Some Inequalities Concerning Power Series and Their Interaction With Univalent Function Theory

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Abstract

Power series are fundamental in the study of Geometric Function Theory. In fact, they constitute a major part in Complex Analysis. The purpose of this dissertation is to employ analytic functions defined by power series in two different directions. The first of these discussed in Part I, mainly contributes to Analytic Inequalities in Real and Complex Analysis, while the second direction discussed in Part II, deals with Analytic and Univalent Function Theory.

Part I of this dissertation is devoted to some inequalities concerning power series with real or nonnegative coefficients and convergent on an open disk. The main purpose of this part is to derive new inequalities for functions defined by convergent power series, which are related to the celebrated classical inequalities in Real and Complex Analysis such as the Cauchy-Bunyakovsky-Schwarz, Young's, Hölder's and Jensen type inequalities. In particular, we obtain new and better inequalities for the power series functions, which provide generalizations, refinements and improvements on the earlier results published by Cerone and Dragomir [89], Dragomir and Ionescu [138], Dragomir and Sándor [136] and others. Further, particular inequalities are obtained by applying the results for the fundamental functions such as exponential, logarithm, trigonometric and hyperbolic functions. Some inequalities involving special functions such as polylogarithm, hypergeometric, Bessel and modified Bessel functions of the first kind are established as well.

Part II is mainly concerned with analytic functions in a unit disk and normalized by the conditions $f(0) = f'(0) - 1 = 0$. The functions in the class S, which are analytic, univalent and normalized in a unit disk and have power series representations, are the center of the study of Univalent Function Theory. The properties of functions in the class S and its subclasses such as the starlike, spirallike, convex, close-to-convex, etc., have been widely investigated by various researchers in the past three decades. In the other direction, new subclasses of functions, which are defined involving operators, were introduced. The Sălăgean operator [383] is one of the famous differential operators, which was used by

Opoola [343] to introduce new subclasses of S. To provide some new properties for certain subclasses of functions introduced by Opoola is the main objective throughout the second part of this dissertation. In this study, we introduce the special subclass of Opoola's functions and then investigate some properties and coefficients inequalities of functions in this class. Finally, we conclude Part II by providing the Fekete-Szegő theorem concerning subclasses of analytic and univalent functions.

Declaration

"I, Alawiah Ibrahim, declare that the PhD thesis entitled "*Some Inequalities Concerning Power Series and Their Interaction with Univalent Function Theory*" is no more than 100,000 words in length including quotes and exclusive of tables, figures, appendices, bibliography, references and footnotes. This thesis contains no material that has been submitted previously, in whole or in part, for the award of any other academic degree or diploma. Except where otherwise indicated, this thesis is my own work".

Signature Date

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List of Publications

This dissertation contains the results from a number of the author's research papers that have been published in refereed publications or have been submitted for publications:

(1) Chapter 3 of this dissertation contains some of the results from the following research papers:

- "Power series inequalities via Buzano's result and applications" with S. S. Dragomir, *Integral Transforms and Special Functions*, Vol. **22**, Issue 12, 2011, pp. 867-878.

- "Power series inequalities via a refinement of the Schwarz inequality" with S. S. Dragomir, *Integral Transforms and Special Functions*, Vol. **23**, Issue 10, 2012, pp. 769-781.

- "Some inequalities for power series with applications" with S. S. Dragomir and M. Darus, *Integral Transforms and Special Functions*, Vol. **24**, No. 5, 2013, pp. 364-376.

(2) Chapter 4 of this dissertation contains some of the results from the following research papers:

- "Power series inequalities via Young's inequality with applications" with S. S. Dragomir and M. Darus, Journal of Inequalities and Application **2013**, 2013, pp. 314 - 326.

- "Power series inequalities related to Young's inequality and applications" with S. S. Dragomir and M. Darus, *Integral Transforms and Special Functions*, Vol. **24**, No. 9, 2013, pp. 700-714.

- "Inequalities for power series with positive coefficients" with S. S. Dragomir, P. Cerone and M. Darus, *Journal of Inequalities and Special Functions,* Vol. **3**, Issue 2, 2012, pp. 1-15.

(3) Chapter 5 of this dissertation contains some of the results from the author's research papers, which are mentioned in (1) and (2) above.

(4) All the results that contain in Chapter 3 and some of the results from Chapter 5, also have been incorporated in the following survey paper:

- "A survey on Cauchy-Bunyakovsky-Schwarz inequality for power series" with S. S. Dragomir. This paper has been accepted as a chapter in the book of H. M. Srivastava Honorary Volume by Springer.

(5) Chapter 7 of this dissertation contains the results from the following author's research papers:

- "Coefficient inequalities for a certain subclass of analytic functions" with M. Darus, S. S. Dragomir and O. A. F-. Joseph, *Journal of Inequalities and Special Functions* (accepted). Preprint available at: *RGMIA Research Report Collection,* Vol. **14**, 2011, Article no. 89, 9 pp, http://www.rgmia.-org/issues.php.

- "Fekete-Szegő problem for concave univalent functions defined by Sălăgean operator" with M. Darus and S. S. Dragomir (submitted). Preprint available at: *RGMIA Research Report Collection,* Vol. **14**, 2011, Article no. 88, 11 pp, http://www.rgmia.org/issues.php.

- "Some properties for a certain subclass of univalent functions" with M. Darus, S. S. Dragomir and O. A. F-. Joseph, to appear.

List of Symbols

The following lists are standard symbols, notations and abbreviations that are used in this dissertation.

