Asynchronous H∞ filtering for switched stochastic systems with time-varying delay

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Jie Lian † Chunwei Mu † and Peng Shi ‡

Abstract– This paper considers the H_∞ filtering problem of discrete-time switched delay systems. Attention is focused on the design of an exponentially mean-square stable filter taking the asynchronous switching and missing measurements into account. New results on exponential mean-square stability and a weighted l_2-gain analysis for filtering error system are given, where the system is allowed to be unstable during the unmatched interval, in which the switching signal of filter is different from that of the system. By using the average dwell time (ADT) method and the Lyapunov-Krasovskii function method, delay-dependent sufficient conditions for the desired H_∞ filter are derived in terms of linear matrix inequalities (LMIs). A numerical example is provided to demonstrate the effectiveness of the proposed design approach.

Keywords: Asynchronous switching, H_∞ filtering, Discrete-time switched systems, Exponential mean-square stability, time-varying delay.

1 Introduction

Switched system is one of the most important classes of hybrid systems in engineering applications. It consists of a family of subsystems operated by a particular type of switching rule. According to this switching rule, one of these subsystems will be activated along the system trajectory at each instant of time [1]. Due to the theoretical development as well as practical applications, analysis and synthesis of switched systems have recently gained considerable attention [2-4]. Since time delay frequently appears in the real systems and is a source of the poor performance and even instability, switched delay system has been extensively investigated [5-7].

On the other hand, it is very difficult to know precisely the statistics of the additive noise actuating

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in the systems, the noise sources are always arbitrary deterministic signals with bounded energy, or bounded average power. Thus, this paper resorts H_∞ filter, which is concerned with the design of estimators ensuring that the stability and the l_2-gain of the filtering error system. In addition, H_∞ filtering is insensitive to uncertainty in the exogenous signal statistics as well as that in dynamic models. H_∞ filtering problem can be described as follow: given a dynamic system with exogenous inputs and measured outputs, design a filter to estimate an unmeasured output such that the mapping from the exogenous input to the estimation error is minimized or no larger than some prescribed level in terms of the norm [8]. Recently, some attempts on the H_∞ filtering problem have been investigated for switched systems [9-13].

While, when considering the filtering problem of switched system, a very common assumption is that the filter is switched synchronously with the switching of system modes, which is quite unpractical. In reality, it takes time to identify the system modes and active the matched filter. So the phenomena of asynchronous switching between system modes and filter candidates generally exist. The necessities of considering asynchronous switching for efficient control design have been shown in mechanical or chemical systems [14]. Recently, the asynchronous switching problem has been investigated and various methodologies have been developed [15-20]. The stabilization of asynchronous linear system has been included in [15]. Stability, l_2-gain and asynchronous H_∞ control of discrete-time switched systems are considered in [17]. Then, the results are expanded to filtering in [20], which discusses the stability and l_2-gain of switched systems.

In almost all the works mentioned above, the hypothesis of consecutive measurements have been made implicitly. Unfortunately, in many practical applications, such a hypothesis does not hold. For example, due to sensor temporal failures or network transmission delay/loss, at certain time points, the system measurements may contain noise only, indicating that the real signal is missing. Switched system with missing measurements has received much attention during the past few years. Using binary switching sequence, the missing measurement phenomena can be modeled. The binary is specified by a conditional probability distribution taking its values of 0 and 1. Much work has been done on such model [21-28]. However, if the asynchronously switching and missing measurements happen simultaneously in the systems, it is hard to deal with the stability. All of these motivate us to shorten such a gap in the present investigation.

This paper investigates the asynchronous H_∞ filtering problem for a class of discrete-time switched delay systems with missing measurements. Based on the average dwell time approach, delay-dependent sufficient conditions on exponential mean-square stability and a weighted l_2-gain are developed for the filtering error system. It is noted that system is allowed to be unstable within a bounded unmatched interval. Then, the corresponding condition for the existence of desired filter is established in terms of LMIs. Finally, a numerical example is given to demonstrate the effectiveness of the proposed design approach.

The remainder of this paper is organized as follows. The asynchronous H_∞ filtering of switched systems is formulated in Section 2. Section 3 presents our main results. A numerical example is given in Section 4, and then we conclude this paper in Section 5.

Notation: The notations used throughout the paper are standard. The superscript 'T' stands for matrix transposition; R^n denotes the n-dimensional Euclidean space; $\mathbb{N}$ represents the set of nonnegative integers; the notation $P > 0$ means that P is real symmetric and positive definite; $l_2\left[0, \infty\right)$ is the space of square-integrable vector functions over $[0, \infty)$; diag$\{\cdots\}$ stands for a block-diagonal matrix; $\lambda_{min}(P)(\lambda_{max}(P)\ )$ denotes the minimum (maximum) eigenvalue of symmetric matrix P; $\|\cdot\|$ denotes the Euclidean norm of a vector and its induced norm of a matrix. In symmetric matrices or long matrix expressions, we use a star $(*)$ to represent a term that is induced by symmetry.

2 Problem description and preliminaries

Consider a class of discrete-time switched delay systems given by

$$x_{k+1} = A_\sigma x_k + A_{d\sigma} x_{k-d_k} + B_\sigma \omega_k,$$

$$z_k = C_\sigma x_k + C_{d\sigma} x_{k-d_k} + D_\sigma \omega_k\ ,$$

$$\tilde{y}_k = C_{2\sigma} x_k + C_{2d\sigma} x_{k-d_k} + D_{2\sigma} \omega_k, \tag{1}$$

where $x_k \in R^n$ is the state vector, $\omega_k \in R^p$ is the disturbance input which belongs to $l_2\left[0, \infty\right)$, $z_k \in R^q$ is the signal to be estimated. σ is a piecewise constant function of time k called a switching signal, which takes its values in the finite set $\mathcal{I} = \{1, \cdots, N\}$, and $N > 1$ is the number of subsystems. The positive integer d_k denotes the time-varying delay satisfying

$$d_m < d_k < d_M. \tag{2}$$

where d_m and d_M denotes the lower bound and upper bound of the time-varying delay, respectively.

In system (1), $\tilde{y}_k$ is the ideal system output that always contains the real signal. However, in practical engineering systems, the system output usually contains probabilistic missing data. Then, the actual system output can be described by

$$y_k = \gamma_k (C_{2i} x_k + C_{2di} x_{k-d_k}) + D_{2i} \omega_k, i \in \mathcal{I}, \tag{3}$$

where the stochastic variable γ_k is a Bernoulli distributed white sequence specified by the following probabilities:

$$Prob\{\gamma_k = 1\} = E\{\gamma_k\} = p, \tag{4}$$

$$Prob\{\gamma_k = 0\} = 1 - E\{\gamma_k\} = 1 - p, \tag{5}$$

with a known constant $p > 0$. Obviously, for a stochastic variable γ_k , we have the mean value $E\{\gamma_k\} = p$ and variance $q^2 = p(1-p)$.

Next, we are interested in designing a full-order filter described by

$$\begin{aligned} \hat{x}_{k+1} &= A_{ci}\hat{x}_k + B_{ci}y_k, \\ \hat{z}_k &= C_{ci}\hat{x}_k + D_{ci}y_k, \end{aligned} \tag{6}$$

where $\hat{x}_k \in R^n$ is the state estimate; $\hat{z}_k \in R^q$ is an estimate for z_k; A_{ci}, B_{ci}, C_{ci} and D_{ci} are matrices to be determined.

It is assumed that the subsystem is activated at the switching instant $k_l, \forall l \in \mathbb{N}$. Owing to the real switching time of filters exceeds or lags behind that of the subsystems, so the switching instant of the filter is $k_l + \mathcal{T}_{(k_{l+1},k_l)}, \forall l \in \mathbb{N}$, where $\mathcal{T}_{(k_{l+1},k_l)}$ represents the intervals during which the switching signals of filter are different from that of the subsystem. Also, we use $\mathcal{T}_{(k_{l+1}-k_l)}$ to denote the length of $\mathcal{T}_{(k_{l+1},k_l)}$.

