1[I^n], \end{cases}$$

where $\tau_n\left(\vec{a},\vec{x},\vec{b}\right)$ *is as defined in (2.33),*

$$(q+1)\,P_i\,(q) = (x_i - a_i)^{q+1} + (b_i - x_i)^{q+1}\,, \tag{2.35}$$

$$\theta_i = \frac{b_i - a_i}{2} + \left|x_i - \frac{a_i + b_i}{2}\right|. \tag{2.36}$$

2.5 Numerical Results

In this section the inequalities developed by the author and given by theorem 2.13 in the previous section are used to approximate the double integral. In the following example we select the integrand for which integrating in each direction is straightforward, but not so for the double integral.

Example 2.1.

$$\int_0^1 \int_0^1 \left(1 - e^{-xy}\right) dxdy = 0.203400400702947. \tag{2.37}$$

Namely, $\int_0^1 \left(1 - e^{-xy}\right)\, dx = \frac{y+e^{-y}-1}{y}$ *and* $\int_0^1 \left(1 - e^{-xy}\right)\, dy = \frac{x+e^{-x}-1}{x}$.

Example 2.1 was chosen also because the integrand $f(x,y)$ is infinitely smooth and its L_∞-norm becomes smaller with each successive derivative, because

$$\begin{array}{ll} f_x(x,y) = ye^{-xy} & f_y(x,y) = xe^{-xy} \\ f_{xx}(x,y) = -y^2 e^{-xy} & f_{yy}(x,y) = -x^2 e^{-xy} \\ \vdots & \vdots \\ \frac{\partial^n f(x,y)}{\partial x^n} = (-1)^{n+1} y^n e^{-xy} & \frac{\partial^n f(x,y)}{\partial y^n} = (-1)^{n+1} x^n e^{-xy} \end{array}$$

as we see, $\forall y \in [0,1)$ the derivative with respect to x tends to 0 as n tends to ∞, and also, $\forall x \in [0,1)$ the derivative with respect to $y \to 0$ as $n \to \infty$. This indicates that the higher order error bound (accompanied by a higher order rule) will give better results.

Example 2.2.

$$\int_0^1 \int_1^2 \frac{y}{x^2} e^{-y/x} dxdy = 0.1548181217. \tag{2.38}$$

The integrand in Example 2.2 was chosen because its L_∞-norm blows up rapidly with successive derivatives. That is, $\forall y \in [0,1)$ the derivative with respect to x tends to ∞ as n tends to ∞, and also, $\forall x \in [1,2)$ the derivative with respect to y tends to ∞ as n tends to ∞. This indicates that the higher order error bounds (accompanied by a lower order rule) will give better results.

γ_1	γ_2	actual error	L_∞-estimated error	L_2-estimated error	L_1-estimated error
0	0	1.5(-3)	6.3(-2)	5.7(-2)	1.6(-1)
$\frac{1}{3}$	$\frac{1}{3}$	5.4(-7)	1.9(-2)	1.9(-2)	7.1(-2)
0.5	0.5	4.3(-4)	1.6(-2)	1.4(-2)	3.9(-2)
1	1	6.5(-3)	6.3(-2)	5.7(-2)	1.6(-1)

Table 2.1: The actual and estimated errors in computing (2.37) using (2.24) with $x_1 = x_2 = 0.5$ and various values of γ_1, γ_2 in the $\|.\|_\infty$ norm, $\|.\|_2$ norm and $\|.\|_1$ norm respectively .

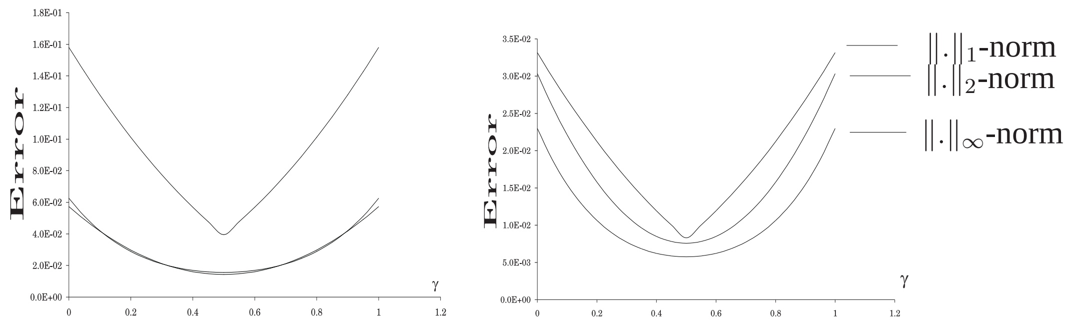
γ_1	γ_2	actual error	L_∞-estimated error	L_2-estimated error	L_1-estimated error
0	0	2.5(-3)	2.2(-2)	3(-2)	3.3(-1)
$\frac{1}{3}$	$\frac{1}{3}$	1.5(-5)	7.1(-3)	1(-2)	1.5(-2)
0.5	0.5	8.6(-4)	5.7(-3)	7.6(-3)	8.3(-3)
1	1	1.9(-2)	2.2(-2)	3(-2)	3.3(-1)

Table 2.2: The actual and estimated errors in computing (2.38) using (2.24), with $x_1 = x_2 = 0.5$ and various values of γ_1, γ_2 in the $\|.\|_\infty$ norm, $\|.\|_2$ norm and $\|.\|_1$ norm.

From this point of view we find that the actual error is much smaller than the theoretical one and is smallest when Simpson's rule is applied ($\gamma_1 = \gamma_2 = \frac{1}{3}$). The optimal theoretical bound is attained when $\gamma_1 = \gamma_2 = \frac{1}{2}$. It should be noted that $\gamma_1 = \gamma_2 = 0$ approximates (2.37) and (2.38) with the "mid-point" rule and employs one function evaluation (at the mid-point of the region) and two integrals (along the bi-sectors). The "trapezoidal" rule uses four sample points (the boundary corners) and four integrals (along the boundary). All other values, that is $\gamma_1, \gamma_2 \in (0,1)$, produce a rule that is a linear combination of the above and results in the use of nine sample points and six integrals.

Furthermore, Simpson's rule ($\gamma_1 = \gamma_2 = \frac{1}{3}$, nine sample points) is more accurate than the mid-point rule ($\gamma_1 = \gamma_2 = 0$, one sample point) which in turn is more accurate than the trapezoidal rule ($\gamma_1 = \gamma_2 = 1$, four sample points). We note that the estimated errors are symmetric about $\gamma_1 = \gamma_2 = \frac{1}{2}$ as in the Tables 2.1 and 2.2.

Cleary we observe from Figure 2.1 that the bound is convex in $\gamma_i \in [0,1]$ for $i = 1,2$. The sharpest occurs at $\gamma_i = \frac{1}{2}$ for $i = 1,2$. The harshest bound is achieved when γ_i are



(a) The estimated error as a function of γ in evaluating (2.37) with $x_1 = x_2 = 0.5$ and various values of γ_1, γ_2 in the $\|.\|_\infty$ norm (the first inequality in equation (2.24)), $\|.\|_2$ norm (the second inequality in equation (2.24)) and $\|.\|_1$ norm (the third inequality in equation (2.24)).

(b) The estimated error as a function of γ in evaluating (2.38) with $x_1 = x_2 = 0.5$ and various values of γ_1, γ_2 in the $\|.\|_\infty$ norm (the first inequality in equation (2.24)), $\|.\|_2$ norm (the second inequality in equation (2.24)) and $\|.\|_1$ norm (the third inequality in equation (2.24)).

Figure 2.1: Diagrammatic representation for the estimated error

taken at either of their boundary points.

Next we will employ the composite rules to explore the numerical results for both Example 2.1 and Example 2.2 respectively and produce briefly the actual and estimated errors in applying the mid-point cubature rules to evaluate the double integral (2.37) and (2.38) for an increasing number of intervals for the different norms.

Cleary, we notice that the actual error ratio in both tables suggests that the composite rule in each case has convergence

$$|R| \sim O\left(\frac{1}{m^2n^2}\right).$$

Also, from Table 2.3 and Table 2.4 we gather that the estimated error predicts a convergence rate of

- $|R| \leq \frac{\left\|f''_{t_1,t_2}\right\|_\infty}{16mn}$, $\left\|f''_{t_1,t_2}\right\|_\infty = 1$ (Example 2.1) and $\left\|f''_{t_1,t_2}\right\|_\infty = .37$ (Example 2.2),
- $|R| \leq \frac{\left\|f''_{t_1,t_2}\right\|_2}{12\sqrt{mn}}$, $\left\|f''_{t_1,t_2}\right\|_2 = .69$ (Example 2.1) and $\left\|f''_{t_1,t_2}\right\|_2 = .36$ (Example 2.2) ,
- $|R| \leq \frac{\left\|f''_{t_1,t_2}\right\|_1}{4mn}$, $\left\|f''_{t_1,t_2}\right\|_1 = .63$ (Example 2.1) and $\left\|f''_{t_1,t_2}\right\|_1 = .13$ (Example 2.2).

n	m	Actual Error	Err ratio	L_∞-estimated error	L_2-estimated error	L_1-estimated error
1	1	1.5(-3)	...	6.3(-2)	5.7(-2)	1.5(-1)
2	2	1.0(-4)	14.51	1.6(-2)	2.9(-2)	4.0(-2)
4	4	6.7(-6)	15.61	3.9(-3)	1.4(-2)	9.9(-3)
8	8	4.2(-7)	15.90	1.0(-3)	7.2(-3)	2.5(-3)
16	16	2.6(-8)	15.98	2.0(-4)	3.6(-3)	6.2(-4)
32	32	1.6(-9)	15.99	6.1(-5)	1.8(-3)	1.5(-4)
64	64	1.0 (-10)	16.00	1.5(-5)	8.9(-4)	3.9(-5)
128	128	6.6 (-12)	16.00	3.8(-6)	4.5(-4)	9.6(-6)

Table 2.3: The actual and estimated errors in evaluating (2.37) using a composite rule, for various values of n, m. Sampling occurs at the mid-point of each region.

n	m	Actual Error	Err ratio	L_∞-estimated error	L_2-estimated error	L_1-estimated error
1	1	2.5(-3)	...	2.2(-2)	1.2(-2)	3.3(-2)
2	2	2.1(-4)	12.32	5.7(-3)	6.1(-2)	8.2(-3)
4	4	1.4(-5)	14.68	1.4(-3)	3.0(-2)	2.1(-3)
8	8	8.9(-7)	15.62	3.5(-4)	1.5(-2)	5.1(-4)
16	16	5.6(-8)	15.90	8.9(-5)	7.6(-3)	1.3(-4)
32	32	3.5(-9)	15.97	2.2(-5)	3.8(-3)	3.2(-5)
64	64	2.2 (-10)	16.00	5.6(-6)	1.9(-3)	8.1(-6)
128	128	1.3 (-11)	16.00	1.4(-6)	9.5(-4)	2.0(-6)

Table 2.4: The actual and estimated errors in evaluating (2.38) using a composite rule, for various values of n, m. Sampling occurs at the mid-point of each region.

2.6 Weighted Integral Inequalities in Two Dimensions

A powerful approximation tool is that in Hanna and Roumeliotis (2005), in which they combined and extended the work of Hanna *et al.* (2000) and Cerone *et al.* (2000) and developed weighted first and second order double integral inequalities. Particular attention has been paid to the influence of the two dimensional weight function on the error bound and explored this influence for different weights and weight null-spaces.

Hanna and Roumeliotis (2005) considered the following identities:

Lemma 2.1. *Let* $f : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ *be bounded and integrable and whose first partial derivatives exist and are also bounded and integrable. Furthermore, let* $w : (a_1, b_1) \times (a_2, b_2) \to (0, \infty)$ *be integrable. Then the following identity holds*

$$I = \int_{a_1}^{b_1}\int_{a_2}^{b_2} \left[f\left(x_1, x_2\right) - f\left(x_1, t_2\right) - f\left(t_1, x_2\right) + f\left(t_1, t_2\right)\right] w\left(t_1, t_2\right)\, dt_2 dt_1$$
$$= \int_{a_1}^{b_1}\int_{a_2}^{b_2} P(t_1, t_2)\frac{\partial^2 f}{\partial t_1 \partial t_2}\, dt_2 dt_1 \quad (2.39)$$

where $x_1 \in [a_1, b_1]$*,* $x_2 \in [a_2, b_2]$ *and*

$$P(t_1, t_2) = \begin{cases} \displaystyle\int_{a_2}^{t_2} p\left(t_1, u_2\right) du_2, & a_2 \le t_2 \le x_2, \\ \displaystyle\int_{b_2}^{t_2} p\left(t_1, u_2\right) du_2, & x_2 < t_2 \le b_2, \end{cases} \quad (2.40)$$

with

$$p(t_1, t_2) = \begin{cases} \displaystyle\int_{a_1}^{t_1} w\left(u_1, t_2\right) du_1, & a_1 \le t_1 \le x_1, \\ \displaystyle\int_{b_1}^{t_1} w\left(u_1, t_2\right) du_1, & x_1 < t_1 \le b_1. \end{cases} \quad (2.41)$$

The upper bound of the integration rule will depend on P. Below, we detail some properties of P that will be subsequently used in the analysis of the bound (see Hanna and Roumeliotis (2005)).

Lemma 2.2. *The kernel* $P : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ *as defined in Lemma 2.1 has the following properties:*

1. *P vanishes on the boundary of the rectangle $[a_1, b_1] \times [a_2, b_2]$,*

2. *$P(t_1, \cdot) : (a_2, b_2) \to \mathbb{R}$ is monotonic increasing for all $t_1 \in (a_1, x_1)$,*

3. *$P(t_1, \cdot) : (a_2, b_2) \to \mathbb{R}$ is monotonic decreasing for all $t_1 \in (x_1, b_1)$,*

4. *P is positive on $(a_1, x_1) \times (a_2, x_2)$ and $(x_1, b_1) \times (x_2, b_2)$,*

5. *P is negative on $(a_1, x_1) \times (x_2, b_2)$ and $(x_1, b_1) \times (a_2, x_2)$.*

for all $(x_1, x_2) \in (a_1, b_1) \times (a_2, b_2)$.

In Figure 2.2, we plot the surface and contours of (2.40) for two different weights. The plots exhibit the properties discussed in Lemma 2.2. It is obvious that the kernel achieves its maximum deviation of its branches at the discontinuous point (x_1, x_2). In the following theorem we state the main result by employing the identity in Lemma 2.1 to produce second order weighted double integral inequalities (see Hanna and Roumeliotis (2005)). In contrast with the inequalities of the previous section, the upper bound here is comprised of just one term.

THEOREM 2.16. *Let the conditions of Lemma 2.1 hold. The following double integral inequalities involving the usual Lebesgue norms of the first mixed partial derivative of f hold:*

$$|I| \leq \left\| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right\|_\infty \int_{a_1}^{b_1} \int_{a_2}^{b_2} |x_1 - t_1| \, |x_2 - t_2| \, w\,(t_1, t_2) \, dt_2 dt_1, \tag{2.42}$$

if $\frac{\partial^2 f}{\partial t_1 \partial t_2} \in L_\infty\left([a_1, b_1] \times [a_2, b_2]\right)$, and

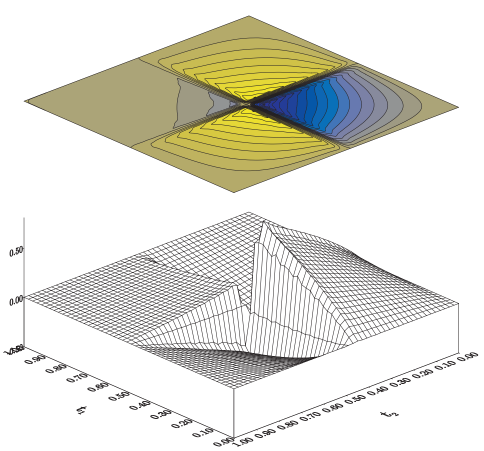
$$|I| \leq \left\| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right\|_1 \max\left\{ \int_{a_1}^{x_1} \int_{a_2}^{x_2} w(t_1, t_2) \, dt_2 dt_1, \int_{a_1}^{x_1} \int_{x_2}^{b_2} w(t_1, t_2) \, dt_2 dt_1, \right. \tag{2.43}$$

$$\left. \int_{x_1}^{b_1} \int_{a_2}^{x_2} w(t_1, t_2) \, dt_2 dt_1, \int_{x_1}^{b_1} \int_{x_2}^{b_2} w(t_1, t_2) \, dt_2 dt_1 \right\} \tag{2.44}$$

if $\frac{\partial^2 f}{\partial t_1 \partial t_2} \in L_1\left([a_1, b_1] \times [a_2, b_2]\right)$, where I is defined in equation (2.39).

Theorem 2.16 can form the basis of a cubature formula for weighted double integrals. That is, we can form a mesh and apply equation (2.42) to each grid rectangle.

(a)



(b)

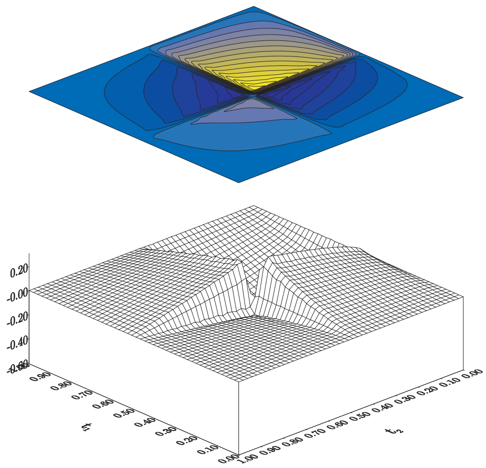


Figure 2.2: Surface and contour plots of the Peano type kernels P defined in (2.40) for different weights. (a) $w(t_1, t_2) = -\ln(t_1 t_2)$ over the unit square and $x_1 = x_2 = 0.5$, (b) $w(t_1, t_2) = \sqrt{t_1/t_2}$ over the unit square and $x_1 = x_2 = 0.5$.

Numerous other developments, extensions and generalisations of Ostrowski inequality have been carried out in various directions (see, Wang *et al.* (2006), Liu (2006), Barnett and Dragomir (2006)).

The analysis discussed in this chapter is used in the next, where we consider the Taylor theorem to extend the Ostrowski results for developing cubature and higher dimension rules. Thus, in the next chapter the Taylor's formula with the Lagrange type remainder will be obtained as well as Taylor expansion of two variables defined on a rectanglular plane. We also utilize Korkine's identity to derive a Grüss type inequality for double integrals that will be employed to obtain perturbed cubature rules which are sometimes more accurate than the unperturbed rules.

CHAPTER 3

NEW TAYLOR LIKE EXPANSIONS FOR FUNCTIONS OF TWO VARIABLES AND ERROR ESTIMATES

In this chapter, some sharp bounds are obtained for new Taylor-like expansions of functions of two variables utilising an integral remainder in which Korkine's identity is used to derive a Grüss type inequality for double integrals.

The chapter is arranged in the following manner. In Section 3.2 , Taylor's theorem with an integral remainder is recalled to obtain a Taylor's formula with a Lagrange type remainder. In Section 3.3, the theorem obtained by Sard (1963) to develop an inequality for Taylor's expansion of two variables defined on a rectangle plane is utilized. Section 3.4 is reserved for a Grüss type inequality for double integrals where Korkine's identity is applied. Finally, the result obtained in this section will be used in Section 3.5 to develop a perturbed version of the Taylor expansion. An application for this expansion is illustrated numerically and plots of the resulting approximation is given.

3.1 Introduction

Taylor's theorem is a popular vehicle for developing cubature and higher dimension rules. Stroud (1971) uses Taylor's expansion to develop cubature rules. Recently, a number of authors have obtained generalizations of the traditional Taylor's series expansion of a function $f(x)$ about a point a assuming sufficient differentiability. The drawback of this approach is in the size of the error bound. For two dimensions, an n-th order rule has a Taylor remainder of $n+1$ terms. Minimizing the error in any rule with order greater than one would be extremely difficult. Thus, the work in this chapter, turns to utilising Korkine's identity, that is

$$\frac{1}{b-a}\int_a^b u(t)v(t)dt - \frac{1}{b-a}\int_a^b u(t)dt \cdot \frac{1}{b-a}\int_a^b v(t)dt \tag{3.1}$$
$$= \frac{1}{2(b-a)^2}\int_a^b\int_a^b (u(t)-u(s))(v(t)-v(s))dtds,$$

provide that $u, v : [a,b] \to \mathbb{R}$ are measurable and all the involved integrals exist. We use identity 3.1 to produce perturbed generalizations of the normal and traditional Taylor expansion of a function $f(x)$ about a point assuming adequate differentiability (Hanna *et al.* (2002b)). Also, Grüss type inequalities are used to provide a means of approximating the integral of the product in terms of the product of integrals.

3.2 A Taylor Like Formula for Mappings of Two Variables for a Rectangular Plane

A number of authors have recently considered generalisations of the traditional Taylor series expansion of a function $f(x)$.

Milovanović (1975) utilized the multiple variable Taylor formula to generalise the Ostrowski inequality to multiple dimensions. As per the Ostrowski (1938) result, the inequality was expressed in terms of the first partial derivatives of the integrand.

Matić *et al.* (1999) derived an estimation using a perturbed generalized one-dimensional Taylor's formula.

Guo and Qi (2003) obtained an integral estimation using the L_p norm of the $(n+1)$-th derivative of its integrand.

Ujević (2003b) developed a perturbation of the classical Taylor formula where lower and upper error bounds are established. One may consider the papers along this line which have been written by Dragomir *et al.* (2001), Cerone (2003b), Barnett *et al.* (2002), Hanna *et al.* (2002b), Bougoffa (2003) and Dah-Yan (2004).

The concern in this section is directed at approximating the remainder using a perturbed generalised one-dimensional Taylor's formula. For example, the following theorem is well known in the literature as Taylor's theorem with an integral remainder.

THEOREM 3.1. *Let $I \subset \mathbb{R}$ be a closed interval, let $a \in I$ and let n be a positive integer. If $f : I \to \mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous, then for each $x \in I$*

$$f(x) = T_n(f; a, x) + R_n(f; a, x), \tag{3.2}$$

where $T_n(f; a, x)$ is the Taylor's polynomial, i.e.,

$$T_n(f; a, x) := \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) \tag{3.3}$$

(note that $f^{(0)} = f$ and $0! = 1$), and the remainder is given by

$$R_n(f; a, x) := \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t)\, dt. \tag{3.4}$$

A simple proof of this theorem can be accomplished by mathematical induction using the integration by parts formula. The following corollary concerning the estimation of the remainder is useful when we want to approximate specific functions by Taylor's expansions.

Corollary 3.1.1. *With the assumptions of Theorem 3.1, we have the estimation:*

$$|R(f; a, x)| \le \frac{(x-a)^n}{n!} \int_a^x \left| f^{(n+1)}(t) \right| dt \tag{3.5}$$

or

$$|R(f; a, x)| \le \frac{1}{n!} \cdot \frac{(x-a)^{n+\frac{1}{q}}}{(nq+1)^{\frac{1}{q}}} \left(\int_a^x \left| f^{(n+1)}(t) \right|^p dt \right)^{\frac{1}{p}} \tag{3.6}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, or the bound

$$|R(f; a, x)| \le \frac{(x-a)^{n+1}}{(n+1)!} \max_{t \in (a,x)} \left| f^{(n+1)}(t) \right| \tag{3.7}$$

for all $x \ge a$, $a \in I \subset \mathbb{R}$.

The case of a multivariable function can be stated as follows (see Sard (1963)):

THEOREM 3.2. *Let f be in the class $C^{(q)}(\mathbb{R}^n)$ and x, $x_0 \in D \subset \mathbb{R}^n$ so that the line segment joining x and x_0 is contained in D and let $h = x - x_0$. Then we have Taylor's formula with a Lagrange type remainder:*

$$\begin{aligned} f(x) &= f(x_0) + \sum_{i=1}^{n} f_i(x_0) h^i + \frac{1}{2!} \sum_{i,j=1}^{n} f_{ij}(x_0) h^i h^j + \dots \\ &\quad + \frac{1}{(q-1)!} \sum_{i_1,\dots,i_{q-1}=1}^{n} f_{i_1,\dots,i_{q-1}}(x_0) h^{i_1} \dots h^{i_{q-1}} + R_q(x), \end{aligned}$$

where $h^i = x^i - x_0^i$ and

$$R_q(x) = \frac{1}{q!} \sum_{i_1,\dots,i_q=1}^{n} f_{i_1,\dots,i_q}(x_0 + sh) h^{i_1} \dots h^{i_q}$$

with $s \in (0,1)$.

3.3 Sard - Stroud Results

The study of Taylor's formula has a rich literature and a long history. A.H. Stroud has pointed out in his celebrated book (Stroud (1971)) that one of the most important tools in the numerical integration of double integrals is the following theorem due to A. Sard (see Sard (1963) and Stroud (1971, p. 138) for the proof).

3.3.1 Sard Linear Approximation

THEOREM 3.3. *Let $n, m \in \mathbb{N}$ and I, J be two closed intervals and $f : I \times J \to \mathbb{R}$ be a mapping so that the following partial derivatives $\frac{\partial^{i+m+1} f(a,\cdot)}{\partial x^i \partial y^{m+1}}$ $(i = 0, ..., n)$, $\frac{\partial^{j+n+1} f(\cdot,b)}{\partial x^{n+1} \partial y^j}$ $(j = 0, ..., m)$ and $\frac{\partial^{n+m+2} f(\cdot,\cdot)}{\partial x^{n+1} \partial y^{m+1}}$ exist on the intervals I, J and $I \times J$ respectively, where $a \in I$ and $b \in J$ are given. Let $x \in I$ and $y \in J$ and assume that $\frac{\partial^{i+m+1} f(x,\cdot)}{\partial x^i \partial y^{m+1}}$ are continuous on $[b, y]$ for $(i = 0, ..., n)$, $\frac{\partial^{j+n+1} f(\cdot,y)}{\partial x^{n+1} \partial y^j}$ are continuous on $[a, x]$ (for $j = 0, ..., m$) and*

$\frac{\partial^{n+m+2}f(\cdot,\cdot)}{\partial x^{n+1}\partial y^{m+1}}$ *is continuous on* $[a,x]\times[b,y]$*. Then we have the representation:*

$$\begin{aligned} f(x,y) = &\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{(x-a)^i}{i!}\cdot\frac{(y-b)^j}{j!}\cdot\frac{\partial^{i+j}f(a,b)}{\partial x^i\partial y^j} \\ &+\frac{1}{m!}\sum_{i=0}^{n}\frac{(x-a)^i}{i!}\cdot\int_b^y (y-s)^m\frac{\partial^{i+m+1}f(a,s)}{\partial x^i\partial y^{m+1}}ds \\ &+\frac{1}{n!}\sum_{j=0}^{m}\frac{(y-b)^j}{j!}\cdot\int_a^x (x-t)^n\frac{\partial^{j+n+1}f(t,b)}{\partial x^{n+1}\partial y^j}dt \\ &+\frac{1}{n!m!}\int_a^x\int_b^y (x-t)^n(y-s)^m\frac{\partial^{n+m+2}f(t,s)}{\partial x^{n+1}\partial y^{m+1}}dsdt. \quad (3.8) \end{aligned}$$

Proof. For the sake of completeness we give here a short proof. Apply Taylor's formula (3.2) for the mapping $f(\cdot,y)$ to get

$$f(x,y)=\sum_{i=0}^{n}\frac{(x-a)^i}{i!}\cdot\frac{\partial^i f(a,y)}{\partial x^i}+\frac{1}{n!}\int_a^x (x-t)^n\frac{\partial^{n+1}f(t,y)}{\partial x^{n+1}}dt. \quad (3.9)$$

Also, by (3.2) applied for the partial derivatives $\frac{\partial^i f(a,\cdot)}{\partial x^i}$ $(i=0,...,n)$ we can state that

$$\frac{\partial^i f(a,y)}{\partial x^i}=\sum_{j=0}^{m}\frac{(y-b)^j}{j!}\cdot\frac{\partial^{j+i}f(a,b)}{\partial x^i\partial y^j}+\frac{1}{m!}\int_b^y (y-s)^m\frac{\partial^{i+m+1}f(a,s)}{\partial x^i\partial y^{m+1}}ds. \quad (3.10)$$

Similarly, we have

$$\frac{\partial^{n+1}f(t,y)}{\partial x^{n+1}}=\sum_{j=0}^{m}\frac{(y-b)^j}{j!}\cdot\frac{\partial^{j+n+1}f(t,b)}{\partial x^{n+1}\partial y^j}+\frac{1}{m!}\int_b^y (y-s)^m\frac{\partial^{n+m+2}f(t,s)}{\partial x^{n+1}\partial y^{m+1}}ds. \quad (3.11)$$

Using (3.10) and (3.11), equation (3.9) becomes

$$
\begin{aligned}
&f(x,y) \\
&= \sum_{i=0}^{n} \frac{(x-a)^i}{i!} \left[\sum_{j=0}^{m} \frac{(y-b)^j}{j!} \cdot \frac{\partial^{j+i} f(a,b)}{\partial x^i \partial y^j} + \frac{1}{m!} \int_b^y (y-s)^m \cdot \frac{\partial^{i+m+1} f(a,s)}{\partial x^i \partial y^{m+1}}\, ds \right] \\
&\quad + \frac{1}{n!} \int_a^x (x-t)^n \left[\sum_{j=0}^{m} \frac{(y-b)^j}{j!} \cdot \frac{\partial^{j+n+1} f(t,b)}{\partial x^{n+1} \partial y^j} \right. \\
&\quad \left. + \frac{1}{m!} \int_b^y (y-s)^m \frac{\partial^{n+m+2} f(t,s)}{\partial x^{n+1} \partial y^{m+1}}\, ds \right] dt \\
&= \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{(x-a)^i}{i!} \cdot \frac{(y-b)^j}{j!} \cdot \frac{\partial^{i+j} f(a,b)}{\partial x^i \partial y^j} \\
&\quad + \frac{1}{m!} \sum_{i=0}^{n} \frac{(x-a)^i}{i!} \cdot \int_b^y (y-s)^m \frac{\partial^{i+m+1} f(a,s)}{\partial x^i \partial y^{m+1}}\, ds \\
&\quad + \frac{1}{n!} \sum_{j=0}^{m} \frac{(y-b)^j}{j!} \cdot \int_a^x (x-t)^n \frac{\partial^{j+n+1} f(t,b)}{\partial x^{n+1} \partial y^j}\, dt \\
&\quad + \frac{1}{n!m!} \int_a^x \int_b^y (x-t)^n (y-s)^m \frac{\partial^{n+m+2} f(t,s)}{\partial x^{n+1} \partial y^{m+1}}\, ds dt \qquad (3.12)
\end{aligned}
$$

and the theorem is completely proved upon simplification. □

3.3.2 The L_p, $p \in [1,\infty]$ Bounds for the Remainder in Sard's Theorem

Now using the above theorem, we can point out the following inequality which provides error bounds in terms of the Lebesgue norm of some partial derivatives (see Hanna *et al.* (2002b)).