$\mathbb{R},\ \mathbb{R}^{+},\ \mathbb{R}^{-}$	The set of real, positive real and negative real numbers
$\mathbb{C}$	The complex numbers
$\mathbb{Z},\ \mathbb{Z}^{+},\ \mathbb{Z}^{-}$	The set of integers, positive integers and negative integers
$\mathbb{Z}_0^{+},\ \mathbb{Z}_0^{-}$	The set of nonnegative integers and nonpositive integers
$\mathbb{N}$	The set of natural numbers
$\mathbb{N}_0$	The set of zero and natural numbers
$\mathbb{K}$	The field of real or complex number
$\mathbf{U}$	The convex set
$\mathbf{X}$	The linear space
D	The open unit disk
$D(0,R)$	The open disk
$D(z_0,R)$	The open disk with center at z_0 and radius R
$\partial D,\ \overline{D}$	The boundary of D, the closure of D
$\Gamma(s)$	The gamma function
$\Psi(s)$	The digamma (psi) function
$\Psi_n(s)$	The polygamma function
$\gamma(\alpha,z)$	The incomplete gamma function
$\Gamma(\alpha,z)$	The complementary incomplete gamma function
$B(x,y)$	The incomplete beta function
$\zeta(s)$	The Riemann zeta function
$\beta(s)$	The beta function
$\eta(s)$	The eta function
$\zeta(s,q)$	The Hurwitz zeta function
$Li_n(s)$	The polylogarithm function
${}_2F_1$	The Gauss hypergeometric function
${}_pF_q$	The generalized hypergeometric function
$J_\alpha(z)$	The Bessel function of the first kind

$I_\alpha(z)$	The modified Bessel function of the first kind
$(a)_n$	The Pochhammer symbol
B_n	The Bernoulli numbers
$E_{n,k}$	The Eulerian numbers
$H_{n,r}$	The generalized harmonic number
Z	The Apèry's constant
G	The Catalan's constant
$n!, n!!$	The factorial function, double factorial function
$\binom{a}{n}$	The binomial coefficient
$sgn(x)$	The signum (sign) function
$\mathcal{C}[a,b]$	The function space
$\langle\cdot,\cdot\rangle$	The inner product
$(V,\langle\cdot,\cdot\rangle)$	The inner product spac
$\mathbf{X}$	The linear space
$\operatorname{Re}(z)$	The real part of z
$\operatorname{Im}(z)$	The imaginary part of z
$\mathcal{P}$	The class of functions with a positive real part
$\mathcal{H}(D)$	The class of analytic functions on the unit disk D
A	The class of analytic and normalized functions on a unit disk
S	The class of analytic, univalent and normalized on a unit disk
S^*	The class of starlike functions
C	The class of convex functions
$\mathcal{K}$	The class of close-to-convex functions
$k(z)$	The Koebe function
$m(z)$	The Möbius transformation
γ	The Jordan curve
$S^*(\alpha)$	The class of starlike functions of order α
$C(\alpha)$	The class of convex functions of order α
C_α	The class of α-convex functions
$\mathcal{B}(\alpha)$	The class of Bazilevč functions of order α
$\mathcal{B}(\alpha,\beta)$	The class of Bazilevč functions or order α and type β
C_α	The class of α-convex functions
$S^*(\alpha)$	The class of starlike functions of order α
$C(\alpha)$	The class of convex functions of order α
$D^n f$	The Sălăgean differential operator
$D_\lambda^n f$	The Al-Oboudi differential operator
$\blacksquare$	The end of proof

Contents

Chapter 1

Introduction

Power series is a special type of series of a function, that is of fundamental importance in Complex Analysis. The fact that an analytic function in an open disk can be represented by convergent power series, makes it an important element in the study of Geometric Function Theory. This chapter begins with a concise introduction to the *power series*. Then, it follows by an introduction to *Inequalities Theory* in real and complex analysis and *Univalent Function Theory*, which are the two main parts of this dissertation. Basic properties of the power series and their representations for some fundamental functions are given in the following sections. Finally, this chapter presents the main motivation for writing this dissertation and provides an overview of the outline and content of this dissertation.

1.1 General Background

1.1.1 Power Series

An *infinite series* is, informally speaking, the sum of the terms of an infinite sequence. One of the most important infinite series, which appears in almost all areas of pure and applied mathematics, is called *power series*.

A power series is a series, where a 'variable' is involved in the terms of the series. Roughly speaking, a power series is an infinite series that can be written in a systematic pattern of change in the powers of variable, and the terms of the series are often produced according to the certain rules. For example,

$$
\begin{aligned}
&\text{(i)} \quad 1 + x + x^2 + x^3 + x^4 + \dots \\
&\text{(ii)} \quad x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots . \\
&\text{(iii)} \quad (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \dots
\end{aligned}
\tag{1.1}
$$

are the power series, which are specifically known as the *geometric series*, *Maclaurin's series* and *Taylor's series*, respectively.

The terminology of the infinite series was introduced in the last third of the 17th century. It probably appeared in print for the first time in 1668, in the second of James Gregory's manuscript entitled "*Exercitationes Geometricae*", published in London (cf. Gregory [172, p. 10-12]). However, the first mention of the notion of the infinite series dates back to antiquity. Historically, Greek mathematician Archimedes (c. 287 BC - c. 212 BC) produced the first summation of the infinite series with a method that is still used in the area of calculus today. He used the method of exhaustion[1] and Egyption fraction[2] to develop the infinite series by calculating the area under the arc of a parabola. Hence, he obtained a remarkable approximation value of π ($223/71 < \pi < 22/7$), see ([31, p. 233-252], [252, p. 35-37], [340]). The development of the infinite series techniques then commenced with the computing of some more accurate values of π by many mathematicians until the 17th century. Chinese mathematicians, Liu Hui (around 265 AD) and Zu Chongzhi (around 480 AD) created the same method based on the polygon iterative algorithm, but they considered the different number of polygon sides in the circle, to obtain the approximation values of π (see [32, p. 177-178], [73, p. 202-203], [329, p. 66, 100-101]). The infinite series were also exploited for π, most notably by European mathematicians such

[1]The method of exhaustion is a method of finding area of shape by inscribing inside it a sequence of polygons whose areas converge to the area containing shape. If the sequence is correctly constructed, the difference in area between the nth polygon and the containing shape will become arbitrary small as n become large[a]. [a]Adopted from http://en.wikipedia.org/Method_of_exhaustion, on 15 May 2013.

[2]Egyption fraction is the sum of distinct unit fractions, where each fraction has a numerator equal to 1 and a denominator is a positive integer[b]. [b]Adopted from http://en.wikipedia.org/Egyption_ fraction, on 15 May 2013.

as François Viète (see [32, p. 187], [428, p. 397-398]), John Wallis [429, p. 1-52, 753-754], James Gregory (see [171, p. 34-39], [172, p. 10-12]), Gottfried Wilhelm Leibniz [32, p. 188-189] and Nicholas Mercator [307]. They found the approximation values of π by developing the infinite series to solve problems under the particular curves. Indeed, they showed that the infinite series techniques were the useful tool in solving the geometrical problems until the 17th century, that is before the rise of calculus. However, in those times, to understand that a sum with infinite number of summands could have a finite result was a great philosophical challenge.