Therefore, from (1) and (6), we can get the resulting filtering error system as follows:

$$\begin{cases} \begin{aligned} \tilde{x}_{k+1} &= \bar{A}_i\tilde{x}_k + \bar{A}_{di}H\tilde{x}_{k-d_k} + \bar{B}_i\omega_k, \\ \tilde{z}_k &= \bar{C}_i\tilde{x}_k + \bar{C}_{di}H\tilde{x}_{k-d_k} + \bar{D}_i\omega_k, \end{aligned} & \forall k \in (k_l, k_l + \mathcal{T}_{(k_{l+1},k_l)}) \\ \begin{aligned} \tilde{x}_{k+1} &= \tilde{A}_i\tilde{x}_k + \tilde{A}_{di}H\tilde{x}_{k-d_k} + \tilde{B}_i\omega_k, \\ \tilde{z}_k &= \tilde{C}_i\tilde{x}_k + \tilde{C}_{di}H\tilde{x}_{k-d_k} + \tilde{D}_i\omega_k, \end{aligned} & \forall k \in (k_l + \mathcal{T}_{(k_{l+1},k_l)}, k_{l+1}) \end{cases} \tag{7}$$

where,

$$\tilde{x}_k = \begin{bmatrix} x_k^T & \hat{x}_k^T \end{bmatrix}^T, \ \tilde{z}_k = z_k - \hat{z}_k, \ H = \begin{bmatrix} I & 0 \end{bmatrix},$$

$$\bar{A}_i = \begin{bmatrix} A_i & 0 \\ \gamma_k B_{cj} C_{2i} & A_{cj} \end{bmatrix}, \tilde{A}_i = \begin{bmatrix} A_i & 0 \\ \gamma_k B_{ci} C_{2i} & A_{ci} \end{bmatrix}, \bar{A}_{di} = \begin{bmatrix} A_{di} \\ \gamma_k B_{cj} C_{2di} \end{bmatrix}, \tilde{A}_{di} = \begin{bmatrix} A_{di} \\ \gamma_k B_{ci} C_{2di} \end{bmatrix},$$

$$\bar{B}_i = \begin{bmatrix} B_i \\ B_{cj} D_{2i} \end{bmatrix}, \tilde{B}_i = \begin{bmatrix} B_i \\ B_{ci} D_{2i} \end{bmatrix} \bar{C}_i = \begin{bmatrix} C_i - \gamma_k D_{cj} C_{2i} & -C_{cj} \end{bmatrix}, \bar{C}_{di} = C_{di} - \gamma_k D_{cj} C_{2di},$$

$$\tilde{C}_i = \begin{bmatrix} C_i - \gamma_k D_{ci} C_{2i} & -C_{ci} \end{bmatrix}, \tilde{C}_{di} = C_{di} - \gamma_k D_{ci} C_{2di}, \bar{D}_i = D_i - D_{cj} D_{2i}, \tilde{D}_i = D_i - D_{ci} D_{2i},$$

For convenience, we denote

$$\bar{A}_{1i}=\begin{bmatrix} A_i & 0 \\ pB_{cj}C_{2i} & A_{cj} \end{bmatrix}, \bar{A}_{2i}=\begin{bmatrix} 0 & 0 \\ B_{cj}C_{2i} & 0 \end{bmatrix}, \bar{A}_{1di}=\begin{bmatrix} A_{di} \\ pB_{cj}C_{2di} \end{bmatrix}, \bar{A}_{2di}=\begin{bmatrix} 0 \\ B_{cj}C_{2di} \end{bmatrix},$$

$$\bar{C}_{1i}=\begin{bmatrix} C_i-pD_{cj}C_{2i} & -C_{cj} \end{bmatrix}, \bar{C}_{2i}=\begin{bmatrix} D_{cj}C_{2i} & 0 \end{bmatrix}, \bar{C}_{1di}=C_{di}-pD_{cj}C_{2di}, \bar{C}_{2di}=D_{cj}C_{2di},$$

$$\tilde{A}_{1i}=\begin{bmatrix} A_i & 0 \\ pB_{ci}C_{2i} & A_{ci} \end{bmatrix}, \tilde{A}_{2i}=\begin{bmatrix} 0 & 0 \\ B_{ci}C_{2i} & 0 \end{bmatrix}, \tilde{A}_{1di}=\begin{bmatrix} A_{di} \\ pB_{ci}C_{2di} \end{bmatrix}, \tilde{A}_{2di}=\begin{bmatrix} 0 \\ B_{ci}C_{2di} \end{bmatrix},$$

$$\tilde{C}_{1i}=\begin{bmatrix} C_i-pD_{ci}C_{2i} & -C_{ci} \end{bmatrix}, \tilde{C}_{2i}=\begin{bmatrix} D_{ci}C_{2i} & 0 \end{bmatrix}, \tilde{C}_{1di}=C_{di}-pD_{ci}C_{2di}, \tilde{C}_{2di}=D_{ci}C_{2di}.$$

We give the following definitions, which will play important roles in deriving our main results subsequently.

Definition 1 [2]: For any $T_1 > T_2 > 0$, let $N(T_1,T_2)$ be the switching number of σ over $[T_1,T_2)$. If $N(T_1,T_2) \leq N_0 + (T_1-T_2)/\tau_a$ hold for $N_0 \geq 0$ and $\tau_\alpha > 0$, then N_0 and τ_α are called chatter bound and average dwell time, respectively. As commonly used in the literature, we choose $N_0 = 0$.

Definition 2: Consider the filtering error system (7), suppose that there exist constants $c > 0$, $d \in (0,1)$ and $f > 1$ such that $E\{\| x_k\|^2\} \leq cd^k E\{\| x_{k_0}\|^2\}$ and $\sum\limits_{k=k_0}^{\infty} f^{-k}E\left\{\tilde{z}_k^T\tilde{z}_k\right\} < \gamma^2 \sum\limits_{k=k_0}^{\infty} \omega_k^T\omega_k$ hold, then the filtering error system is said to be exponentially mean-square stable with $\omega_k = 0$ under switching signal σ and has a weighted l_2-gain no greater than γ.

3 Main results

3.1 Stability and H_∞ performance analysis

In this section, delay-dependent sufficient conditions on exponential mean-square stability with a weighted l_2-gain are derived for the filtering error system (7) via the average dwell time approach.

Theorem 1: Given scalars $0 < \alpha < 1$, $\beta > 0$ and $\gamma > 0$, the filtering error system (7) is exponentially mean-square stable with a weighted l_2-gain $\gamma_s = \gamma\sqrt{p_1\theta^{\mathcal{T}_{\max}}(1-\tilde{\alpha}p_1^{1/\tau_a})\Big/(1-\tilde{\alpha}\theta^{\mathcal{T}_{\max}/\tau_a}p_1^{1/\tau_a})}$ under ADT switching signals σ, if there exist symmetric and positive definite matrices P_i, Q_i, R_{bi}, Z_{ci}, and matrices Ω_i, M_{ci}, N_{ci}, $b=1,2$, $c=1,2,3$, such that the following inequalities hold

$$\begin{bmatrix} \psi_1-\tilde{\alpha}P_i & H^TM_{2i}^T+H^TN_{2i}^T & H^TM_{3i}^T+H^TN_{3i}^T & \tilde{\Xi}_{11} & \tilde{\Xi}_1 \\ * & -\tilde{\alpha}^{d_M}Q_i & 0 & \tilde{\Xi}_{12} & \tilde{\Xi}_2 \\ * & * & -\gamma^2 I & \tilde{\Xi}_{13} & \tilde{\Xi}_3 \\ * & * & * & \tilde{\Xi}_4 & 0 \\ * & * & * & * & \tilde{\Xi}_5 \end{bmatrix} < 0, \tag{8}$$

$$
\begin{bmatrix}
\psi_1-\bar{\beta}P_i & H^TM_{2i}^T+H^TN_{2i}^T & H^TM_{3i}^T+H^TN_{3i}^T & \bar{\Xi}_{11} & \bar{\Xi}_1 \\
* & -\bar{\beta}^{d_m}Q_i & 0 & \bar{\Xi}_{12} & \bar{\Xi}_2 \\
* & * & -\gamma^2 I & \bar{\Xi}_{13} & \bar{\Xi}_3 \\
* & * & * & \bar{\Xi}_4 & 0 \\
* & * & * & * & \bar{\Xi}_5
\end{bmatrix} < 0, \tag{9}
$$

where,

$$
\tilde{\Xi}_{1c} = \begin{bmatrix} -N_{ci} & -M_{ci} & \tau_{11}M_{ci} & \tau_{12}N_{ci} \end{bmatrix}, \tilde{\Xi}_1 = \begin{bmatrix} \tilde{C}_{1i}^T & q\tilde{C}_{2i}^T & \tilde{A}_{1i}^T\Omega_i^T & q\tilde{A}_{2i}^T\Omega_i^T & H^T(A_i^T-I)Z_{3i} \end{bmatrix},
$$