THEOREM 3.4. *Assume that the mapping $f : I \times J \to \mathbb{R}$ fulfills the hypotheses from Theorem 3.3. Then for $x \geq a$ and $y \geq b$ we have the inequality*

$$
\begin{aligned}
&\left| f(x,y) - \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{(x-a)^i}{i!} \cdot \frac{(y-b)^j}{j!} \cdot \frac{\partial^{i+j} f(a,b)}{\partial x^i \partial y^j} \right. \\
&\quad - \frac{1}{m!} \sum_{i=0}^{n} \frac{(x-a)^i}{i!} \cdot \int_b^y (y-s)^m \frac{\partial^{i+m+1} f(a,s)}{\partial x^i \partial y^{m+1}} ds \\
&\quad \left. - \frac{1}{n!} \sum_{j=0}^{m} \frac{(y-b)^j}{j!} \cdot \int_a^x (x-t)^n \frac{\partial^{j+n+1} f(t,b)}{\partial x^{n+1} \partial y^j} dt \right|
\end{aligned}
$$

$$\leq \begin{cases} \frac{1}{(n+1)!(m+1)!}(x-a)^{n+1}(y-b)^{m+1}\left\|\frac{\partial^{n+m+2}f}{\partial x^{n+1}\partial y^{m+1}}\right\|_{\infty;[a,x]\times[b,y]} \\ if\,\frac{\partial^{n+m+2}f}{\partial x^{n+1}\partial y^{m+1}} \in L_\infty\Big[[a,x]\times[b,y]\Big]; \\ \frac{1}{n!(nq+1)^{\frac{1}{q}}m!(mq+1)^{\frac{1}{q}}}(x-a)^{n+\frac{1}{q}}(y-b)^{m+\frac{1}{q}}\left\|\frac{\partial^{n+m+2}f}{\partial x^{n+1}\partial y^{m+1}}\right\|_{p;[a,x]\times[b,y]} \\ if\,\frac{\partial^{n+m+2}f}{\partial x^{n+1}\partial y^{m+1}} \in L_p\Big[[a,x]\times[b,y]\Big] \qquad p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{n!m!}(x-a)^n(y-b)^m\left\|\frac{\partial^{n+m+2}f}{\partial x^{n+1}\partial y^{m+1}}\right\|_{1;[a,x]\times[b,y]} \\ if\,\frac{\partial^{n+m+2}f}{\partial x^{n+1}\partial y^{m+1}} \in L_1\Big[[a,x]\times[b,y]\Big]. \end{cases} \tag{3.13}$$

where $\|\cdot\|_{p,[a,x]\times[b,y]}$ *is the usual* p*-norm (*$p\in[1,\infty]$*) on the region* $[a,x]\times[b,y]$*.*

Proof. Using the representation (3.8) and the property of the modulus we have

$$\begin{aligned} &\Bigg| f(x,y) - \sum_{i=0}^{n}\sum_{j=0}^{m} \frac{(x-a)^i}{i!}\cdot\frac{(y-b)^j}{j!}\cdot\frac{\partial^{i+j}f(a,b)}{\partial x^i\partial y^j} \\ &\quad -\frac{1}{m!}\sum_{i=0}^{n}\frac{(x-a)^i}{i!}\cdot\int_b^y (y-s)^m\frac{\partial^{i+m+1}f(a,s)}{\partial x^i\partial y^{m+1}}ds \\ &\quad -\frac{1}{n!}\sum_{j=0}^{m}\frac{(y-b)^j}{j!}\cdot\int_a^x (x-t)^n\frac{\partial^{j+n+1}f(t,b)}{\partial x^{n+1}\partial y^j}dt\Bigg| \\ &\leq \frac{1}{n!m!}\int_a^x\int_b^y |x-t|^n|y-s|^m\left|\frac{\partial^{n+m+2}f(t,s)}{\partial x^{n+1}\partial y^{m+1}}\right|dsdt \\ &=: M(x,y). \end{aligned}$$

It is easy to see that

$$\begin{aligned} &M(x,y) \\ &\quad\leq \left\|\frac{\partial^{n+m+2}f}{\partial x^{n+1}\partial y^{m+1}}\right\|_{\infty;[a,x]\times[b,y]}\frac{1}{n!m!}\int_a^x\int_b^y (x-t)^n(y-s)^m\,dsdt \\ &\quad= \frac{1}{n!m!}\left[\frac{-(x-t)^{n+1}}{n+1}\Bigg|_a^x\right]\left[\frac{-(y-s)^{n+1}}{m+1}\Bigg|_b^y\right]\left\|\frac{\partial^{n+m+2}f}{\partial x^{n+1}\partial y^{m+1}}\right\|_{\infty;[a,x]\times[b,y]} \\ &\quad= \frac{1}{(n+1)!(m+1)!}(x-a)^{n+1}(y-b)^{m+1}\left\|\frac{\partial^{n+m+2}f}{\partial x^{n+1}\partial y^{m+1}}\right\|_{\infty;[a,x]\times[b,y]} \end{aligned}$$

and the first inequality in (3.13) is proved. Using Hölder's inequality for double integrals, we have

$$\begin{aligned}
&M(x,y)\\
&\leq \frac{1}{n!m!}\left\|\frac{\partial^{n+m+2}f}{\partial x^{n+1}\partial y^{m+1}}\right\|_{p;[a,x]\times[b,y]}\left[\int_a^x\int_b^y (x-t)^{nq}(y-s)^{mq}\,dsdt\right]^{\frac{1}{q}}\\
&= \frac{1}{n!m!}\left\|\frac{\partial^{n+m+2}f}{\partial x^{n+1}\partial y^{m+1}}\right\|_{p;[a,x]\times[b,y]}\left[\frac{(x-a)^{nq+1}(y-b)^{mq+1}}{(nq+1)(mq+1)}\right]^{\frac{1}{q}}\\
&= \frac{1}{n!(nq+1)^{\frac{1}{q}}\,m!(mq+1)^{\frac{1}{q}}}\\
&\times (x-a)^{n+\frac{1}{q}}(y-b)^{m+\frac{1}{q}}\left\|\frac{\partial^{n+m+2}f}{\partial x^{n+1}\partial y^{m+1}}\right\|_{p;[a,x]\times[b,y]}
\end{aligned}$$

and the second inequality in (3.13) is proved. Finally, we have

$$\begin{aligned}
&M(x,y)\\
&\leq \frac{1}{n!m!}\sup_{(t,s)\in[a,x]\times[b,y]}\left[(x-t)^n(y-s)^m\right]\left\|\frac{\partial^{n+m+2}f}{\partial x^{n+1}\partial y^{m+1}}\right\|_{1;[a,x]\times[b,y]}\\
&= \frac{1}{n!m!}(x-a)^n(y-b)^m\left\|\frac{\partial^{n+m+2}f}{\partial x^{n+1}\partial y^{m+1}}\right\|_{1;[a,x]\times[b,y]}
\end{aligned}$$

and the theorem is proved. □

The following approximation of the mapping $f(x,y)$ in terms of

$$\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{(x-a)^i}{i!}\cdot\frac{(y-b)^j}{j!}\cdot\frac{\partial^{i+j}f(a,b)}{\partial x^i\partial y^j}$$

also holds.

THEOREM 3.5. *Assume that the mapping $f: I\times J\to\mathbb{R}$ fulfills the hypotheses from Theorem 3.3. Then for $x\geq I$ and $y\geq J$ we have the inequality*

$$\begin{aligned}
&\left|f(x,y)-\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{(x-a)^i}{i!}\cdot\frac{(y-b)^j}{j!}\cdot\frac{\partial^{i+j}f(a,b)}{\partial x^i\partial y^j}\right|\\
&\leq \frac{1}{(m+1)!}(y-b)^{m+1}\sum_{i=0}^{n}\frac{(x-a)^i}{i!}\left\|\frac{\partial^{i+m+1}f(a,\cdot)}{\partial x^i\partial y^{m+1}}\right\|_{\infty,[b,y]}\\
&+\frac{1}{(n+1)!}(x-a)^{n+1}\sum_{j=0}^{m}\frac{(y-b)^j}{j!}\left\|\frac{\partial^{j+n+1}f(\cdot,b)}{\partial x^{n+1}\partial y^j}\right\|_{\infty,[a,x]}\\
&+\frac{1}{(n+1)!(m+1)!}(x-a)^{n+1}(y-b)^{m+1}\left\|\frac{\partial^{n+m+2}f}{\partial x^{n+1}\partial y^{m+1}}\right\|_{\infty;[a,x]\times[b,y]}. \qquad (3.14)
\end{aligned}$$

The proof follows from the formula (3.8) however details will not be covered.

Similar bounds in terms of the other norms may be established, but we omit the details (see Hanna *et al.* (2002b)).

3.4 A Grüss Type Inequality for Double Integrals

It is well known that Grüss type inequalities provide a useful means of approximating the integral of the product in terms of the product of integrals.

In this section we will state, with complete proof, the following lemma representing a Grüss type inequality for double integrals (Hanna *et al.* (2002b)).

Lemma 3.1. *We assume that*

$$\left|f\left(x,y\right)-f\left(u,v\right)\right|\leq M_{1}\left|x-u\right|^{\alpha_{1}}+M_{2}\left|y-v\right|^{\alpha_{2}}, \tag{3.15}$$

where $M_1, M_2 > 0, \alpha_1, \alpha_2 \in (0,1]$ *and*

$$\left|g\left(x,y\right)-g\left(u,v\right)\right|\leq N_{1}\left|x-u\right|^{\beta_{1}}+N_{2}\left|y-v\right|^{\beta_{2}}, \tag{3.16}$$

where $N_1,\ N_2 > 0,\ \beta_1,\ \beta_2 \in (0,1]$ *for all* $(x,y),\ (u,v) \in [a,b]\times[c,d]$*, then we have the following inequality:*

$$\begin{aligned}
&\left|\frac{1}{\left(b-a\right)\left(d-c\right)}\int_{a}^{b}\int_{c}^{d}f\left(x,y\right)g\left(x,y\right)dydx\right. \qquad (3.17)\\
&\left.-\frac{1}{\left(b-a\right)\left(d-c\right)}\int_{a}^{b}\int_{c}^{d}f\left(x,y\right)dydx\times\frac{1}{\left(b-a\right)\left(d-c\right)}\int_{a}^{b}\int_{c}^{d}g\left(x,y\right)dydx\right|\\
&\leq\left[M_{1}N_{1}\frac{\left(b-a\right)^{\alpha_{1}+\beta_{1}}}{\left(\alpha_{1}+\beta_{1}+1\right)\left(\alpha_{1}+\beta_{1}+2\right)}+M_{1}N_{2}\frac{2\left(b-a\right)^{\alpha_{1}}\left(d-c\right)^{\beta_{2}}}{\left(\alpha_{1}+1\right)\left(\beta_{2}+1\right)\left(\alpha_{1}+2\right)\left(\beta_{2}+2\right)}\right.\\
&\left.+M_{2}N_{1}\frac{2\left(b-a\right)^{\beta_{1}}\left(d-c\right)^{\alpha_{2}}}{\left(\alpha_{2}+1\right)\left(\alpha_{2}+2\right)\left(\beta_{1}+1\right)\left(\beta_{1}+2\right)}+M_{2}N_{2}\frac{\left(d-c\right)^{\alpha_{2}+\beta_{2}}}{\left(\alpha_{2}+\beta_{2}+1\right)\left(\alpha_{2}+\beta_{2}+2\right)}\right].
\end{aligned}$$

Proof. Multiplying (3.15) and (3.16), we get

$$\begin{aligned}
&\left|\left(f\left(x,y\right)-f\left(u,v\right)\right)\left(g\left(x,y\right)-g\left(u,v\right)\right)\right|\\
\leq\ &M_{1}N_{1}\left|x-u\right|^{\alpha_{1}+\beta_{1}}+M_{1}N_{2}\left|x-u\right|^{\alpha_{1}}\left|y-v\right|^{\beta_{2}}\\
&+M_{2}N_{1}\left|y-v\right|^{\alpha_{2}}\left|x-u\right|^{\beta_{1}}+M_{2}N_{2}\left|y-v\right|^{\alpha_{2}+\beta_{2}}.
\end{aligned}$$

Integrating on $([a,b]\times[c,d])^2$ over (x,y) and (u,v) , we obtain

$$\int_a^b \int_c^d \int_a^b \int_c^d |(f(x,y) - f(u,v))(g(x,y) - g(u,v))| \, dydxdvdu \tag{3.18}$$

$$\begin{aligned}
&\leq M_1 N_1 \int_a^b \int_c^d \int_a^b \int_c^d |x-u|^{\alpha_1+\beta_1} \, dydxdvdu \\
&+ M_1 N_2 \int_a^b \int_c^d \int_a^b \int_c^d |x-u|^{\alpha_1} |y-v|^{\beta_2} \, dydxdvdu \\
&+ M_2 N_1 \int_a^b \int_c^d \int_a^b \int_c^d |y-v|^{\alpha_2} |x-u|^{\beta_1} \, dydxdvdu \\
&+ M_2 N_2 \int_a^b \int_c^d \int_a^b \int_c^d |y-v|^{\alpha_2+\beta_2} \, dydxdvdu
\end{aligned}$$

$$= M_1 N_1 I_1 + M_1 N_2 I_2 + M_2 N_1 I_3 + M_2 N_2 I_4. \tag{3.19}$$

Applying Korkine's identity (see, Mitrinović *et al.* (1993, p. 242)) to the left side of (3.18) gives

$$\begin{aligned}
&\frac{1}{2} \int_a^b \int_c^d \int_a^b \int_c^d [(f(x,y) - f(u,v))(g(x,y) - g(u,v))] dydxdvdu \\
&= \frac{1}{2} \int_a^b \int_c^d \int_a^b \int_c^d [f(x,y) g(x,y) - f(x,y) g(u,v) \\
&\quad - f(u,v) g(x,y) + f(u,v) g(u,v)] dydxdvdu \\
&= \frac{1}{2} \int_a^b \int_c^d \int_a^b \int_c^d f(x,y) g(x,y) \, dydxdvdu \\
&\quad - \frac{1}{2} \int_a^b \int_c^d \int_a^b \int_c^d f(x,y) g(u,v) \, dydxdvdu \\
&\quad - \frac{1}{2} \int_a^b \int_c^d \int_a^b \int_c^d f(u,v) g(x,y) \, dydxdvdu \\
&\quad + \frac{1}{2} \int_a^b \int_c^d \int_a^b \int_c^d f(u,v) g(u,v) \, dydxdvdu
\end{aligned}$$

$$\begin{aligned}
&= (b-a)(d-c)\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx \\
&\quad -\int_a^b\int_c^d\int_a^b\int_c^d f(x,y)\,g(u,v)\,dydxdvdu \\
&= (b-a)(d-c)\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx \\
&\quad -\int_a^b\int_c^d f(x,y)\,dydx\int_a^b\int_c^d g(x,y)\,dydx,
\end{aligned}$$

For the right side of (3.18) the following Lemma 3.2 proved by P. Cerone in (Budimir *et al.* 2001) will be used. Namely,

Lemma 3.2. *Let a,b,c,d* $\in R$ *with* $a < b$ *and* $c < d$. *Define*

$$C_\theta(a,b,c,d) := \int_a^b\int_c^d |x-y|^\theta dydx, \quad \theta \ge 0, \tag{3.20}$$

then

$$(\theta+1)(\theta+2)C_\theta(a,b,c,d) \tag{3.21}$$

$$= |b-c|^{\theta+2} - |b-d|^{\theta+2} + |d-a|^{\theta+2} - |c-a|^{\theta+2}.$$

If $c = a$ and $d = b$, then from (3.21)

$$\begin{aligned}
D_\theta(a,b) = C_\theta(a,b,a,b) &= \int_a^b\int_a^b |x-y|^\theta dydx, \quad \theta \ge 0 \\
&= \frac{2(b-a)^{\theta+2}}{(\theta+1)(\theta+2)}\,.
\end{aligned} \tag{3.22}$$

Now, utilizing the result of Lemma 3.2 and returning to (3.19) we find that:

$$\begin{aligned}
I_1 &= \int_a^b\int_c^d\int_a^b\int_c^d |x-u|^{\alpha_1+\beta_1}\,dydxdvdu \\
&= (d-c)^2\int_a^b\int_a^b |x-u|^{\alpha_1+\beta_1}\,dxdu \\
&= (d-c)^2 D_{\alpha_1+\beta_1}(a,b).
\end{aligned} \tag{3.23}$$

(3.24)

and using (3.22) gives

$$I_1 = \frac{2\left(d-c\right)^2\left(b-a\right)^{\alpha_1+\beta_1+2}}{\left(\alpha_1+\beta_1+1\right)\left(\alpha_1+\beta_1+2\right).},$$

Further, from (3.19) and using (3.21) gives

$$\begin{aligned} I_2 &= \int_a^b \int_c^d \int_a^b \int_c^d |x-u|^{\alpha_1} |y-v|^{\beta_2} \, dydxdvdu \\ &= \int_a^b \int_a^b |x-u|^{\alpha_1} \, dxdu \int_c^d \int_c^d |y-v|^{\beta_2} \, dydv \\ &= D_{\alpha_1}(a,b) D_{\beta_2}(c,d) \end{aligned} \tag{3.25}$$

and using (3.22) produces

$$I_2 = \frac{4\left(b-a\right)^{\alpha_1+2}\left(d-c\right)^{\beta_2+2}}{\left(\beta_1+1\right)\left(\beta_1+2\right)\left(\alpha_2+1\right)\left(\alpha_2+2\right)}. \tag{3.26}$$

Using a similar procedure we get for I_3 and I_4 as defined in (3.21),

$$I_3 = \frac{4\left(b-a\right)^{\alpha_2+2}\left(d-c\right)^{\beta_1+2}}{\left(\alpha_2+1\right)\left(\beta_1+1\right)\left(\alpha_1+2\right)\left(\beta_2+2\right)}, \tag{3.27}$$

and

$$I_4 = \frac{2\left(b-a\right)^2\left(d-c\right)^{\alpha_2+\beta_2+2}}{\left(\alpha_2+\beta_2+1\right)\left(\alpha_2+\beta_2+2\right)}. \tag{3.28}$$

Thus, using (3.19), (3.25), (3.26), (3.27), (3.28) and Korkine's identities in (3.18), we get

$$\begin{aligned} &(b-a)(c-d) \int_a^b \int_c^d f(x,y)\, g(x,y)\, dydx - \int_a^b \int_c^d f(x,y)\, dydx \int_a^b \int_c^d g(x,y)\, dydx \\ &\le \frac{1}{2}\left[M_1 N_1 \frac{2\left(d-c\right)^2\left(b-a\right)^{\alpha_1+\beta_1+2}}{\left(\alpha_1+\beta_1+1\right)\left(\alpha_1+\beta_1+2\right)} + M_1 N_2 \frac{4\left(b-a\right)^{\alpha_1+2}\left(d-c\right)^{\beta_2+2}}{\left(\alpha_1+1\right)\left(\alpha_1+2\right)\left(\beta_2+1\right)\left(\beta_2+2\right)} \right. \\ &\qquad \left. + M_2 N_1 \frac{4\left(b-a\right)^{\beta_1+2}\left(d-c\right)^{\alpha_2+2}}{\left(\alpha_2+1\right)\left(\alpha_2+2\right)\left(\beta_1+1\right)\left(\beta_1+2\right)} + M_2 N_2 \frac{2\left(b-a\right)^2\left(d-c\right)^{\alpha_2+\beta_2+2}}{\left(\alpha_2+\beta_2+1\right)\left(\alpha_2+\beta_2+2\right)} \right] \end{aligned}$$

from which, upon dividing both sides by $(b-a)^2(d-c)^2$ the proof is completed. □

Corollary 3.5.1. *(see also Mitrinović* et al. *(1993, p. 305)) When* $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 1$, *we have*

$$\begin{aligned} |f(x,y) - f(u,v)| &\le L_1 |x-u| + L_2 |y-v|, \\ |g(x,y) - g(u,v)| &\le K_1 |x-u| + K_2 |y-v|, \end{aligned}$$

and then (3.17) becomes

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx\right.$$

$$\left.-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\times\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g(x,y)\,dydx\right|$$

$$\leq\left[L_1K_1\frac{(b-a)^2}{12}+L_1K_2\frac{(b-a)(d-c)}{18}+L_2K_1\frac{(b-a)(d-c)}{18}+L_2K_2\frac{(d-c)^2}{12}\right].$$

Corollary 3.5.2. *Let the conditions of Corollary 3.5.1 hold*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f^2(x,y)\,dx\,dy\right.$$
$$\left.-\left[\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dx\,dy\right]^2\right|$$
$$\leq\left[L_1^2\frac{(b-a)^2}{12}+L_1L_2\frac{(b-a)(d-c)}{9}+L_2^2\frac{(d-c)^2}{12}\right].$$

Proof. In (3.17) let $\alpha_1=\alpha_2=\beta_1=\beta_2=1$ and $f(\cdot,\cdot)=g(\cdot,\cdot)$. □

3.5 An Application for Taylor's Expansion

The above result will be used to obtain a perturbed version of the Taylor's expansion. We may now state the following result (Hanna *et al.* (2002b)).

THEOREM 3.6. *With the conditions as in Theorem 3.5 and assuming that*

$$\left|\frac{\partial^{n+m+2}f(t,s)}{\partial x^{n+1}\partial y^{m+1}}-\frac{\partial^{n+m+2}f(u,v)}{\partial x^{n+1}\partial y^{m+1}}\right|\leq L_1|t-u|+L_2|s-v|,$$

we have the inequality

$$|R_{nm}(f,a,x,b,y)|\leq\frac{(x-a)^{n+1}(y-b)^{m+1}}{6n!m!}G(n,m)\,p(a,x,b,y)' \tag{3.29}$$

where

$$G(n,m)=\left[\frac{1}{(2n+1)(2m+1)}-\frac{1}{(n+1)^2(m+1)^2}\right]^{\frac{1}{2}}$$

and

$$p(a,x,b,y)=\left[3L_1^2(x-a)^2+4L_1L_2(x-a)(y-b)+3L_2^2(y-b)^2\right]^{\frac{1}{2}},$$

with

$$R_{nm}\left(f,a,x,b,y\right)=f\left(x,y\right)-\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\left(x-a\right)^{i}}{i!}\cdot\frac{\left(y-b\right)^{j}}{j!}\cdot\frac{\partial^{i+j}f\left(a,b\right)}{\partial x^{i}\partial y^{j}} \tag{3.30}$$

$$-\frac{1}{m!}\sum_{i=0}^{n}\frac{\left(x-a\right)^{i}}{i!}\cdot\int_{b}^{y}\left(y-s\right)^{m}\frac{\partial^{i+m+1}f\left(a,s\right)}{\partial x^{i}\partial y^{m+1}}ds$$

$$-\frac{1}{n!}\sum_{j=0}^{m}\frac{\left(y-b\right)^{j}}{j!}\cdot\int_{a}^{x}\left(x-t\right)^{n}\frac{\partial^{j+n+1}f\left(t,b\right)}{\partial x^{n+1}\partial y^{j}}dt$$

$$-\frac{\left(x-a\right)^{n}\left(y-b\right)^{m}}{\left(n+1\right)!\left(m+1\right)!}\cdot\int_{a}^{x}\int_{b}^{y}\frac{\partial^{n+m+2}f\left(t,s\right)}{\partial x^{n+1}\partial y^{m+1}}dsdt.$$

Proof. We can write (3.8) as

$$\begin{aligned} f\left(x,y\right) &= \sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\left(x-a\right)^{i}}{i!}\cdot\frac{\left(y-b\right)^{j}}{j!}\cdot\frac{\partial^{i+j}f\left(a,b\right)}{\partial x^{i}\partial y^{j}} \\ &\quad +\frac{1}{m!}\sum_{i=0}^{n}\frac{\left(x-a\right)^{i}}{i!}\cdot\int_{b}^{y}\left(y-s\right)^{m}\frac{\partial^{i+m+1}f\left(a,s\right)}{\partial x^{i}\partial y^{m+1}}ds \\ &\quad +\frac{1}{n!}\sum_{j=0}^{m}\frac{\left(y-b\right)^{j}}{j!}\cdot\int_{a}^{x}\left(x-t\right)^{n}\frac{\partial^{j+n+1}f\left(t,b\right)}{\partial x^{n+1}\partial y^{j}}dt \\ &\quad +R\left(f,a,x,b,y\right), \end{aligned} \tag{3.31}$$

where

$$R\left(f,a,x,b,y\right)=\frac{1}{n!m!}\int_{a}^{x}\int_{b}^{y}\left(x-t\right)^{n}\left(y-s\right)^{m}\cdot\frac{\partial^{n+m+2}f\left(t,s\right)}{\partial x^{n+1}\partial y^{m+1}}dsdt. \tag{3.32}$$

Now, let

$$h\left(t,s\right)=\left(x-t\right)^{n}\left(y-s\right)^{m}$$

and

$$g\left(t,s\right)=\frac{\partial^{n+m+2}f\left(t,s\right)}{\partial x^{n+1}\partial y^{m+1}},$$

then, recalling Korkine's identity, we have from (3.32) that

$$R(f,a,x,b,y) \tag{3.33}$$

$$= \frac{1}{n!m!}\int_a^x\int_b^y h(t,s)\,g(t,s)\,dsdt$$

$$= \frac{1}{n!m!(x-a)(y-b)}\int_a^x\int_b^y h(t,s)\,dsdt\int_a^x\int_b^y g(t,s)\,dsdt + R_1(f,a,x,b,y),$$

where

$$R_1(f,a,x,b,y) \tag{3.34}$$
$$= \frac{1}{2n!m!(x-a)(y-b)}\int_\Omega (h(t,s)-h(u,v))(g(t,s)-g(u,v))\,dtdsdudv$$

with

$$\Omega = [[a,x]\times[b,y]]^2. \tag{3.35}$$

In addition, applying the Cauchy-Schwartz inequality (Dragomir (1999b)) for (3.34) we get

$$|R_1| \tag{3.36}$$
$$= \left|\frac{1}{2n!m!(x-a)(y-b)}\int_\Omega (h(t,s)-h(u,v))(g(t,s)-g(u,v))\,dtdsdudv\right|$$

$$\le \frac{1}{2n!m!(x-a)(y-b)}\sqrt{\int_\Omega (h(t,s)-h(u,v))^2\,dtdsdudv}$$

$$\times\sqrt{\int_\Omega (g(t,s)-g(u,v))^2\,dtdsdudv}.$$

Simple computation shows that

$$\int_\Omega (h(t,s)-h(u,v))^2\,dtdsdudv \tag{3.37}$$

$$= \int_\Omega ((x-t)^n(y-s)^m-(x-u)^n(y-v)^m)^2\,dtdsdudv$$

$$= 2(x-a)^{2n+2}(y-b)^{2m+2}\left[\frac{1}{(2n+1)(2m+1)}-\frac{1}{(n+1)^2(m+1)^2}\right].$$

Now, we let

$$I = \int_{\Omega} \left(g\left(t,s\right) - g\left(u,v\right)\right)^2 dtdsdudv \tag{3.38}$$

$$= \int_{\Omega} \left(\frac{\partial^{n+m+2} f\left(t,s\right)}{\partial x^{n+1}\partial y^{m+1}} - \frac{\partial^{n+m+2} f\left(u,v\right)}{\partial x^{n+1}\partial y^{m+1}}\right)^2 dtdsdudv,$$

then,

$$I = 2\left[\left(x-a\right)\left(y-b\right)\int_a^x \int_b^y \left(\frac{\partial^{n+m+2} f\left(t,s\right)}{\partial x^{n+1}\partial y^{m+1}}\right)^2 dsdt \right. \tag{3.39}$$

$$\left. - \left(\int_a^x \int_b^y \frac{\partial^{n+m+2} f\left(u,v\right)}{\partial x^{n+1}\partial y^{m+1}} dvdu\right)^2\right].$$

Applying Corollary 3.5.2, we have the following inequality

$$|I| \leq 2\left(x-a\right)^2\left(y-b\right)^2 \tag{3.40}$$

$$\times \left[\frac{L_1^2\left(x-a\right)^2}{12} + L_1 L_2 \frac{\left(x-a\right)\left(y-b\right)}{9} + \frac{L_2^2\left(y-b\right)^2}{12}\right].$$

Utilising (3.37), (3.40), and (3.33) and substituting in (3.31), the theorem is proved. □

3.5.1 Numerical Experiments

In this section the perturbed Taylor's expansion developed in equation (3.30) is used for different values of m and n to approximate some functions with different behaviours as shown in the following examples.

Example 3.1.

$$f(x,y) = e^{-x^2-y^2}, \qquad 0 \leq x, y \leq 1. \tag{3.41}$$

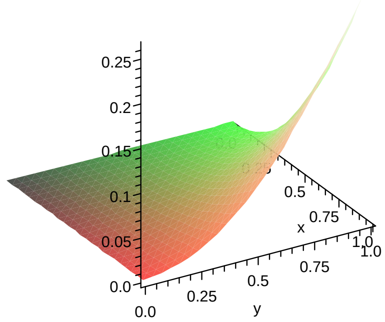
Example 3.1 was chosen because the function is infinitely smooth and the partial differentiation for it blows up quickly with successive derivatives. This indicates that the higher order error bounds will give better results.

The plots of the error bound $|R_{nm}(f,a,x,b,y)|$ given by (3.30) for the function in Example 3.1 are shown in Figure 3.1.
We may notice from the subfigures in Figures 3.1 that doubling the values of m and n results in squaring the value of the error bound. Notice, that the numerical noise in subfigure (e) are due to the computing limit.

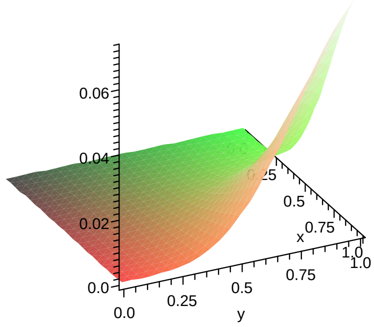
Example 3.2.

$$f(x,y) = e^{-xy} \qquad 0 \leq x, y \leq 1. \tag{3.42}$$

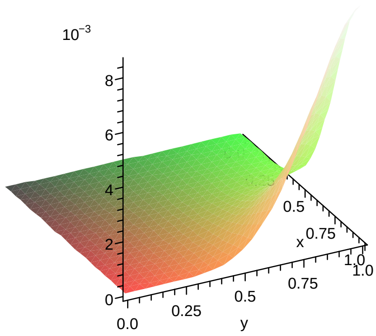
Example 3.2 was chosen because the function is not separable as a product of two functions of one variable in contrast to Example 3.1 and the partial differentiation for it becomes smaller on $[0,1] \times [0,1]$ with successive derivatives. This indicates that the higher order error bounds will give better results. The plots of the error bounds $|R_{nm}(f,a,x,b,y)|$ for the function in Example 3.2 are shown in Figure 3.2.
By looking at Figure 3.2 we see a resemblance to that of Figures 3.1 where the results in error bound have been squared as a result of doubling the values of m and n.