Obtaining the sum of a particular series of numbers is of relatively little interest in comparison with the expanding of a function into a series, whose terms depend on a variable. In modern terminology, this is called the *power series expansion.* The idea of the infinite series expansion of functions was first conceived in India, in the 14th century by Madhava of Sangamagrama (c. 1350 - c. 1425). He laid down the precursors of the modern conception of the power series and discovered a number of the Taylor series expansion of functions, such as sine, cosine, tangent and arctangent (see [43], [106], [168]). Alongside his discovering, Madhava also developed the convergence criteria of the infinite series; his students and followers at the Kerala School of Astronomy and Mathematics, India, further expanded his works and obtained various type of series expansions until the 16th century (see [399], [409, p. 173], [420]).

Madhava's formula of arctangent was rediscovered, in a different way, by the Scottish mathematician James Gregory in 1671, a few years after he published the formal terminology of the infinite series. This discovery is attributed to him, though he also formulated the expansion of power series for the trigonometric functions such as sine, cosine, arcsine and arcosine, and published several Maclaurin's series as well [73, p. 439-445]. After Issac Newton developed the formal methods of modern calculus in 1660s, then Leibniz was able to find the series expansions of sine, arcsine and explored many more infinite series including the Taylor series (see [73, p. 446-447], [278], [279]). However, a general method for constructing the Taylor series expansion for all functions was formally provided by Brook Taylor in 1715, after whom the series are now named (cf. Taylor [416, p. 21-23]). A Maclaurin series was named after Scottish mathematician Colin Maclaurin, who made an extensive use of the special case of the Taylor series in the 18th century, when the series is expanded around zero. A remarkable contribution on the development of the Series Theory also goes to Swiss Leonhard Euler, who made some important discoveries in this field. He directly proved the power series expansions for the exponential and inverse tangent functions [73, p. 496-497], and provided the ways to express various logarithmic functions by using the power series method. Notably, he introduced the theory of *hypergeometric series* and $q-$*series.*

The discovery of the mathematical quantities such as the exponential, logarithm, trigonometric functions of sine, cosine, tangent, etc. that could be written as the power series expansions, was one of the great mathematical achievements in the mid 17th century. Together with the 'invention' of calculus by English scientist Newton and German mathematician Leibniz in the 1660s, then it led to the development of many important power series in analysis. Later, the theory of infinite series was thoroughly developed and used to work out many significant problems that had eluded solutions with any other approach in mathematics and various fields.

In this section, we state the formal definition of the power series in one complex variable. The definition is adopted from the standard texts of Ahlfors [6, p. 38], Agarwal [5, p. 151], Ponnusamy and Silverman [359, p. 153] and other references which are cited therein. We start with the definition of series as follows.

Definition 1 *Let* $(a_0, a_1, a_2, \ldots)$ *be an infinite sequence. Then, the sum*

$$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + \cdots \;, \tag{1.2}$$

is called an infinite series or simply a series.

Definition 2 *A power series is an infinite series of the form*

$$\sum_{k=0}^{\infty} a_k \left(z - z_0\right)^k = a_0 + a_1 \left(z - z_0\right) + a_2 \left(z - z_0\right)^2 + \cdots \;, \tag{1.3}$$

where $z,\ z_0 \in \mathbb{C}$*,* a_k*,* $k \in \{0, 1, 2, \ldots\}$ *represents the coefficient of the* k*-th term of the series,* z_0 *is an any fixed point in* $\mathbb{C}$ *and* z *varies around* z_0*.*

If z_0 is equal to zero (i.e., $z_0 = 0$), then the power series (1.3) takes the simpler form as follows:

$$\sum_{k=0}^{\infty} a_k z^k = a_0 + a_1 z + a_2 z^2 + \cdots \ . \tag{1.4}$$

The series (1.3) is often called the power series about z_0 or the power series with respect to the center at z_0. It usually arises as the *Taylor series* of some known functions [4, p. 65-89]. In other words, many fundamental functions in a real or complex variable can be represented as the Taylor series expansion (as we shall show later in Section 1.3), whereas, the series in (1.4), which involves the simple powers of z, is known as *Maclaurin's series.* It is clearly seen that Maclaurin's series (1.4) is the Taylor series (1.3) about the origin. The simplest form of the power series is called the *geometric series,* namely

$$\sum_{k=0}^{\infty} z^k = 1 + z + z^2 + \cdots. \tag{1.5}$$

It is the special case of Maclaurin's series (1.4), where the coefficients $a_k = 1$ for all $k \in \{0, 1, 2, \ldots\}$.

The power series such as the geometric, Maclaurin and Taylor series, are frequently used in mathematical analysis. They provide powerful ways of representing some of the most important functions in a wide range of mathematics, physics, engineering, computer science and other fields. The crucial properties such as the algebraic operations (including the differentiation and integration) that can be done much more readily on the power series form, thus make such series particularly easy to study. In analysis, the solution of linear and nonlinear of ordinary or partial differential equations are often raised in the form of power series (see for instance [2], [33], [100], [113], [163], [200], [276]). Power series also arise under the name of *generating function* in the study of Combinatorics Theory (see [92], [97, Chapt. 2], [178]), while, the concept of *$p-$adic number,* which is closed related to the power series, occurs in the field of Number Theory (see [28], [199], [413]).

Much physical modelling requires mathematical approximations and often involves the series expansions. This method plays an important role in many diverse areas of applied mathematics including physics and engineering. Particularly, the Taylor series expansions are widely used for approximating functions with a large or a small parameter. Hence, it always can be found in the study of the wave equations (see [258], [290]), the dynamical systems (see [38], [94], [177]), the kinetics equations (see [242], [315], [325], [326]), the quantum mechanics (see [230], [247], [424, p. 96-104]) and the neural networks (see[231], [417]). In addition, the power series method is simpler and it is able to reduce the complicated linear or nonlinear equations in order to find their exact solutions.

The applications of the power series also can be found in the field of computer sciences as well. The excellent way of representing some complicated functions, and also be able to serve the quick approximation and evaluation of functions at the given points, makes the power series a powerful tool in many computer algorithms developments. This means that, the power series expansions serve as a workhorse in many computer algorithms, in which the program runs faster or maybe even is possible. In fact, the numerical algorithm developed by using the power series expansion is much simpler, easier and more efficient with less error determination [205]. In other words, the approximation technique using the power series allows the computer algorithm to approximate functions with more speed, and in a short computational time with a high degree of accuracy (see [251], [256], [257], [275], [323]).