$$
\tilde{\Xi}_2 = \begin{bmatrix} \tilde{C}_{1di}^T & q\tilde{C}_{2di}^T & \tilde{A}_{1di}^T\Omega_i^T & q\tilde{A}_{2di}^T\Omega_i^T & A_{di}^TZ_{3i} \end{bmatrix}, \tilde{\Xi}_3 = \begin{bmatrix} \tilde{D}_i^T & 0 & \tilde{B}_i^T\Omega_i^T & 0 & B_i^TZ_{3i} \end{bmatrix},
$$

$$
\tilde{\Xi}_4 = diag\left\{ -\tilde{\alpha}^{d_m}R_{1i} \quad -\tilde{\alpha}^{d_M}R_{2i} \quad -\tau_{11}Z_{1i} \quad -\tau_{12}Z_{2i} \right\}, \tilde{\Xi}_5 = diag\left\{ -I \quad -I \quad \tilde{\psi}_2 \quad \tilde{\psi}_2 \quad -Z_{3i} \right\},
$$

$$
\bar{\Xi}_{1c} = \begin{bmatrix} -N_{ci} & -M_{ci} & \tau_{21}M_{ci} & \tau_{22}N_{ci} \end{bmatrix}, \bar{\Xi}_1 = \begin{bmatrix} \bar{C}_{1i}^T & q\bar{C}_{2i}^T & \bar{A}_{1i}^T\Omega_j^T & q\bar{A}_{2i}^T\Omega_j^T & H^T(A_i^T-I)Z_{3i} \end{bmatrix},
$$

$$
\bar{\Xi}_2 = \begin{bmatrix} \bar{C}_{1di}^T & q\bar{C}_{2di}^T & \bar{A}_{1di}^T\Omega_j^T & q\bar{A}_{2di}^T\Omega_j^T & A_{di}^TZ_{3i} \end{bmatrix}, \bar{\Xi}_3 = \begin{bmatrix} \bar{D}_i^T & 0 & \bar{B}_i^T\Omega_j^T & 0 & B_i^TZ_{3i} \end{bmatrix},
$$

$$
\bar{\Xi}_4 = diag\left\{ -\bar{\beta}^{d_m}R_{1i} \quad -\bar{\beta}^{d_M}R_{2i} \quad -\tau_{21}Z_{1i} \quad -\tau_{22}Z_{2i} \right\}, \bar{\Xi}_5 = diag\left\{ -I \quad -I \quad \bar{\psi}_2 \quad \bar{\psi}_2 \quad -Z_{3i} \right\},
$$

$$
\psi_1 = H^T(\tau Q_i + R_{1i} + R_{2i})H + M_{1i}H + H^TM_{1i}^T + N_{1i}H + H^TN_{1i}^T,\ \tilde{\psi}_2 = P_i - \Omega_i^T - \Omega_i,
$$

$$
\bar{\psi}_2 = P_i - \Omega_j^T - \Omega_j,\ Z_{3i} = d_MZ_{1i} + d_mZ_{2i},\ \tau_{11} = (\tilde{\alpha}^{-d_M}-1)\Big/\alpha,\ \tau_{12} = (\tilde{\alpha}^{-d_m}-1)\Big/\alpha,
$$

$$
\tau_{21} = (1-\bar{\beta}^{-d_M})\Big/\beta,\ \tau_{22} = (1-\bar{\beta}^{-d_m})\Big/\beta,\ \tau = d_M - d_m + 1,\ \bar{\beta} = 1+\beta,\ \tilde{\alpha} = 1-\alpha,
$$

and the average dwell time τ_α

$$
\tau_a > \tau_a^* = -(\mathcal{T}_{\max}\ln\theta + \ln p_1p_2)/\ln\tilde{\alpha} \tag{10}
$$

where, $\theta = \bar{\beta}/\tilde{\alpha},\ p_1 = \max\limits_{i\in\mathcal{I}}\{\kappa_{2i}/\kappa_{4i}\},\ \kappa_{4i} = \lambda_{\min}(P_i) + d_m\tau\lambda_{\min}(Q_i) + d_m\lambda_{\max}(R_{1i}) + d_M\lambda_{\max}(R_{2i}),$

$$
p_2 = \max_{i,j\in\mathcal{I},i\neq j}\{\kappa_{1i}/\kappa_{3j}\},\ \kappa_{3i} = \lambda_{\min}(P_i) + d_m\tau\lambda_{\min}(Q_i) + d_m\tilde{\alpha}^{d_m}\lambda_{\max}(R_{1i}) + d_M\tilde{\alpha}^{d_M}\lambda_{\max}(R_{2i}),
$$

$$
\kappa_{1i} = \lambda_{\max}(P_i) + d_M\tau\bar{\beta}^{d_M}\lambda_{\max}(Q_i) + d_M\bar{\beta}^{d_M}\lambda_{\max}(R_{2i}) + d_M^2\bar{\beta}^{d_M}\lambda_{\max}(Z_{1i}) + d_m^2\bar{\beta}^{d_m}\lambda_{\max}(Z_{2i}),
$$

$$
\kappa_{2i} = \lambda_{\max}(P_i) + d_M\tau\lambda_{\max}(Q_i) + d_M\lambda_{\max}(R_{2i}) + d_M^2\lambda_{\max}(Z_{1i}) + d_m^2\lambda_{\max}(Z_{2i}), \mathcal{T}_{\max} \triangleq \max_{\forall l\in\mathbb{N}}\mathcal{T}_{(k_{l+1}-k_l)}.
$$

Proof: Denote $\eta_l = x_{l+1} - x_l$ and choosing the following Lyapunov-Krasovskii function as

$$
\begin{cases}
\bar{V}_i(k) = \bar{V}_{1i}(k) + \bar{V}_{2i}(k) + \bar{V}_{3i}(k) + \bar{V}_{4i}(k), & \forall k\in[k_l, k_l + \mathcal{T}_{(k_{l+1},k_l)}) \\
\tilde{V}_i(k) = \tilde{V}_{1i}(k) + \tilde{V}_{2i}(k) + \tilde{V}_{3i}(k) + \tilde{V}_{4i}(k), & \forall k\in[k_l + \mathcal{T}_{(k_{l+1},k_l)}, k_{l+1})
\end{cases} \tag{11}
$$

where, $\bar{V}_{1i}(k) = \tilde{V}_{1i}(k) = \tilde{x}_k^TP_i\tilde{x}_k,$

$$
\bar{V}_{2i}(k) = \sum_{l=k-d_k}^{k-1}\bar{\beta}^{k-l-1}x_l^TQ_ix_l + \sum_{j=k-d_M+1}^{k-d_m}\sum_{l=j}^{k-1}\bar{\beta}^{k-l-1}x_l^TQ_ix_l,
$$

$$\bar{V}_{3i}(k)=\sum_{l=k-d_m}^{k-1}\bar{\beta}^{k-l-1}x_l^TR_{1i}x_l+\sum_{l=k-d_M}^{k-1}\bar{\beta}^{k-l-1}x_l^TR_{2i}x_l,$$

$$\bar{V}_{4i}(k)=\sum_{j=k-d_M}^{k-1}\sum_{l=j}^{k-1}\bar{\beta}^{k-l-1}\eta_l^TZ_{1i}\eta_l+\sum_{j=k-d_m}^{k-1}\sum_{l=j}^{k-1}\bar{\beta}^{k-l-1}\eta_l^TZ_{2i}\eta_l$$

$$\tilde{V}_{2i}(k)=\sum_{l=k-d_k}^{k-1}\tilde{\alpha}^{k-l-1}x_l^TQ_ix_l+\sum_{j=k-d_M+1}^{k-d_m}\sum_{l=j}^{k-1}\tilde{\alpha}^{k-l-1}x_l^TQ_ix_l,$$

$$\tilde{V}_{3i}(k)=\sum_{l=k-d_m}^{k-1}\tilde{\alpha}^{k-l-1}x_l^TR_{1i}x_l+\sum_{l=k-d_M}^{k-1}\tilde{\alpha}^{k-l-1}x_l^TR_{2i}x_l,$$

$$\tilde{V}_{4i}(k)=\sum_{j=k-d_M}^{k-1}\sum_{l=j}^{k-1}\tilde{\alpha}^{k-l-1}\eta_l^TZ_{1i}\eta_l+\sum_{j=k-d_m}^{k-1}\sum_{l=j}^{k-1}\tilde{\alpha}^{k-l-1}\eta_l^TZ_{2i}\eta_l,$$