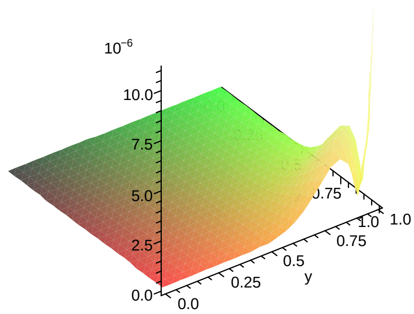
(a) error graph for $n = m = 1$



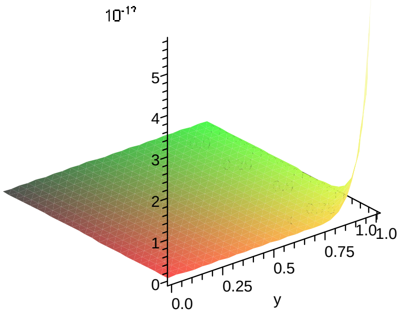
(b) error graph for $n = m = 2$



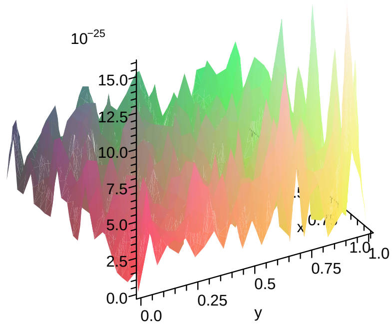
(c) error graph for $n = m = 4$



(d) error graph for $n = m = 8$

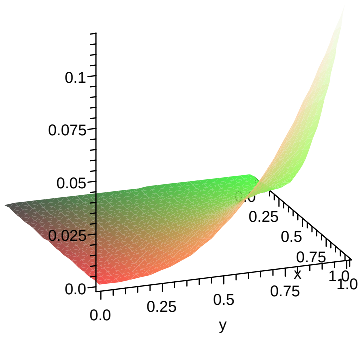


(e) error graph for $n = m = 16$

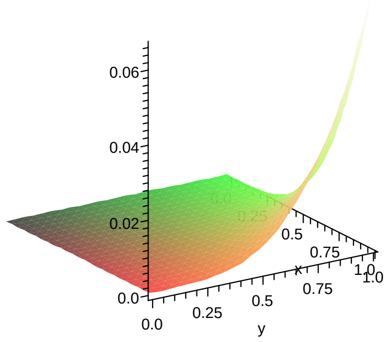


(f) error graph for $n = m = 32$

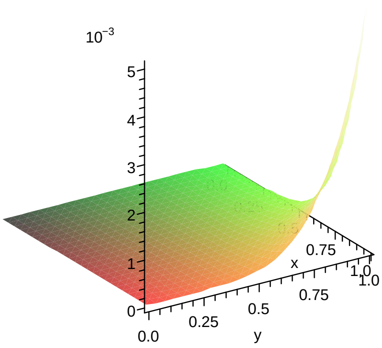
Figure 3.1: Plots of the error $|R_{nm}(f, a, x, b, y)|$ for $f(x, y) = e^{-x^2-y^2}, \ x, y \in [0, 1]$.



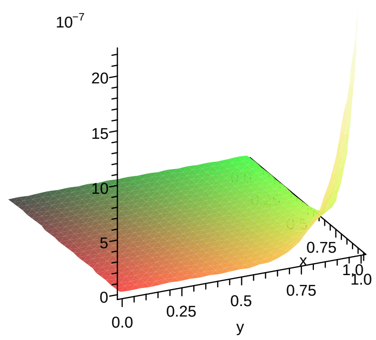
(a) error graph for $n = m = 1$



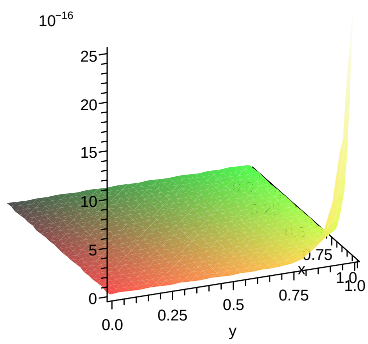
(b) error graph for $n = m = 2$



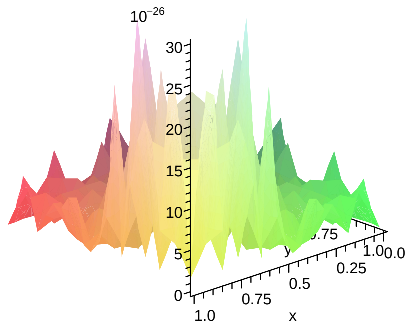
(c) error graph for $n = m = 4$



(d) error graph for $n = m = 8$



(e) error graph for $n = m = 16$



(f) error graph for $n = m = 32$

Figure 3.2: Plots of the error $|R_{nm}(f, a, x, b, y)|$ for $f(x, y) = e^{-xy},\ x, y \in [0, 1]$.

In the next chapter we introduce some Appell type polynomials and typical examples such as a convex combination of the end points, Bernoulli polynomials and Euler polynomials. We also represent the remainder in an integral form which will allow a better estimation using the Theory of Integral Inequalities (including Grüss type inequalities).

Furthermore, in the next chapter we will extend the work of this chapter to explore a new Taylor's expansion which is comprised of the product of two polynomials, each of which satisfies the Appell condition and also, obtain a generalisation of the Taylor like formula.

CHAPTER 4

A GENERAL TAYLOR LIKE EXPANSION IN TERMS OF TWO DIFFERENT APPELL POLYNOMIALS

In this chapter, generalised Taylor's formulae are developed utilising an integral remainder in which the kernel is comprised of a product of two polynomials, each of which satisfies the Appell condition (Appell (1880))

$$\frac{\partial P_n(t,x)}{\partial t} = P_{n-1}(t,x), \;\; P_0(t,x) = 1 \text{ for all } (t,x) \in \mathbb{R}^2 \text{ and } n \geq 1. \tag{4.1}$$

Bounds are determined in terms of Lebesgue norms. Furthermore, some of the previous results are shown to be recaptured as special cases of the current work. An application and numerical experimentation is undertaken to demonstrate the developments.

The material in this chapter is presented in the following order. In Section 4.2, a review of previous work and results is given. In Section 4.3, a generalisation of the Taylor-like formula for two Appell polynomials is obtained and its impact on the numerical integration of double integrals is studied. Some estimates for the remainder of the generalized Taylor-like formula are given in Section 4.4. Finally, in Section 4.5 attention is focused on the symbolic computation of Appell polynomials using the computer algebra system "Maple". The computation is also illustrated by using some numerical experiments to plot the theoretical results obtained in this chapter.

4.1 Introduction

This chapter aims to extend the work of Chapter 3 to explore a new Taylor's expansion which is comprised of a product of two polynomials, each of which satisfies the Appell condition (Appell (1880)). The methodology to be followed involves:

- The development of a general Taylor like expansion, in terms of two different Appell polynomials incorporating Sard's result, Sard (1963).
- The representation of the remainder in an integral form, which will allow a better estimation using the theory of integral inequalities (including Grüss type inequalities).
- The provision of new tools for the numerical evaluation of double integrals via Bernoulli and Euler polynomials, the properties of which are well documented in the literature.
- Achievement of a sharper analysis of the error bounds.

There are many examples of Appell polynomials, the following are some (see also Matić *et al.* (1999)):

(a) $P_n^{(1)}(t,x) \triangleq \frac{1}{n!}(t-x)^n, n \in \mathbb{N};$

(b) $P_n^{(2)}(t,x) \triangleq \frac{1}{n!}(t-\frac{a+x}{2})^n, n \in \mathbb{N};$ (or, more generally),
$P_{n,\lambda}^{(2)}(t,x) = \frac{1}{n!}(t-(\lambda a+(1-\lambda)x))^n,$ where $\lambda \in [0,1]$

(c) $P_n^{(3)}(t,x) \triangleq \frac{1}{n!}(x-a)^n B_n(\frac{t-a}{x-a}), n \geq 1$, with, $P_0^{(3)}(t,x) \triangleq 1$,
where $B_n(\cdot)$ are Bernoulli polynomials (Abramowitz and Stegun (1972));

(d) $P_n^{(4)}(t,x) \triangleq \frac{1}{n!}(x-a)^n E_n(\frac{t-a}{x-a}), n \geq 1$, with, $P_0^{(4)}(t,x) \triangleq 1,$
where $E_n(\cdot)$ are Euler polynomials (Abramowitz and Stegun (1972)).

Using a generalization of integration by parts, it will be shown below that any double integral $\int_a^b \int_c^d f(t,s)dsdt$ may be expressed as follows (Dragomir *et al.* (2005)),

$$\int_a^b \int_c^d f(t,s)dsdt = A_{n,m}(f,P_n,Q_m) + B_{n,m}(f,P_n,Q_m) + R_{n,m}(f,P_n,Q_m), \quad (4.2)$$

where $A_{n,m}(f,P_n,Q_m)$ can always be numerically evaluated for different choices of Appell polynomials $P_n,\ Q_m$ and $B_{n,m}(f,P_n,Q_m)$ is a linear combination of some univariate integrals. Further, $R_{n,m}(f,P_n,Q_m)$ is a double integral involving two Appell polynomials $P_n(.,.)$ and $Q_m(.,.)$ and the partial derivatives of the function f, where

$$R_{n,m}(f,P_n,Q_m) \triangleq (-1)^{n+m} \int_a^b \int_c^d P_n(t,b)Q_m(s,d)\frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m} dsdt. \quad (4.3)$$

In cases where the univariate integrals in $B_{n,m}(f,P_n,Q_m)$ can be computed exactly or easily approximated using any univariate quadrature, we can approximate the double integral as

$$\int_a^b \int_c^d f(t,s)dsdt \approx A_{n,m}(f,P_n,Q_m) + B_{n,m}(f,P_n,Q_m)$$

and hence $R_{n,m}(f,P_n,Q_m)$ represents the error. When the univariate integrals in $B_{n,m}(f,P_n,Q_m)$ are not easily approximated, then the double integral is approximated by

$$\int_a^b \int_c^d f(t,s)dsdt \approx A_{n,m}(f,P_n,Q_m)$$

and hence $B_{n,m}(f,P_n,Q_m) + R_{n,m}(f,P_n,Q_m)$ represents the error.

One of the main aims of this chapter is to study the error bounds for either the simple remainder $R_{n,m}(f,P_n,Q_m)$, when the term $B_{n,m}(f,P_n,Q_m)$ is known or easily computable, or the extended remainder, $B_{n,m}(f,P_n,Q_m)+R_{n,m}(f,P_n,Q_m)$, when $B_{n,m}(f,P_n,Q_m)$ is difficult to compute, for different particular classes of Appell polynomials as shown above (a) - (d). It is well known that Grüss type inequalities provide a useful means of approximating the integral of the product in terms of the product of integrals. Thus, using Grüss type integral inequalities for the remainder $R_{n,m}\left(f,P_n,Q_m\right)$ we are able to obtain perturbed versions of (4.2) and have different estimates for the new remainder, including the one in terms of the upper and lower bounds of the partial derivatives $\frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m}$ on $[a,b]\times[a,b]$ which will improve the classical bounds in terms of the sup-norm $\left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_\infty$. which are based on the lower and upper bounds of the functions within the In a similar fashion Hölder's inequality (see Mitrinović *et al.* (1993)) for double integrals will also be used. We will develop the approximation of (4.2) in which we will give different estimates of the remainder which is expected to improve the classical bounds. Extension to higher order integrals will also be studied. Numerical comparison of the results with existing procedures will be provided.

4.2 Some Recent Results

In this section, some of the previous and recent results are shown to be recaptured as special cases of the current work to obtain a generalisation of the Taylor like formula (4.5) for two Appell polynomials.

Let $x \in [a,b]$ and $y \in [c,d]$. If $f(x,y)$ is a function of two variables we shall adopt the following notation for partial derivatives of $f(x,y)$:

$$\begin{aligned} f^{(i,j)}(x,y) &\triangleq \frac{\partial^{i+j} f(x,y)}{\partial x^i \partial y^j}, \\ f^{(0,0)}(x,y) &\triangleq f(x,y), \\ f^{(i,j)}(\alpha,\beta) &\triangleq f^{(i,j)}(x,y)|_{(x,y)=(\alpha,\beta)} \end{aligned} \tag{4.4}$$

for $i,j \geq 0$ and $(\alpha,\beta) \in [a,b] \times [c,d]$.

A. H. Stroud has pointed out in Stroud (1971) that one of the most important tools in the numerical integration of double integrals is the following Taylor's formula (Stroud 1971, p. 138 and p. 157) due to Sard (1963).

THEOREM 4.1. *If $f(x,y)$ satisfies the condition that all the derivatives $f^{(i,j)}(x,y)$ for $i+j \leq m$ are defined and continuous on $[a,b] \times [c,d]$, then $f(x,y)$ has the expansion*

$$\begin{aligned} f(x,y) = &\sum_{i+j<m} \frac{(x-a)^i}{i!} \frac{(y-c)^j}{j!} f^{(i,j)}(a,c) \\ &+ \sum_{j<q} \frac{(y-c)^j}{j!} \int_a^x \frac{(x-u)^{m-j-1}}{(m-j-1)!} f^{(m-j,j)}(u,c)\,du \\ &+ \sum_{i<p} \frac{(x-a)^i}{i!} \int_c^y \frac{(y-v)^{m-i-1}}{(m-i-1)!} f^{(i,m-i)}(a,v)\,dv \\ &+ \int_a^x \int_c^y \frac{(x-u)^{p-1}}{(p-1)!} \frac{(y-v)^{q-1}}{(q-1)!} f^{(p,q)}(u,v)\,du\,dv, \end{aligned} \tag{4.5}$$

where i,j are nonnegative integers; p,q are positive integers and $m \triangleq p+q \geq 2$.

Essentially, the representation (4.5) is used for obtaining the fundamental Kernel Theorems and Error Estimates in numerical integration of double integrals (Stroud 1971, p. 142, p. 145 and p. 158). This representation has both an important theoretical and practical value in the whole domain.

Definition 1. *A sequence of polynomials $\{P_i(x)\}_{i=0}^{\infty}$ is called harmonic (Matić* et al. *(1999)) if it satisfies the following recursive formula*

$$P_i'(x) = P_{i-1}(x) \tag{4.6}$$

for $i \in \mathbb{N}$ and $P_0(x) = 1$.

A slightly different concept that specifies the connection between the variables is the following one.

Definition 2. *We say that a sequence of polynomials $\{P_i(t,x)\}_{i=0}^{\infty}$ satisfies the Appell condition (Appell (1880)) if*

$$\frac{\partial P_i(t,x)}{\partial t} = P_{i-1}(t,x) \tag{4.7}$$

and $P_0(t,x) = 1$ for all defined (t,x) and $i \in \mathbb{N}$.

It is well-known that the Bernoulli polynomials $B_i(t)$ can be defined by the following expansion

$$\frac{xe^{tx}}{e^x - 1} = \sum_{i=0}^{\infty} \frac{B_i(t)}{i!} x^i, \quad |x| < 2\pi, \quad t \in \mathbb{R}. \tag{4.8}$$

It can be shown that the polynomials $B_i(t)$, $i \in \mathbb{N}$, are uniquely determined by the following two formulae

$$B_i'(t) = iB_{i-1}(t), \quad B_0(t) = 1; \tag{4.9}$$

$$\text{and} \quad B_i(t+1) - B_i(t) = it^{i-1}. \tag{4.10}$$

Euler polynomials can be defined by the expansion

$$\frac{2e^{tx}}{e^x + 1} = \sum_{i=0}^{\infty} \frac{E_i(t)}{i!} x^i, \quad |x| < \pi, \quad t \in \mathbb{R}. \tag{4.11}$$

It can also be shown that the polynomials $E_i(t)$, $i \in \mathbb{N}$, are uniquely determined by the following two properties

$$E_i'(t) = iE_{i-1}(t), \quad E_0(t) = 1; \tag{4.12}$$

$$\text{and} \quad E_i(t+1) + E_i(t) = 2t^i. \tag{4.13}$$

For further details about Bernoulli polynomials and Euler polynomials, refer to Abramowitz and Stegun (1972), (sections 23.1.5 and 23.1.6).

In Matić *et al.* (1999), the following generalized Taylor's formula was established.

THEOREM 4.2. *Let* $\{P_i(x)\}_{i=0}^{\infty}$ *be Appell polynomials. Further, let* $I \subset \mathbb{R}$ *be a closed interval and* $a \in I$*. If* $f : I \to \mathbb{R}$ *is any function such that* $f^{(n)}(x)$ *is absolutely continuous for some* $n \in \mathbb{N}$*, then, for any* $x \in I$*, we have*

$$f(x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \left[P_k(x) f^{(k)}(x) - P_k(a) f^{(k)}(a) \right] + R_n(f; a, x), \tag{4.14}$$

where

$$R_n(f; a, x) = (-1)^n \int_a^x P_n(t) f^{(n+1)}(t)\, dt. \tag{4.15}$$

In the following section we utilize equation (4.5) and Theorem 2 in Matić *et al.* (1999) to develop a procedure and obtain a generalisation of the Taylor like formula (4.5) for two Appell polynomials effectively and efficiently.

4.3 Two New Taylor Like Expansions

The advantage of the method in this section is that polynomial and polynomial-like approximations can be developed in a straightforward manner with their accuracy incorporated into the formulation. That is, we aim to devise and investigate new multiple integration formulae and provide *a priori* error information as well.
Following a similar argument to the proof of Theorem 2 in Matić *et al.* (1999) (which is presented as Theorem 4.2 in Section 4.2), we obtain the following result.

THEOREM 4.3. *If $g : [a,b] \to \mathbb{R}$ is such that $g^{(n-1)}$ is absolutely continuous on $[a,b]$ and $\{P_i(t,x)\}_{i=0}^{\infty}$ an Appell polynomial, then we have the generalised integration by parts formula for $x \in [a,b]$*

$$\begin{aligned}\int_a^b g(t)\,dt &= \sum_{k=1}^{n}(-1)^{k+1}\left[P_k(b,x)g^{(k-1)}(b) - P_k(a,x)g^{(k-1)}(a)\right] \\ &\quad + (-1)^n \int_a^b P_n(t,x)g^{(n)}(t)\,dt.\end{aligned} \tag{4.16}$$

Proof. By integration by parts we obtain

$$\begin{aligned}&(-1)^n \int_a^b P_n(t,x)g^{(n)}(t)\,dt \\ &= (-1)^n P_n(t,x)g^{(n-1)}(t)\Big|_a^b + (-1)^{n-1}\int_a^b P_{n-1}(t,x)g^{(n-1)}(t)\,dt \qquad (4.17) \\ &= (-1)^n\left[P_n(b,x)g^{(n-1)}(b) - P_n(a,x)g^{(n-1)}(a) - \int_a^b P_{n-1}(t,x)g^{(n-1)}(t)\,dt\right].\end{aligned}$$

Clearly, the same procedure can be used for the term $\int_a^b P_{n-1}(t,x)g^{(n-1)}(t)\,dt$. Therefore, formula (4.16) follows from successive integration by parts. □

The version for functions of two independent variables is incorporated in the following result (Dragomir *et al.* (2005)).

THEOREM 4.4. *Let D be a domain in $\mathbb{R}^2$ and the point $(a,b) \in D$, let $\{P_i(t,x)\}_{i=0}^{\infty}$ and $\{Q_j(s,y)\}_{j=0}^{\infty}$ be two Appell polynomials. If $f : D \to \mathbb{R}$ is such that $f^{(i,j)}(x,y)$ are* *continuous on D for all $0 \le i \le n$ and $0 \le j \le m$, then*

$$f(x,y) = f(a,b)+C(f,P_n,Q_m)+D(f,P_n,Q_m)+S(f,P_n,Q_m)+T(f,P_n,Q_m), \quad (4.18)$$

where

$$\begin{aligned}
C(f,P_n,Q_m) &= \sum_{k=1}^{n}(-1)^{k+1}\big[P_k(x,x)f^{(k,0)}(x,b) - P_k(a,x)f^{(k,0)}(a,b)\big] \\
&\quad + \sum_{i=1}^{m}(-1)^{i+1}\big[Q_i(y,y)f^{(0,i)}(a,y) - Q_i(b,y)f^{(0,i)}(a,b)\big], \\
D(f,P_n,Q_m) &= \sum_{k=1}^{n}\sum_{i=1}^{m}(-1)^{k+i}P_k(x,x)\big[Q_i(y,y)f^{(k,i)}(x,y) - Q_i(b,y)f^{(k,i)}(x,b)\big] \\
&\quad - \sum_{k=1}^{n}\sum_{i=1}^{m}(-1)^{k+i}P_k(a,x)\big[Q_i(y,y)f^{(k,i)}(a,y) - Q_i(b,y)f^{(k,i)}(a,b)\big], \\
S(f,P_n,Q_m) &= (-1)^{n}\int_a^x P_n(t,x)f^{(n+1,0)}(t,b)\,dt + (-1)^{m}\int_b^y Q_m(s,y)f^{(0,m+1)}(a,s)\,ds \\
&\quad + \sum_{k=1}^{n}(-1)^{m+k+1}\int_b^y Q_m(s,y)\big[P_k(x,x)f^{(k,m+1)}(x,s) - P_k(a,x)f^{(k,m+1)}(a,s)\big]\,ds \\
&\quad + \sum_{i=1}^{m}(-1)^{n+i+1}\int_a^x P_n(t,x)\big[Q_i(y,y)f^{(n+1,i)}(t,y) - Q_i(b,y)f^{(n+1,i)}(t,b)\big]\,dt
\end{aligned}$$

and

$$T(f,P_n,Q_m) = (-1)^{m+n}\int_a^x\int_b^y P_n(t,x)Q_m(s,y)f^{(n+1,m+1)}(t,s)\,ds\,dt. \quad (4.19)$$

Proof. Let $P_n(t,x)$ be an Appell polynomial. Applying formula (4.14) to the function

$f(x,y)$ with respect to variable x yields

$$f(x,y) = f(a,y) + \sum_{k=1}^{n}(-1)^{k+1}\big[P_k(x,x)f^{(k,0)}(x,y) - P_k(a,x)f^{(k,0)}(a,y)\big] + (-1)^n \int_a^x P_n(t,x)f^{(n+1,0)}(t,y)\,dt. \tag{4.20}$$

Similarly, we have

$$f^{(k,0)}(x,y) = f^{(k,0)}(x,b) + (-1)^m \int_b^y Q_m(s,y)f^{(k,m+1)}(x,s)\,ds + \sum_{i=1}^{m}(-1)^{i+1}\big[Q_i(y,y)f^{(k,i)}(x,y) - Q_i(b,y)f^{(k,i)}(x,b)\big], \tag{4.21}$$

$$f^{(k,0)}(a,y) = f^{(k,0)}(a,b) + (-1)^m \int_b^y Q_m(s,y)f^{(k,m+1)}(a,s)\,ds + \sum_{i=1}^{m}(-1)^{i+1}\big[Q_i(y,y)f^{(k,i)}(a,y) - Q_i(b,y)f^{(k,i)}(a,b)\big], \tag{4.22}$$

$$f^{(n+1,0)}(t,y) = f^{(n+1,0)}(t,b) + (-1)^m \int_b^y Q_m(s,y)f^{(n+1,m+1)}(a,s)\,ds + \sum_{i=1}^{m}(-1)^{i+1}\big[Q_i(y,y)f^{(n+1,i)}(t,y) - Q_i(b,y)f^{(n+1,i)}(t,b)\big], \tag{4.23}$$

$$f(a,y) = f(a,b) + (-1)^m \int_b^y Q_m(s,y)f^{(0,m+1)}(a,s)\,ds + \sum_{i=1}^{m}(-1)^{i+1}\big[Q_i(y,y)f^{(0,i)}(a,y) - Q_i(b,y)f^{(0,i)}(a,b)\big]. \tag{4.24}$$

Substituting formulae (4.21)–(4.24) into (4.20) produces

$$\begin{aligned} f(x,y) = f(a,b) &+ \sum_{k=1}^{n}(-1)^{k+1}\big[P_k(x,x)f^{(k,0)}(x,b) - P_k(a,x)f^{(k,0)}(a,b)\big] \\ &+ \sum_{i=1}^{m}(-1)^{i+1}\big[Q_i(y,y)f^{(0,i)}(a,y) - Q_i(b,y)f^{(0,i)}(a,b)\big] \\ &+ \sum_{k=1}^{n}\sum_{i=1}^{m}(-1)^{k+i}P_k(x,x)\big[Q_i(y,y)f^{(k,i)}(x,y) - Q_i(b,y)f^{(k,i)}(x,b)\big] \\ &- \sum_{k=1}^{n}\sum_{i=1}^{m}(-1)^{k+i}P_k(a,x)\big[Q_i(y,y)f^{(k,i)}(a,y) - Q_i(b,y)f^{(k,i)}(a,b)\big] \\ &+ (-1)^n \int_a^x P_n(t,x)f^{(n+1,0)}(t,b)\,dt + (-1)^m \int_b^y Q_m(s,y)f^{(0,m+1)}(a,s)\,ds \\ &+ \sum_{k=1}^{n}(-1)^{m+k+1}\int_b^y Q_m(s,y)\big[P_k(x,x)f^{(k,m+1)}(x,s) \end{aligned} \tag{4.25}$$

$$- P_k(a,x) f^{(k,m+1)}(a,s)\big]\, ds$$
$$+ \sum_{i=1}^{m} (-1)^{n+i+1} \int_a^x P_n(t,x) \big[Q_i(y,y) f^{(n+1,i)}(t,y) - Q_i(b,y) f^{(n+1,i)}(t,b)\big]\, dt$$
$$+ (-1)^{m+n} \int_a^x \int_b^y P_n(t,x) Q_m(s,y) f^{(n+1,m+1)}(t,s)\, \mathrm{d}s\, \mathrm{d}t.$$

The proof of Theorem 4.4 is thus complete. □

Remark 4.4.1. *If we take*

$$P_i(t,x) = \frac{1}{i!}(t - (\lambda a + (1-\lambda))x)^i, \quad Q_j(s,y) = \frac{1}{j!}(s - (\lambda b + (1-\lambda))y)^j \tag{4.26}$$

for $0 \le i \le n$*,* $0 \le j \le m$ *and* $\lambda, \mu \in [0,1]$ *in Theorem 4.4, then*

$$C(f, P_n, Q_m) = \sum_{k=1}^{n} \frac{(a-x)^k}{k!} \big[(\lambda-1)^k f^{(k,0)}(a,b) - \lambda^k f^{(k,0)}(x,b)\big]$$
$$+ \sum_{i=1}^{m} \frac{(b-y)^i}{i!} \big[(\mu-1)^i f^{(0,i)}(a,b) - \mu^i f^{(0,i)}(a,y)\big], \tag{4.27}$$

$$D(f, P_n, Q_m) = \sum_{k=1}^{n} \sum_{i=1}^{m} \frac{\lambda^k (a-x)^k (b-y)^i}{k! \cdot i!} \big[\mu^i f^{(k,i)}(x,y) - (\mu-1)^i f^{(k,i)}(x,b)\big]$$
$$- \sum_{k=1}^{n} \sum_{i=1}^{m} \frac{(\lambda-1)^k (a-x)^k (b-y)^i}{k! \cdot i!} \big[\mu^i f^{(k,i)}(a,y) - (\mu-1)^i f^{(k,i)}(a,b)\big], \tag{4.28}$$

$$S(f, P_n, Q_m) = (-1)^n \int_a^x \frac{[t - (\lambda a + (1-\lambda)x)]^n}{n!} f^{(n+1,0)}(t,b)\, \mathrm{d}t$$
$$+ (-1)^m \int_b^y \frac{[s - (\mu b + (1-\mu)y)]^m}{m!} f^{(0,m+1)}(a,s)\, \mathrm{d}s$$
$$+ \sum_{k=1}^{n} \int_b^y \frac{[\mu b + (1-\mu)y - s]^m (a-x)^k}{m! \cdot k!} \big[(\lambda-1)^k f^{(k,m+1)}(a,s) - \lambda^k f^{(k,m+1)}(x,s)\big]\, ds$$
$$+ \sum_{i=1}^{m} \int_a^x \frac{[\lambda a + (1-\lambda)x - t]^n (b-y)^i}{n! \cdot i!} \big[(\mu-1)^i f^{(n+1,i)}(t,b) - \mu^i f^{(n+1,i)}(t,y)\big]\, dt, \tag{4.29}$$

and

$$T(f, P_n, Q_m) = \int_a^x \int_b^y \frac{[(\lambda a + (1-\lambda)x) - t]^n [(\mu b + (1-\mu)y) - s]^m}{n! \cdot m!} f^{(n+1,m+1)}(t,s)\, ds\, dt. \tag{4.30}$$

If further taking $\lambda = 0$ and $\mu = 0$ in (4.26)*, then we can deduce Theorem 4.1 from Theorem 4.4.*

The other choices of Appell type polynomials will provide generalizations of Theorem 4.1.

The following approximation of double integrals in terms of Appell polynomials holds (Dragomir *et al.* (2005)).

THEOREM 4.5. *Let $\{P_i(t,x)\}_{i=0}^{\infty}$ and $\{Q_j(s,y)\}_{j=0}^{\infty}$ be two Appell polynomials and $f : [a,b] \times [c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ such that $f^{(i,j)}(x,y)$ are continuous on $[a,b] \times [c,d]$ for all $0 \le i \le n$ and $0 \le j \le m$. Then we have*

$$\int_a^b \int_c^d f(t,s)\,ds\,dt = A(f,P_n,Q_m) + B(f,P_n,Q_m) + R(f,P_n,Q_m), \tag{4.31}$$

where

$$\begin{aligned} A(f,P_n,Q_m) = &\sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{i+j}P_j(a,b)\big[Q_i(d,d)f^{(i-1,j-1)}(a,d) - Q_i(c,d)f^{(i-1,j-1)}(a,c)\big] \\ &- \sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{i+j}P_j(b,b)\big[Q_i(d,d)f^{(i-1,j-1)}(b,d) - Q_i(c,d)f^{(i-1,j-1)}(b,c)\big], \\ B(f,P_n,Q_m) = &\sum_{i=1}^{m}(-1)^iQ_i(c,d)\int_a^b f^{(0,i-1)}(t,c)\,dt - \sum_{i=1}^{m}(-1)^iQ_i(d,d)\int_a^b f^{(0,i-1)}(t,d)\,dt \\ &+ \sum_{j=1}^{n}(-1)^jP_j(a,b)\int_c^d f^{(j-1,0)}(a,s)\,ds - \sum_{j=1}^{n}(-1)^jP_j(b,b)\int_c^d f^{(j-1,0)}(b,s)\,ds \end{aligned}$$

and

$$R(f,P_n,Q_m) = (-1)^{m+n}\int_a^b \int_c^d P_n(t,b)Q_m(s,d)f^{(n,m)}(t,s)\,ds\,dt.$$

Proof. The proof is obvious by using the generalized integration by parts formula consecutively, and we omit the details. □

Remark 4.5.1. *As usual, let B_i, $i \in \mathbb{N}$, denote Bernoulli numbers. From properties* (4.9) *and* (4.10), (4.12) *and* (4.13) *of Bernoulli and Euler polynomials respectively, we can obtain easily that, for $i \ge 1$,*

$$B_{i+1}(0) = B_{i+1}(1) = B_{i+1}, \quad B_1(0) = -B_1(1) = -\frac{1}{2}, \tag{4.32}$$

and, for $j \in \mathbb{N}$*,*

$$E_j(0) = -E_j(1) = -\frac{2}{j+1}(2^{j+1} - 1)B_{j+1}. \tag{4.33}$$

It is also a well known fact that $B_{2i+1} = 0$ *for all* $i \in \mathbb{N}$*.*

Taking $P_i(t,x) = P_{i,B}(t,x;a)$ *and* $Q_j(s,y) = P_{j,E}(s,y;c)$ *for* $0 \leq i \leq n$ *and* $0 \leq j \leq m$ *in Theorem 4.5 and considering* (4.32) *and* (4.33) *yields*

$$\begin{aligned} A(f,P_n,Q_m) = & \sum_{i=1}^{m}\sum_{j=2}^{n} \frac{(a-b)^j (c-d)^i}{j!\cdot i!} \cdot \frac{2(2^{i+1}-1)}{i+1} B_j B_{i+1} \\ & \times \left[f^{(i-1,j-1)}(a,d) + f^{(i-1,j-1)}(a,c) - f^{(i-1,j-1)}(b,d) - f^{(i-1,j-1)}(b,c)\right] \\ & + (b-a)\sum_{i=1}^{m} \frac{(2^{i+1}-1)(c-d)^i}{(i+1)!} B_{i+1} \\ & \times \left[f^{(i-1,0)}(a,d) + f^{(i-1,0)}(a,c) + f^{(i-1,0)}(b,d) + f^{(i-1,0)}(b,c)\right], \end{aligned} \tag{4.34}$$

$$\begin{aligned} B(f,P_n,Q_m) = & \, 2\sum_{i=1}^{m} \frac{(1-2^{i+1})(c-d)^i}{(i+1)!} B_{i+1} \int_a^b \left[f^{(0,i-1)}(t,c) + f^{(0,i-1)}(t,d)\right] dt \\ & + \sum_{j=2}^{n} \frac{(a-b)^j}{j!} B_j \int_c^d \left[f^{(j-1,0)}(a,s) - f^{(j-1,0)}(b,s)\right] ds \\ & + \frac{b-a}{2}\int_c^d \left[f(a,s) + f(b,s)\right] ds, \end{aligned} \tag{4.35}$$

and

$$R(f,P_n,Q_m) = \frac{(a-b)^n (c-d)^m}{n!\cdot m!} \int_a^b \int_c^d B_n\left(\frac{t-a}{b-a}\right) B_m\left(\frac{s-c}{d-c}\right) f^{(n,m)}(t,s)\, ds\, dt. \tag{4.36}$$

In Section 4.5 we will discuss numerical experiment that relates to Remark 4.5.1 and utilize the perturbed Taylor's expansion developed in equations (4.34) and (4.35) for different values of m and n to approximate the functions given in example 3.2 in Chapter 3.