Power series are also useful in chemistry as well as in mathematics, physics, engineering and computer science for a number of reasons. First, the power series in truncated polynomial form provide an excellent tool for fitting experimental data, when there is no model formula available (see [95, Chap. 2], [96, Chap. 1], [258], [290]). Second, the great needs of the power series is for geometric optimization of physical systems: for instance, the quasi-newton method makes use of a two variables Taylor series to approximate the equilibrium geometry of a cluster of atoms [76]. In the field of economics and finance, the geometric series are widely used to represent the present and future value of an annuity[3], and to estimate the present value of expected stock dividends or the terminal value of security (see [29], [197], [226, p. 175], [444]). For other applications of the power series in the Theory of Financial, see ([156], [184, p. 416-423], [367, p. 186-191], [425]).

[3] is a series of payments made at fixed intervals of time.

In general, the importance of the power series representation is because of its extensive and very effective applications in various fields of pure and applied mathematics. Their effectiveness in error determination, function optimization and definite integral resolution are the evidence of the power series being an enormous tool in physical sciences and computational science, as well as an effective way of representing complicated functions in mathematical analysis.

1.1.2 Inequalities Theory

In [414], Tanner pointed out that "*Mathematics begins with inequality*". He said in a common language, the basic ideas of "*more*" and "*less*" are mathematical ideas of unequal and they are more primitive than the ideas of equal and numerals. However, these basic ideas were not symbolized until the mathematical sign of equality "=" appeared in the 16th century, and the sign of inequalities "<" and ">" in the 17th century. Tanner [414] also believes that to find an equality in a real practice is much more difficult than inequality . His argument seems to be relevant to the statement from the pioneers in Inequalities Theory, Beckenbach and Bellman [55, p. 3], who say in their book "*An Introduction to Inequalities*" that the fundamental results of mathematics are often *inequalities* rather than *equalities.*

The Theory of Inequalities plays an important role in mathematical analysis for finding approximations of numbers, functions, integrals, etc. Around the beginning of the 20th century, numerous inequalities were derived and applied in various branches of mathematics as well as in science and engineering. The pioneers in this field were credited to Hardy, Littlewood and Pólya [185], who transformed this field of inequalities from a collection of isolated formulas into a systematic discipline through the book "*Inequalities*", which was originally published in 1934. This book is the first devoted solely to the subject of inequalities, which presents a lot of fundamental ideas, problems, results, methods of proving and applications for a large variety of classical and new inequalities. Hence, it has had much influence on research in various branches of mathematical analysis. Some other notable books in this area are "*An Introduction to Inequalities*" by Beckenbach and Bellman [55], which was published in 1961 and the early one of Mitrinovič [316], "*Analytic Inequalities*" in 1970. These major books have made considerable contributions to the field of Inequalities Theory and serve as important references for many mathematicians and to those who use analysis seriously. Since them, an enormous amount of effort has been devoted to the discovery of new types of inequalities and their applications in many part of mathematical analysis, see for instance the books of Mitrinovič, Pečarić and Fink [318], Kapur [237], Kazarinoff [239], Garling [161] and recently, Bullen [79], Cerone and Dragomir [91].

Nowadays, a large number of classical and new inequalities can be found in the literature. The classical inequalities are those associated with the names Gauss, Cauchy, Schwarz, Bunyakovsky[4], Young, Hölder, Hilbert, Hardy, Littlewood, Pólya, Minkowski, Jensen, Čebyšev[5], Bessel, Ostrowski, Grüss, Hadamard and others. These type of inequalities have been explored by various researchers, hence new inequalities are created in which to provide their generalizations, refinements, improvements, etc., see for instance ([9], [19], [128], [157], [158], [160], [192], [271], [299], [302], [310], [317], [445], [449]) and the references which are cited therein. The studies related to these classical inequalities remain the active fields and have grown into substantial areas of research with many important applications in various fields of Modern Mathematics.

One of the most important inequalities in analysis, that has attracted the great attention of a large number of researchers in the last few decades, is the famous *Cauchy-Bunyakovsky-Schwarz inequality.* This inequality is also known in the literature as *Cauchy's, Schwarz's* or *Cauchy-Schwarz's inequality,* and it has a long history connected with these three names of the famous mathematicians. Starting with the French mathematician A. L. Cauchy [88], who established the elementary form of the Cauchy-Bunyakovsky-Schwarz inequality for real numbers. His inequality, later called Cauchy's inequality, is contained in his book "*Cour d'Analyse Algébrique*", published in Paris in 1821. Then, in 1859, the Cauchy's student, V. Y. Bunyakovsky [81] derived a corresponding inequality for classical integrals. After around three decades later, in 1888, the German mathematician H. A. Schwarz [388] independently rediscovered and obtained the same result of Cauchy's inequality for Lebesque integrals, without any reference

[4]There are several different spellings of Bunyakovsky's name can be found in the literature, for example Bunyakovskii, Buniakowski, Bounjakowsky, etc., see MatSciNet.

[5]The other spellings for Čebyšev's name are Chebyshev, Čebišev, Tchebycheff, etc., see MatSciNet.

to Bunyakovsky's work. Meanwhile, in the year 1885, Schwarz [388] had already obtained the generalization of Cauchy's inequality in inner product spaces, which is popularly known in the literature as Schwarz's or Cauchy-Schwarz's inequality. The classical Cauchy inequality and its generalizations for integral and in inner product spaces, therefore shows up under the *Cauchy-Bunyakovsky-Schwarz inequality*. For a short history of these type inequalities, see ([387, p. 64-70], [407, p. 10-12]).

The generalization of the classical Cauchy-Bunyakovsky-Schwarz inequality for sums was established by O. L. Hölder [201] in 1889, and it is called *Hölder's inequality*. Although this inequality was first derived in 1888 by L. J. Rogers [378], then it was proved in another way a year later by Hölder through his monograph "*Über einen Mittelwertsatz*". Hölder's inequality, which is sometimes referred as Rogers's inequality, was built around the two positive real numbers p and q with the conditions that $p > 1$ and $1/p + 1/q = 1$. The Cauchy-Bunyakovsky-Schwarz inequality is the special case of Hölder's inequality when $p = q = 2$. There are several proofs of Hölder's inequality can be found in the literature, where one of them is the proof via a standard *Young's inequality* for the product of two real numbers.