When $k\in(k_l+\mathcal{T}_{(k_{l+1},k_l)},k_{l+1})$, denote $\Delta\tilde{V}_i(k)=\tilde{V}_i(k+1)-\tilde{V}_i(k)$, we can get

$$E\left\{\Delta\tilde{V}_{1i}(k)+\alpha\tilde{V}_{1i}(k)\right\}=E\left\{\tilde{x}_{k+1}^TP_i\tilde{x}_{k+1}-\tilde{\alpha}\tilde{x}_k^TP_i\tilde{x}_k\right\}. \tag{12}$$

Note that

$$E\left\{\sum_{l=k+1-d_{k+1}}^{k}\tilde{\alpha}^{k-l}x_l^TQ_ix_l\right\}\le E\left\{\sum_{l=k+1-d_k}^{k}\tilde{\alpha}^{k-l}x_l^TQ_ix_l+\sum_{l=k+1-d_M}^{k-d_m}\tilde{\alpha}^{k-l}x_l^TQ_ix_l\right\},$$

Thus, we can get

$$\begin{aligned}E\left\{\Delta\tilde{V}_{2i}(k)+\alpha\tilde{V}_{2i}(k)\right\}\le E&\left\{\tau x_k^TQ_ix_k-\tilde{\alpha}^{d_k}x_{k-d_k}^TQ_ix_{k-d_k}+\sum_{l=k+1-d_M}^{k-d_m}\tilde{\alpha}^{k-l}x_l^TQ_ix_l\right.\\&\left.-\sum_{l=k+1-d_M}^{k-d_m}\tilde{\alpha}^{k-l}x_l^TQ_ix_l\right\}=E\left\{\tau x_k^TQ_ix_k-\tilde{\alpha}^{d_M}x_{k-d_k}^TQ_ix_{k-d_k}\right\}.\end{aligned} \tag{13}$$

Similarly, we can obtain

$$E\left\{\Delta\tilde{V}_{3i}(k)+\alpha\tilde{V}_{3i}(k)\right\}=E\left\{x_k^T(R_{1i}+R_{2i})x_k-\tilde{\alpha}^{d_m}x_{k-d_m}^TR_{1i}x_{k-d_m}\ -\tilde{\alpha}^{d_M}x_{k-d_M}^TR_{2i}x_{k-d_M}\right\}. \tag{14}$$

$$\begin{aligned}E\left\{\Delta\tilde{V}_{4i}(k)+\alpha\tilde{V}_{4i}(k)\right\}=E&\left\{d_M\eta_k^TZ_{1i}\eta_k+d_m\eta_k^TZ_{2i}\eta_k-\sum_{l=k-d_M}^{k-1}\tilde{\alpha}^{k-l}\eta_l^TZ_{1i}\eta_l\right.\\&\left.-\sum_{l=k-d_m}^{k-1}\tilde{\alpha}^{k-l}\eta_l^TZ_{2i}\eta_l\right\}.\end{aligned} \tag{15}$$

On the other hand, by means of the Newton-Leibniz formula, it gives rise to

$$x_k-x_{k-d_M}-\sum_{l=k-d_M}^{k-1}\eta_l=0,\ x_k-x_{k-d_m}-\sum_{l=k-d_m}^{k-1}\eta_l=0.$$

Then, we have

$$2\xi_k^T M_i\left[x_k - x_{k-d_M} - \sum_{l=k-d_M}^{k-1}\eta_l\right] = 0,\ 2\xi_k^T N_i\left[x_k - x_{k-d_m} - \sum_{l=k-d_m}^{k-1}\eta_l\right] = 0, \tag{16}$$

where $\xi_k = \begin{bmatrix} \tilde{x}_k^T & x_{k-d_k}^T & \omega_k^T & x_{k-d_m}^T & x_{k-d_M}^T \end{bmatrix}^T$, $M_i = \begin{bmatrix} M_{1i}^T & M_{2i}^T & M_{3i}^T & 0 & 0 \end{bmatrix}^T$, $N_i = \begin{bmatrix} N_{1i}^T & N_{2i}^T & N_{3i}^T & 0 & 0 \end{bmatrix}^T$.

From (12)-(16), we can get the following matrix inequality

$$\begin{aligned}
&E\left\{\Delta\tilde{V}_i(k) + \alpha\tilde{V}_i(k) + \tilde{z}_k^T\tilde{z}_k - \gamma^2\omega_k^T\omega_k\right\} \\
&= E\Bigg\{\Delta\tilde{V}_i(k) + \alpha\tilde{V}_i(k) + \tilde{z}_k^T\tilde{z}_k - \gamma^2\omega_k^T\omega_k + 2\xi_k M_i\left[x_k - x_{k-d_M} - \sum_{l=k-d_M}^{k-1}\eta_l\right] \\
&\quad + \ 2\xi_k N_i\left[x_k - x_{k-d_m} - \sum_{l=k-d_m}^{k-1}\eta_l\right]\Bigg\} \\
&\le E\Big\{\xi_k^T(\Phi_i + \tau_{11}M_i Z_{1i}^{-1}M_i^T + \tau_{12}N_i Z_{2i}^{-1}N_i^T)\xi_k + \tilde{z}_k^T\tilde{z}_k + x_{k+1}^T P_i\tilde{x}_{k+1} + \eta_k^T Z_{3i}\eta_k \\
&\quad - \sum_{l=k-d_M}^{k-1}\left[\xi_k^T M_i + \tilde{\alpha}^{k-l}\eta_l^T Z_{1i}\right]\tilde{\alpha}^{l-k}Z_{1i}^{-1}\left[M_i^T\xi_k + \tilde{\alpha}^{k-l}Z_{1i}\eta_l\right] \\
&\quad - \sum_{l=k-d_m}^{k-1}\left[\xi_k^T N_i + \tilde{\alpha}^{k-l}\eta_l^T Z_{2i}\right]\tilde{\alpha}^{l-k}Z_{2i}^{-1}\left[N_i^T\xi_k + \tilde{\alpha}^{k-l}Z_{2i}\eta_l\right]\Bigg\},
\end{aligned} \tag{17}$$

where

$$\Phi_i = \begin{bmatrix}
\psi_1 - \tilde{\alpha}P_i & H^T M_{2i}^T + H^T N_{2i}^T & H^T M_{3i}^T + H^T N_{3i}^T & -N_{1i} & -M_{1i} \\
* & -\tilde{\alpha}^{d_M}Q_i & 0 & -N_{2i} & -M_{2i} \\
* & * & -\gamma^2 I & -N_{3i} & -M_{3i} \\
* & * & * & -\tilde{\alpha}^{d_m}R_{1i} & 0 \\
* & * & * & * & -\tilde{\alpha}^{d_M}R_{2i}
\end{bmatrix}.$$

Since $Z_{1i} > 0$ and $Z_{2i} > 0$, the last two terms are all non-positive definite. By Schur complement, we have

$$E\left\{\Delta\tilde{V}_i(k) + \alpha\tilde{V}_i(k)\right\} \le 0\,. \tag{18}$$

$$E\left\{\Delta\tilde{V}_i(k) + \alpha\tilde{V}_i(k) + \tilde{z}_k^T\tilde{z}_k - \gamma^2\omega_k^T\omega_k\right\} \le 0\,. \tag{19}$$

if the following inequality holds,

$$\begin{bmatrix}
\psi_1 - \tilde{\alpha}P_i & H^T M_{2i}^T + H^T N_{2i}^T & H^T M_{3i}^T + H^T N_{3i}^T & \tilde{\Xi}_{11} & \tilde{\Theta}_1 \\
* & -\tilde{\alpha}^{d_M}Q_i & 0 & \tilde{\Xi}_{12} & \tilde{\Theta}_2 \\
* & * & -\gamma^2 I & \tilde{\Xi}_{13} & \tilde{\Theta}_3 \\
* & * & * & \tilde{\Xi}_4 & 0 \\
* & * & * & * & \tilde{\Theta}_4
\end{bmatrix} < 0, \tag{20}$$

where, $\tilde{\Theta}_1 = \begin{bmatrix} \tilde{C}_{1i}^T & q\tilde{C}_{2i}^T & \tilde{A}_{1i}^T & q\tilde{A}_{2i}^T & H^T(A_i^T - I)Z_{3i} \end{bmatrix}, \tilde{\Theta}_3 = \begin{bmatrix} \tilde{D}_i^T & 0 & \tilde{B}_i^T & 0 & B_i^T Z_{3i} \end{bmatrix},$