4.4 Estimates of the Remainders

In this section, we will give some bounds for the remainders of expansions in Theorems 4.4 and 4.5.

We firstly need to introduce some notation.

For a function $\ell : [a,b] \times [c,d] \to \mathbb{R}$, then for any $x, y \in [a,b]$, $z, u \in [c,d]$ we denote

$$\|\ell\|_{[x,y]\times[z,u],\infty} \triangleq ess\sup\{|\ell(t,s)|\}, \quad t \in [x,y] \text{ or } [y,x] \text{ and } s \in [z,u] \text{ or } [u,z]$$

and

$$\|\ell\|_{[x,y]\times[z,u],p} \triangleq \left|\int_x^y \int_z^u |h(t,s)|^p \, dsdt\right|^{\frac{1}{p}}, \quad p \geq 1.$$

The following result in establishing bounds for the remainder in the Taylor-like formula (4.18) holds (Dragomir *et al.* (2005)).

THEOREM 4.6. *Assume that* $\{P_i(t,x)\}_{i=0}^{\infty}$, $\{Q_j(s,y)\}_{j=0}^{\infty}$ *and* f *satisfy the assumptions of Theorem 4.4. Then we have the representation (4.18) and the remainder satisfies the estimate*

$$|T(f,P_n,Q_m)| \leq \begin{cases} \|P_n(\cdot,x)\|_{[a,x],\infty}\|Q_m(\cdot,y)\|_{[b,y],\infty}\left\|f^{(n+1,m+1)}\right\|_{[a,x]\times[b,y],1}, \\ \|P_n(\cdot,x)\|_{[a,x],q}\|Q_m(\cdot,y)\|_{[b,y],q}\left\|f^{(n+1,m+1)}\right\|_{[a,x]\times[b,y],p}, \\ \qquad \text{where } p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \|P_n(\cdot,x)\|_{[a,x],1}\|Q_m(\cdot,y)\|_{[b,y],1}\left\|f^{(n+1,m+1)}\right\|_{[a,x]\times[b,y],\infty}. \end{cases} \tag{4.37}$$

The proof follows on using Hölder's inequality applied for the integral representation of the remainder $T(f,P_n,Q_m)$ provided by equation (4.37). We omit the details.

The integral remainder in the cubature formula (4.31) may be estimated in the following manner.

THEOREM 4.7. *Assume that* $\{P_i(t,x)\}_{i=0}^{\infty}$, $\{Q_j(s,y)\}_{j=0}^{\infty}$ *and* f *satisfy the assumptions in Theorem 4.5. Then one has the cubature formula (4.31) and the remainder*

$R(f, P_n, Q_m)$ *satisfies the estimate:*

$$|R(f, P_n, Q_m)| \leq \begin{cases} \|P_n(\cdot, b)\|_{[a,b],\infty} \|Q_m(\cdot, d)\|_{[c,d],\infty} \left\|f^{(n,m)}\right\|_{[a,b]\times[c,d],1}, \\ \|P_n(\cdot, b)\|_{[a,b],q} \|Q_m(\cdot, d)\|_{[c,d],q} \left\|f^{(n,m)}\right\|_{[a,b]\times[c,d],p}, \\ \qquad \text{where } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \|P_n(\cdot, b)\|_{[a,b],1} \|Q_m(\cdot, d)\|_{[c,d],1} \left\|f^{(n,m)}\right\|_{[a,b]\times[c,d],\infty}. \end{cases} \tag{4.38}$$

Remark 4.7.1. *If we consider the particular instances of Appell polynomials provided by (4.11), (4.12) and (4.13), then a number of particular formulae may be obtained. Their remainder may be estimated by the use of Theorems 4.6 and 4.7, providing a two-dimensional version of the results in Matić* et al. *(1999).*

For instance, if we consider

$$P_{n,\lambda}(t, x; a) = \frac{[t - (\lambda\, a + (1-\lambda)\, x)]^n}{n!} \tag{4.39}$$

$$\text{and} \quad Q_{m,\mu}(s, y; b) = \frac{[s - (\mu\, b + (1-\mu)\, y)]^m}{m!} \tag{4.40}$$

then we obtain the following result:

THEOREM 4.8. *Let* $\{P_{n,\lambda}(t, x; a)\}_{n=0}^{\infty}$, $\{Q_{m,\mu}(s, y; b)\}_{m=0}^{\infty}$ *and* f *satisfy the assumptions of Theorem 4.4. Then we have the representation (4.18) and the remainder satisfies the estimate*

$$|T(f, P_{n,\lambda}, Q_{m,\mu})| \leq \begin{cases} \frac{(x-a)^n (y-b)^m}{n!m!} \lambda_\infty\, \mu_\infty \left\|f^{(n+1,m+1)}\right\|_{[a,x]\times[b,y],1}, \\ \frac{1}{n!m!} \left[\frac{(x-a)^{nq+1}(y-b)^{mq+1}}{(nq+1)(mq+1)}\right]^{\frac{1}{q}} \lambda_q\, \mu_q \left\|f^{(n+1,m+1)}\right\|_{[a,x]\times[b,y],p}, \\ \qquad \text{where } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(x-a)^{n+1}(y-b)^{m+1}}{(n+1)!(m+1)!} \lambda_1 \mu_1 \left\|f^{(n+1,m+1)}\right\|_{[a,x]\times[b,y],\infty}. \end{cases} \tag{4.41}$$

where

$$\lambda_1 = \left[\lambda^{n+1} + (1-\lambda)^{n+1}\right] \quad , \quad \mu_1 = \left[\mu^{m+1} + (1-\mu)^{m+1}\right]$$
$$\lambda_q = \left[\lambda^{nq+1} + (1-\lambda)^{nq+1}\right]^{\frac{1}{q}} \quad , \quad \mu_q = \left[\mu^{mq+1} + (1-\mu)^{mq+1}\right]^{\frac{1}{q}}$$
$$\lambda_\infty = \left[\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right]^n \quad , \quad \mu_\infty = \left[\frac{1}{2} + \left|\mu - \frac{1}{2}\right|\right]^n .$$

Proof. Utilizing equations (4.39) and (4.40) and using Hölder's inequality for double integrals and the properties of the modulus on equation (4.37), then we have that

$$\left|\int_a^x \int_b^y T(f, P_{n,\lambda}, Q_{m,\mu})\right|$$
$$= \left|\int_a^x \int_b^y P_{n,\lambda}(t,x;a)\, Q_{m,\mu}(s,y;b) f^{(n+1,m+1)}\, ds\, dt \right|$$
$$\le \int_a^x \int_b^y \left|P_{n,\lambda}(t,x;a)\, Q_{m,\mu}(s,y;b)\right| \left|f^{(n+1,m+1)}\right|\, ds\, dt$$
$$\le \begin{cases} \sup\limits_{(t,s)\in[a,b]\times[c,d]} \left|P_{n,\lambda}(t,x;a) Q_{m,\mu}(s,y;b)\right| \left\|f^{(n+1,m+1)}\right\|_{[a,x]\times[b,y],1} . \\ \\ \left(\int_a^x \int_b^y \left|P_{n,\lambda}(t,x;a) Q_{m,\mu}(s,y;b)\right|^q dt\, ds\right)^{\frac{1}{q}} \left\|f^{(n+1,m+1)}\right\|_{[a,x]\times[b,y],p} , \\ \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \int_a^x \int_b^y \left|P_{n,\lambda}(t,x;a) Q_{m,\mu}(s,y;b)\right| dt\, ds \left\|f^{(n+1,m+1)}\right\|_{[a,x]\times[b,y],\infty} . \end{cases} \tag{4.42}$$

Now, the result in equation (4.42) can be further simplified by application of equations (4.39) and (4.40), given that,

$$\alpha = (1-\lambda)\, x + \lambda\, a \quad \text{and} \quad \beta = (1-\mu)\, y + \mu\, b.$$

It follows

$$\sup_{(t,s)\in[a,x]\times[b,y]} \left|P_{n,\lambda}(t,x;a) Q_{m,\mu}(s,y;b)\right|$$
$$= \sup_{t\in[a,b]} \left|P_{n,\lambda}(t,x;a)\right| \sup_{s\in[c,d]} \left|Q_{m,\mu}(s,y;b)\right|$$
$$= \max\left\{\frac{(\alpha-a)^n}{n!}, \frac{(x-\alpha)^n}{n!}\right\} \times \max\left\{\frac{(\beta-b)^m}{m!}, \frac{(y-\beta)^m}{m!}\right\}$$
$$= \frac{(x-a)^n\,(y-b)^m}{n!m!} \left[\max\{(1-\lambda), \lambda\}\right]^n \times \left[\max\{(1-\mu), \mu\}\right]^m$$
$$= \frac{(x-a)^n\,(y-b)^m}{n!m!} \left[\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right]^n \times \left[\frac{1}{2} + \left|\mu - \frac{1}{2}\right|\right]^m$$

giving the first inequality in (4.41) where we have used the fact that

$$\max\{X,Y\} = \frac{X+Y}{2} + \left|\frac{Y-X}{2}\right|.$$

Further, we have

$$\begin{aligned}
&\left(\int_a^x \int_b^y |P_{n,\lambda}(t,x;a)Q_{m,\mu}(s,y;b)|^q \, ds\, dt\right)^{\frac{1}{q}} \\
= &\left(\int_a^x |P_{n,\lambda}(t,x;a)|^q \, dt\right)^{\frac{1}{q}} \left(\int_b^y |Q_{m,\mu}(s,y;b)|^q \, ds\, dt\right)^{\frac{1}{q}} \\
= &\frac{1}{n!m!}\left[\int_a^\alpha (\alpha - t)^{nq}\, dt + \int_\alpha^x (t-\alpha)^{nq}\, dt\right]^{\frac{1}{q}} \\
&\times \left[\int_b^\beta (\beta - s)^{mq}\, ds + \int_\beta^y (s-\beta)^{mq}\, ds\right]^{\frac{1}{q}} \\
= &\frac{1}{n!m!}\left[\frac{(x-a)^{nq+1}(y-b)^{mq+1}}{(nq+1)(mq+1)}\right]^{\frac{1}{q}} \lambda_q\, \mu_q
\end{aligned}$$

producing the second inequality in (4.41).

Finally,

$$\begin{aligned}
&\int_a^x \int_b^y |P_{n,\lambda}(t,x;a)Q_{m,\mu}(s,y;b)|\, dt\, ds \\
= &\int_a^x \left|\frac{(t-\alpha)^n}{n!}\right| dt \int_b^y \left|\frac{(s-\beta)^m}{m!}\right| ds \\
= &\left[\int_a^\alpha \frac{(\alpha - t)^n}{n!} dt + \int_\alpha^x \frac{(t-\alpha)^n}{n!} dt\right] \times \left[\int_b^\beta \frac{(\beta - s)^m}{m!} ds + \int_\beta^y \frac{(s-\beta)^m}{m!} ds\right] \\
= &\frac{(x-a)^{n+1}(y-b)^{m+1}}{(n+1)!\,(m+1)!}\left[(1-\lambda)^{n+1} + \lambda^{n+1}\right] \times \left[(1-\mu)^{m+1} + \mu^{m+1}\right]
\end{aligned}$$

gives the last inequality in (4.41). Thus the theorem is completely proved. □

Remark 4.8.1. *By taking $\lambda = \mu = 0$ or 1, we recapture the result obtained by* Hanna *et al.* (2002b).

In a similar fashion, we can bound the remainder $R(f, P_{n,\lambda}, Q_{m,\mu})$ in the cubature formula (4.31) as in the following

THEOREM 4.9. *Let* $\{P_{n,\lambda}(t,x;a)\}_{n=0}^{\infty}$, $\{Q_{m,\mu}(s,y)\}_{m=0}^{\infty}$ *and* f *satisfy the assumptions of Theorem 4.5, then the remainder* $R(f,P_{n,\lambda},Q_{m,\mu})$ *estimate in the cubature formula (4.31) satisfies the following*

$$|R(f,P_{n,\lambda},Q_{m,\mu})| \leq \begin{cases} \frac{(b-a)^n(d-c)^m}{n!m!}\lambda_\infty\,\mu_\infty \left\|f^{(n,m)}\right\|_{[a,b]\times[c,d],1}, \\ \frac{1}{n!m!}\left[\frac{(b-a)^{nq+1}(d-c)^{mq+1}}{(nq+1)(mq+1)}\right]^{\frac{1}{q}}\lambda_q\,\mu_q\left\|f^{(n,m)}\right\|_{[a,b]\times[c,d],p}, \\ \qquad \text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^{n+1}(d-c)^{m+1}}{(n+1)!(m+1)!}\lambda_1\,\mu_1\left\|f^{(n,m)}\right\|_{[a,b]\times[c,d],\infty}. \end{cases} \tag{4.43}$$

The proof is similar to the one in Theorem 4.8 applied on the interval $[a,b]\times[c,d]$, and we omit the details.

In the next section we will apply those equations that are presented in (4.34) and (4.35) in Remark 4.5.1 to approximating general double integrals.

4.5 Numerical Experiments

It is often desirable to boost the existing theoretical results with the associated numerical results in order to obtain the desired precision of the estimates. Thus, in this section we discuss a numerical experiment using Remark 4.5.1 and utilize the representation

$$\int_0^x\int_0^y f(t,s)ds\,dt = A(f,P_n,Q_m)+B(f,P_n,Q_m)+R(f,P_n,Q_m) \quad \text{for} \quad 0\leq x,y\leq 1,$$

where $A(f,P_n,Q_m)$ and $B(f,P_n,Q_m)$ are from equations (4.34) and (4.35) respectively with $a=c=0$ and $b=x,\ \ d=y$. This is used to plot the behaviour of the absolute value of the error as a function of $(x,y)\in[0,1]^2$ for different values of m and n when we consider the function

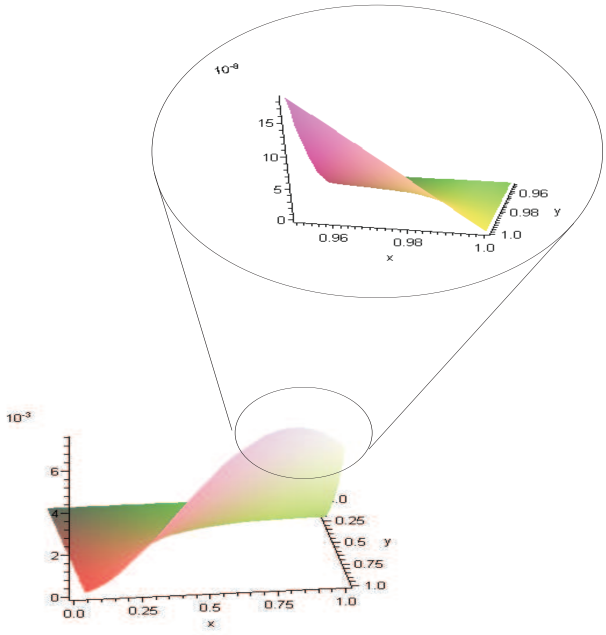
$$f(x,y)=e^{-xy}, \qquad 0\leq x,y\leq 1, \tag{4.44}$$

which is given in Example 3.2 in Chapter 3. Again, this function was chosen because it is not separable as a product of two univariable functions and the partial differentiation for it becomes smaller on $[0,1]\times[0,1]$ with successive derivatives. This indicates that the higher order error bounds will give better results.

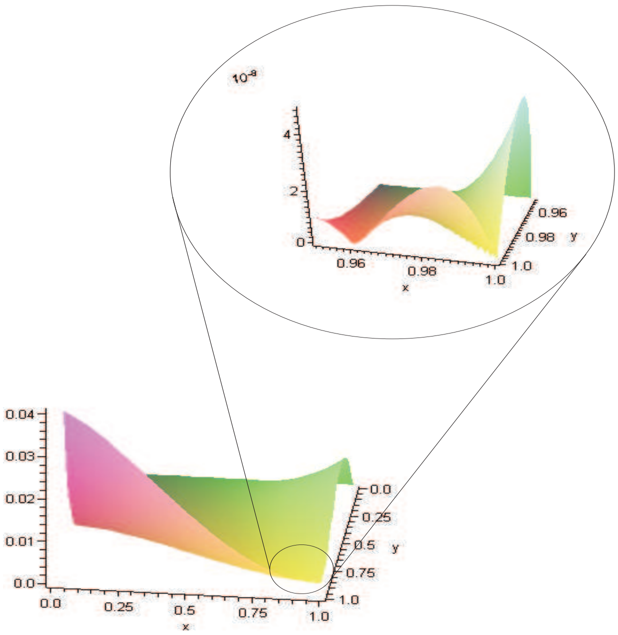
In Figure 4.1 we show the three dimensional plot for the behaviour of the absolute value of the error as a function of $(x,y)\in[0,1]^2$. It is clear that the error is smaller near the right end of the interval in each direction.

The magnified graph demonstrates the behaviour of the absolute error over the interval $.95\leq x,y\leq 1$.

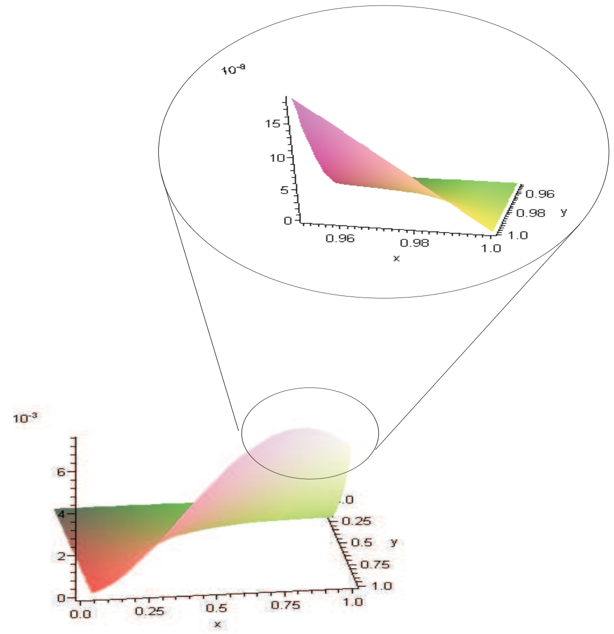
All the algebraic calculations of the previous section have been performed using Maple and the code for this is shown in Appendix A.1.2.



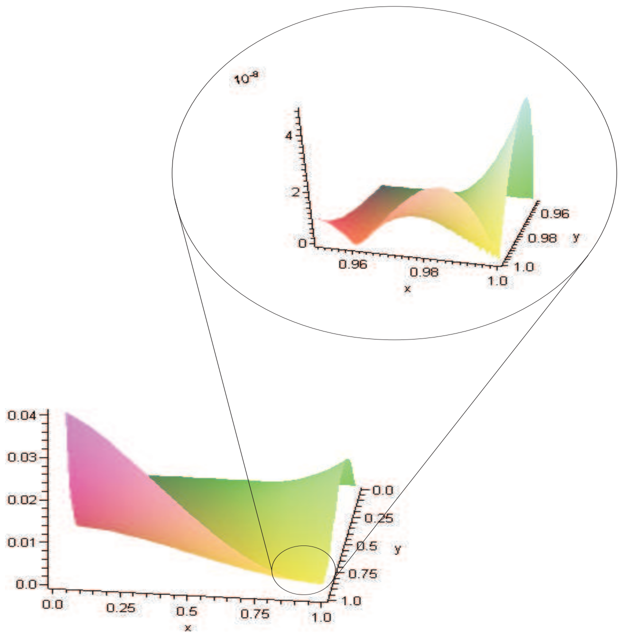
(a) The absolute error of approximating the integral $\int_0^x \int_0^y f(t,s)\ dsdt$ as a function of $(x,y) \in [0,1]^2$ for the function (4.44) using (4.34) and (4.35) for $m = n = 1$



(b) The absolute error of approximating the integral $\int_0^x \int_0^y f(t,s)\ dsdt$ as a function of $(x,y) \in [0,1]^2$ for the function (4.44) using (4.34) and (4.35) for $m = n = 4$



(c) The absolute error of approximating the integral $\int_0^x \int_0^y f(t,s)\ dsdt$ as a function of $(x,y) \in [0,1]^2$ for the function (4.44) using (4.34) and (4.35) for $m = n = 16$



(d) The absolute error of approximating the integral $\int_0^x \int_0^y f(t,s)\ dsdt$ as a function of $(x,y) \in [0,1]^2$ for the function (4.44) using (4.34) and (4.35) for $m = n = 32$

Figure 4.1: The absolute error of approximating the integral $\int_0^x \int_0^y f(t,s)\ dsdt$ as a function of $(x,y) \in [0,1]^2$ for the function (4.44) using (4.34) and (4.35) for various values of m and n.

CHAPTER 5

A REVERSE OF THE CAUCHY-BUNYAKOVSKY-SCHWARZ (CBS) INTEGRAL INEQUALITY

The Cauchy-Bunyakovsky-Schwarz inequality, or for short, the CBS-inequality, plays an important role in different branches of modern mathematics including Hilbert spaces theory, probability and statistics, classical real and complex analysis, numerical analysis, qualitative theory of differential equations and their applications. The main purpose of this chapter is to identify and highlight the discrete inequalities that are connected with the CBS-inequality and provide refinements and reverse results as well as to study some functional properties of certain mappings that can be naturally associated with this inequality.

The chapter is arranged in the following manner. In Section 5.2, reverse results for the CBS-inequality are obtained. The results of Cassels are represented with their original proofs. New results and versions for complex numbers are also obtained. Reverse results of Dragomir *et al.* (2005) are mentioned and some refinements of Cassels results are obtained.

Finally, Section 5.3 is reserved for a pre-Grüss type inequality for double integrals where Korkine's identity is applied.

5.1 Introduction

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, Σ a $\sigma-$algebra of subset of Ω and μ a countably additive and positive measure on Σ with values $\mathbb{R} \cup \{\infty\}$. Let $\rho \geq 0$ be a $\mu-$measurable function on Ω. Denote by $L^2_\rho(\Omega, \mathbb{K})$ the Hilbert space of all real or complex valued functions defined on Ω and $\rho-$integrable on Ω, namely,

$$\int_\Omega \rho(s) |f(s)|^2 d\mu(s) < \infty. \tag{5.1}$$

If $f, g : \Omega \to \mathbb{R}$ are real functions such that there exist the constants $0 < m \leq M < \infty$ with property that

$$m \leq \frac{f(s)}{g(s)} \leq M \quad \text{for } \mu - \text{a.e. } s \in \Omega, \tag{5.2}$$

then we have

$$\frac{\int_\Omega \rho(s) f^2(s) d\mu(s) \int_\Omega \rho(s) g^2(s) d\mu(s)}{\left(\int_\Omega \rho(s) f(s) g(s) d\mu(s)\right)^2} \leq \frac{(M+m)^2}{4mM}. \tag{5.3}$$

Inequality (5.3) (in its discrete version) is known in the literature as the Cassels inequality (see for instance Watson (1955)).

If we assume that there exist constants m_i, M_i $(i = 1, 2)$ such that

$$\begin{aligned} 0 < m_1 \leq f(s) \leq M_1 < \infty \text{ for } \mu - \text{a.e. } s \in \Omega, \\ 0 < m_2 \leq g(s) \leq M_2 < \infty \text{ for } \mu - \text{a.e. } s \in \Omega, \end{aligned} \tag{5.4}$$

then from Cassels' inequality, we deduce the weighted inequality obtained in Pólya and Szego (1925), which is also known in the literature as the Greub-Reinboldt inequality (Greub and Rheinboldt (1959)):

$$\frac{\int_\Omega \rho(s) f^2(s) d\mu(s) \int_\Omega \rho(s) g^2(s) d\mu(s)}{\left(\int_\Omega \rho(s) f(s) g(s) d\mu(s)\right)^2} \leq \frac{(M_1 M_2 + m_1 m_2)^2}{4 m_1 m_2 M_1 M_2}. \tag{5.5}$$

In the recent works Dragomir (2003), Dragomir *et al.* (2005), the authors obtained the following extension for real or complex-valued functions of the Cassels inequality (see Proposition 4, Dragomir (2003)):

Let $f, g \in L^2_\rho(\Omega, \mathbb{K})$ and $\gamma, \Gamma \in \mathbb{K}$ such that $Re(\Gamma\bar{\gamma}) \geq 0$. If either

$$Re\left[(\Gamma g(s) - f(s))\left(\overline{f(s)} - \bar{\gamma}\overline{g(s)}\right)\right] \geq 0 \text{ for } \mu-\text{a.e. } s \in \Omega \tag{5.6}$$

or equivalently

$$\left|f(x) - \frac{\Gamma+\gamma}{2} \cdot g(s)\right| \leq \frac{1}{2}|\Gamma-\gamma||g(s)| \text{ for } \mu-\text{a.e. } s \in \Omega, \tag{5.7}$$

then we have the inequality

$$\begin{aligned} &\int_\Omega \rho(s)|f(s)|^2 d\mu(s) \int_\Omega \rho(s)|g(s)|^2 d\mu(s) \\ &\leq \frac{1}{4} \cdot \frac{\left\{Re\left[(\bar{\Gamma}+\bar{\gamma})\right]\int_\Omega \rho(s) f(s)\overline{g(s)}d\mu(s)\right\}^2}{Re(\Gamma\bar{\gamma})} \\ &\leq \frac{1}{4} \cdot \frac{|\Gamma+\gamma|^2}{Re(\Gamma\bar{\gamma})}\left|\int_\Omega \rho(s) f(s)\overline{g(s)}d\mu(s)\right|^2. \end{aligned} \tag{5.8}$$

The constant $\frac{1}{4}$ is best possible in both inequalities.

If (5.6) or (5.7) holds true, then the following additive version of (5.8) also holds

$$\begin{aligned} 0 &\leq \int_\Omega \rho(s)|f(s)|^2 d\mu(s) \int_\Omega \rho(s)|g(s)|^2 d\mu(s) - \left|\int_\Omega \rho(s) f(s)\overline{g(s)}d\mu(s)\right|^2 \\ &\leq \frac{1}{4} \cdot \frac{|\Gamma-\gamma|^2}{Re(\Gamma\bar{\gamma})}\left|\int_\Omega \rho(s) f(s)\overline{g(s)}d\mu(s)\right|^2. \end{aligned} \tag{5.9}$$

Here $\frac{1}{4}$ is also the best possible constant.

5.2 Some Reverses of the CBS-Inequality

We start with the following lemma that is of interest in itself (see Dragomir *et al.* (2005)).

Lemma 5.1. *Let $f, g \in L^2_\rho(\Omega, \mathbb{K})$ with $g(s) \neq 0$ for μ–a.e. $s \in \Omega$. If there exist the constants $\alpha \in \mathbb{K}$ and $r > 0$ such that*

$$\frac{f(s)}{g(s)} \in \bar{D}(\alpha, r) := \{z \in \mathbb{K} | |z-\alpha| \leq r\}, \tag{5.10}$$

then we have the inequality

$$\begin{aligned}&\int_{\Omega}\rho(s)|f(s)|^2d\mu(s)+\left(|\alpha|^2-r^2\right)\int_{\Omega}\rho(s)|g(s)|^2d\mu(s)\\&\leq 2Re\left[\bar{\alpha}\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s)\right]\\&\leq 2|\alpha|\left|\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s)\right|.\end{aligned}\tag{5.11}$$

The constant 2 in the right side of (5.11) is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. From (5.10) we have

$$|f(s)-\alpha g(s)|^2\leq r|g(s)|^2$$

for $\mu-$a.e. $s\in\Omega$, which is clearly equivalent to

$$|f(s)|^2+\left(|\alpha|^2-r^2\right)|g(s)|^2\leq 2Re\left[\bar{\alpha}\left(f(s)\overline{g(s)}\right)\right]\tag{5.12}$$

for $\mu-$a.e. $s\in\Omega$.

Multiplying (5.12) with $g(s)\geq 0$ and integrating on Ω, we deduce the first inequality in (5.11). The second inequality is obvious by the fact that $Re(z)\leq|z|$ for $z\in\mathbb{C}$.

To prove the sharpness of the constant 2, assume that under the hypothesis of the theorem, there exists a constant $C>0$ such that

$$\begin{aligned}&\int_{\Omega}\rho(s)|f(s)|^2d\mu(s)+\left(|\alpha|^2-r^2\right)\int_{\Omega}\rho(s)|g(s)|^2d\mu(s)\\&\qquad\leq CRe\left[\bar{\alpha}\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s)\right]\end{aligned}\tag{5.13}$$

provided $\frac{f(s)}{g(s)}\in\bar{D}(\alpha,r)$ for $\mu-$a.e. $s\in\Omega$.