Jensen's inequality is another important result in Real and Complex Analysis. It was developed by Danish mathematician J. L. W. V. Jensen [223] in 1906 based on the concept of *convexity* of functions (see also [222]). Due to the fact that Jensen's inequality provides a source of deriving for many classical results in analysis such as the triangle inequality, arithmetic mean-geometric mean inequality, Hölder's inequality, Minkowski's inequality, etc., through judicious choice of convex or concave function, this type of inequality thus becomes one of the landmarks in the study of Inequalities Theory, and in Convex Analysis as a whole.

The classical inequalities of the Cauchy-Bunyakovsky-Schwarz, Jensen, Young and Hölder inequalities play a crucial role in the Theory of Inequalities. They have attracted the attention of a large number of researchers, and stimulated new research directions and influenced various aspects of mathematical analysis and their applications.

1.1.3 Univalent Function Theory

In Complex Analysis, a geometric function is a function whose range describes certain geometries. A Geometric Function Theory is one of the branches of Complex Analysis, which deals and studies the geometric properties of complex analytic functions. The Theory of Univalent Function is one of the most important subjects in the Geometric Function Theory, where the primary interest in this field is the functions that are analytic[6] and univalent in a certain complex domain. This theory was founded around the turn of the 20th century, when the first important paper appeared in this area by Koebe [253] in 1907. It followed by Alexander ([15], [16]) and Bieberbach [67] in 1915 and 1916, respectively. Koebe [253] introduced the notion of univalent mapping through his monograph '*Über die uniformisierung beliebiger analytischer kurven',* which gives a great contribution to the origin of the Univalent Function Theory.

The concept of univalent mapping has been explored by Alexander [16] in his PhD dissertation entitled '*Functions Which Map the Interior of the Unit Circle Upon Single Region*' at Princeton University, USA in 1915. Since then, the study of Univalent Function Theory has attracted the great attention and efforts from many mathematicians: for instance, Lindelöf [287], Plemelj [356], Gronwall [174], Löwner ([293], [294]), Privalov [362], Nevanlinna [332], Landau [274] and Littlewood [288] are among others, who continue the study in this area around those decades (see also [8], [60], [145], [150], [181], [190], [264], [309], [343], [384], [385], [397], [398]) and ([39], [188], [207], [229], [243], [344], [348], [403], [426]) for some further studies and recent results in this field.

Many expositions of a multitude of interesting results in the Theory of Univalent Functions have been published by Montel [324] in his monograph entitled "*Leçons sur les Fonctions Univalentes ou Multivalentes"*, which was published in Paris in 1933. He noted in [324] that the origin of the Theory of Univalent Function comes from the general problem of conformal mapping, which maps the simply connected domain to another simply connected domain in a complex plane $\mathbb{C}$. which is a powerful impetus to the "*Verzerrungssats*" of Koebe's and Bieberbach's work. The volumes of material in the filed of Univalent Function

[6] The terns holomorphic and regular are also frequently used for analytic complex-valued functions.

Theory increased rapidly around the middle of the 20th century. According to the two bibliography manuscripts, which were prepared by Bernardi [61] in 1966 and 1975 respectively, there have been 570 authors extensively working in the field of Univalent and Multivalent Functions Theory since 1907. Hence, they were able to contribute 1694 research articles and monographs, with around 1563 of the research papers published within the following ten years. These two volumes of bibliography have provided valuable references to the many researchers on this subject. The "*Les Functions Multivalents*" is another remarkable survey monograph concerned with univalent and multivalent functions was produced in Paris, in 1983 by Biernacki [68].

Along with these, there are also many texts contributing to the field of Univalent Functions Theory and its related topics. For instance, the books of Goluzin [175], Nehari [330], Jenkins [221], Hayman [189] and Ahlfors [7], which cover this field in great detail by providing comprehensive surveys and some open problems. Other recent books that cover this subject in depth are the three different books entitled "*Univalent Functions*", and produced individually by Pommerenke [357] in 1975, Goodman [169] and Duren [144] in 1983. Other useful monographs that contribute to this area are (see for instance [183], [308], [320], [406]).

One of the most classical and remarkable results in Geometric Functions Theory is *Riemann Mapping Theorem*. Earlier than Koebe, in 1851, Riemann [374] has provided this important result in Geometric Function Theory by proving that there always exists a unique analytic function, which maps a simply connected domain onto another simply connected domain in a complex plane. Koebe [253] initiated the study of univalent functions in 1907, then in view of the Riemann Mapping Theorem, the study of the properties of analytic and univalent functions began.

The class A of all functions that are analytic in a simply connected domain in a complex plane and satisfy certain conditions of normalization, was introduced. Further, the the class of functions $S \subset A$ consisting of all analytic, univalent and normalized functions in the domain of definition, becomes the center of the study of Univalent Function Theory. This class of functions has attracted great attention from various researchers around the world. In order to fulfill the requirement of the Riemann Mapping Theorem, the domain of definition is generally selected as the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$, which is one of the simplest examples of simply connected domain in a complex plane. All of the analytic functions are conformal, in the sense that their angles are preserved under the mapping. Thus, we simply understand that all the analytic functions in the class S, map the unit disk D conformally and one-to-one onto another simply connected domain in a complex plane. The most interesting about this function is that, it can be represented by a power series expansion with real or complex coefficients and convergent on the unit disk D.

The crucial property of such functions is that, the image domain $f(D)$ will describe various nice geometries, such as *starlike* (or star-shaped), *spirallike*, *close-to-star*, *convex*, *close-to-convex*, etc. The functions map the unit disk D onto those geometrical domains with some interesting properties, which can be characterized in several ways, such as by analytical and geometrical conditions. The subclasses of S; such as the class of starlike, convex, close-to-convex functions, etc., were defined and some properties of functions in this subclasses exhaustively studied by several authors in the second half of the 20th century. In [144, p. 40-44], Duren provided an analytical description of functions in the class of starlike and convex (see also Pommerenke [357, p. 42-44]). Kaplan [235] introduced the class of close-to-convex functions and studied their properties of functions in this class. Some more general subclasses of S, such as the class of $\lambda-$convex, the class of close-to-convex of order α and type β, the class of Bazilević functions, etc., have been studied by numerous authors (see for instance [54], [108], [110], [164], [182], [219], [206], [309], [319], [328], [336], [376], [400], [418]). Libera [283] introduced an integral operator and investigated some properties of starlike functions under this operator. Meanwhile, Sălăgean [383] studied the class of analytic functions defined by differential operator. These works opened new ways to study the operators in Geometric Function Theory. Hence, afterwards, many studies have been conducted by many authors, which attempt to generalize and define various subclasses of analytic functions involving the integral and differential operators (see for instance [62], [337], [343], [344], [382], [383]).