$\tilde{\Theta}_2 = \begin{bmatrix} \tilde{C}_{1di}^T & q\tilde{C}_{2di}^T & \tilde{A}_{1di}^T & q\tilde{A}_{2di}^T & A_{di}^T Z_{3i} \end{bmatrix}, \tilde{\Theta}_4 = diag\left\{ -I \quad -I \quad -P_i^{-1} \quad -P_i^{-1} \quad -Z_{3i} \right\}.$

From the fact $(P_i - \Omega_i)P_i^{-1}(P_i - \Omega_i)^T > 0$, we have the following inequalities: $-\Omega_i P_i^{-1}\Omega_i^T < P_i - \Omega_i - \Omega_i^T$. Then pre- and post-multiplying (20) by $diag\left\{ I \quad I \quad I \quad I \quad I \quad I \quad I \quad I \quad I \quad \Omega_i \quad \Omega_i \quad I \right\}$ and $diag\left\{ I \quad I \quad I \quad I \quad I \quad I \quad I \quad I \quad I \quad \Omega_i^T \quad \Omega_i^T \quad I \right\}$ respectively, then we can get (8). This means that if (8) holds, (20) is true.

When $k \in (k_l, k_l + \mathcal{T}_{(k_{l+1},k_l)})$, following the similar proof line, from (9), we obtain

$$E\left\{\Delta\bar{V}_i(k) - \beta\bar{V}_i(k)\right\} \leq 0\,. \tag{21}$$

$$E\left\{\Delta\bar{V}_i(k) - \beta\bar{V}_i(k) + \tilde{z}_k^T\tilde{z}_k - \gamma^2\omega_k^T\omega_k\right\} \leq 0\,. \tag{22}$$

From (11), we can get

$$\tilde{V}_{\sigma_{k_l}}(k_l + \mathcal{T}_{(k_{l+1},k_l)}) \leq p_1\bar{V}_{\sigma_{k_l}}(k_l + \mathcal{T}_{(k_{l+1},k_l)}). \tag{23}$$

$$\bar{V}_{\sigma_{k_l}}(k_l) \leq p_2\tilde{V}_{\sigma_{k_{l-1}}}(k_l). \tag{24}$$

Combining with (18), (21) and (23)-(24) we have

$$\begin{aligned} & E\left\{\tilde{V}_{\sigma_k}(x_k)\right\} \leq E\left\{\tilde{\alpha}^{(k-k_l-\mathcal{T}_{(k-k_l)})}\tilde{V}_{\sigma_{k_l}}(x_{k_l+\mathcal{T}_{(k,k_l)}})\right\} \\ & \leq E\left\{\tilde{\alpha}^{(k-k_l-\mathcal{T}_{(k-k_l)})}p_1\bar{V}_{\sigma_{k_l}}(x_{k_l+\mathcal{T}_{(k,k_l)}})\right\} \leq E\left\{\tilde{\alpha}^{(k-k_l)}\theta^{\mathcal{T}_{(k-k_l)}}p_1\bar{V}_{\sigma_{k_l}}(x_{k_l})\right\} \\ & \leq E\left\{\tilde{\alpha}^{(k-k_l)}\theta^{\mathcal{T}_{(k-k_l)}}p_1p_2\tilde{V}_{\sigma_{k_{l-1}}}(x_{k_l})\right\} \leq E\left\{\tilde{\alpha}^{(k-k_{l-1})}\theta^{\mathcal{T}_{(k-k_{l-1})}}(p_1p_2)^2\tilde{V}_{\sigma_{k_{l-2}}}(x_{k_{l-1}})\right\} \\ & \leq \cdots \leq E\left\{\tilde{\alpha}^{(k-k_0)}\theta^{(N_\sigma(k,k_0)+1)\mathcal{T}_{\max}}(p_1p_2)^{N_\sigma(k,k_0)}p_1\bar{V}_{\sigma_{k_0}}(x_{k_0})\right\} \\ & \leq E\left\{p_1\theta^{\mathcal{T}_{\max}}\left(\tilde{\alpha}\theta^{\mathcal{T}_{\max}/\tau_a}(p_1p_2)^{1/\tau_a}\right)^{k-k_0}\bar{V}_{\sigma_{k_0}}(x_{k_0})\right\}. \end{aligned} \tag{25}$$

Then, we can obtain

$$\min_{i\in\mathcal{I}}\kappa_{3i}E\left\{\|\ x_k\|^2\right\} \leq E\left\{V_i(k)\right\} \leq \max_{i\in\mathcal{I}}\kappa_{1i}p_1\theta^{\mathcal{T}_{\max}}(\tilde{\alpha}\theta^{\mathcal{T}_{\max}/\tau_a}(p_1p_2)^{1/\tau_a})^{k-k_0}E\left\{\|\ x_{k_0}\|^2\right\}. \tag{26}$$

From (10), we can obtain $\tilde{\alpha}\theta^{\mathcal{T}_{\max}/\tau_a}(p_1p_2)^{1/\tau_a} < 1$. Therefore, according to Definition 2, the filtering error system (7) is exponentially mean-square stable.

Next, we will analysis the H_∞ performance of the filtering error system (7).

We denote $\Gamma_s = \tilde{z}_s^T\tilde{z}_s - \gamma^2\omega_s^T\omega_s$, and consider (19), (22)-(24), we can get

$$\begin{aligned} & E\{\tilde{V}_{\sigma_{k_l}}(x_k)\} \leq E\left\{p_1\tilde{\alpha}^{k-k_l-\mathcal{T}_{k-k_l}}\tilde{V}_{\sigma_{k_l}}(x_{k_l+\mathcal{T}_{(k_l,k)}}) - \sum_{s=k_l+\mathcal{T}_{(k,k_l)}}^{k-1}\tilde{\alpha}^{k-s-1}\Gamma_s\right\} \\ & \leq E\left\{\tilde{\alpha}^{k-k_l}\theta^{\mathcal{T}_{(k-k_l)}}p_1p_2\tilde{V}_{\sigma_{k_{l-1}}}(x_{k_l}) - \sum_{s=k_l}^{k_l+\mathcal{T}_{(k,k_l)}-1}\tilde{\alpha}^{k-s-1}\theta^{k_l+\mathcal{T}_{(k-k_l)}-s-1}p_1\Gamma_s - \sum_{s=k_l+\mathcal{T}_{(k,k_l)}}^{k-1}\tilde{\alpha}^{k-s-1}\Gamma_s\right\}. \end{aligned}$$

Then, we can get

$$E\{\tilde{V}_{\sigma_{k_l}}(x_k)\} \le E\Big\{\tilde{\alpha}^{k-k_0}\theta^{\mathcal{T}_{(k-k_0)}}(p_1p_2)^{N(k,k_0)}p_1V_{\sigma_{k_0}}(x_{k_0})$$
$$-\sum_{s=k_0}^{k_0+\mathcal{T}_{(k_1,k_0)}-1}\tilde{\alpha}^{k-s-1}\theta^{k_0+\mathcal{T}_{(k-k_0)}-s-1}(p_1p_2)^{N(k,k_0)}p_1\Gamma_s-\sum_{s=k_0+\mathcal{T}_{(k_1,k_0)}}^{k_1-1}\tilde{\alpha}^{k-s-1}\theta^{\mathcal{T}_{(k-k_1)}}(p_1p_2)^{N(k,k_0)}\Gamma_s$$
$$-\cdots-\sum_{s=k_{l-1}}^{k_{l-1}+\mathcal{T}_{(k_l,k_{l-1})}-1}\tilde{\alpha}^{k-s-1}\theta^{k_{l-1}+\mathcal{T}_{(k-k_{l-1})}-s-1}p_1^2p_2\Gamma_s-\sum_{s=k_{l-1}+\mathcal{T}_{(k_l,k_{l-1})}}^{k_l-1}\tilde{\alpha}^{k-s-1}\theta^{\mathcal{T}_{(k-k_l)}}p_1p_2\Gamma_s$$
$$-\sum_{s=k_l}^{k_l+\mathcal{T}_{(k,k_l)}-1}\tilde{\alpha}^{k-s-1}\theta^{k_l+\mathcal{T}_{(k-k_l)}-s-1}p_1\Gamma_s-\sum_{s=k_l+\mathcal{T}_{(k,k_l)}}^{k-1}\tilde{\alpha}^{k-s-1}\Gamma_s\Big\}$$