If we choose ρ such that $\int_{\Omega}\rho(s)d\mu(s)=1$, $f(s)=2r$, $g(s)=1$ and $\alpha=r$, $r>0$, then we have $\frac{f(s)}{g(s)}=2r\in\bar{D}(r,r)$, and by (5.13) we deduce

$$4r^2\leq 2Cr^2$$

giving $C\geq 2$. □

The case where the disk $\bar{D}(\alpha, r)$ does not contain the origin, i.e., $|\alpha| > r > 0$, provides the following interesting reverse of the CBS-inequality (see Dragomir *et al.* (2005)).

THEOREM 5.1. *Let f, g, ρ be as in Lemma 5.1 and assume that $|\alpha| > r > 0$. Then we have the inequality*

$$\begin{aligned} &\int_\Omega \rho(s)|f(s)|^2 d\mu(s) \int_\Omega \rho(s)|g(s)|^2 d\mu(s) \\ &\le \frac{1}{|\alpha|^2 - r^2}\left[Re\left\{\bar{\alpha}\int_\Omega \rho(s) f(s)\overline{g(s)}d\mu(s)\right\}\right]^2 \\ &\le \frac{|\alpha|^2}{|\alpha|^2 - r^2}\left|\int_\Omega \rho(s) f(s)\overline{g(s)}d\mu(s)\right|^2. \end{aligned} \tag{5.14}$$

The constant 1 in the first and second inequalities is the best possible in the sense that it cannot be replaced by a smaller quantity.

Proof. Since $|\alpha| > r$, we may divide (6.3) by $\sqrt{|\alpha|^2 - r^2} > 0$ to obtain

$$\begin{aligned} &\int_\Omega \rho(s)|f(s)|^2 d\mu(s) + \sqrt{|\alpha|^2 - r^2}\int_\Omega \rho(s)|g(s)|^2 d\mu(s) \\ &\qquad \le \frac{2}{\sqrt{|\alpha|^2 - r^2}} Re\left[\bar{\alpha}\int_\Omega \rho(s) f(s)\overline{g(s)}d\mu(s)\right]. \end{aligned} \tag{5.15}$$

On the other hand, by the use of the following elementary inequality

$$\frac{1}{\beta}p + \beta q \ge 2\sqrt{pq} \quad \text{for } \beta > 0 \text{ and } p, q \ge 0, \tag{5.16}$$

we may state that

$$\begin{aligned} &2\left(\int_\Omega \rho(s)|f(s)|^2 d\mu(s)\right)^{\frac{1}{2}}\left(\int_\Omega \rho(s)|g(s)|^2 d\mu(s)\right)^{\frac{1}{2}} \\ &\quad \le \frac{1}{\sqrt{|\alpha|^2 - r^2}}\int_\Omega \rho(s)|f(s)|^2 d\mu(s) + \sqrt{|\alpha|^2 - r^2}\int_\Omega \rho(s)|g(s)|^2 d\mu(s). \end{aligned} \tag{5.17}$$

Utilising (5.15) and (5.17), we deduce

$$\begin{aligned} &\left(\int_\Omega \rho(s)|f(s)|^2 d\mu(s)\right)^{\frac{1}{2}}\left(\int_\Omega \rho(s)|g(s)|^2 d\mu(s)\right)^{\frac{1}{2}} \\ &\qquad \le \frac{1}{\sqrt{|\alpha|^2 - r^2}} Re\left[\bar{\alpha}\int_\Omega \rho(s) f(s)\overline{g(s)}d\mu(s)\right], \end{aligned}$$

which is clearly equivalent to the first inequality in (5.14).

To prove the sharpness of the constant, assume that (5.14) holds with a constant $C > 0$, that is,

$$\int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s) \leq \frac{C}{|\alpha|^2 - r^2} \left[Re \left\{ \bar{\alpha} \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) \right\} \right]^2 \quad (5.18)$$

provided $\frac{f(s)}{g(s)} \in \bar{D}(\alpha, r)$ and $|\alpha| > r$.

Assume that $\mathbb{K} = \mathbb{R}$, $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 \neq \emptyset$, Ω_1, Ω_2 are measurable sets, $\int_{\Omega} \rho(s) d\mu(s) = \frac{1}{2}$ and $f(s) = \alpha + r$, $s \in \Omega_1$, $f(s) = \alpha - r$, $s \in \Omega_2$, $g(s) = 1$, $s \in \Omega$, $\alpha > r$. Then $\frac{f(s)}{g(s)} \in \bar{D}(\alpha, r)$ for any $s \in \Omega$ and

$$\int_{\Omega} \rho(s) (f(s))^2 d\mu(s) = \int_{\Omega_1} \rho(s) (\alpha + r)^2 d\mu(s) + \int_{\Omega_2} \rho(s) (\alpha - r)^2 d\mu(s) = \frac{1}{2} \left[(\alpha + r)^2 + (\alpha - r)^2 \right] = \alpha^2 + r^2,$$

$$2 \int_{\Omega} \rho(s) f(s) g(s) d\mu(s) = \int_{\Omega_1} \rho(s) (\alpha + r) d\mu(s) + \int_{\Omega_2} \rho(s) (\alpha - r) d\mu(s) = \frac{1}{2} [\alpha + r + \alpha - r] = \alpha$$

and then, by (5.18), we deduce

$$\alpha^2 + r^2 \leq \frac{C\alpha^4}{\alpha^2 - r^2} \text{ for } \alpha > r,$$

which is clearly equivalent to

$$(C - 1)\alpha^4 + r^4 \geq 0 \text{ for any } 0 < \alpha < 2.$$

If in this inequality we choose $\alpha = 1$, $r = q \in (0, 1)$ and let $q \to 0+$, then we deduce $C \geq 1$. □

The following corollary is a natural consequence of the above theorem.

Corollary 5.1.1. *Under the assumptions of Theorem 5.1, we have the following additive reverse of the CBS-inequality:*

$$\begin{aligned} 0 &\leq 2\int_\Omega \rho(s)|f(s)|^2 d\mu(s) \int_\Omega \rho(s)|g(s)|^2 d\mu(s) \\ &\quad - \left|\int_\Omega \rho(s) f(s)\overline{g(s)} d\mu(s)\right|^2 \\ &\leq \frac{r^2}{|\alpha|^2 - r^2}\left|\int_\Omega \rho(s) f(s)\overline{g(s)} d\mu(s)\right|^2 . \end{aligned} \tag{5.19}$$

The constant 1, the coefficient of the bound, is best possible in the sense mentioned above.

Remark 5.1.1. *If in Theorem 5.1, we assume that $|\alpha| = r$, then we obtain the inequality:*

$$\begin{aligned} \int_\Omega \rho(s)|f(s)|^2 d\mu(s) &\leq 2Re\left[\bar{\alpha}\int_\Omega \rho(s) f(s)\overline{g(s)} d\mu(s)\right] \\ &\leq 2|\alpha|\left|\int_\Omega \rho(s) f(s)\overline{g(s)} d\mu(s)\right| . \end{aligned} \tag{5.20}$$

The constant 2 is sharp in both inequalities.

We also remark that if $r > |\alpha|$, then (5.11) may be written as

$$\begin{aligned} &\int_\Omega \rho(s)|f(s)|^2 d\mu(s) \\ &\leq \left(r^2 - |\alpha|^2\right)\int_\Omega \rho(s)|g(s)|^2 d\mu(s) + 2Re\left[\bar{\alpha}\int_\Omega \rho(s) f(s)\overline{g(s)} d\mu(s)\right] \\ &\leq \left(r^2 - |\alpha|^2\right)\int_\Omega \rho(s)|g(s)|^2 d\mu(s) + 2|\alpha|\left|\int_\Omega \rho(s) f(s)\overline{g(s)} d\mu(s)\right| . \end{aligned} \tag{5.21}$$

The following particular case of interest also holds.

Corollary 5.1.2. *Let $f, g \in L^2_\rho(\Omega, \mathbb{K})$ with $g(s) \neq 0$ for $\mu-$a.e. $s \in \Omega$. If there exist the constants $\gamma, \Gamma \in \mathbb{K}$ with $Re(\Gamma\bar{\gamma}) > 0$ and $\Gamma \neq \gamma$, so that either:*

$$\left|\frac{f(s)}{g(s)} - \frac{\gamma + \Gamma}{2}\right| \leq \frac{1}{2}|\Gamma - \gamma| \quad \text{for } \mu - \text{a.e. } s \in \Omega \tag{5.22}$$

or, equivalently,

$$Re\left[\left(\Gamma - \frac{f(s)}{g(s)}\right)\left(\overline{\frac{f(s)}{g(s)}} - \gamma\right)\right] \geq 0 \;\; \text{for } \mu - \text{a.e. } s \in \Omega \tag{5.23}$$

holds, then we have the inequalities

$$\begin{aligned}
&\int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s) \qquad (5.24)\\
&\leq \frac{1}{2Re(\Gamma\bar{\gamma})} \left\{ Re\left[\left(\bar{\gamma} + \bar{\Gamma}\right) \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) \right] \right\}^2 \\
&\leq \frac{|\Gamma + \gamma|}{4Re(\Gamma\bar{\gamma})} \left| \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) \right|^2.
\end{aligned}$$

The constants $\frac{1}{2}$ and $\frac{1}{4}$ are best possible in (5.24).

Proof. The fact that the relations (5.22) and (5.23) are equivalent follows by the fact that for $z, u, U \in \mathbb{C}$, the following inequalities are equivalent

$$\left| z - \frac{u+U}{2} \right| \leq \frac{1}{2} |U - u|$$

and

$$Re\left[(u - z)(\bar{z} - \bar{u})\right] \geq 0.$$

Define $\alpha := \frac{\gamma+\Gamma}{2}$ and $r = \frac{1}{2}|\Gamma - \gamma|$. Then

$$|\alpha|^2 - r^2 = \frac{|\Gamma + \gamma|^2}{4} - \frac{|\Gamma - \gamma|^2}{4} = Re(\Gamma\bar{\gamma}) > 0.$$

Consequently, we may apply Theorem 5.1, and the inequalities (5.24) are proved.

The sharpness of the constants may be proven in a similar manner to that in proof of Theorem 5.1, and we omit the details. □

Remark 5.1.2. *Note that the above result is due to Dragomir (2003) and has been obtained in a different manner in the above mentioned reference.*

If $\gamma = m, \Gamma = M$ and $M > m > 0$, then from (5.24) we also recapture Cassels' result (5.3).

The following additive version is of interest (see also eqation (5.9)).

Corollary 5.1.3. *With the assumptions of Corollary 5.1.2, we have the inequalities:*

$$\begin{aligned} 0 &\leq \int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) \int_{\Omega} \rho(s) |g(s)|^2 d\mu(s) \\ &\quad - \left| \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) \right|^2 \\ &\leq \frac{1}{4} \cdot \frac{|\Gamma + \gamma|}{Re(\Gamma\bar{\gamma})} \left| \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) \right|^2 . \end{aligned} \tag{5.25}$$

The constant $\frac{1}{4}$ is best possible in (5.25) in the sense that it cannot be replaced by a smaller constant.

5.3 A Pre Grüss Type Inequality

The following result provides an inequality of Grüss type that may be useful in applications when one of the factors is known and some bounds for the second factor are provided (see Dragomir *et al.* (2005)).

THEOREM 5.2. *Let $\rho : \Omega \to [0, \infty)$ be a $\mu-$measurable function on Ω with the property that $\int_{\Omega} \rho(s) d\mu(s) = 1$. If $f, g \in L^2_{\rho}(\Omega, \mathbb{K})$ and there exist the constants $\varphi \in \mathbb{K}$ and $\delta > 0$ such that $f(s) \in \bar{D}(\varphi, \delta)$ for $\mu-$a.e. $s \in \Omega$, then we have the inequality:*

$$\begin{aligned} &\left| \int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) - \int_{\Omega} \rho(s) f(s) d\mu(s) \int_{\Omega} \rho(s) \overline{g(s)} d\mu(s) \right| \\ &\qquad \leq \left[\int_{\Omega} \rho(s) |g(s)|^2 d\mu(s) - \left| \int_{\Omega} \rho(s) \overline{g(s)} d\mu(s) \right|^2 \right]^{\frac{1}{2}} \\ &\qquad\qquad \times \frac{\delta}{\sqrt{|\varphi|^2 - \delta^2}} \left| \int_{\Omega} \rho(x) f(s) d\mu(s) \right| . \end{aligned} \tag{5.26}$$

The multiplicative constant 1 in the bound is best possible.

Proof. We know, by Korkine's identity, that

$$\begin{aligned} &\int_{\Omega} \rho(s) f(s) \overline{g(s)} d\mu(s) - \int_{\Omega} \rho(s) f(s) d\mu(s) \int_{\Omega} \rho(s) \overline{g(s)} d\mu(s) \\ &\qquad = \frac{1}{2} \int_{\Omega} \int_{\Omega} \rho(s) \rho(t) [f(s) - f(t)] \left[\overline{g(s)} - \overline{g(t)} \right] d\mu(s) d\mu(t) . \end{aligned}$$

Applying the Schwarz integral inequality for double integrals we also have

$$\begin{aligned}
&\left|\int_\Omega\int_\Omega \rho(s)\rho(t)\left[f(s)-f(t)\right]\left[\overline{g(s)}-\overline{g(t)}\right]d\mu(s)\,d\mu(t)\right| \qquad (5.27)\\
&\leq\left(\int_\Omega\int_\Omega \rho(s)\rho(t)\left|f(s)-f(t)\right|^2 d\mu(s)\,d\mu(t)\right)^{\frac{1}{2}}\\
&\quad\times\left(\int_\Omega\int_\Omega \rho(s)\rho(t)\left|g(s)-g(t)\right|^2 d\mu(s)\,d\mu(t)\right)^{\frac{1}{2}}\\
&=\left(\int_\Omega \rho(s)\left|f(s)\right|^2 d\mu(s)-\left|\int_\Omega \rho(s)f(s)\,d\mu(s)\right|^2\right)^{\frac{1}{2}}\\
&\quad\times\left(\int_\Omega \rho(s)\left|g(s)\right|^2 d\mu(s)-\left|\int_\Omega \rho(s)\overline{g(s)}d\mu(s)\right|^2\right)^{\frac{1}{2}},
\end{aligned}$$

and, for the last identity, we also have used Korkine's identity for one function $(f=g)$.

Applying Corollary 5.1.1 for the function $g(s)=1$, $s\in\Omega$ and taking into account the fact that $f(s)\in\bar{D}(\rho,\delta)$ for $\mu-$a.e. $s\in\Omega$, then we can state that

$$\begin{aligned}
0&\leq\int_\Omega \rho(s)\left|f(s)\right|^2 d\mu(s)-\left|\int_\Omega \rho(s)f(s)\,d\mu(s)\right|^2 \qquad (5.28)\\
&\leq\frac{\delta}{\left|\varphi\right|^2-\delta^2}\left|\int_\Omega \rho(s)f(s)\,d\mu(s)\right|.
\end{aligned}$$

Utilising (5.26) and (5.28), we deduce the desired result (5.26).

The fact that the multiplication constant of the bound 1 is the best constant is obvious by Corollary 5.1.1 and we omit the details. □

The following corollary is of interest in itself.

Corollary 5.2.1. *Let* ρ *be as in Theorem 5.2. If* $f,g\in L^2_\rho(\Omega,\mathbb{K})$ *and there exist the constants* $\varphi,\Phi\in\mathbb{K}$ *with* $Re(\Phi\bar{\varphi})>0$ *and*

$$\left|f(s)-\frac{\varphi+\Phi}{2}\right|\leq\frac{1}{2}\left|\Phi-\varphi\right| \text{ for } \mu-a.e.\ s\in\Omega \qquad (5.29)$$

or, equivalently,

$$Re\left[\left(\Phi-f(s)\right)\left(\overline{f(s)}-\overline{\varphi}\right)\right]\geq 0 \text{ for } \mu-a.e.\ s\in\Omega, \qquad (5.30)$$

then we have the inequality

$$\left|\int_{\Omega}\rho(s)f(s)\overline{g(s)}d\mu(s)-\int_{\Omega}\rho(s)f(s)d\mu(s)\cdot\int_{\Omega}\rho(s)\overline{g(s)}d\mu(s)\right|$$
$$\leq\frac{1}{2}|\Phi-\varphi|\left|\int_{\Omega}\rho(s)f(s)d\mu(s)\right|\left[\int_{\Omega}\rho(s)|g(s)|^{2}d\mu(s)-\left|\int_{\Omega}\rho(s)\overline{g(s)}d\mu(s)\right|^{2}\right]^{\frac{1}{2}}. \tag{5.31}$$

The constant $\frac{1}{2}$ is best possible in (5.31).

In what follows we briefly mention some approximation results for the finite Fourier transform whose proofs have employed recent techniques and facts from the theory of integral inequalities of Ostrowski type.

CHAPTER 6

APPROXIMATIONS OF THE FINITE FOURIER TRANSFORM FOR FUNCTIONS OF ONE VARIABLE

The reason the Fourier transform is so widely used is that it offers specific computational advantages over other mathematical approaches. The Fourier transform therefore provides a computationally versatile tool to analyze complex functions arising from experimental measurements by decomposing them into simpler wave functions which can be used to determine experimental unknowns.

The material in this chapter is presented in the following order. In Section 6.2, some new inequalities for the Fourier transform of function of bounded variation are given. In Section 6.3, some numerical quadrature formulas are developed. The pre-Grüss inequality which was developed in previous chapter is used to form an integral inequality for complex-valued functions in 6.4. Finally, in Section 6.5 attention is focused on the symbolic computation of the Fourier Transform using the “Maple" computer algebra system. This is also illustrated by using some numerical experiments to plot the theoretical results obtained in this chapter.

6.1 Introduction

The Fourier transform has long been a principal analytical tool in such diverse fields as linear systems, optics, random process modeling, probability theory, quantum physics, and boundary-value problems (Brigham (1988)). In particular, it has been very successfully applied to the restoration of astronomical data (Brault and White (1971)). The Fourier transform is a pervasive and versatile tool, which has been used in many fields of science as a mathematical or physical tool to alter a problem into one that can be more easily solved. Some scientists understand Fourier transform as a physical phenomenon, not simply as a mathematical tool. In some branches of science, the Fourier transform of one function may yield another physical function (Bracwell (1965)).

Utilizing some integral identities and inequalities developed in Barnett and Dragomir (2001), Barnett and Dragomir (2002), Dragomir (2001b), Barnett *et al.* (2004) (see also Dragomir and Rassias (2002)), we point out some approximations of the one dimensional finite-Fourier transform in terms of the complex exponential mean $E(z,w)$ and estimate the error of approximation for different classes of mappings of bounded variation defined on finite intervals.

Let $g:[a,b]\to\mathbb{R}$ be a Lebesgue integrable mapping defined on the finite interval $[a,b]$ and $\mathcal{F}(g)$ its finite Fourier transform, namely,

$$\mathcal{F}(g)(t):=\int_a^b g(s)\,e^{-2\pi its}ds.$$

The inverse finite Fourier transform of g will also be considered, and will be defined by

$$\mathcal{F}^{-1}(g)(t):=\int_a^b g(s)\,e^{2\pi its}ds.$$

The following result was obtained in Barnett and Dragomir (2002).

THEOREM 6.1. *Let* $g:[a,b]\to\mathbb{R}$ *be an absolutely continuous mapping on* $[a,b]$. *Then* *we have the inequality*

$$\left|\mathcal{F}(g)(x) - E(-2\pi ixa, -2\pi ixb)\int_a^b g(t)\,dt\right|$$
$$\leq \begin{cases} \frac{1}{3}\|g'\|_\infty (b-a)^2, & \text{if } g' \in L_\infty[a,b], \\ \\ \frac{2^{\frac{1}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}}(b-a)^{1+\frac{1}{q}}\|g'\|_p, & \text{if } g' \in L_p[a,b]; \frac{1}{p}+\frac{1}{q}=1, p>1, \\ \\ (b-a)\|g'\|_1 \end{cases};$$

for all $x \in [a,b], x \neq 0$, *where* E *is the exponential mean of two complex numbers, that is,*

$$E(z,w) := \begin{cases} \frac{e^z - e^w}{z-w} & \text{if } z \neq w \\ \\ \exp(w) & \text{if } z = w \end{cases}, \quad z, w \in \mathbb{C}. \tag{6.1}$$

The following inequality for a more general class of functions was pointed out in (Dragomir *et al.* 2003).

THEOREM 6.2. *Let* $g : [a,b] \to \mathbb{R}$ *be a measurable mapping on* $[a,b]$, *then we have the inequality:*

$$\left|\mathcal{F}(g)(x) - E(-2\pi ixa, -2\pi ixb)\int_a^b g(s)\,ds\right|$$
$$\leq \begin{cases} \frac{2\pi}{3}|x|(b-a)^2\|g\|_\infty & \text{if} \quad g \in L_\infty[a,b]; \\ \\ \frac{2^{1+\frac{1}{q}}\pi(b-a)^{1+\frac{1}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}}|x|\,\|g\|_p & \text{if} \quad g \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ 2\pi|x|(b-a)\|g\|_1 & \text{if} \quad g \in L_1[a,b]; \end{cases}$$

for all $x \in [a,b]$, $x \neq 0$.

The main aim of this chapter is to point out some new inequalities for the finite Fourier transform of functions of bounded variation. Error bounds for some associated quadrature formulae are also established. Numerical experiments by utilizing the Maple package are conducted as well.

6.2 Inequalities for the Fourier Transform of Functions of Bounded Variation.

The following inequality holds (Barnett *et al.* (2004)):

THEOREM 6.3. *Let* $g : [a, b] \to \mathbb{R}$ *be a mapping of bounded variation on* $[a, b]$, *then we have the inequality*

$$\left| \mathfrak{F}(g)(x) - E(-2\pi ixa, -2\pi ixb) \int_a^b g(s)\, ds \right| \leq \frac{3}{4}(b-a) \bigvee_a^b (g) \tag{6.2}$$

for all $x \in [a, b]$, $x \neq 0$, *where* $\bigvee_a^b (g)$ *is the total variation of* g *on* $[a, b]$, *defined as follow:*

$$\bigvee_a^b (g) = \sup_{\triangle n} \sum_{i=0}^{n-1} \left| g\left(x_{i+1}^{(n)}\right) - \left(x_i^{(n)}\right) \right|$$

and $\triangle n$ *is any partition of* $[a, b]$

Proof. Using integration by parts for the Riemann-Stieltjes integral, we have (see also (Dragomir 2001b)) that

$$\int_a^t (s-a)\, dg(s) = (t-a)\, g(t) - \int_a^t g(s)\, ds \tag{6.3}$$

and

$$\int_t^b (s-b)\, dg(s) = (b-t)\, g(t) - \int_t^b g(s)\, ds, \tag{6.4}$$

for all $t \in [a, b]$.

Adding (6.3) and (6.4) and dividing by $(b-a)$, we deduce the representation (Dragomir 2001b):

$$g(t) = \frac{1}{b-a} \int_a^b g(s)\, ds + \frac{1}{b-a} \int_a^t (s-a)\, dg(s) + \frac{1}{b-a} \int_t^b (s-b)\, dg(s), \tag{6.5}$$

for all $t \in [a, b]$, which is itself of interest.

Assume that $x \in [a,b]$, $x \neq 0$, then utilizing (6.5) we have

$$\begin{aligned} \mathcal{F}(g)(x) &= \int_a^b g(t)\, e^{-2\pi ixt} dt \qquad (6.6) \\ &= \int_a^b \left[\frac{1}{b-a} \int_a^b g(s)\, ds + \frac{1}{b-a} \int_a^t (s-a)\, dg(s) \right. \\ &\qquad \left. + \frac{1}{b-a} \int_t^b (s-b)\, dg(s) \right] e^{-2\pi ixt} dt \\ &= \frac{1}{b-a} \int_a^b g(s)\, ds \int_a^b e^{-2\pi ixt} dt \\ &\qquad + \frac{1}{b-a} \int_a^b \left(\int_a^t (s-a)\, dg(s) \right) e^{-2\pi ixt} dt \\ &\qquad + \frac{1}{b-a} \int_a^b \left(\int_t^b (s-b)\, dg(s) \right) e^{-2\pi ixt} dt \\ &= E(-2\pi ixa, -2\pi ixb) \int_a^b g(s)\, ds \\ &\qquad + \frac{1}{b-a} \int_a^b \left(\int_a^t (s-a)\, dg(s) \right) e^{-2\pi ixt} dt \\ &\qquad + \frac{1}{b-a} \int_a^b \left(\int_t^b (s-b)\, dg(s) \right) e^{-2\pi ixt} dt, \end{aligned}$$

where we have used the notation (6.1) and

$$\int_a^b e^{-2\pi ixt} dt = (b-a)\, E(-2\pi ixa, -2\pi ixb).$$

Using the properties of the modulus, we have, by (6.6), that

$$\begin{aligned} &\left| \mathcal{F}(g)(x) - E(-2\pi ixa, -2\pi ixb) \int_a^b g(s)\, ds \right| \qquad (6.7) \\ &\le \frac{1}{b-a} \left| \int_a^b \left(\int_a^t (s-a)\, dg(s) \right) e^{-2\pi ixt} dt \right| \\ &\qquad + \frac{1}{b-a} \left| \int_a^b \left(\int_t^b (s-b)\, dg(s) \right) e^{-2\pi ixt} dt \right| \\ &\le \frac{1}{b-a} \int_a^b \left| \int_a^t (s-a)\, dg(s) \right| \left| e^{-2\pi ixt} \right| dt \\ &\qquad + \frac{1}{b-a} \int_a^b \left| \int_t^b (s-b)\, dg(s) \right| \left| e^{-2\pi ixt} \right| dt \\ &= \frac{1}{b-a} \int_a^b \left| \int_a^t (s-a)\, dg(s) \right| dt + \frac{1}{b-a} \int_a^b \left| \int_t^b (s-b)\, dg(s) \right| dt. \end{aligned}$$

Now, it is well known that if $p : [c,d] \to \mathbb{R}$ is continuous and $v : [c,d] \to \mathbb{R}$ is of bounded variation on $[c,d]$, then the Riemann-Stieltjes integral $\int_c^d p(x)\,dv(x)$ exists and

$$\left|\int_c^d p(x)\,dv(x)\right| \le \sup_{x\in[c,d]} |p(x)| \bigvee_c^d (v). \tag{6.8}$$

Applying (6.8) on the intervals $[a,t]$ and $[t,b]$, we deduce that

$$\left|\int_a^t (s-a)\,dg(s)\right| \le (t-a)\bigvee_a^t (g),$$
$$\left|\int_t^b (s-b)\,dg(s)\right| \le (b-t)\bigvee_t^b (g)$$

and further that,

$$\begin{aligned}\left|\int_a^t (s-a)\,dg(s)\right| + \left|\int_t^b (s-b)\,dg(s)\right| &\le (t-a)\bigvee_a^t (g) + (b-t)\bigvee_t^b (g)\\ &\le \max_{t\in[a,b]}\{t-a,b-t\}\left[\bigvee_a^t (g) + \bigvee_t^b (g)\right]\\ &= \left[\frac{1}{2}(b-a) + \left|t-\frac{a+b}{2}\right|\right]\bigvee_a^b (g).\end{aligned}$$

Using (6.7),

$$\begin{aligned}&\left|\mathfrak{F}(g)(x) - E(-2\pi ixa,-2\pi ixb)\int_a^b g(s)\,ds\right|\\ &\le \frac{1}{b-a}\int_a^b \left[\frac{1}{2}(b-a) + \left|t-\frac{a+b}{2}\right|\right]dt\bigvee_a^b (g)\\ &= \frac{3}{4}(b-a)\bigvee_a^b (g),\end{aligned}$$

since a simple calculation shows that

$$\int_a^b \left|t-\frac{a+b}{2}\right|dt = \frac{(b-a)^2}{4},$$

and the theorem is completely proved. □

Remark 6.3.1. *If we consider the inverse Fourier transform*

$$\mathfrak{F}^{-1}(g)(x) = \int_a^b g(t)\,e^{2\pi ixt}dt,$$

then, by a similar argument, we can prove that

$$\left|\mathfrak{F}^{-1}(g)(x) - E(2\pi ixa, 2\pi ixb)\int_a^b g(s)\,ds\right| \leq \frac{3}{4}(b-a)\bigvee_a^b(g),\ x\in[a,b],\ x\neq 0. \tag{6.9}$$

The constant $\frac{3}{4}$ *is the best possible.*

The following corollaries are natural consequences of the above results.

Corollary 6.3.1. *Let* $g:[a,b]\to\mathbb{R}$ *be a monotonic mapping on* $[a,b]$. *Then we have the inequality*

$$\left|\mathfrak{F}(g)(x) - E(-2\pi ixa, -2\pi ixb)\int_a^b g(s)\,ds\right| \leq \frac{3}{4}(b-a)\,|g(b)-g(a)|, \tag{6.10}$$

for all $x\in[a,b]$, $x\neq 0$.

The proof is obvious by Theorem 6.3, taking into account that every monotonic mapping is of bounded variation and $\bigvee_a^b(g) = |g(b)-g(a)|$.

Corollary 6.3.2. *Let* $g:[a,b]\to\mathbb{R}$ *be an* $L-$*Lipschitzian mapping on* $[a,b]$, *i.e.,*

$$|g(t)-g(s)| \leq L\,|t-s| \text{ for all } t,s\in[a,b]. \tag{L}$$

Then we have the inequality

$$\left|\mathfrak{F}(g)(x) - E(-2\pi ixa, -2\pi ixb)\int_a^b g(s)\,ds\right| \leq \frac{3}{4}L(b-a)^2. \tag{6.11}$$

The proof is obvious by Theorem 6.3, taking into account that if $g:[a,b]\to\mathbb{R}$ is $L-$Lipschitzian, then L is of bounded variation on $[a,b]$ and $\bigvee_a^b(g)\leq L(b-a)$.

6.3 A Numerical Quadrature Formula

Let $I_n: a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$, put $h_k := x_{k+1} - x_k$ $(k = 0,...,n-1)$ and $\nu(h) := \max\{h_k | k = 0,...,n-1\}$. Define the sum (see also Barnett and Dragomir (2002) and Cho *et al.* (2003))

$$\mathcal{E}(g, I_n, x) := \sum_{k=0}^{n-1} E(-2\pi i x x_k, -2\pi i x x_{k+1}) \times \int_{x_k}^{x_{k+1}} g(t)\,dt, \tag{6.12}$$

where $x \in [a, b]$, $x \neq 0$.

The following approximation theorem holds ((Barnett *et al.* 2004)).