Generally, an analytic and univalent function, which is defined in a unit disk of a complex domain, lie in a complex plane by various properties and conditions. Therefore, a general problem in the study of Geometric Function Theory is to investigate the properties of such functions in various subclasses of S. As a part of the study of geometric functions, the necessary and sufficient conditions for the existing functions to be univalent in the unit disk, is one of the fundamental studies in the Theory of Univalent Function.

1.2 Basic Properties of Power Series

One of the most important applications of the power series is that, it provides a useful way of representing some fundamental functions in real and complex analysis. Therefore, the convergent property of the power series plays an important role to guarantee that it converges back to the given function in its domain. Generally, the power series (1.4) may be converge or diverge in a certain domain in a complex plane.

Theorem 3 ([5, p. 38]) *Suppose that the power series is given by (1.4). Then, one of the following happens:*

(a) The series (1.4) converges only when $z = 0$.

(b) The series (1.4) converges absolutely for all $z \in \mathbb{C}$.

(c) There is a number R, $R > 0$ *such that the series (1.4) converges absolutely for all* $z \in \mathbb{C}$ *with* $|z| < R$, *and it diverges for all* $|z| > R$.

The number R described in Theorem 3 is known as the *radius of convergence* of the power series (1.4). Obviously, by allowing $R = 0$ in the case (i) and $R = \infty$ in the case (ii) as above, we can consider that every power series has a positive radius of convergence. The set of all $z \in \mathbb{C}$ such that $|z| < R$ is called the *domain* or *disk of convergence* of the series (1.4). The set of all $z \in \mathbb{C}$ with $|z| = R$ is called the *circle of convergence* of the series (1.4), and nothing is claimed about the convergence on this circle, only the continuity of the series along the line segment from the origin to the point z.

Power series are classified not only whether they converge or diverge, but also by the properties of the terms a_k (i.e., absolute or conditional convergence) and by the types of convergence of the series (i.e., pointwise or uniform). The following results describe various properties of the power series.

Theorem 4 *The power series (1.4) is said to be convergent absolutely if the series of absolute values, i.e.,* $\sum_{k=0}^{\infty} \left|a_k z^k\right|$ *converges. It is said to be conditionally convergent (or semi convergent) if the series (1.4) is convergent but not absolutely convergent.*

Theorem 5 ([359, p. 174]) *If the power series (1.4) converges when* $z = z_1$ $(z_1 \neq 0)$*, then it is absolutely convergent at each point* z *in the open disk* $|z| < |z_1|$.

Theorem 6 ([5, p. 151]) *If* z_1 *is a point inside the circle of convergence* $|z| = R$ *of the power series (1.4), then the series converges absolutely and uniformly in the close disk* $|z| < |z_1|$.

Theorem 7 ([166, p. 28]) *Let (1.4) be a power series with radius of convergence* R*,* $R > 0$*. If* $0 < \rho < R$*, then the power series (1.4) converges normally, absolutely and uniformly for* $|z| \leq \rho$*. The power series (1.4) diverges for* $|z| > \rho$.

Theorem 8 ([166, p. 28]) *Let (1.4) be a power series and convergent on* $|z| < R$*,* $R > 0$*. Then the function* $f(z)$ *which is represented by the power series (1.4) continuous all* $z \in \mathbb{C}$ *with* $|z| < R$.

Several methods to determine the convergent of the infinite series are discussed in the literature, for instance, the comparison test, the integral test, the ratio test, the roots test, the Cauchy-Hadamard formula, etc., (see [5, p. 152], [70, p. 451-461], [98, p. 125-134], [166, p. 31], [358, p. 331-336]). The *Ratio Test* is commonly used, fairly simple and applicable in the case of the power series with real or nonnegative coefficients. The following result provides a rule for the convergence test of the power series by using this test. The power series given by (1.4) converges absolutely if

$$\lim_{k\to\infty} \left|\frac{a_{k+1} z^{k+1}}{a_k z^k}\right| < 1, \tag{1.6}$$

provided that the limit exists. This property is equivalent to

$$|z| \leq R = \lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right|, \tag{1.7}$$

where R is the radius of convergence of the power series (1.4). Thus, we can determine the value R of radius of convergence as well through this result (1.7).

Remark 9 *Throughout this study, we mostly considered the simplest and the most widely used form of the power series (1.4), which is convergent in the open disk* $|z| < R$, $R > 0$.

The algebraic operations (such as addition, multiplication, division), differentiation and integration, can be applied for the power series as well, where the resulting new power series are convergent inside of their common domain of convergence. Let, the two power series $f(z)$ be of the form (1.4) and $g(z) = \sum_{k=0}^{\infty} b_k z^k$, which are convergent for all $|z| < R_f$ and $|z| < R_g$, respectively. Then, the following rules of algebra apply (see [4, p. 15], [5, p. 155], [26, p. 14], [93], [276], [359, p. 186-189]):

(i) The two convergent power series are equal if their corresponding terms are identical. That is, if $f(z) = g(z)$, wherever the two series converge, then $a_k = b_k$ for each $k \in \{0, 1, 2, \ldots\}$.

(ii) The two or more power series can be combined through termwise by addition or subtraction, that is,

$$f(z) \pm g(z) = \sum_{k=0}^{\infty} (a_k \pm b_k) z^k, \tag{1.8}$$

for all $z \in \mathbb{C}$ such that $|z| < \min\{R_f, R_g\}$. For a real number λ, the multiple of a series converges with the same radius of convergence and

$$\lambda f(z) = \sum_{k=0}^{\infty} \lambda a_k z^k. \tag{1.9}$$

(iii) If $g(z) \neq 0$ and $b_0 \neq 0$ in the disk $|z| < R_g$, then the quotient

$$\frac{f(z)}{g(z)} = \sum_{k=0}^{\infty} c_k z^k, \tag{1.10}$$

where the coefficients satisfy the equation $a_k = \sum_{j=0}^{\infty} b_{k-j} c_j$, and the radius of convergence of the series (1.10) is $R = \min\{R_f, R_g\}$.