Under zero initial condition $x(k_0)=0$, we know that

$$E\Big\{\sum_{s=k_0}^{k_0+\mathcal{T}_{(k_1,k_0)}-1}\tilde{\alpha}^{k-s-1}\theta^{k_0+\mathcal{T}_{(k-k_0)}-s-1}(p_1p_2)^{N(k,k_0)}p_1z_s^Tz_s+\sum_{s=k_0+\mathcal{T}_{(k_1,k_0)}}^{k_1-1}\tilde{\alpha}^{k-s-1}\theta^{\mathcal{T}_{(k-k_1)}}(p_1p_2)^{N(k,k_0)}$$
$$\times z_s^Tz_s+\cdots+\sum_{s=k_{l-1}}^{k_{l-1}+\mathcal{T}_{(k_l,k_{l-1})}-1}\tilde{\alpha}^{k-s-1}\theta^{k_{l-1}+\mathcal{T}_{(k-k_{l-1})}-s-1}p_1^2p_2z_s^Tz_s+\sum_{s=k_{l-1}+\mathcal{T}_{(k_l,k_{l-1})}}^{k_l-1}\tilde{\alpha}^{k-s-1}\theta^{\mathcal{T}_{(k-k_l)}}p_1p_2$$
$$\times z_s^Tz_s+\sum_{s=k_l}^{k_l+\mathcal{T}_{(k,k_l)}-1}\tilde{\alpha}^{k-s-1}\theta^{k_l+\mathcal{T}_{(k-k_l)}-s-1}p_1z_s^Tz_s-\sum_{s=k_l+\mathcal{T}_{(k,k_l)}}^{k-1}\tilde{\alpha}^{k-s-1}z_s^Tz_s\Big\}$$
$$\le\sum_{s=k_0}^{k_0+\mathcal{T}_{(k_1,k_0)}-1}\tilde{\alpha}^{k-s-1}\theta^{k_0+\mathcal{T}_{(k-k_0)}-s-1}(p_1p_2)^{N(k,k_0)}p_1\gamma^2\omega_s^T\omega_s+\sum_{s=k_0+\mathcal{T}_{(k_1,k_0)}}^{k_1-1}\tilde{\alpha}^{k-s-1}\theta^{\mathcal{T}_{(k-k_1)}}(p_1p_2)^{N(k,k_0)}$$
$$\times\gamma^2\omega_s^T\omega_s+\cdots+\sum_{s=k_{l-1}}^{k_{l-1}+\mathcal{T}_{(k_l,k_{l-1})}-1}\tilde{\alpha}^{k-s-1}\theta^{k_{l-1}+\mathcal{T}_{(k-k_{l-1})}-s-1}p_1^2p_2\gamma^2\omega_s^T\omega_s+\sum_{s=k_{l-1}+\mathcal{T}_{(k_l,k_{l-1})}}^{k_l-1}\tilde{\alpha}^{k-s-1}p_1p_2$$
$$\times\theta^{\mathcal{T}_{(k-k_l)}}\gamma^2\omega_s^T\omega_s+\sum_{s=k_l}^{k_l+\mathcal{T}_{(k,k_l)}-1}\tilde{\alpha}^{k-s-1}\theta^{k_l+\mathcal{T}_{(k-k_l)}-s-1}p_1\gamma^2\omega_s^T\omega_s-\sum_{s=k_l+\mathcal{T}_{(k,k_l)}}^{k-1}\tilde{\alpha}^{k-s-1}\gamma^2\omega_s^T\omega_s$$

Therefore, we can obtain that

$$E\Big\{\sum_{s=k_0}^{k-1}\bar{\alpha}^{k-s-1}(p_1p_2)^{N(k,s)}\tilde{z}_s^T\tilde{z}_s\Big\}=E\Big\{\sum_{s=k_0}^{k_1-1}\bar{\alpha}^{k-s-1}(p_1p_2)^{N(k,s)}\tilde{z}_s^T\tilde{z}_s+\cdots+\sum_{s=k_l}^{k-1}\bar{\alpha}^{k-s-1}\tilde{z}_s^T\tilde{z}_s\Big\}$$
$$\le E\Big\{\sum_{s=k_0}^{k_0+\mathcal{T}_{(k_1,k_0)}-1}\tilde{\alpha}^{k-s-1}\theta^{k_0+\mathcal{T}_{(k-k_0)}-s-1}(p_1p_2)^{N(k,k_0)}p_1z_s^Tz_s+\sum_{s=k_0+\mathcal{T}_{(k_1,k_0)}}^{k_1-1}\tilde{\alpha}^{k-s-1}\theta^{\mathcal{T}_{(k-k_1)}}(p_1p_2)^{N(k,k_0)}$$
$$\times z_s^Tz_s+\cdots+\sum_{s=k_{l-1}}^{k_{l-1}+\mathcal{T}_{(k_l,k_{l-1})}-1}\tilde{\alpha}^{k-s-1}\theta^{k_{l-1}+\mathcal{T}_{(k-k_{l-1})}-s-1}p_1^2p_2z_s^Tz_s+\sum_{s=k_{l-1}+\mathcal{T}_{(k_l,k_{l-1})}}^{k_l-1}\tilde{\alpha}^{k-s-1}\theta^{\mathcal{T}_{(k-k_l)}}p_1p_2$$
$$\times z_s^Tz_s+\sum_{s=k_l}^{k_l+\mathcal{T}_{(k,k_l)}-1}\tilde{\alpha}^{k-s-1}\theta^{k_l+\mathcal{T}_{(k-k_l)}-s-1}p_1z_s^Tz_s+\sum_{s=k_l+\mathcal{T}_{(k,k_l)}}^{k-1}\tilde{\alpha}^{k-s-1}z_s^Tz_s\Big\}$$
$$\le\sum_{s=k_0}^{k_0+\mathcal{T}_{(k_1,k_0)}-1}\tilde{\alpha}^{k-s-1}\theta^{k_0+\mathcal{T}_{(k-k_0)}-s-1}(p_1p_2)^{N(k,k_0)}p_1\gamma^2\omega_s^T\omega_s+\sum_{s=k_0+\mathcal{T}_{(k_1,k_0)}}^{k_1-1}\tilde{\alpha}^{k-s-1}\theta^{\mathcal{T}_{(k-k_1)}}(p_1p_2)^{N(k,k_0)}$$
$$\times\gamma^2\omega_s^T\omega_s+\cdots+\sum_{s=k_{l-1}}^{k_{l-1}+\mathcal{T}_{(k_l,k_{l-1})}-1}\tilde{\alpha}^{k-s-1}\theta^{k_{l-1}+\mathcal{T}_{(k-k_{l-1})}-s-1}p_1^2p_2\gamma^2\omega_s^T\omega_s+\sum_{s=k_{l-1}+\mathcal{T}_{(k_l,k_{l-1})}}^{k_l-1}\tilde{\alpha}^{k-s-1}p_1p_2$$
$$\times\theta^{\mathcal{T}_{(k-k_l)}}\gamma^2\omega_s^T\omega_s+\sum_{s=k_l}^{k_l+\mathcal{T}_{(k,k_l)}-1}\tilde{\alpha}^{k-s-1}\theta^{k_l+\mathcal{T}_{(k-k_l)}-s-1}p_1\gamma^2\omega_s^T\omega_s-\sum_{s=k_l+\mathcal{T}_{(k,k_l)}}^{k-1}\tilde{\alpha}^{k-s-1}\gamma^2\omega_s^T\omega_s$$
$$\le\sum_{s=k_0}^{k-1}\tilde{\alpha}^{k-s-1}\theta^{(N(k,s)+1)\mathcal{T}_{\max}}(p_1p_2)^{N(k,s)}p_1\gamma^2\omega_s^T\omega_s$$