THEOREM 6.4. *Let* $g : [a, b] \to \mathbb{R}$ *be a mapping of bounded variation on* $[a, b]$. *Then we have the quadrature rule*

$$\mathcal{F}(g)(x) = \mathcal{E}(g, I_n, x) + R(g, I_n, x); \tag{6.13}$$

where $\mathcal{E}(g, I_n, x)$ *is as defined in* (6.12) *and the remainder* $R(g, I_n, x)$ *satisfies the estimate*

$$|R(g, I_n, x)| \leq \frac{3}{4}\nu(h) \bigvee_a^b (g). \tag{6.14}$$

Proof. Applying Theorem 6.3 on every subinterval $[x_k, x_{k+1}]$, we can state that

$$\left| \int_{x_k}^{x_{k+1}} g(t) e^{-2\pi i x t} dt - E(-2\pi i x x_k, -2\pi i x x_{k+1}) \times \int_{x_k}^{x_{k+1}} g(t)\,dt \right| \leq \frac{3}{4} h_k \bigvee_{x_k}^{x_{k+1}} (g),$$

for all $k \in \{0, ..., n-1\}$ and $x \in [a, b]$, $x \neq 0$.
Summing over i from 0 to $n-1$ and using the generalized triangle inequality produces

$$\begin{aligned} |R(g, I_n, x)| &= |\mathcal{F}(g)(x) - \mathcal{E}(g, I_n, x)| \\ &\leq \frac{3}{4} \sum_{k=0}^{n-1} h_k \bigvee_{x_k}^{x_{k+1}} (g) \leq \frac{3}{4}\nu(h) \sum_{k=0}^{n-1} \bigvee_{x_k}^{x_{k+1}} (g) \\ &= \frac{3}{4}\nu(h) \bigvee_a^b (g), \end{aligned}$$

and the theorem is proved. □

In practical applications, it is more convenient to consider the equidistant partitioning of the interval $[a, b]$. Thus, let

$$I_n : x_j = a + j \cdot \frac{b-a}{n}, \; j = 0, ..., n;$$

be an equidistant partition of $[a,b]$, and define the sum (see also Barnett and Dragomir (2002) and Barnett *et al.* (2004))

$$\mathcal{E}_n(g,x) := \sum_{k=0}^{n-1} E\left[-2\pi i x\left(a+k\cdot\frac{b-a}{n}\right), -2\pi i x\left(a+(k+1)\cdot\frac{b-a}{n}\right)\right] \times \int_{a+k\cdot\frac{b-a}{n}}^{a+(k+1)\cdot\frac{b-a}{n}} g(t)\,dt. \quad (6.15)$$

The following corollary of Theorem 6.4 holds.

Corollary 6.4.1. *Let g be as defined in Theorem 6.4. Then we have*

$$\mathcal{F}(g)(x) = \mathcal{E}_n(g,x) + R_n(g,x), \quad (6.16)$$

where $\mathcal{E}_n(g,x)$ approximates the Fourier transform at any point $x \in [a,b]$, $x \neq 0$. The error of approximation $R_n(g,x)$ satisfies the bound

$$|R_n(g,x)| \leq \frac{3}{4n}(b-a)\bigvee_a^b(g), \quad (6.17)$$

for all $x \in [a,b]$, $x \neq 0$.

Remark 6.4.1. *If we know the total variation $\bigvee_a^b(g)$ of g on $[a,b]$ and would like to approximate the Fourier transform $F(g)(x)$ by the adaptive quadrature formula $\mathcal{E}_n(g,x)$ with an error less than a given $\varepsilon > 0$, we have to divide the interval $[a,b]$ into at least $n_\varepsilon \in \mathbb{N}$ points, where*

$$n_\varepsilon := \left[\frac{3(b-a)}{4\varepsilon}\bigvee_a^b(g)\right] + 1,$$

and $[r]$ denotes the integer part of $r \in \mathbb{R}$.

The following corollaries of Theorem 6.4 also hold.

Corollary 6.4.2. *Let $g : [a,b] \to \mathbb{R}$ be a monotonic mapping on $[a,b]$. Then we have the quadrature formula (7.15) where the remainder satisfies the estimate*

$$|R(g,I_n,x)| \leq \frac{3}{4}\nu(h)\,|g(b)-g(a)|, \; x \in [a,b], \; x \neq 0. \quad (6.18)$$

In particular, if I_n is taken to be equidistant, then we have the formula (6.16), where the remainder $R_n(g,x)$ satisfies the estimate

$$|R_n(g,x)| \leq \frac{3(b-a)}{4n}|g(b)-g(a)|, \; x \in [a,b], \; x \neq 0. \quad (6.19)$$

A similar result holds for Lipschitzian mappings.

Corollary 6.4.3. *Let* $g : [a,b] \to \mathbb{R}$ *be a Lipschitzian mapping with the constant* $L > 0$. *Then we have the quadrature formula* (7.15) *where the remainder is bounded by*

$$|R(g, I_n, x)| \leq \frac{3}{4} L \sum_{i=0}^{n-1} h_i^2 \leq \frac{3}{4} L (b-a) \nu(h). \tag{6.20}$$

In particular, if I_n *is chosen to be equidistant, then we have the formula* (6.16) *where the remainder* $R_n(g, x)$ *satisfies the inequality*

$$|R_n(g, x)| \leq \frac{3L(b-a)^2}{4n}. \tag{6.21}$$

6.4 Integral Inequality for Complex Valued Functions

The following result illustrates the usefulness of the pre-Grüss inequality (5.31), at this point, that $\mathbb{K}$ denotes the field of real complex numbers.

THEOREM 6.5. *Let $g : [a,b] \to \mathbb{K}$ be a real or complex-valued function with $g \in L^2([a,b];\mathbb{K})$ and there exist the constants $\varphi, \phi \in \mathbb{K}$ with the property that, either*

$$\left| g(s) - \frac{\varphi + \Phi}{2} \right| \le \frac{1}{2} |\Phi - \varphi| \text{ for a.e. } s \in [a,b] \tag{6.22}$$

or, equivalently,

$$Re\left[(\Phi - g(s)) \left(\overline{g(s)} - \bar{\varphi} \right) \right] \ge 0 \text{ for a.e. } s \in [a,b], \tag{6.23}$$

holds. Then we have the inequality:

$$\begin{aligned} & \left| \mathcal{F}(g)(x) - E(-2\pi i x a, -2\pi i x b) \int_a^b g(s)\, ds \right| \\ & \le \frac{1}{2} |\Phi - \varphi| \left[1 - \frac{\sin^2[\pi x (b-a)]}{(b-a)^2 \pi^2 x^2} \right]^{\frac{1}{2}} \left| \int_a^b g(s)\, ds \right| \\ & \le \frac{1}{2} |\Phi - \varphi| \left[1 - \frac{\sin^2[\pi x (b-a)]}{(b-a)^2 \pi^2 x^2} \right]^{\frac{1}{2}} \\ & \quad \times \begin{cases} (b-a) \|g\|_{\infty,[a,b]} & \text{if } g \in L_\infty[a,b]; \\ (b-a)^{\frac{1}{2}} \|g\|_{p,[a,b]} & \text{if } g \in L_p[a,b], \\ & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \|g\|_{1,[a,b]}, & \text{if } g \in L_1[a,b] \end{cases} \end{aligned} \tag{6.24}$$

for each $x \in [a,b]$ $(x \neq 0)$, where $E(\cdot,\cdot)$ is the exponential mean defined in (6.1).

Proof. We apply the pre-Grüss inequality (5.31) to get:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b g(s) e^{-2\pi i x s} ds - \frac{1}{b-a} \int_a^b e^{-2\pi i x s} ds \cdot \frac{1}{b-a} \int_a^b g(s)\, ds \right| \\ & \quad \le \frac{1}{2} |\Phi - \varphi| \left| \int_a^b g(s)\, ds \right| \times \left[\frac{1}{b-a} \int_a^b \left| e^{-2\pi i x s} \right|^2 ds - \left| \frac{1}{b-a} \int_a^b e^{-2\pi i x s} ds \right|^2 \right]. \end{aligned} \tag{6.25}$$

However

$$\int_a^b e^{-2\pi ixs} ds = (b-a) E(-2\pi ixa, -2\pi ixb),$$

$$\left|e^{-2\pi ixs}\right|^2 = 1,$$

$$\int_a^b e^{2\pi ixs} ds = \frac{1}{2\pi ix}\left[e^{2\pi ixb} - e^{2\pi ixa}\right]$$

and

$$\begin{aligned}
\left|\int_a^b e^{2\pi ixs} ds\right|^2 &= \frac{1}{(2\pi x)^2}\left[\left|e^{2\pi ixb}\right|^2 - 2Re\left[e^{2\pi ixb} \cdot e^{-2\pi ixa}\right] + \left|e^{2\pi ixa}\right|^2\right] \\
&= \frac{1}{2\pi^2 x^2}\left[1 - Re\left[e^{2\pi ix(b-a)}\right]\right] \\
&= \frac{1}{2\pi^2 x^2}\left[1 - Re\left[\cos(2\pi x(b-a)) + i\sin(2\pi x(b-a))\right]\right] \\
&= \frac{1}{2\pi^2 x^2}\left[1 - \cos(2\pi x(b-a))\right] \\
&= \frac{1}{2\pi^2 x^2}\left[1 - \left(1 - 2\sin^2(\pi x(b-a))\right)\right] \\
&= \frac{\sin^2[\pi x(b-a)]}{\pi^2 x^2}.
\end{aligned}$$

Using (6.25) multiplied with $b-a>0,$ we deduce the first result (6.24). The second part is obvious by Hölder's inequality. □

Remark 6.5.1. *If g takes real values, then the condition (6.22) may be replaced by the equivalent condition (for $\Phi > \varphi$)*

$$\varphi \le g(s) \le \Phi \text{ for a.e. } s \in [a,b]. \tag{6.26}$$

6.5 Some Numerical Experiments

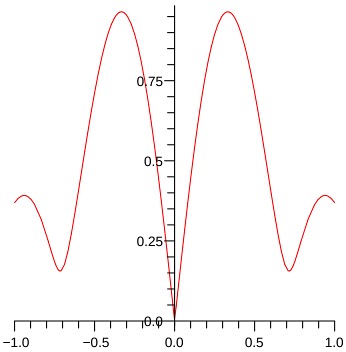
In the following we numerically illustrate the approximation for the finite Fourier transform provided by

$$\begin{aligned}
&\mathcal{E}(g, I_n, x) \\
:&= \sum_{k=0}^{n-1} E\left(2\pi ix\left(a + k\cdot\frac{b-a}{n}\right), -2\pi\, i\left(a + (k+1)\frac{b-a}{n}\right)\right) \\
&\times \int_{a+k\cdot\frac{b-a}{n}}^{a+(k+1)\frac{b-a}{n}} g(t)\, dt.
\end{aligned} \tag{6.27}$$

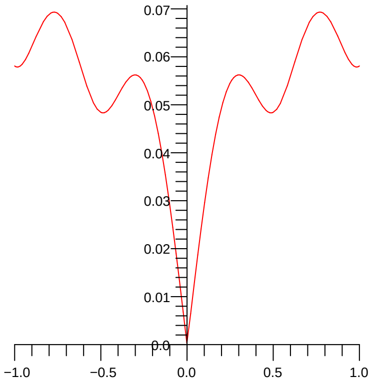
In Figures 6.1- 6.6 the error $r_n(t) := |R_n(g,t)|$ for $n = 1,\ n = 4,\ n = 16,\ n = 64$ and $n = 128$ is plotted for the different functions $g(t) = e^t,\ t \in [-1,1],\ g(t) = e^{-t^2},\ t \in [-\pi,\pi],\ g(t) = \ln t,\ t \in (0,1],\ g(t) = \cosh t,\ t \in [-\pi,\pi],\ g(t) = \sinh t,\ t \in [-\pi,\pi]$ and $g(t) = \sin(2t)\, e^t,\ t \in [-\pi,\pi]$.

These functions were used because they demonstrate different behaviour types. Therefore, this will allow for an adequate examination of the method.

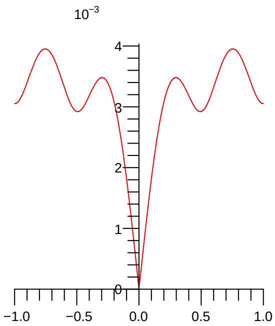
The approximates finite Fourier transform when applying equation (6.27) on the functions listed above are illustrated by the following graphs.



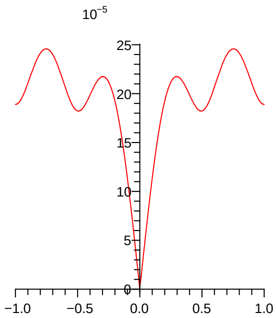
(a) $r_n(x)$ for $n = 1$



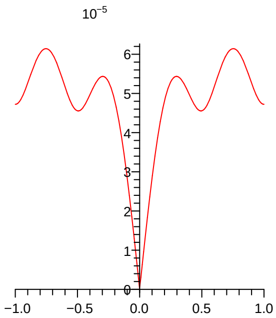
(b) $r_n(x)$ for $n = 4$



(c) $r_n(x)$ for $n = 16$

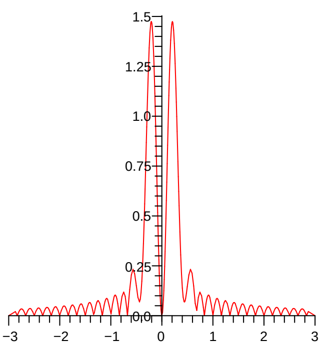


(d) $r_n(x)$ for $n = 64$

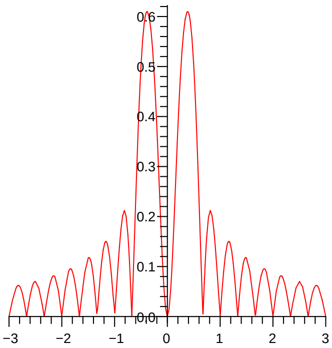


(e) $r_n(x)$ for $n = 128$

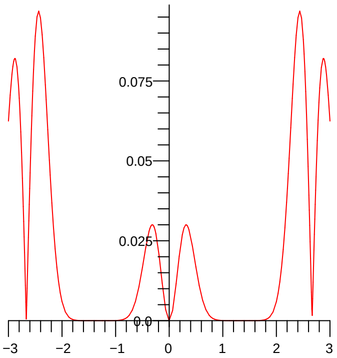
Figure 6.1: Plots of the bound on error $r_n(x) := |R_n(g, x)|$ for the function $g(x) = e^x,\ x \in [-1, 1]$.



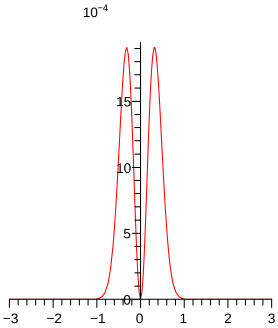
(a) $r_n(x)$ for $n = 1$



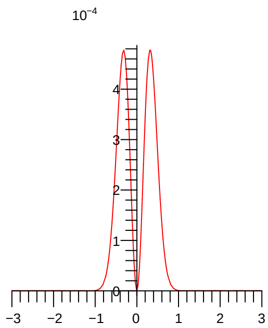
(b) $r_n(x)$ for $n = 4$



(c) $r_n(x)$ for $n = 16$

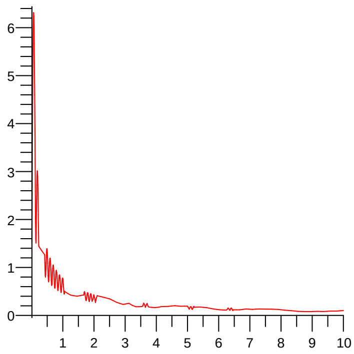


(d) $r_n(x)$ for $n = 64$

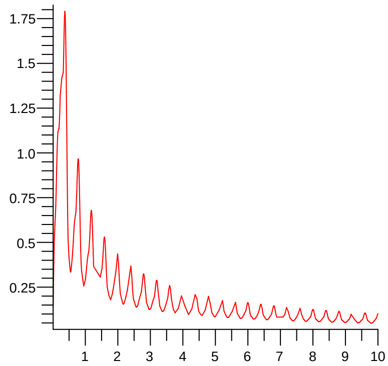


(e) $r_n(x)$ for $n = 128$

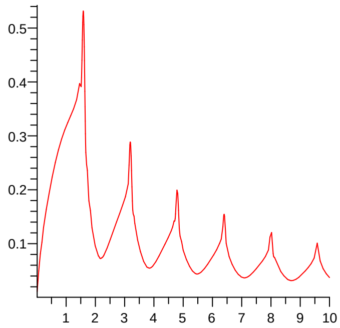
Figure 6.2: Plots of the bound on the error $r_n\left(x\right) := \left|R_n\left(g, x\right)\right|$ for $g(x) = e^{-x^2},\ x \in [-1, 1]$.



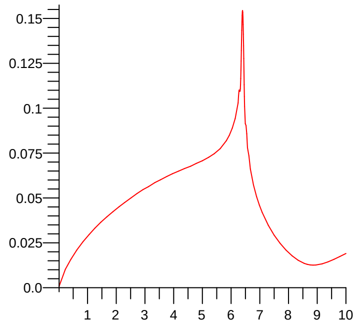
(a) $r_n(x)$ for $n = 1$



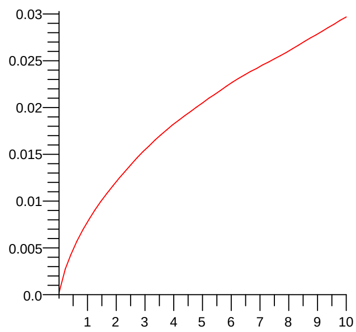
(b) $r_n(x)$ for $n = 4$



(c) $r_n(x)$ for $n = 16$

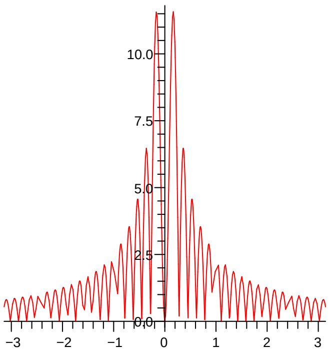


(d) $r_n(x)$ for $n = 64$

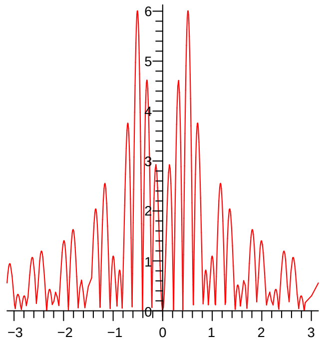


(e) $r_n(x)$ for $n = 128$

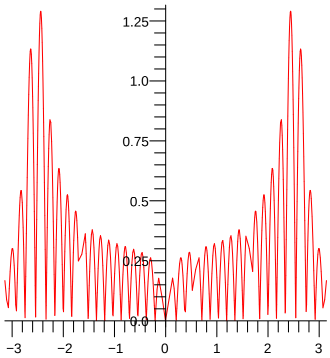
Figure 6.3: Plots of the bound on the error $r_n(x) := |R_n(g, x)|$ for $g(x) = \ln x, \ x \in (0, 1]$.



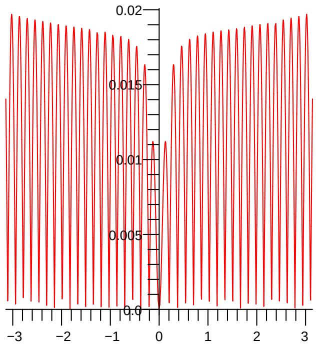
(a) $r_n(x)$ for $n = 1$



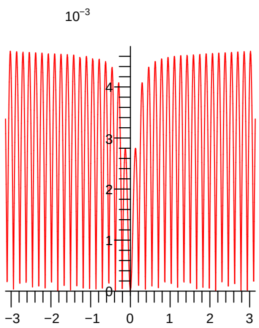
(b) $r_n(x)$ for $n = 4$



(c) $r_n(x)$ for $n = 16$

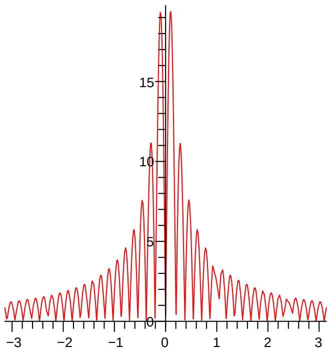


(d) $r_n(x)$ for $n = 64$

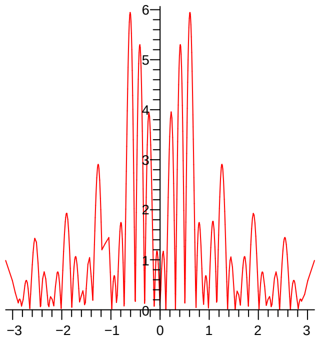


(e) $r_n(x)$ for $n = 128$

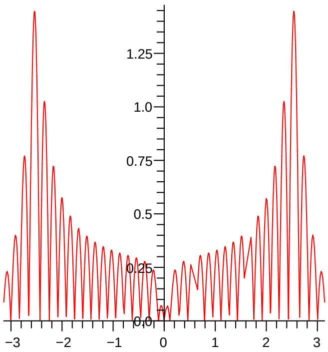
Figure 6.4: Plots of the bound on the error $r_n(x) := |R_n(g, x)|$ for $g(x) = \cosh x,\ x \in [-\pi, \pi]$.



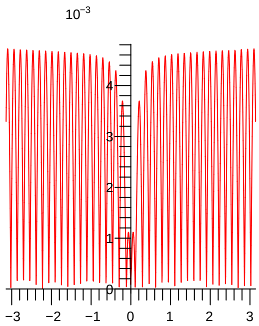
(a) $r_n(x)$ for $n = 1$



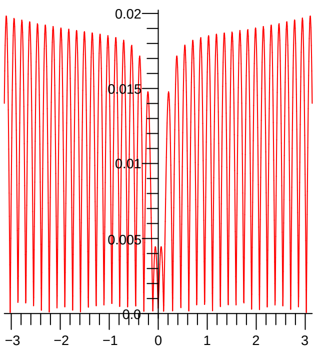
(b) $r_n(x)$ for $n = 4$



(c) $r_n(x)$ for $n = 16$

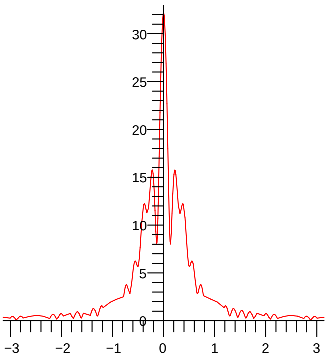


(d) $r_n(x)$ for $n = 64$

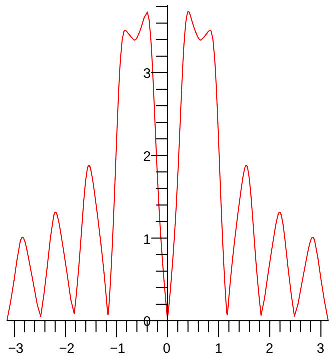


(e) $r_n(x)$ for $n = 128$

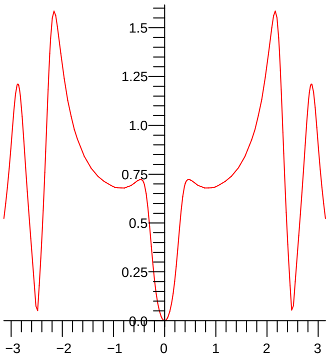
Figure 6.5: Plots of the bound on the error $r_n(x) := |R_n(g, x)|$ for $g(x) = \sinh x,\ x \in [-\pi, \pi]$.



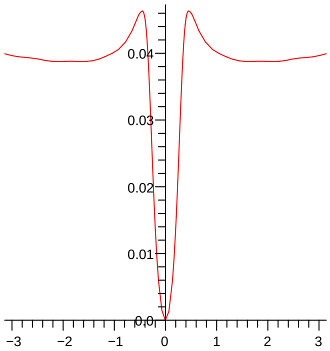
(a) $r_n(x)$ for $n = 1$



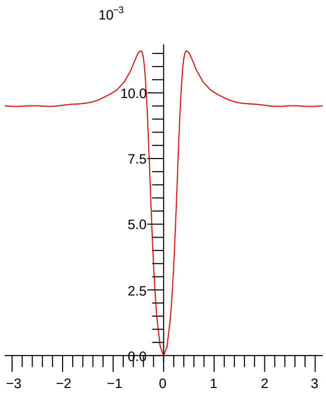
(b) $r_n(x)$ for $n = 4$



(c) $r_n(x)$ for $n = 16$



(d) $r_n(x)$ for $n = 64$



(e) $r_n(x)$ for $n = 128$

Figure 6.6: Plots of the bound on the error $r_n\,(x) := |R_n\,(g, x)|$ for $g(x) = \sin(2x)\ e^x,\ x \in [-\pi, \pi]\,.$

In conclusion, it is evident that the approximate Fourier transformation achieves single precision accuracy when the values n increase. The results also represent that a doubling of the mesh size leads to a squaring of the error bound.

In the next chapter we will discuss some approximation results for the two-dimensional finite Fourier transform and develop inequalities for the estimation of the two dimensional Fourier transform.

CHAPTER 7

APPROXIMATION OF THE FINITE FOURIER TRANSFORM FOR FUNCTIONS OF TWO VARIABLES.

Since Fourier series and Fourier transforms are important tools in applied mathematics, it is not surprising that there is a great deal of interest in their discrete approximation. For a more recent survey of finite Fourier analysis, see Henrici (1998, Chapter 13).

The Fourier transform can be generalized to higher dimensions. For example, many signals $f(x, y)$ are functions of 2D space defined in a plane.

The material in this chapter is presented in the following manner:

In Section 7.2, some new inequalities for the estimation of the two-dimensional Fourier transform are developed.

Some numerical cubature formulas are developed and applied to provide some numerical experiments in Section 7.3.

The pre Grüss inequality which was developed in Chapter 6 is used to form some integral inequalities for complex-valued functions of two variables as shown in Section 7.4.

Finally, in Section 7.4.2 attention is focused on the symbolic computation of Fourier Transform using the “Maple" computer algebra system . This is also illustrated by using some numerical experiments to plot the theoretical results obtained in this chapter.

7.1 Introduction

In this chapter we point out some approximations of the two dimensional finite Fourier transform in terms of the complex exponential mean $E(z,w)$ and estimate the error of approximation for different classes of mappings defined on finite intervals.

In this chapter $f : [a,b]\times[c,d] \to \mathbb{R}$ will be a continuous mapping defined on the finite interval $[a,b]\times[c,d]$ and $\mathcal{F}(f)$ its finite Fourier transform. That is

$$\mathcal{F}(f)(u,v;a,b,c,d) = \int_a^b \int_c^d f(x,y)\, e^{-2\pi i(ux+vy)} dy dx, \tag{7.1}$$

$(u,v) \in [a,b]\times[c,d]$. For a function of one variable we have used the notation

$$\mathcal{F}(g)(u;a,b) = \int_a^b g(x)\, e^{-2\pi iux} dx.$$

7.2 Some Integral Inequalities

In this section we employ an identity obtained in Barnett and Dragomir (2001) and develop inequalities for the estimation of the two dimensional Fourier transform. The following inequality holds (Hanna *et al.* (2002)).

THEOREM 7.1. *Let* $f : [a,b]\times[c,d] \to \mathbb{R}$ *be an absolutely continuous mapping on* $[a,b]\times[c,d]$ *and assume that* $f''_{x,y} := \frac{\partial^2 f}{\partial x \partial y}$ *exists on* $(a,b)\times(c,d)$, *then we have the inequality*

$$\begin{aligned}
&\left|\mathcal{F}(f)(u,v;a,b,c,d) - \mathcal{I}_1 - \mathcal{I}_2 + \mathcal{I}_3\right| \\
&\quad \le \begin{cases}
\dfrac{(b-a)^2(d-c)^2}{9}\left\|f''_{x,y}\right\|_\infty, & \text{if } f''_{x,y} \in L_\infty([a,b]\times[c,d]); \\[2ex]
\left[\dfrac{2\left[(b-a)(d-c)\right]^{\frac{q+2}{2}}}{(q+1)(q+2)}\right]^{\frac{2}{q}}\left\|f''_{x,y}\right\|_p, & \text{if } f''_{x,y} \in L_p([a,b]\times[c,d]), \\
& \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\[2ex]
(b-a)(d-c)\left\|f''_{x,y}\right\|_1, & \text{if } f''_{x,y} \in L_1([a,b]\times[c,d])
\end{cases}
\end{aligned} \tag{7.2}$$

for all $(u,v) \in [a,b] \times [c,d]$, *where*

$$\begin{aligned}
\mathcal{I}_1 &:= \mathcal{I}_1(u,v;a,b,c,d) = E(u;a,b) \int_a^b \mathcal{F}(f(s,\cdot))(v;c,d)\,ds, \\
\mathcal{I}_2 &:= \mathcal{I}_2(u,v;a,b,c,d) = E(v;c,d) \int_c^d \mathcal{F}(f(\cdot,t))(u;a,b)\,dt, \\
\mathcal{I}_3 &:= \mathcal{I}_3(u,v;a,b,c,d) = E(u;a,b)\,E(v;c,d) \int_a^b \int_c^d f(s,t)\,dtds
\end{aligned}$$

with

$$\begin{aligned}
E(u;a,b) &:= E(-2\pi iub, -2\pi iua), \\
E(v;c,d) &:= E(-2\pi ivd, -2\pi ivc), \qquad (7.3)
\end{aligned}$$

where $E(\cdot,\cdot)$ *is the exponential mean of complex numbers as defined in (6.1). Furthermore we define the usual Lebesgue norms on two dimensional space by*

$$\left\|f''_{x,y}\right\|_\infty = \sup_{(s,t)\in[a,b]\times[c,d]} \left|\frac{\partial^2 f(s,t)}{\partial x \partial y}\right| < \infty, \text{ and}$$
$$\left\|f''_{x,y}\right\|_p = \left(\int_a^b \int_c^d \left|\frac{\partial^2 f(s,t)}{\partial x \partial y}\right|^p dtds\right)^{\frac{1}{p}}, 1 \le p < \infty.$$

Proof. A Montgomery type identity obtained by Barnett and Dragomir (2001), can be obtained as follow:

$$\begin{aligned}
f(x,y) = &\frac{\int_a^b f(s,y)\,ds}{b-a} + \frac{\int_c^d f(x,t)\,dt}{d-c} \\
&- \frac{\int_a^b \int_c^d f(s,t)\,dtds}{(b-a)(d-c)} \\
&+ \frac{\int_a^b \int_c^d P(x,s)\,Q(y,t)\,f''_{x,y}(s,t)\,dtds}{(b-a)(d-c)}, \qquad (7.4)
\end{aligned}$$

provided that f is continuous on $[a,b] \times [c,d]$ and

$$P(x,s) = \begin{cases} s-a, & a \le s \le x \\ s-b, & x < s \le b \end{cases} \quad \text{and} \quad Q(y,t) = \begin{cases} t-c, & c \le t \le y \\ t-d, & y < t \le d. \end{cases}$$

If we replace $f(x,y)$ in (7.1) by its representation from (7.4), we get

$$\begin{aligned}
&\mathcal{F}(f)(u,v;a,b,c,d) \\
&= \int_a^b \int_c^d \left(\frac{e^{-2\pi i(ux+vy)}}{b-a} \int_a^b f(s,y)\,ds \right) dydx \\
&+ \int_a^b \int_c^d \left(\frac{e^{-2\pi i(ux+vy)}}{d-c} \int_c^d f(x,t)\,dt \right) dydx \\
&- \int_a^b \int_c^d \left(\frac{e^{-2\pi i(ux+vy)}}{(b-a)(d-c)} \int_a^b \int_c^d f(s,t)\,dtds \right) dydx \\
&+ R(f,u,v;a,b,c,d),
\end{aligned} \tag{7.5}$$

where

$$\begin{aligned}
&R(f,u,v;a,b,c,d) \\
&= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left(e^{-2\pi i(ux+vy)} \right) \\
&\times \left[\int_a^b \int_c^d P(x,s)\,Q(y,t)\,f''_{x,y}(s,t)\,dtds \right] dydx.
\end{aligned} \tag{7.6}$$

Let

$$\mathcal{I}_1 = \int_a^b \int_c^d \left(\frac{e^{-2\pi i(ux+vy)}}{b-a} \int_a^b f(s,y)\,ds \right) dydx,$$

then

$$\begin{aligned}
\mathcal{I}_1 &= \int_a^b \frac{e^{-2\pi iux}}{b-a} dx \left(\int_c^d e^{-2\pi ivy} \left(\int_a^b f(s,y)\,ds \right) dy \right) \\
&= \frac{e^{-2\pi iub} - e^{-2\pi iua}}{-2\pi iu(b-a)} \int_a^b \left(\int_c^d e^{-2\pi ivy} f(s,y)\,dy \right) ds \\
&= E(u;a,b) \int_a^b \mathcal{F}(f(s,\cdot))(v;c,d)\,ds.
\end{aligned}$$

In a similar fashion we obtain

$$\begin{aligned}
\mathcal{I}_2 &= \int_a^b \int_c^d \left(\frac{e^{-2\pi i(ux+vy)}}{d-c} \int_c^d f(x,t)\,dt \right) dydx \\
&= E(v;c,d) \int_c^d \mathcal{F}(f(\cdot,t))(u;a,b)\,dt
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{I}_3 &= \int_a^b \int_c^d \left(\frac{e^{-2\pi i(ux+vy)}}{(b-a)(d-c)} \cdot \int_a^b \int_c^d f(s,t)\,dtds \right) dydx \\
&= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(s,t)\,dtds \times \int_a^b \int_c^d e^{-2\pi iux} \cdot e^{-2\pi ivy} dydx \\
&= E(u;a,b)\,E(v;c,d) \int_a^b \int_c^d f(s,t)\,dtds.
\end{aligned} \tag{7.7}$$

Using the properties of the modulus, then from (7.5) we have

$$\begin{aligned}
&\left|\mathcal{F}(f)(u,v;a,b,c,d)-\mathcal{I}_1-\mathcal{I}_2+\mathcal{I}_3\right| \\
&=\left|\int_a^b\int_c^d\Big(\int_a^b\int_c^d \frac{e^{-2\pi i(ux+vy)}}{(b-a)(d-c)}\cdot P(x,s)\,Q(y,t)\times f''_{x,y}(s,t)dtds\Big)dydx\right| \\
&\leq \int_a^b\int_c^d\int_a^b\int_c^d \left|\frac{e^{-2\pi i(ux+vy)}}{(b-a)(d-c)}\right| |P(x,s)|\,|Q(y,t)|\times\left|f''_{x,y}(s,t)\right| dtdsdydx \qquad (7.8)\\
&=\int_a^b\int_c^d\int_a^b\int_c^d \frac{|P(x,s)|\,|Q(y,t)|}{(b-a)(d-c)}\times\left|f''_{x,y}(s,t)\right| dtdsdydx. \qquad (7.9)
\end{aligned}$$

Now, we observe that

$$\begin{aligned}
&\int_a^b\int_c^d\int_a^b\int_c^d |P(x,s)|\,|Q(y,t)|\times\left|f''_{x,y}(s,t)\right| dtdsdydx \qquad (7.10)\\
&\quad\leq \left\|f''_{x,y}\right\|_\infty\left[\int_a^b\left(\int_a^b|P(x,s)|\,ds\right)dx\int_c^d\left(\int_c^d|Q(y,t)|\,dt\right)dy\right]\\
&\quad= \left\|f''_{x,y}\right\|_\infty\left[\int_a^b\left\{\left.\frac{(s-a)^2}{2}\right|_a^x+\left.\frac{(b-s)^2}{2}\right|_x^b\right\}dx\right.\\
&\qquad\qquad\left.\times\int_c^d\left\{\left.\frac{(t-c)^2}{2}\right|_c^y+\left.\frac{(d-t)^2}{2}\right|_y^d\right\}dy\right]\\
&\quad= \left\|f''_{x,y}\right\|_\infty\left[\left(\int_a^b\frac{(x-a)^2}{2}dx+\int_a^b\frac{(b-x)^2}{2}dx\right)\right.\\
&\qquad\qquad\left.\times\left(\int_c^d\frac{(y-c)^2}{2}dy+\int_c^d\frac{(d-y)^2}{2}dy\right)\right]\\
&\quad= \left\|f''_{x,y}\right\|_\infty\left[\frac{(b-a)^3}{3}\cdot\frac{(d-c)^3}{3}\right].
\end{aligned}$$

Substituting (7.10) in (7.9), we obtain the first inequality in (7.2).