(iv) The *Cauchy products* of $f(z)$ and $g(z)$ is defined by

$$f(z)g(z) = \sum_{k=0}^{\infty} \left[\sum_{j=0}^{k} a_j b_{k-j} \right] z^k = \sum_{k=0}^{\infty} \left[\sum_{j=0}^{k} a_{k-j} b_k \right] z^k, \tag{1.11}$$

which converges with the radius of convergence $R = \min\{R_f, R_g\}$. The double sum of the infinite series of the coefficients,

$$\sum_{k=0}^{\infty}\sum_{j=0}^{k} a_j b_{k-j} = \sum_{k=0}^{\infty}\sum_{j=0}^{k} a_{k-j} b_k, \tag{1.12}$$

is known as the *convolution* of the sequences $\{a_k\}$ and $\{b_k\}$, $k \in \{0, 1, 2, \ldots\}$. Some other basic rules of the multiplication of the series of coefficients are

$$\begin{aligned}
&\text{(a)}\quad c\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty}(ca_k), \quad c \in \mathbb{R},\\
&\text{(b)}\quad \left(\sum_{j=0}^{\infty} a_j\right)\left(\sum_{k=0}^{\infty} b_k\right) = \sum_{j=0}^{\infty} a_j\left(\sum_{k=0}^{\infty} b_k\right) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_j b_k,\\
&\text{(c)}\quad \left(\sum_{j=0}^{\infty} a_j\right)^2 = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_j a_k.
\end{aligned} \tag{1.13}$$

(v) For a convergent power series, it always converges to a continuous function. Thus, the power series can always be differentiated and integrated termwise within in its domain of convergence, that is [5, p. 153],

$$\frac{df(z)}{dz} = \sum_{k=1}^{\infty} ka_k z^{k-1} = \sum_{k=0}^{\infty}(k+1)\,a_{k+1}z^k, \quad |z| < R_f \tag{1.14}$$

and

$$\int_0^z f(t)dt = \sum_{k=0}^{\infty}\frac{a_k}{k+1}z^{k+1} = \sum_{k=1}^{\infty}\frac{a_{k-1}}{k}z^k, \quad |z| < R_f, \tag{1.15}$$

by assuming that the summation $\sum$, differentiation $\frac{d}{dz}$ and integration $\int$ are interchanged. Clearly, both of the newly derived series in (1.14) and (1.15) have the same radius of convergence as the original series (1.4).

(vi) Let $f(z)$ and $g(z)$ be the two power series, which are both convergent in the unit disk $|z| < 1$. Then, the *convolution* or *Hadamard product* of $f(z)$ and $g(z)$ is denoted by $f * g$, and it is defined by the power series

$$(f * g)(z) = \sum_{k=1}^{\infty} a_k b_k z^k, \quad |z| < 1, \tag{1.16}$$

where, the geometric series

$$h(z) = \sum_{k=1}^{\infty} z^k = \frac{z}{1-z}, \quad |z| < 1 \tag{1.17}$$

is the identity function under this convolution (1.16), i.e., $(f * h)(z) = f(z)$. The *integral convolution*, which is denoted by $f \circledast g$, is defined by

$$(f \circledast g)(z) = \sum_{k=1}^{\infty} \frac{a_k b_k}{k} z^k = \int_0^z \frac{h(t)}{t} dt, \tag{1.18}$$

see ([144, p. 246-247], [357, p. 49]).

1.3 Power Series Representations

There are many fundamental real or complex functions that can be represented by the power series expansions with real or complex coefficients. In this section, we give some examples of the functions and their corresponding power series expansions in complex variable.

(1) Geometric series and their variants:

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k, \tag{1.19}$$

$$\frac{1}{1+z} = \sum_{k=0}^{\infty} (-1)^k z^k, \tag{1.20}$$

$$\frac{z}{1-z} = \sum_{k=1}^{\infty} z^k, \tag{1.21}$$

$$\frac{1}{(1-z)^2} = \sum_{k=0}^{\infty} k z^{k-1}, \tag{1.22}$$

$$\frac{z}{(1-z)^2} = \sum_{k=0}^{\infty} k z^k, \tag{1.23}$$

$$\sqrt{1+z} = \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{2^{2k} (1-2k) (k!)^2} z^k, \tag{1.24}$$

for all $z \in \mathbb{C}$ such that $|z| < 1$, where $k!$ denotes the *factorial function.*

(2) Exponential functions and their variants:

$$\exp(z) = \sum_{k=0}^{\infty} \frac{1}{k!} z^k, \quad \text{for any } z \in \mathbb{C}, \tag{1.25}$$

$$\exp(-z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} z^k, \quad \text{for any } z \in \mathbb{C}, \tag{1.26}$$

$$\exp(\sqrt{z}) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} z^k, \quad \text{for any } z \in \mathbb{C}, \tag{1.27}$$

$$z\exp(z) = \sum_{k=0}^{\infty} \frac{1}{(k-1)!} z^k, \quad \text{for any } z \in \mathbb{C}, \tag{1.28}$$

$$\left(z + z^2\right)\exp(z) = \sum_{k=0}^{\infty} \frac{k}{(k-1)!} z^k, \quad \text{for any } z \in \mathbb{C}. \tag{1.29}$$

(3) Natural logarithmic functions:

$$\ln\left(\frac{1}{1+z}\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} z^k, \quad \text{for } |z| < 1, \tag{1.30}$$

$$\ln\left(\frac{1}{1-z}\right) = \sum_{k=1}^{\infty} \frac{1}{k} z^k, \quad \text{for } |z| < 1. \tag{1.31}$$

(4) Trigonometric and the inverse of trigonometric functions:

$$\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}, \quad \text{for all } z \in \mathbb{C}, \tag{1.32}$$

$$\cos(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}, \quad \text{for all } z \in \mathbb{C}, \tag{1.33}$$

$$\tan(z) = \sum_{k=0}^{\infty} \frac{(4)^k \left(4^k - 1\right) |B_{2k}|}{(2k)!} z^{2k-1}, \quad \text{for } |z| < \frac{\pi}{2}, \tag{1.34}$$

$$\arcsin(z) = \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k} (k!)^2 (2k+1)} z^{2k+1}, \quad \text{for } |z| \leq 1, \tag{1.35}$$

$$\arccos(z) = \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2(2k+1)} z^{2n+1}, \quad \text{for } |z| < 1, \tag{1.36}$$

$$\arctan(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} z^{2k+1}, \quad \text{for } |z| < 1, \tag{1.37}$$

where B_{2k} denotes the *Bernoulli numbers*[7] (see [25, p. 12], [102, p. 201], [305, p. 50-53]).