Multiplying both sides by p_2^{-k/τ_α} yields

$$E\left\{\sum_{s=k_0}^{k-1}\tilde{\alpha}^{k-s-1}p_1^{k-s/\tau_a}p_2^{-s/\tau_a}\tilde{z}_s^T\tilde{z}_s\right\}\leq\sum_{s=k_0}^{k-1}\tilde{\alpha}^{k-s-1}\theta^{(k-s)\mathcal{T}_{\max}/\tau_a+\mathcal{T}_{\max}}p_1^{k-s/\tau_a}p_2^{-s/\tau_a}p_1\gamma^2\omega_s^T\omega_s$$
$$\leq\sum_{s=k_0}^{k-1}\tilde{\alpha}^{k-s-1}\theta^{(k-s)\mathcal{T}_{\max}/\tau_a+\mathcal{T}_{\max}}p_1^{k-s/\tau_a}p_1\gamma^2\omega_s^T\omega_s.$$

This is equal to the following inequality

$$E\left\{\sum_{s=k_0}^{\infty}\sum_{k=s}^{\infty}\tilde{\alpha}^{k-s-1}p_1^{k-s/\tau_a}p_2^{-s/\tau_a}\tilde{z}_s^T\tilde{z}_s\right\}\leq\sum_{s=k_0}^{\infty}\sum_{k=s}^{\infty}\tilde{\alpha}^{k-s-1}\theta^{(k-s)\mathcal{T}_{\max}/\tau_a}p_1^{k-s/\tau_a}\theta^{\mathcal{T}_{\max}}p_1\gamma^2\omega_s^T\omega_s. \quad (27)$$

Then, from (10), we can get that $\tilde{\alpha}\theta^{\mathcal{T}_{\max}/\tau_a}p_1^{1/\tau_a}<1$. Then

$$E\left\{\sum_{s=k_0}^{\infty}\frac{p_2^{-s/\tau_a}}{(1-\tilde{\alpha}p_1^{1/\tau_a})\tilde{\alpha}}\tilde{z}_s^T\tilde{z}_s\right\}\leq\sum_{s=k_0}^{\infty}\frac{1}{1-\tilde{\alpha}\theta^{\mathcal{T}_{\max}/\tau_a}p_1^{1/\tau_a}}\frac{p_1\theta^{\mathcal{T}_{\max}}}{\tilde{\alpha}}\gamma^2\omega_s^T\omega_s.$$

Finally, we can get

$$\sum_{s=k_0}^{\infty}p_2^{-s/\tau_a}E\left\{\tilde{z}_s^T\tilde{z}_s\right\}\leq\frac{p_1\theta^{\mathcal{T}_{\max}}(1-\tilde{\alpha}p_1^{1/\tau_a})}{1-\tilde{\alpha}p_1^{1/\tau_a}\theta^{\mathcal{T}_{\max}/\tau_a}}\gamma^2\sum_{s=k_0}^{\infty}\omega_s^T\omega_s. \quad (28)$$

Therefore, according to Definition 2, the filtering error system has a weighted l_2-gain

$$\gamma_s=\gamma\sqrt{p_1\theta^{\mathcal{T}_{\max}}(1-\tilde{\alpha}p_1^{1/\tau_a})\Big/(1-\tilde{\alpha}\theta^{\mathcal{T}_{\max}/\tau_a}p_1^{1/\tau_a})}.$$

This completes the proof.

Remark 1: Note that the switched systems activate in the intervals constituting of matched intervals and unmatched intervals. And the system maybe unstable in the unmatched intervals, in other words, the Lyapunov function maybe increased. However, the possible increment will be compensated by the more specific decrement (by limiting the lower bound of ADT), therefore, the system energy is decreasing from a whole perspective. Thus, we can get the filter error system is exponential mean-square stability with a weighted l_2-gain γ_s.

Remark 2: The proof of disturbance attenuation level is different from [20], in which the result is got under zero conditon $V_i(x_{k_l})=0$. In our paper, we provided a better result about weighted l_2-gain under zero initial conditon $V_i(x_{k_0})=0$; besides the result is suitable for any positive number $\mathcal{T}_{\max}$, which has no the limit of $\mathcal{T}_{\max}>1$ in [20]. On the other hand, we can get the result of [13] under the condition without considering the missing measurement and asynchronous switching.

3.2 H_∞ filter design

Now, based on the conditions on exponential mean-square stability with a weighted l_2-gain in Theorem 1, sufficient conditions for the existence of filter (6) are presented in the following theorem. Then, the admissible H_∞ filter parameters can be given.

Theorem 2: Given scalars $0 < \alpha < 1$ and $\beta > 0$, an H_∞ filter (6) can be designed such that the filter error system (7) is exponentially mean-square stable with a weighted l_2-gain γ_s under an average dwell time switching satisfying (10), if there exist symmetric and positive-definite matrices P_i, Q_i, R_{bi}, Z_{ci} and matrices M_{ci}, N_{ci}, X_i, Y_i, Z_i, $\mathbb{A}_i$, $\mathbb{B}_i$, $\mathbb{C}_i$ and $\mathbb{D}_i$ satisfying the following inequalities

$$\begin{bmatrix} \psi_1 - \tilde{\alpha} P_i & H^T M_{2i}^T + H^T N_{2i}^T & H^T M_{3i}^T + H^T N_{3i}^T & \tilde{\Xi}_{11} & \tilde{\Sigma}_1 \\ * & -\tilde{\alpha}^{d_M} Q_i & 0 & \tilde{\Xi}_{12} & \tilde{\Sigma}_2 \\ * & * & -\gamma^2 I & \tilde{\Xi}_{13} & \tilde{\Sigma}_3 \\ * & * & * & \tilde{\Xi}_4 & 0 \\ * & * & * & * & \tilde{\Xi}_5 \end{bmatrix} < 0, \tag{29}$$

$$\begin{bmatrix} \psi_1 - \bar{\beta} P_i & H^T M_{2i}^T + H^T N_{2i}^T & H^T M_{3i}^T + H^T N_{3i}^T & \bar{\Xi}_{11} & \bar{\Sigma}_1 \\ * & -\bar{\beta}^{d_m} Q_i & 0 & \bar{\Xi}_{12} & \bar{\Sigma}_2 \\ * & * & -\gamma^2 I & \bar{\Xi}_{13} & \bar{\Sigma}_3 \\ * & * & * & \bar{\Xi}_4 & 0 \\ * & * & * & * & \bar{\Xi}_5 \end{bmatrix} < 0, \tag{30}$$

where, $\tilde{\Sigma}_1 = \begin{bmatrix} \tilde{C}_{1i}^T & q\tilde{C}_{2i}^T & \tilde{\varphi}_1 & q\tilde{\varphi}_4 & H^T(A_i^T - I)Z_{3i} \end{bmatrix}$, $\tilde{\Sigma}_2 = \begin{bmatrix} \tilde{C}_{1di}^T & q\tilde{C}_{2di}^T & \tilde{\varphi}_2 & q\tilde{\varphi}_5 & A_{di}^T Z_{3i} \end{bmatrix}$,

$$\tilde{\Sigma}_3 = \begin{bmatrix} \tilde{D}_i^T & 0 & \tilde{\varphi}_3 & 0 & B_i^T Z_{3i} \end{bmatrix}, \bar{\Sigma}_1 = \begin{bmatrix} \bar{C}_{1i}^T & q\bar{C}_{2i}^T & \bar{\varphi}_1 & q\bar{\varphi}_4 & H^T(A_i^T - I)Z_{3i} \end{bmatrix},$$

$$\bar{\Sigma}_2 = \begin{bmatrix} \bar{C}_{1di}^T & q\bar{C}_{2di}^T & \bar{\varphi}_2 & q\bar{\varphi}_5 & A_{di}^T Z_{3i} \end{bmatrix}, \ \bar{\Sigma}_3 = \begin{bmatrix} \bar{D}_i^T & 0 & \bar{\varphi}_3 & 0 & B_i^T Z_{3i} \end{bmatrix},$$

$$\tilde{\varphi}_1^T = \begin{bmatrix} X_i A_i + p\mathbb{B}_i C_{2i} & \mathbb{A}_i \\ Z_i A_i + p\mathbb{B}_i C_{2i} & \mathbb{A}_i \end{bmatrix}, \ \tilde{\varphi}_2^T = \begin{bmatrix} X_i A_{di} + p\mathbb{B}_i C_{2di} \\ Z_i A_{di} + p\mathbb{B}_i C_{2di} \end{bmatrix}, \ \tilde{\varphi}_3^T = \begin{bmatrix} X_i B_i + \mathbb{B}_i D_{2i} \\ Z_i B_i + \mathbb{B}_i D_{2i} \end{bmatrix},$$