Further, applying Hölder's integral inequality for double integrals, we get

$$
\begin{aligned}
&\int_a^b \int_c^d \int_a^b \int_c^d |P(x,s)\,Q(y,t)| \left|f''_{x,y}(s,t)\right| dtdsdydx \\
&\quad \le \left(\int_a^b \int_c^d \int_a^b \int_c^d |P(x,s)\,Q(y,t)|^q \,dtdsdydx\right)^{\frac{1}{q}} \\
&\quad \times \left(\int_a^b \int_c^d \int_a^b \int_c^d \left|f''_{x,y}(s,t)\right|^p dtdsdydx\right)^{\frac{1}{p}} \qquad (7.11)\\
&\quad = \left\|f''_{x,y}\right\|_p ((b-a)(d-c))^{\frac{1}{p}} \times \left(\int_a^b \left(\int_a^b |P(x,s)|^q\,ds\right) dx\right)^{\frac{1}{q}} \\
&\qquad \times \left(\int_c^d \left(\int_c^d |Q(y,t)|^q\,dt\right) dy\right)^{\frac{1}{q}} \\
&\quad = \left\|f''_{x,y}\right\|_p ((b-a)(d-c))^{\frac{1}{p}} \times \left(\int_a^b \left(\frac{(x-a)^{q+1}}{q+1} + \frac{(b-x)^{q+1}}{q+1}\right) dx\right)^{\frac{1}{q}} \\
&\qquad \times \left(\int_c^d \left(\frac{(y-c)^{q+1}}{q+1} + \frac{(d-y)^{q+1}}{q+1}\right) dy\right)^{\frac{1}{q}} \\
&\quad = \left\|f''_{x,y}\right\|_p \left[\frac{2^{\frac{2}{q}}(b-a)^{1+\frac{2}{q}}(d-c)^{1+\frac{2}{q}}}{((q+1)(q+2))^{\frac{2}{q}}}\right]. \qquad (7.12)
\end{aligned}
$$

Utilizing (7.9) with (7.12), we get the second inequality of (7.2).

Finally, we obtain that

$$
\begin{aligned}
&\int_a^b \int_c^d \int_a^b \int_c^d |P(x,s)\,Q(y,t)| \times \left|f''_{x,y}(s,t)\right| dtdsdydx \qquad (7.13)\\
&\le \sup_{(x,s)\in[a,b]^2} |P(x,s)| \sup_{(y,t)\in[c,d]^2} |Q(y,t)| \times \int_a^b \int_c^d \int_a^b \int_c^d \left|f''_{x,y}\right| dtdsdydx \\
&= (b-a)(d-c)\int_a^b \int_c^d \int_a^b \int_c^d \left|f''_{x,y}\right| dtdsdydx \\
&= \left\|f''_{x,y}\right\|_1 (b-a)^2 (d-c)^2 .
\end{aligned}
$$

Substituting in (7.13) into (7.9) gives the final inequality in (7.2), where we have used the fact that

$$\max\{X,Y\} = \frac{X+Y}{2} + \left|\frac{Y-X}{2}\right|.$$

Thus the theorem is completely proved. □

7.3 A Numerical Cubature Formula

To illustrate the use of a cubature formula, we form a composite rule from the inequality (7.2).

Let us consider the arbitrary divisions $I_m : a = x_0 < x_1 < \cdots < x_m = b$ on $[a, b]$ and $J_n : c = y_0 < y_1 < \cdots < y_n = d$ on $[c, d]$, define the sum

$$\mathfrak{F}(f, I_m, J_n, u, v) = \sum_{k=0}^{m-1}\sum_{l=0}^{n-1} \mathcal{I}_1(\mathcal{SD}) + \sum_{k=0}^{m-1}\sum_{l=0}^{n-1} \mathcal{I}_2(\mathcal{SD}) - \sum_{k=0}^{m-1}\sum_{l=0}^{n-1} \mathcal{I}_3(\mathcal{SD}) \tag{7.14}$$

where

$$(\mathcal{SD}) := (u, v; x_k, x_{k+1}, y_l, y_{l+1});$$

$h_k := x_{k+1} - x_k \ (k = 0, 1, 2, \cdots, m-1)$ and $\nu_l := y_{l+1} - y_l \ (l = 0, 1, \cdots, n-1)$

Under the above assumptions the following theorem can be obtained (Hanna *et al.* (2002)).

THEOREM 7.2. *Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a continuous mapping on $[a, b] \times [c, d]$, then we have the cubature formula*

$$\mathcal{F}(f)(u, v; a, b, c, d) = \mathfrak{F}(f, I_m, J_n, u, v) + R(f, I_m, J_n, u, v), \tag{7.15}$$

where $\mathfrak{F}(f, I_m, J_n, \cdot, \cdot)$ approximates the Fourier Transform $\mathcal{F}(f)$ at every point $(u, v) \in [a, b] \times [c, d]$, and the remainder term $R(f, I_m, J_n, \cdot, \cdot)$ satisfies the bounds

$$|R(f, I_m, J_n, u, v)| \leq \begin{cases} \frac{1}{9}\left(\sum_{k=0}^{m-1}\sum_{l=0}^{n-1} h_k^2 \nu_l^2\right) \|f''_{x,y}\|_{\mathbb{B}nfty}, \\ \left[\frac{2}{(q+1)(q+2)}\right]^{\frac{2}{q}} \left(\sum_{k=0}^{m-1}\sum_{l=0}^{n-1} h_k \nu_l\right)^{\frac{q+1}{q}} \|f''_{x,y}\|_p, \\ \kappa(h)\tau(\nu)\|f''_{x,y}\|_1, \end{cases} \tag{7.16}$$

where $\kappa(h) :=$ max $\{h_k |\ k = 0, \cdots, m-1\}$, and $\tau(\nu) :=$ max $\{\nu_l |\ l = 0, \cdots, n-1\}$.

Proof. Applying Theorem 7.1 over every subinterval $[x_k, x_{k+1}]$ and $[y_l, y_{l+1}]$, we can state that

$$\left|\mathcal{F}(f)(\mathcal{SD})-\mathcal{I}_1(\mathcal{SD})-\mathcal{I}_2(\mathcal{SD})+\mathcal{I}_3(\mathcal{SD})\right|$$
$$\leq\begin{cases}\frac{1}{9}h_k^2v_l^2\sup\limits_{(s,t)\in[x_k,x_{k+1}]\times[y_l,y_{l+1}]}\left|\frac{\partial^2 f(s,t)}{\partial x\partial y}\right|\\ \\ \left[\frac{2\left[h_k\nu_l\right]^{\frac{q+1}{2}}}{(q+1)(q+2)}\right]^{\frac{2}{q}}\mathcal{DIS}\\ \\ h_kv_l\int_{x_k}^{x_{k+1}}\int_{y_l}^{y_{l+1}}\left|\frac{\partial^2 f(s,t)}{\partial x\partial y}\right|^p dtds\end{cases}$$

where

$$\mathcal{DIS}:=\left(\int_{x_k}^{x_{k+1}}\int_{y_l}^{y_{l+1}}\left|\frac{\partial^2 f(s,t)}{\partial x\partial y}\right|^p dtds\right)^{\frac{1}{p}},$$

Summing over k from 0 to $m-1$ and l from 0 to $n-1$, and using the triangle inequality, we obtain

$$\begin{aligned}&|R(f,I_m,J_n,u,v)|\\&=|\mathcal{F}(f)(u,v;a,b,c,d)-\mathfrak{F}(f,I_m,J_n,u,v)|\\&\leq\begin{cases}\frac{1}{9}\sum_{k=0}^{m-1}\sum_{l=0}^{n-1}\sup\limits_{(s,t)\in[x_k,x_{k+1}]\times[y_l,y_{l+1}]}\left|\frac{\partial^2 f(s,t)}{\partial x\partial y}\right|h_k^2\nu_l^2\\ \\ \left[\frac{2\left(\sum_{k=0}^{m-1}\sum_{l=0}^{n-1}\left[h_kv_l\right]^{q+1}\right)^{\frac{1}{2}}}{(q+1)(q+2)}\right]^{\frac{2}{q}}\mathcal{DIS}\\ \\ \sum_{k=0}^{m-1}\sum_{l=0}^{n-1}h_kv_l\int_{x_k}^{x_{k+1}}\int_{y_l}^{y_{l+1}}\left|\frac{\partial^2 f(s,t)}{\partial x\partial y}\right|^p dtds,\end{cases}\end{aligned}$$

where

$$\sup_{(s,t)\in[x_k,x_{k+1}]\times[y_l,y_{l+1}]}\left|\frac{\partial^2 f(s,t)}{\partial x\partial y}\right|\leq\sup_{(s,t)\in[a,b]\times[c,d]}\left|\frac{\partial^2 f(s,t)}{\partial x\partial y}\right|=\|f''_{x,y}\|_\infty,$$

thus the first inequality in (7.16) is obtained. Using Hölder's discrete inequality, we have

$$\begin{aligned}
&\sum_{k=0}^{m-1}\sum_{l=0}^{n-1}[h_k\nu_l]^{\frac{q+1}{q}}\left(\int_{x_k}^{x_{k+1}}\int_{y_l}^{y_{l+1}}\left|\frac{\partial^2 f(s,t)}{\partial x\partial y}\right|^p dtds\right)^{\frac{1}{p}}\\
&\qquad\le\left[\sum_{k=0}^{m-1}\sum_{l=0}^{n-1}\left([h_k\nu_l]^{\frac{q+1}{q}}\right)^q\right]^{\frac{1}{q}}\\
&\qquad\quad\times\left[\sum_{k=0}^{m-1}\sum_{l=0}^{n-1}\left[\left(\int_{x_k}^{x_{k+1}}\int_{y_l}^{y_{l+1}}\left|\frac{\partial^2 f(s,t)}{\partial x\partial y}\right|^p dtds\right)^{\frac{1}{p}}\right]^p\right]^{\frac{1}{p}}\\
&\qquad=\left(\sum_{k=0}^{m-1}\sum_{l=0}^{n-1}(h_k\nu_l)^{q+1}\right)^{\frac{1}{q}}\|f''_{x,y}\|_p
\end{aligned}$$

which proves the second inequality in (7.16).

For the last inequality, we observe that

$$\begin{aligned}
&\sum_{k=0}^{m-1}\sum_{l=0}^{n-1}h_k\nu_l\int_{x_k}^{x_{k+1}}\int_{y_l}^{y_{l+1}}\left|\frac{\partial^2 f(s,t)}{\partial x\partial y}\right|dtds\\
&\le\kappa(h)\tau(\nu)\sum_{k=0}^{m-1}\sum_{l=0}^{n-1}\int_{x_k}^{x_{k+1}}\int_{y_l}^{y_{l+1}}\left|\frac{\partial^2 f(s,t)}{\partial x\partial y}\right|dtds\\
&=\kappa(h)\tau(\nu)\int_a^b\int_c^d\left|\frac{\partial^2 f(s,t)}{\partial x\partial y}\right|dtds\\
&=\kappa(h)\tau(\nu)\|f''_{x,y}\|_1
\end{aligned}$$

and the theorem is completely proved. □

In practical applications, it is convenient to consider the equidistant partitioning of the region $[a,b]\times[c,d]$. Thus let

$$\begin{aligned}
&I_m: x_k=a+k\cdot\frac{b-a}{m},\quad k=0,1,\cdots,m\quad\text{and}\\
&J_n: y_l=c+l\cdot\frac{d-c}{n},\quad l=0,1,\cdots,n,
\end{aligned}$$

and we define the sum

$$\begin{aligned}
&\mathfrak{F}_{\mathfrak{m},\mathfrak{n}}(f,I_m,J_n,u,v)\\
&\qquad=\sum_{k=0}^{m-1}\sum_{l=0}^{n-1}\mathcal{I}_1(\mathcal{ES})+\sum_{k=0}^{m-1}\sum_{l=0}^{n-1}\mathcal{I}_2(\mathcal{ES})-\sum_{k=0}^{m-1}\sum_{l=0}^{n-1}\mathcal{I}_3(\mathcal{ES})
\end{aligned}\tag{7.17}$$

where $(\mathcal{ES}):=(u,v;a+k\cdot\frac{b-a}{m},a+(k+1)\cdot\frac{b-a}{m},c+l\cdot\frac{d-c}{n},c+(l+1)\cdot\frac{d-c}{n})$.
Then the following corollary of Theorem 7.2 holds:

Corollary 7.2.1. *Let f be as defined in Theorem 7.2. Then we have*

$$\mathcal{F}(f)(u,v;a,b,c,d) = \mathfrak{F}_{\mathfrak{n},\mathfrak{m}}(f,I_m,J_n,u,v) + R_{m,n}(f,I_m,J_n,u,v), \tag{7.18}$$

where $\mathfrak{F}_{\mathfrak{n},\mathfrak{m}}(f,I_m,J_n,.,.)$ *approximates the Fourier Transform* $\mathcal{F}(f)$ *at every point* $(u,v) \in [a,b]\times[c,d]$*, and the remainder term* $R_{m,n}(f,I_m,J_n,.,.)$ *satisfies the bounds*

$$|R_{n,m}(f,I_m,J_n,u,v)| \leq \begin{cases} \dfrac{(b-a)^2(d-c)^2}{9mn}\|f''_{x,y}\|_\infty; \\[2em] \left[\dfrac{2[(b-a)(d-c)]^{\frac{q+2}{2}}}{(q+1)(q+2)}\right]^{\frac{2}{q}} \dfrac{\|f''_{x,y}\|_p}{mn}; \\[2em] \dfrac{(b-a)(d-c)}{mn}\|f''_{x,y}\|_1. \end{cases} \tag{7.19}$$

7.3.1 Numerical Experiments

Now, we will employ the cubature from equation (7.14) to approximate the finite Fourier transform of

$$f(x,y) = e^{3x-2y}(x-y), \qquad 0 \leq x,y \leq 1. \tag{7.20}$$

Since $\mathcal{F}(f)$ can be computed analytically we can gauge the performance of the cubature rule as well as compare it to the theoretical error bound (7.19).

The results are shown in Table 1 where n^2 is the number of uniform partitions of the domain $[0,1]\times[0,1]$. It is clearly evident that the cubature rule performs extremely well and achieves single precision accuracy when $n = 16$. Halving the interval size will increase the accuracy by approximately one and a half orders, and a simple analysis shows that the rate of convergence is at least $O((nm)^{-2})$. This contrasts with the theoretical error which is $O(1/(nm))$. Extending the Peano kernel, that is using a higher order identity to that of (7.4), may provide a higher order theoretical error result. This will be investigated in future work.

In Figure 7.1, we show a three dimensional plot of the finite Fourier transform obtained using (7.14) for the example (7.20).

n	Num. Error	Ratio	Th. Error
1	0.32E+00	3.11	0.13E+02
2	0.13E-01	25.28	0.33E+01
4	0.48E-04	267.37	0.82E+00
8	0.16E-05	30.63	0.20E+00
16	0.23E-07	67.49	0.51E-01
32	0.34E-09	68.02	0.13E-01
64	0.77E-11	44.09	0.32E-02

Table 7.1: Numerical error (column 2) and theoretical error (column 4) in approximating the finite Fourier transform of (7.20) using equation (7.14).

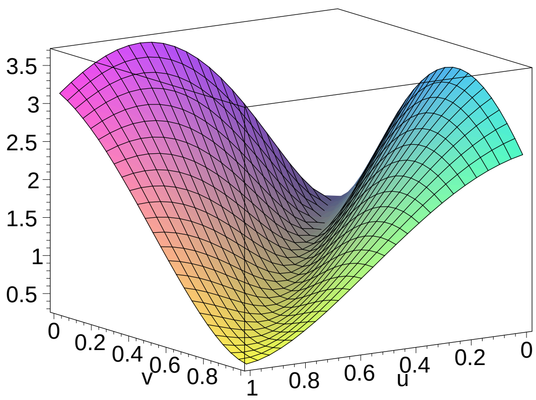


Figure 7.1: Finite Fourier transform of $f(x, y) = e^{3x-2y}(x - y),\ 0 \le x, y \le 1$ evaluated using the rule (7.14).

7.4 A Pre Grüss Type Inequality for Complex Valued Functions

Hanna *et al.* (2004) developed the following theorem which provides the possibility to approximate the integral of the product in terms of the product of integrals.

THEOREM 7.3. *Let* $\rho : \Omega \to [0, \infty)$ *be a* $\mu-$*measurable function on* Ω *with* $\int_\Omega \rho(s)\, d\mu(s) = 1$. *If* $f, g \in L^2_\rho(\Omega, \mathbb{K})$ *and there exist the constants* $\gamma, \Gamma \in \mathbb{K}$ *with the property that either*

$$Re\left[(\Gamma - f(s))\left(\overline{f(s)} - \overline{\gamma}\right)\right] \geq 0 \;\; \textit{for} \;\; \mu - \textit{a.e.}\; s \in \Omega \tag{7.21}$$

or, equivalently,

$$\left| f(s) - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \;\; \textit{for} \;\; \mu - \textit{a.e.}\; s \in \Omega \tag{7.22}$$

holds, then

$$\begin{aligned} &\left| \int_\Omega \rho(s) f(s) \overline{g(s)} d\mu(s) - \int_\Omega \rho(s) f(s)\, d\mu(s) \int_\Omega \rho(s) \overline{g(s)} d\mu(s) \right| \\ &\qquad \leq \frac{1}{2} |\Gamma - \gamma| \left[\int_\Omega \rho(s) |g(s)|^2 d\mu(s) - \left| \int_\Omega \rho(s) g(s)\, d\mu(s) \right|^2 \right]^{\frac{1}{2}}. \end{aligned} \tag{7.23}$$

The above result provides an inequality of Grüss type that may be useful in applications where one of the factors is known and some bounds for the second factor are provided. For more details see (Hanna *et al.* 2004; Barnett *et al.* 2004; Dragomir 1999a; Dragomir and Gomm 2003).

Now, we will apply the above theorem for the two dimensional case namely, $f : [a, b] \times [c, d] \to \mathbb{R}$, and we consider the two examples

$$f(x, s) \to f(t, y)$$
$$\text{and} \;\; g(x, y) = e^{2\pi i(ux+vy)}.$$

We state the following theorem to approximate the Fourier transform $\mathcal{F}(\cdot, \cdot)$ for the two dimensional case.

THEOREM 7.4. *Let* $f : [a, b] \times [c, d] \to \mathbb{K}$ *be a real or complex valued function with* $f \in L^2_\rho(\Omega, \mathbb{K})$, *and there exist the constants* $\Gamma, \gamma \in \mathbb{K}$ *with the property that, either*

$$\left| f(x, y) - \frac{\Gamma + \gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \tag{7.24}$$

or, equivalently

$$Re\left[\left(\Gamma - f\left(x,y\right)\right)\left(\overline{f\left(x,y\right)} - \overline{\gamma}\right)\right] \geq 0 \text{ for } \left(x,y\right) \in [a,b] \times [c,d] \tag{7.25}$$

hold. Then we have the inequality

$$\left|\mathcal{F}\left(f\right)\left(u,v;a,b,c,d\right) - E\left(u;a,b\right)E\left(v;c,d\right)\int_a^b\int_c^d f\left(x,y\right)dydx\right| \leq \frac{1}{2}\left|\Gamma-\gamma\right|\left(b-a\right)\left(d-c\right)\left[1-\frac{\sin^2\left[\pi u\left(b-a\right)\right]}{\pi^2\left|x\right|^2\left(b-a\right)^2}\right]^{\frac{1}{2}}\left[1-\frac{\sin^2\left[\pi v\left(d-c\right)\right]}{\pi^2\left|x\right|^2\left(d-c\right)^2}\right]^{\frac{1}{2}}. \tag{7.26}$$

Proof. Utilizing Theorem 7.3 we can state that

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(x,y\right)e^{-2\pi i(ux+vy)}dydx - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(x,y\right)dydx \cdot\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d e^{-2\pi i(ux+vy)}dydx\right| \leq \frac{1}{2}\left|\Gamma-\gamma\right|K, \tag{7.27}$$

where

$$K = \left[\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\left|e^{-2\pi i(ux+vy)}\right|^2 dydx - \left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d e^{-2\pi i(ux+vy)}dydx\right|^2\right]^{\frac{1}{2}}, \tag{7.28}$$

provided that $\gamma, \Gamma \in \mathbb{K}$ (if they exist) satisfy the property that, either

$$Re\left[\left(\Gamma - f\left(x,y\right)\right)\left(\overline{f\left(x,y\right)} - \overline{\gamma}\right)\right] \geq 0 \text{ for } \left(x,y\right) \in [a,b] \times [c,d], \tag{7.29}$$

or, equivalently,

$$\left|f\left(x,y\right) - \frac{\gamma+\Gamma}{2}\right| \leq \frac{1}{2}\left|\Gamma-\gamma\right| \text{ for } \left(x,y\right) \in [a,b] \times [c,d]. \tag{7.30}$$

Also, observe that

$$\begin{aligned}\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d e^{-2\pi i(ux+vy)}dydx \\ &= \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d e^{-2\pi iux}\cdot e^{-2\pi ivy}dydx \\ &= \frac{1}{(b-a)(d-c)}\left(\int_a^b e^{-2\pi iux}dx\right)\left(\int_c^d e^{-2\pi ivy}dy\right) \\ &= \frac{1}{(b-a)(d-c)}\times\frac{-1}{2\pi iu}\left[e^{-2\pi iux}\right]_a^b\times\frac{-1}{2\pi iv}\left[e^{-2\pi ivy}\right]_c^d \\ &= \frac{e^{-2\pi iub}-e^{-2\pi iua}}{-2\pi iub-2\pi iua}\times\frac{e^{-2\pi ivd}-e^{-2\pi ivc}}{-2\pi ivd-2\pi ivc} \\ &= E(u;a,b)E(v;c,d).\end{aligned} \tag{7.31}$$

Moreover, we have

$$\left|e^{2\pi i(ux+vy)}\right|^2 = 1, \tag{7.32}$$

$$\int_a^b e^{2\pi iux}dx = \frac{1}{2\pi iu}\left[e^{2\pi iub}-e^{2\pi iua}\right],$$

and

$$\left|\int_a^b\int_c^d e^{2\pi i(ux+vy)}dydx\right|^2 = \left|\int_a^b e^{2\pi uix}dx\right|^2\left|\int_c^d e^{2\pi viy}dy\right|^2$$

but

$$\begin{aligned}\left|\int_a^b e^{2\pi uix}dx\right|^2 &= \left(\frac{1}{2\pi|u|}\right)^2\left[\left|e^{2\pi iub}\right|^2-2Re\left[e^{2\pi iu(b-a)}\right]+\left|e^{2\pi iua}\right|^2\right] \\ &= \frac{1}{2\pi^2|u|^2}\left[1-\cos\left(2\pi u\left(b-a\right)\right)\right] \\ &= \frac{1}{2\pi^2|u|^2}\left[1-\left(1-2\sin^2\left(\pi u\left(b-a\right)\right)\right)\right] \\ &= \frac{\sin^2\left[\pi u\left(b-a\right)\right]}{\pi^2|u|^2}.\end{aligned} \tag{7.33}$$

In similar way we have

$$\left|\int_c^d e^{2\pi i(vy)}dy\right|^2 = \frac{\sin^2\left[\pi v\left(d-c\right)\right]}{\pi^2|v|^2}. \tag{7.34}$$

Utilizing (7.31) to (7.34) and substitute in (7.27) we deduce the desired inequality (7.26). Thus the theorem is completely proved. □

7.4.1 Applications to a Cubature Formula

Let us consider the arbitrary divisions as in section (7.3) and define the sum

$$\mathcal{F}(f, I_m, J_n, u, v) = \sum_{k=0}^{m-1} \sum_{l=0}^{n-1} E(-2\pi i u x_{k+1}, -2\pi i u x_k) E(-2\pi i v y_{l+1}, -2\pi i v y_l) \times \int_{x_k}^{x_{k+1}} \int_{y_l}^{y_{l+1}} f(x, y) dy dx \tag{7.35}$$

where $(u, v) \in [a, b] \times [c, d],\ u \neq 0$ and $v \neq 0$.

Then the following theorem can be obtained.

THEOREM 7.5. *Let* $f : [a, b] \times [c, d] \to \mathbb{R}$ *be continuous mapping on* $[a, b] \times [c, d]$*, then we have the cubature formula*

$$\mathcal{F}(f)(u, v; a, b, c, d) = \mathcal{F}(f, I_m, J_n, u, v) + R(f, I_n, J_m, u, v),$$

where

$\mathcal{F}(f, I_n, J_m, \cdot, \cdot)$ *approximates the Fourier Transform* $\mathcal{F}(f)$ *at every point* $(u, v) \in [a, b] \times [c, d]$,

and the remainder term $R(f, I_m, J_n, \cdot, \cdot)$ *satisfies the bounds*

$$|\mathcal{R}(f, I_m, J_n, u, v)| \leq \frac{1}{2} |\Gamma - \gamma| (b-a)^{\frac{1}{2}} (d-c)^{\frac{1}{2}} [\kappa(h)]^{\frac{1}{2}} [\tau(\nu)]^{\frac{1}{2}} \times \left(\sum_{k=0}^{m-1} \sum_{l=0}^{n-1} \left(1 - \frac{\sin^2(\pi u h_k)}{\pi^2 |u|^2 h_k^2}\right)^{\frac{1}{2}} \left(1 - \frac{\sin^2(\pi v \nu_l)}{\pi^2 |v|^2 \nu_l^2}\right)^{\frac{1}{2}} \right). \tag{7.36}$$

Proof. If we apply Theorem 7.4 over every subinterval $[x_k, x_{k+1}]$ and $[y_l, y_{l+1}]$, we can state that

$$\left| \int_{x_k}^{x_{k+1}} \int_{y_l}^{y_{l+1}} f(x, y) e^{-2\pi i (ux+vy)} dy dx - E(u) E(v) \cdot \int_{x_k}^{x_{k+1}} \int_{y_l}^{y_{l+1}} f(x, y) dy dx \right| \leq \frac{1}{2} |\Gamma - \gamma| h_k \nu_l \left[1 - \frac{\sin^2(\pi u h_k)}{\pi^2 |u|^2 h_k^2}\right]^{\frac{1}{2}} \left[1 - \frac{\sin^2(\pi v \nu_l)}{\pi^2 |v|^2 \nu_l^2}\right]^{\frac{1}{2}}.$$

Summing over k from 0 to $m-1$ and l from 0 to $n-1$ using the triangle inequality and utilizing the Cauch-Bunyakovsky-Schwarz inequality

$$\left| \sum_{i=1}^{r} X_i Y_i \right|^2 \leq \sum_{i=1}^{r} |X_i|^2 \sum_{i=1}^{r} |Y_i|^2,$$

where $X_i, Y_i \in \mathbb{K}$, we obtain

$$\begin{aligned} |\mathcal{R}(f, I_m, J_n, u, v)| &= |\mathcal{F}(f)(u, v; a, b, c, d) - \mathcal{F}(f, I_m, J_n, u, v)| \\ &\leq \frac{1}{2}|\Gamma - \gamma| \sum_{k=0}^{m-1} h_k \left[1 - \frac{\sin^2(\pi u h_k)}{\pi^2 |u|^2 h_k^2}\right]^{\frac{1}{2}} \sum_{l=0}^{n-1} \nu_l \left[1 - \frac{\sin^2(\pi v \nu_l)}{\pi^2 |v|^2 \nu_l^2}\right]^{\frac{1}{2}} \\ &\leq \frac{1}{2}|\Gamma - \gamma| \left(\sum_{k=0}^{m-1} h_k^2\right)^{\frac{1}{2}} \left[\sum_{k=0}^{m-1} \left(1 - \frac{\sin^2(\pi u h_k)}{\pi^2 |u|^2 h_k^2}\right)\right]^{\frac{1}{2}} \\ &\times \left(\sum_{l=0}^{n-1} \nu_l^2\right)^{\frac{1}{2}} \left[\sum_{l=0}^{n-1} \left(1 - \frac{\sin^2(\pi v \nu_l)}{\pi^2 |v|^2 \nu_l^2}\right)\right]^{\frac{1}{2}}. \end{aligned}$$

Finally by observing that

$$\sum_{k=0}^{m-1} h_k^2 \leq \kappa(h) \sum_{k=0}^{m-1} h_k = (b-a)\kappa(h) \text{ and } \sum_{l=0}^{n-1} \nu_l^2 \leq \tau(\nu) \sum_{l=0}^{n-1} \nu_l = (d-c)\tau(\nu),$$

we deduce the estimate (7.36).