$$\bar{\varphi}_1^T = \begin{bmatrix} X_j A_i + p\mathbb{B}_j C_{2i} & \mathbb{A}_j \\ Z_j A_i + p\mathbb{B}_j C_{2i} & \mathbb{A}_j \end{bmatrix}, \ \bar{\varphi}_2^T = \begin{bmatrix} X_j A_{di} + p\mathbb{B}_j C_{2di} \\ Z_j A_{di} + p\mathbb{B}_j C_{2di} \end{bmatrix}, \ \bar{\varphi}_3^T = \begin{bmatrix} X_j B_i + \mathbb{B}_j D_{2i} \\ Z_j B_i + \mathbb{B}_j D_{2i} \end{bmatrix},$$

$$\tilde{\varphi}_4^T = \begin{bmatrix} \mathbb{B}_i C_{2i} & 0 \\ \mathbb{B}_i C_{2i} & 0 \end{bmatrix}, \ \tilde{\varphi}_5^T = \begin{bmatrix} \mathbb{B}_i C_{2di} \\ \mathbb{B}_i C_{2di} \end{bmatrix}, \ \bar{\varphi}_4^T = \begin{bmatrix} \mathbb{B}_j C_{2i} & 0 \\ \mathbb{B}_j C_{2i} & 0 \end{bmatrix}, \ \bar{\varphi}_5^T = \begin{bmatrix} \mathbb{B}_j C_{2di} \\ \mathbb{B}_j C_{2di} \end{bmatrix}.$$

Moreover, if feasible solutions exist, the parameters of an admissible filter of (6) are constructed as

$$A_{ci} = Y_i^{-1}\mathbb{A}_i, \ B_{ci} = Y_i^{-1}\mathbb{B}_i, \ C_{ci} = \mathbb{C}_i, \ D_{ci} = \mathbb{D}_i. \tag{31}$$

Proof: We denote matrices $\Omega_i = \begin{bmatrix} X_i & Y_i \\ Z_i & Y_i \end{bmatrix}$, $\forall i \in \mathcal{I}$, then can obtain (29). By the similar proof line, we can also get (30). In addition, the admissible filter parameter matrices are given by (31), the proof is completed.

4 Numerical example

In this section, we give an example to demonstrate the effectiveness of the proposed method.

Example: Considering system (1) with two subsystems, and the parameters of each subsystem are given as follows:

$$A_1 = \begin{bmatrix} 0.33 & -0.12 \\ 0.36 & -0.37 \end{bmatrix}, A_2 = \begin{bmatrix} 0.25 & 0.28 \\ -0.14 & -0.19 \end{bmatrix}, A_{d1} = \begin{bmatrix} 0.07 & -0.02 \\ 0.02 & 0.06 \end{bmatrix}, A_{d2} = \begin{bmatrix} 0.08 & -0.03 \\ 0 & 0.1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -0.09 \\ 0.01 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ -0.01 \end{bmatrix}, C_1 = \begin{bmatrix} 0.64 & -0.79 \end{bmatrix}, C_2 = \begin{bmatrix} -0.16 & -0.02 \end{bmatrix}, D_1 = 0.19,$$

$$D_2 = -0.55, D_{21} = 0.04, D_{22} = -0.55, C_{d1} = \begin{bmatrix} 0.2 & 0.04 \end{bmatrix}, C_{d2} = \begin{bmatrix} -0.39 & 0.04 \end{bmatrix},$$

$$C_{21} = \begin{bmatrix} 0.93 & 0.14 \end{bmatrix}, C_{22} = \begin{bmatrix} 0.23 & 0.74 \end{bmatrix}, C_{2d1} = \begin{bmatrix} 0.58 & 0.49 \end{bmatrix}, C_{2d2} = \begin{bmatrix} -1.11 & 0.18 \end{bmatrix}.$$

Let $d_m = 1, d_M = 2, \alpha = 0.5, \beta = 0.01$ and $\mathcal{T}_{\max} = 2$, we consider the asynchronous switching in the design phase and turn to Theorem 2, by utilizing LMI Toolbox, we can get $\tau_\alpha^* = 7.5258$, $\gamma = 1.9874$ and $\gamma_s = 11.0333$, and the filter parameters are obtained as follow:

$$A_{c1} = \begin{bmatrix} 0.1126 & 0.00237 \\ 0.0371 & -0.1404 \end{bmatrix}, A_{c2} = \begin{bmatrix} 0.0899 & 0.0276 \\ 0.0380 & -0.0947 \end{bmatrix}, B_{c1} = \begin{bmatrix} -0.0745 \\ -0.0881 \end{bmatrix}, B_{c2} = \begin{bmatrix} 0.0954 \\ 0.0310 \end{bmatrix},$$

$$C_{c1} = \begin{bmatrix} 0.0286 & 0.2278 \end{bmatrix}, \ C_{c2} = \begin{bmatrix} 0.1283 & 0.2259 \end{bmatrix}, \ D_{c1} = 0.9347, \ D_{c2} = 0.8880.$$

Using the filter in (31) and given switching sequences with $\tau_\alpha = 10.0$, the state responses of the resulting system are given in Fig.1.(a)-(d). Fig.1.(a)-(d) show the switching signals of system and filter, the output error of H_∞ filter system, the state and estimation of $x(1)$ and $x(2)$,respectively. It can be seen that the designed filter in (31) under the admissible switching signals is effective despite asynchronous switching.

5 Conclusion

In this paper, the H_∞ filtering problem for a class of discrete-time asynchronous switched systems with time-delay and missing measurement has been investigated. By the aid of Lyapunov-Krasovskii and average dwell time method, the H_∞ filter has been designed such that the filter error system is exponential mean-square stable with a weighted l_2-gain. By allowing the system to be unstable within the unmatched intervals, the more general conditions for H_∞ filter has been derived and formulated in terms of LMIs, then the corresponding filter is obtained.

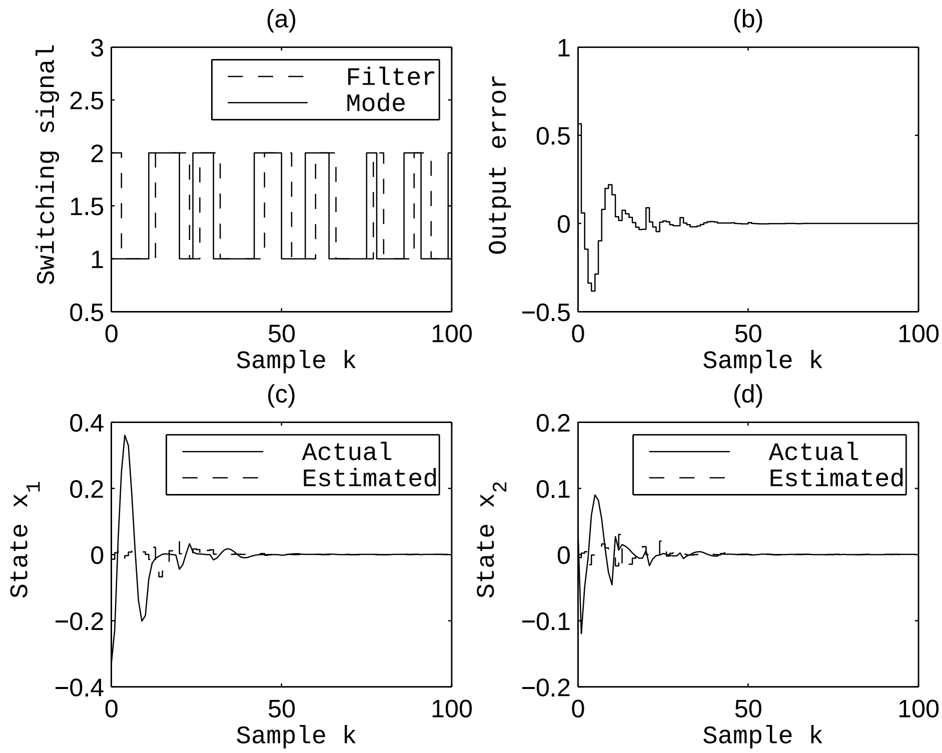

Fig.1. (a) switching signal; (b) output error ; (c) state and its estimation; (d) state and its estimation

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