The proof is thus completed. □

7.4.2 Numerical Experiments

In this section the cubature developed in equation (7.35) is used to approximate the finite Fourier transform of the following functions:

Example 7.1.

$$f(x,y) = e^{x+y}, \qquad 0 \le x,y \le 1. \tag{7.37}$$

The surface and contour plots over different partitions using the equation (7.35) to approximate the finite Fourier transform of the function (7.37) is shown in Figure 7.2.

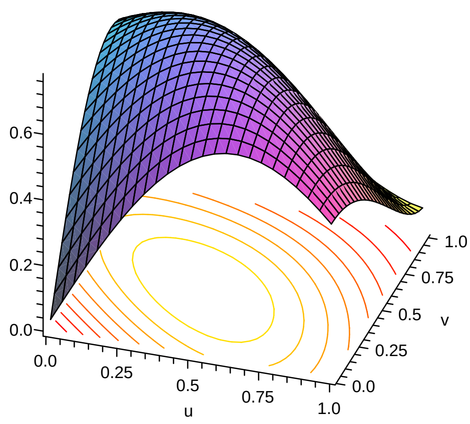
Example 7.2.

$$f(x,y) = e^{-x^2-y^2}, \qquad -0.1 \le x,y \le 0.1 \tag{7.38}$$

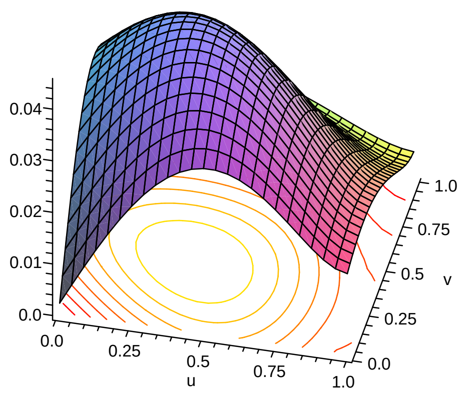
The plots over different partitions using the equation (7.35) to approximate the finite Fourier transform of the function in Example 7.2 as shown in Figure 7.3.

Since $\mathcal{F}(f)$ can be computed analytically we can gauge the performance of the cubature rule as well as compare it to the theoretical error bound (7.36).

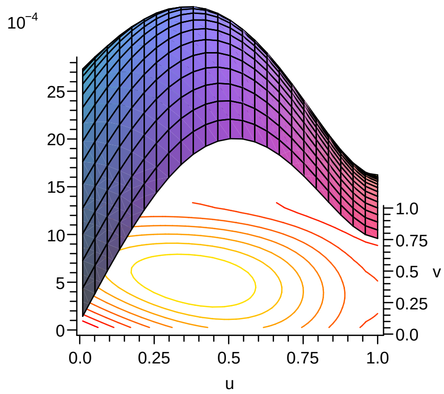
The results are shown in Figures 7.2 and 7.3 where $2n$ and $2m$ are the number of uniform partitions of the domains $[0,1] \times [0,1]$ in Example 7.1, and $[-0.1, 0.1] \times [-0.1, 0.1]$ in Example 7.2. The plots of the approximation of the finite Fourier transform of the two functions for partitions $1 \times 1,\ 4 \times 4,\ 8 \times 8,\ 16 \times 16$, 32×32 and 64×64, respectively, are depicted in Figures 7.2 and 7.3. Clearly, we notice from the figures; 7.2 and 7.3 that the error is always smaller than the error bound when utilizing equation (7.35) to approximate the finite Fourier transform of the above two functions. This seems to be quite typical behaviour, this method is expected to give the same results for any function chosen.



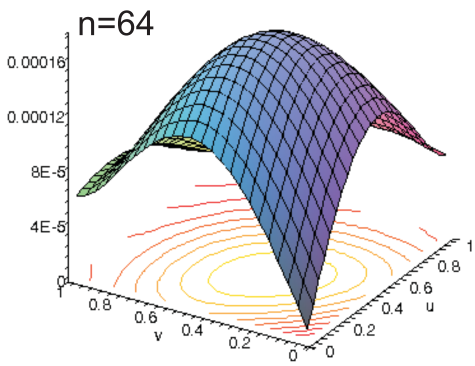
(a) Results using a 1×1 partition



(b) Results using a 4×4 partition

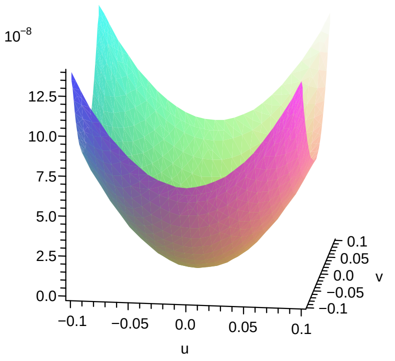


(c) Results using a 16×16 partition

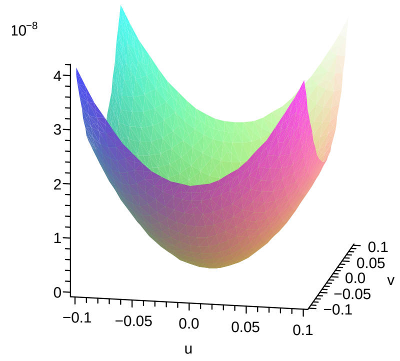


(d) Results using a 64×64 partition

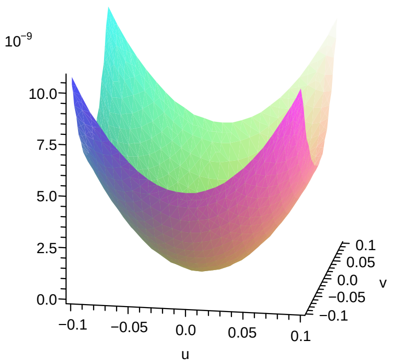
Figure 7.2: Surface and contour plots of the error between over different partitions using the equation (7.35). (The approximation and the finite Fourier transform of $f(x, y) = e^{x+y}, \quad 0 \leq x, y \leq 1$.)



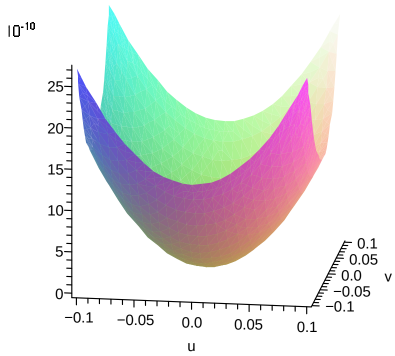
(a) the error over a partition 1×1



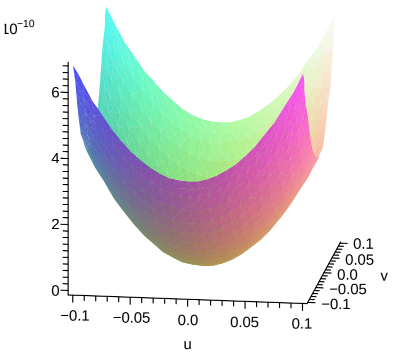
(b) the error over a partition 4×4



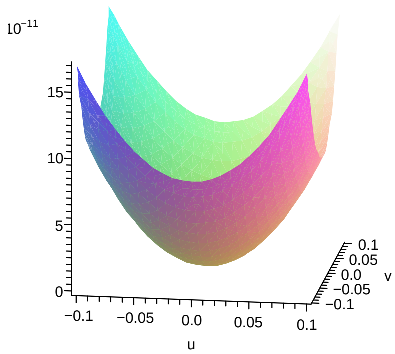
(c) the error over a partition 8×8



(d) the error over a partition 16×16



(e) the error over a partition 32×32



(f) the error over a partition 64×64

Figure 7.3: Plots of the error over different partitions using the equation (7.35). (The approximation and the finite Fourier transform of $f(x,y) = e^{-x^2-y^2}, \quad -0.1 \le x, y \le 0.1$)

APPENDIX A

USING THE MAPLE PACKAGE TO PRODUCE GRAPHICAL RESULTS FOR THE FIGURES

For evaluating single or double integration of a function in one or two variables on a computer, it is generally more efficient in terms of both space and time to have an analytic approximation to the integration rather than to store a table and use interpolations. It is also desirable to use the lowest possible degree of polynomial that will give the desired accuracy in approximating the integration. The following sections give a number of programs using the Maple package for producing the desired approximations for all the numerical application within the thesis.

A.1 Cubature Related Maple Programs

A.1.1 Graphing the numerical results of Chapter 3

Maple program to produce Figure 3.1

The perturbed Taylor expansion developed in equation (3.30) is used for different values of m and n to approximate the error $|R_{nm}(f,a,x,b,y)|$ for the function $f(x,y)=e^{-x^2-y^2},\ x,y\in[0,1]$ as in Figure 3.1.

```
> with (student):
> restart:
```

```
> Digits:=25:

> g:=unapply(exp (-x^2-y^2),x,y):

> appr:= proc (g, a, c, d, m, n)

local sum1, sum2, i, j, dsum, sumd1, sumd2, firstp, sum3, sum4, appro2:
sum1:=0: sum2:=0: sum3:=0:

for j from 0 to n do

   for i from 0 to m do

      dsum:=(((x-a)ˆi)/i!)* (((y-b)ˆj)/j!)*  (D[1$i,2$(j)](g)(a, b)):

      sum1:=sum1 + dsum:

   od:

od:

i:=i: j:=j:

   for i from 0 to m do

      sumd1:=(((x-a)ˆi)/i!)*int (((y-s)ˆn)* (D[1$i,2$(n+1)](g)(a,s)),s=b..y):

      sum2:=sum2+sumd1:

   od: firstp:=(1/n!)* sum2:

i:=i: j:=j:

   for j from 0 to n do

      sumd2:=(((y-b)ˆj)/j!)* int((x-t)ˆn* D[1$m+1,2$(j)](g)(t,b),t=a..x):

      sum3:=sum3+sumd2:

   od:

appro2:=(1/m!)* sum3:

   sum4:=(x-a)ˆm* (y-b)ˆn* int(int(D[1$m+1,2$(n+1)](g)(t,s),s=b..y),t=a..x)/((m+1)!* (n+1)!):

plot3d(abs(g(x,y)-sum1-firstp-appro2-sum4),x=a..c,y=b..d):

end:


> appr( g,0,1,0,1,1,1);
```

Maple program to produce Figure 3.2

The perturbed Taylor expansion developed in equation (3.30) is used for different values of m and n to approximate the error $|R_{nm}(f, a, x, b, y)|$ for the function $f(x, y) = e^{-xy},\ x, y \in [0, 1]$ as in Figure3.2.

```
> with (student):
> restart:
> Digits:=25:
> g:=unapply(exp (-xy),x,y):
> appr:= proc (g, a, c, d, m, n)
local sum1, sum2, i, j, dsum, sumd1, sumd2, firstp, sum3, sum4, appro2:
sum1:=0: sum2:=0: sum3:=0:
for j from 0 to n do
  for i from 0 to m do
    dsum:=(((x-a)^i)/i!)* (((y-b)^j)/j!)*  (D[1$i,2$(j)](g)(a, b)):
    sum1:=sum1 + dsum:
  od: od:
i:=i: j:=j:
  for i from 0 to m do
    sumd1:=(((x-a)^i)/i!)*int (((y-s)^n)* (D[1$i,2$(n+1)](g)(a,s)),s=b..y):
    sum2:=sum2+sumd1:    od:
firstp:=(1/n!)* sum2:
i:=i: j:=j:
  for j from 0 to n do
    sumd2:=(((y-b)^j)/j!)* int((x-t)^n* D[1$m+1,2$(j)](g)(t,b),t=a..x):
    sum3:=sum3+sumd2:
  od:
appro2:=(1/m!)* sum3:
  sum4:=(x-a)^m* (y-b)^n* int(int(D[1$m+1,2$(n+1)](g)(t,s),s=b..y),t=a..x)/((m+1)!* (n+1)!):
plot3d(abs(g(x,y)-sum1-firstp-appro2-sum4),x=a..c,y=b..d):
end:
> appr( g,0,1,0,1,m,n);
```

A.1.2 Graphing the numerical results of Chapter 4

Maple program to produce Figures 4.1

The absolute error of approximating the integral $\int_0^x \int_0^y f(t,s)\; dsdt$ as a function of $(x,y) \in [0,1]^2$ for the function (4.44) using (4.34) and (4.35) for various values of m and n as in Figure 4.1.

```
> restart:
> Digits:=20:
> f:=unapply(exp(-x*y),x,y);
> approxg:=proc(f,a,b,c,d,m,n)
local i,j,sum1,dsum1,sum2,dsum2,A,B1,sumB1,B2,sumB2,B3,sumB3,B4,sumB4,B,sumf,sums,sumth,
 sumfo,sumb1,sumff,sumsf,errorb,doubint,Af,threrr:
sum1:=0: sum2:=0: sumf:=0: sums:=0: sumth:=0: sumfo:=0:
   for i from 1 to m do
      for j from 2 to n do
      dsum1:=(((a-x)^(j)*(c-y)^(i))/((i)!*j!))*
(2*(2^(i+1)-1)/(i+1))*(bernoulli(j)*bernoulli(i+1))*(((D[1(i − 1), 2(j-1)](f)(a, y))
+((D[1(i − 1), 2(j-1)](f)(a, c)))+((D[1(i − 1), 2(j-1)](f)(x, y)))+((D[1(i − 1), 2(j-1)](f)(x, c))))):
      sum1:=(sum1+dsum1):
      od:
   od: i:=i: j:=j:
   for i from 1 to m do
      sumb1:=((2^(i+1)-1)*(c-y)^(i)/(i+1)!)*bernoulli(i+1)*(((D[1(i − 1), 2(0)](f)(a, y))
+((D[1(i − 1), 2(0)](f)(a, c)))+((D[1(i − 1), 2(0)](f)(x, y)))
+((D[1(i − 1), 2(0)](f)(x, c))))):
      sumf := (sumf+sumb1):
      sumff:=(b-a)*sumf:
      od:
i:=i: j:=j:
   for i from 1 to m do
      B2:=(2*(1-2^(i+1))*(c-y)^(i)/(i+1)!)*bernoulli(i+1)*int((D[1(0), 2(i-1)](f)(t, c))
```

```
+(D[1(0), 2(i-1)](f)(t, y)),t=a..x):
    sums:=(sums+B2):
  od:
i:=i: j:=j:
  for j from 2 to n do
    B3:=(((a-x)^j)/j!)*bernoulli(j)*int((D[1(j - 1), 2(0)](f)(a, s))
+(D[1(j - 1), 2(0)](f)(x, s)),s=c..y):
    sumth:=(sumth+B3):
  od:
i:=i: j:=j:      B4:=((b-a)/2)*int((f(a,s)+f(x,s)),s=c..y):
    Af:=sumff+sum1:
    B:=sums+sumth+B4:
    doubint:=evalf((int(int(f(t,s),s=c..y),t=a..x))):
    errorb:=evalf((abs(doubint-Af-B))):
    plot3d(abs(doubint-Af-B),y=c..d,x=a..b,axes=framed,style=patchnogrid):
end:
> approxg(f,0,1,0,1,1,1);
```

A.2 Fourier Related Maple Programs

A.2.1 Graphing the numerical results of Chapter 6

Maple program of Figure 6.1

Maple program applying equation (6.27) to plot the bound on error $r_n\left(x\right) := \left|R_n\left(g,x\right)\right|$ for the function $g(x) = \exp(x),\ x \in [-1,1]$ as in Figure 6.1.

```
> restart:
> gen:=proc(m,a,b)
local g,F,Eg1,Eg3,Eg5,Lg,Sg,
g:=unapply((exp(xˆ 2)),x);
F:=Int(g(x)* exp(-2* Pi* I* ((u*x))),x=a..b):
F:=unapply(int(g(x)* exp(-2* Pi* I* ((u*x))),x=a..b),u):
Eg1:=((exp(-2* Pi* I* u* (a+(1+k)* ((b-a)/m))))-(exp(-2* Pi* I* u* (a+(k)* ((b-a)/m))))):
Eg3:=((-2* Pi* I* u* (a+(1+k)* ((b-a)/m)))-(-2*Pi* I* u* (a+(k)* ((b-a)/m)))):
Eg5:=int(g(x),x=a+k* ((b-a)/m)..a+(1+k)* ((b-a)/m)):
Lg:=unapply((((Eg1)/(Eg3))* (Eg5),k):
Sg:=unapply(sum(Lg(k),k=0..m-1),u):
plot(abs(F(u)-Sg(u)),u=a..b,labels=["",""]):
end proc:
> gen(1,-1,1):
```

Maple program to produce Figure 6.2

Maple program applying equation (6.27) to plot the bound on error $r_n(x) := |R_n(g,x)|$ for the function $g(x) = \exp(-x^2),\ x \in [-1,1]$ as in Figure 6.2.

```
> restart:
> gen:=proc(m,a,b)
local g,F,Eg1,Eg3,Eg5,Lg,Sg,
g:=unapply((exp(xˆ 2)),x);
F:=Int(g(x)* exp(-2* Pi* I* ((u*x))),x=a..b):
F:=unapply(int(g(x)* exp(-2* Pi* I* ((u*x))),x=a..b),u):
Eg1:=((exp(-2* Pi* I* u* (a+(1+k)* ((b-a)/m))))-(exp(-2* Pi* I* u* (a+(k)* ((b-a)/m))))):
Eg3:=((-2* Pi* I* u* (a+(1+k)* ((b-a)/m)))-(-2*Pi* I* u* (a+(k)* ((b-a)/m)))):
Eg5:=int(g(x),x=a+k* ((b-a)/m)..a+(1+k)* ((b-a)/m)):
Lg:=unapply(((Eg1)/(Eg3))* (Eg5),k):
Sg:=unapply(sum(Lg(k),k=0..m-1),u):
plot(abs(F(u)-Sg(u)),u=a..b,labels=["",""]):
end proc:
> gen(1,-1,1):
```

Maple program to produce Figure 6.3

Maple program applying equation (6.27) to plot the bound on error $r_n(x) := |R_n(g, x)|$ for the function $g(x) = \ln(x),\ x \in (0, 1]$ as in Figure 6.3.

```
> restart:
> gen:=proc(m,a,b)
local g,F,Eg1,Eg3,Eg5,Lg,Sg,
g:=unapply((ln(x)),x);
F:=Int(g(x)* exp(-2* Pi* I* ((u*x))),x=a..b):
F:=unapply(int(g(x)* exp(-2* Pi* I* ((u*x))),x=a..b),u):
Eg1:=((exp(-2* Pi* I* u* (a+(1+k)* ((b-a)/m))))-(exp(-2* Pi* I* u* (a+(k)* ((b-a)/m))))):
Eg3:=((-2* Pi* I* u* (a+(1+k)* ((b-a)/m)))-(-2*Pi* I* u* (a+(k)* ((b-a)/m)))):
Eg5:=int(g(x),x=a+k* ((b-a)/m)..a+(1+k)* ((b-a)/m)):
Lg:=unapply(((Eg1)/(Eg3))* (Eg5),k):
Sg:=unapply(sum(Lg(k),k=0..m-1),u):
plot(abs(F(u)-Sg(u)),u=a..b,labels=["",""]):
end proc:
> gen(1,.001,1):
```

Maple program to produce Figure 6.4

Maple program applying equation (6.27) to plot the bound on error $r_n(x) := |R_n(g,x)|$ for the function $g(x) = \cosh(x),\ x \in [-\pi,\pi]$ as in Figure 6.4.

```
> restart:
> gen:=proc(m,a,b)
local g,F,Eg1,Eg3,Eg5,Lg,Sg,
g:=unapply((cosh (x)),x);
F:=Int(g(x)* exp(-2* Pi* I* ((u*x))),x=a..b):
F:=unapply(int(g(x)* exp(-2* Pi* I* ((u*x))),x=a..b),u):
Eg1:=((exp(-2* Pi* I* u* (a+(1+k)* ((b-a)/m))))-(exp(-2* Pi* I* u* (a+(k)* ((b-a)/m))))):
Eg3:=((-2* Pi* I* u* (a+(1+k)* ((b-a)/m)))-(-2*Pi* I* u* (a+(k)* ((b-a)/m)))):
Eg5:=int(g(x),x=a+k* ((b-a)/m)..a+(1+k)* ((b-a)/m)):
Lg:=unapply(((Eg1)/(Eg3))* (Eg5),k):
Sg:=unapply(sum(Lg(k),k=0..m-1),u):
plot(abs(F(u)-Sg(u)),u=a..b,labels=["",""]):
end proc:
> gen(1,-Pi,Pi):
```

Maple program to produce Figure 6.5

Maple program applying equation (6.27) to plot the bound on error $r_n\left(x\right) := \left|R_n\left(g,x\right)\right|$ for the function $g(x) = \sinh(x),\ x \in [-\pi, \pi]$ as in Figure 6.5.

```
> restart:
> gen:=proc(m,a,b)
local g,F,Eg1,Eg3,Eg5,Lg,Sg,
g:=unapply((sinh (x)),x);
F:=Int(g(x)* exp(-2* Pi* I* ((u*x))),x=a..b):
F:=unapply(int(g(x)* exp(-2* Pi* I* ((u*x))),x=a..b),u):
Eg1:=((exp(-2* Pi* I* u* (a+(1+k)* ((b-a)/m))))-(exp(-2* Pi* I* u* (a+(k)* ((b-a)/m))))):
Eg3:=((-2* Pi* I* u* (a+(1+k)* ((b-a)/m)))-(-2*Pi* I* u* (a+(k)* ((b-a)/m)))):
Eg5:=int(g(x),x=a+k* ((b-a)/m)..a+(1+k)* ((b-a)/m)):
Lg:=unapply(((Eg1)/(Eg3))* (Eg5),k):
Sg:=unapply(sum(Lg(k),k=0..m-1),u):
plot(abs(F(u)-Sg(u)),u=a..b,labels=["",""]):
end proc:
> gen(1,-Pi,Pi):
```

Maple program to produce Figure 6.6

Maple program applying equation (6.27) to plot the bound on error $r_n(x) := |R_n(g,x)|$ for the function $g(x) = \sin(2x)\,e^x,\ x \in [-\pi,\pi]$ as in Figure 6.6.

```
> restart:
> gen:=proc(m,a,b)
local g,F,Eg1,Eg3,Eg5,Lg,Sg,
g:=unapply((sin(2x)* exp(x)),x);
F:=Int(g(x)* exp(-2* Pi* I* ((u*x))),x=a..b):
F:=unapply(int(g(x)* exp(-2* Pi* I* ((u*x))),x=a..b),u):
Eg1:=((exp(-2* Pi* I* u* (a+(1+k)* ((b-a)/m))))-(exp(-2* Pi* I* u* (a+(k)* ((b-a)/m))))):
Eg3:=((-2* Pi* I* u* (a+(1+k)* ((b-a)/m)))-(-2*Pi* I* u* (a+(k)* ((b-a)/m)))):
Eg5:=int(g(x),x=a+k* ((b-a)/m)..a+(1+k)* ((b-a)/m)):
Lg:=unapply(((Eg1)/(Eg3))* (Eg5),k):
Sg:=unapply(sum(Lg(k),k=0..m-1),u):
plot(abs(F(u)-Sg(u)),u=a..b,labels=["",""]):
end proc:
> gen(1,-Pi,Pi):
```

A.2.2 Graphing the numerical results of Chapter 7

Maple program to produce Figure 7.1

Maple program for the finite Fourier transform of $f(x,y) = e^{3x-2y}(x-y),\ 0 \leq x, y \leq 1$ evaluated using the rule (7.14) as in Figure 7.1.

```
> restart:
> read ‘fourier.mpl‘:
> with( codegen , fortran ):
> fortran(FinFTp , optimized , precision=double ):
I1 := E1(u,x[k],x[k+1]) * int( F12( f,s,v,y[l],y[l+1] ) , s=x[k]..x[k+1] ):
I1 := combine( evalc( simplify( I1 , power ) ) , trig ):
I1r := coeff( I1 , I , 0 ): I1i := coeff( I1 , I , 1 ):
I2 := E1(v,y[l],y[l+1]) * int( F11( f,t,u,x[k],x[k+1] ) , t=y[l]..y[l+1] ):
I2 := combine( evalc( simplify( I2 , power ) ) , trig ):
I2r := coeff( I2 , I , 0 ): I2i := coeff( I2 , I , 1 ):
I3 := E1(u,x[k],x[k+1])* E1(v,y[l],y[l+1])* int( int( f(m,n) , m=x[k]..x[k+1] ) , n=y[l]..y[l+1] ):
I3 := combine( evalc( simplify( I3 , power ) ) , trig ):
I3r := coeff( I3 , I , 0 ): I3i := coeff( I3 , I , 1 ):
Sumr := Sumr + I1r+I2r-I3r: Sumi := Sumi + I1i+I2i-I3i;
I3 := E1(u,x[k],x[k+1])* E1(v,y[l],y[l+1])* int( int( f(m,n) , m=x[k]..x[k+1] ) , n=y[l]..y[l+1] ):
I3 := combine( evalc( simplify( I3 , power ) ) , trig ):
I3r := coeff( I3 , I , 0 ): I3i := coeff( I3 , I , 1 ):
I3i:
fortran( I2r , optimized , precision=double):
fortran( I2i , optimized , precision=double):
fortran( I3r , optimized , precision=double):
fortran( I3i , optimized , precision=double):
```

Maple program to produce Figure 7.2

Maple program using the equation (7.35) to approximate the finite Fourier transform of $f(x, y) = e^{x+y}, \qquad 0 \leq x, y \leq 1.$ as in Figure 7.2.

```
restart: > Digits:=15:
> with( plots ): with( plottools ):with(student):
> genp:=proc(m,n,a,b)
local g,F,Eg1,Eg2,Eg3,Eg4,Eg5,Lg,Sg:
g:=unapply(exp(x+y),x,y):
F:=Int(Int(g(x,y)* exp(-2* Pi* I* ((u*x)+(v*y))),y=a..b), x=a..b):
F:=unapply(int(int(g(x,y)* exp(-2* Pi* I* ((u*x)+(v*y))),y=a..b), x=a..b),u,v):
Eg1:=((exp(-2* Pi* I* u* (a+(1+k)* ((b-a)/m))))-(exp(-2* Pi* I* u* (a+(k)* ((b-a)/m))))):
Eg2:=((exp(-2* Pi* I* v* (a+(1+l)* ((b-a)/n))))-(exp(-2* Pi* I* v* (a+(l)* ((b-a)/n))))):
Eg3:=((-2* Pi* I* u* (a+(1+k)* ((b-a)/m)))-(-2* Pi* I* u* (a+(k)* ((b-a)/m)))):
Eg4:=((-2* Pi* I* v* (a+(1+l)* ((b-a)/n)))-(-2* Pi* I* v* (a+(l)* ((b-a)/n)))):
Eg5:=(int(int(g(x,y),y=a+(l)* ((b-a)/n)..a+(l+1)* ((b-a)/n)),x=a+k* ((b-a)/m)..a+(1+k)* ((b-a)/m))):
Lg:=unapply(((Eg1* Eg2)/(Eg3* Eg4))* (Eg5),k,l):
Sg:=unapply(sum(sum(Lg(k,l), l=0..n-1),k=0..m-1),u,v):
plot3d(abs(F(u,v)-Sg(u,v)),u=a..b,v=a..b,axes=framed,style=patchnogrid):
end:
> genp(1,1,0,1);
```

Maple program to produce Figure 7.3

> Maple program using the equation (7.35) to approximate the finite Fourier transform of $f(x,y) = e^{-x^2-y^2}, \quad -0.1 \leq x,y \leq 0.1$ as in Figure 7.3.

```
restart: > Digits:=15:
> with( plots ): with( plottools ):with(student):
> genp:=proc(m,n,a,b)
local g,F,Eg1,Eg2,Eg3,Eg4,Eg5,Lg,Sg:
g:=unapply(exp(-xˆ 2-yˆ 2),x,y):
F:=Int(Int(g(x,y)* exp(-2* Pi* I* ((u*x)+(v*y))),y=a..b), x=a..b):
F:=unapply(int(int(g(x,y)* exp(-2* Pi* I* ((u*x)+(v*y))),y=a..b), x=a..b),u,v):
Eg1:=((exp(-2* Pi* I* u* (a+(1+k)* ((b-a)/m))))-(exp(-2* Pi* I* u* (a+(k)* ((b-a)/m))))):
Eg2:=((exp(-2* Pi* I* v* (a+(1+l)* ((b-a)/n))))-(exp(-2* Pi* I* v* (a+(l)* ((b-a)/n))))):
Eg3:=((-2* Pi* I* u* (a+(1+k)* ((b-a)/m)))-(-2* Pi* I* u* (a+(k)* ((b-a)/m)))):
Eg4:=((-2* Pi* I* v* (a+(1+l)* ((b-a)/n)))-(-2* Pi* I* v* (a+(l)* ((b-a)/n)))):
Eg5:=(int(int(g(x,y),y=a+(l)* ((b-a)/n)..a+(l+1)* ((b-a)/n)),x=a+k* ((b-a)/m)..a+(1+k)* ((b-a)/m))):
Lg:=unapply(((Eg1* Eg2)/(Eg3* Eg4))* (Eg5),k,l):
Sg:=unapply(sum(sum(Lg(k,l), l=0..n-1),k=0..m-1),u,v):
plot3d(abs(F(u,v)-Sg(u,v)),u=a..b,v=a..b,axes=framed,style=patchnogrid):
end:
> genp(1,1,-.1,.1);
```

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