(5) Hyperbolic and the inverse of hyperbolic functions:

$$\sinh(z) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} z^{2k+1}, \quad \text{for all } z \in \mathbb{C}, \tag{1.38}$$

$$\cosh(z) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} z^{2k}, \quad \text{for all } z \in \mathbb{C}, \tag{1.39}$$

$$\tanh(z) = \sum_{k=1}^{\infty} \frac{4^k\left(4^k-1\right)B_{2k}}{(2k)!} z^{2k-1}, \quad \text{for } |z| < \frac{\pi}{2}, \tag{1.40}$$

$$\coth(z) = \sum_{k=1}^{\infty} \frac{4^k B_{2k}}{(2k)!} z^{2k-1}, \quad \text{for } |z| < \pi, \tag{1.41}$$

$$\operatorname{ar}\sinh(z) = \sum_{k=0}^{\infty} \frac{(-1)^k(2k)!}{4^k(k!)^2(2k+1)} z^{2k+1}, \quad \text{for } |z| < 1, \tag{1.42}$$

$$\operatorname{ar}\tanh(z) = \sum_{k=0}^{\infty} \frac{1}{2k+1} z^{2k+1}, \quad \text{for } |z| < 1. \tag{1.43}$$

Instead of defining the trigonometric and hyperbolic functions of sine, cosine, sinh and cosh by their power series expansions, it is possible and might be useful to define them directly in terms of the exponential functions as follows:

$$\begin{aligned} \sin(z) &= \frac{1}{2i}\left(e^{iz}-e^{-iz}\right), \qquad \cos(z) = \frac{1}{2}\left(e^{iz}+e^{-iz}\right), \\ \sinh(z) &= \frac{1}{2}\left(e^{z}-e^{-z}\right), \qquad \cosh(z) = \frac{1}{2}\left(e^{z}+e^{-z}\right), \end{aligned} \tag{1.44}$$

[7] The first few Bernoulli numbers are $B_0 = 1$, $B_1 = \pm 1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, $B_5 = 0$, $B_6 = 1/42...$, which are derived from the recursion formula: $B_{2k} = \frac{1}{2} - \frac{1}{2k+1}\sum_{m=0}^{k-1}\binom{2k+1}{2m}B_{2m}$ for $k \geq 1$ and $B_{2k+1} = 0$, $k \geq 1$..

for all $z \in \mathbb{C}$.

Some of the above series can be found in the Handbooks of Abramowitz and Stegun [4, p. 68-69, 81-85] and Bronshtein et al. [77, p. 698-699]. The concept of the power series is also widely used in the Theory of Special Functions, in which, are defined the special functions of Polylogarithm, hypergeometric, Bessel and modified Bessel functions, etc. that we will discuss later in Chapter 5. Nowadays, there are numerous aids of modern computer software, such as MAPLE, MATHEMATICA, MATLAB, etc. that can be easily used in finding the series expansion of a given function.

Some numerical infinite series, which are given in the following might be useful for the subsequence chapters:

$$\begin{aligned}
&\text{(a)}\quad \sum_{k=0}^{\infty}\frac{1}{2^k}=2, &&\text{(b)}\quad \sum_{k=0}^{\infty}\frac{(-1)^k}{2^k}=\frac{2}{3},\\
&\text{(c)}\quad \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}=\ln 2, &&\text{(d)}\quad \sum_{k=0}^{\infty}\frac{1}{(k+1)^2}=\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6},\\
&\text{(e)}\quad \sum_{k=1}^{\infty}\frac{1}{k^4}=\frac{\pi^4}{90}, &&\text{(f)}\quad \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)}=\frac{\pi}{4}\\
&\text{(g)}\quad \sum_{k=1}^{\infty}\frac{1}{k^6}=\frac{\pi^6}{945}, &&\text{(h)}\quad \sum_{k=1}^{\infty}\frac{1}{(nk)^2}=\frac{\pi^2}{6n^2},\ n>0.
\end{aligned} \tag{1.45}$$

The following section discusses the main motivation for writing this dissertation.

1.4 Motivation

In the early 1820s, Cauchy [88] published his famous inequality involving two sequences of real numbers, which is popularly known in the literature as Cauchy's inequality. Long after Cauchy, in 1885, Schwarz [389] provided the generalization of Cauchy's inequality in an inner product space, which is later known as Schwarz's or Cauchy-Schwarz's inequality. In 1888, Bunyakovsky interfered and proved the corresponding version of Cauchy's inequality for integrals. Since them, the inequalities, which are now referred to the Cauchy-Bunyakovsky-Schwarz type inequalities, have attracted the attention of a large number of researchers, and hence, have stimulated new research directions and influenced various aspects of mathematical analysis and applications.

As noted in the celebrated monograph "*Analytic Inequalities*" written by Mitrinović [316] in 1970, various remarkable results related to the celebrated Cauchy-Bunyakovsky-Schwarz inequality were established. Also, a large number of research papers have been published since 1970 providing various improvements, refinements and generalizations of this type of the famous inequality; see for instance [19], [20], [78], [116], [128], [138], [147], [269], [302], [408], [445] and the references cited therein. All those results that have been established in the literature, are mainly concerned with the finite sequences and finite sums. Consequently, much effort has been made by several authors to extend the inequalities to infinite sequences and sums, functions and integrals, in order to provide more general results that are useful in various fields of Modern Mathematics and its applications (see [22], [49], [52], [173]).

A revival of the interest in this area occurred in recent years with several results in analytic inequalities involving functions defined by the convergent power series. The approach of the power series functions in the Inequalities Theory has been explored by several researchers, see particularly [89], [122], [123], [173]. This research is mainly motivated by Cerone and Dragomir's work in [89] in 2007, who established some inequalities concerning functions defined by the convergent power series with real or nonnegative coefficients by utilising a refinement of the Cauchy-Bunyakovsky-Schwarz inequality, which is known in the literature as the de Bruijn inequality. They also obtained some particular inequalities by applying the results for fundamental functions of interest such as the exponential, logarithm, trigonometric and hyperbolic functions. Applications for special function such as polylogarithm functions are provided as well in [89].

In this study, we employ the techniques developed in [89], to derive new and better inequalities for functions defined by the power series with real or nonnegative coefficients, and convergent on an open disk. Utilising the classical results that have been available in the literature such as Buzano's inequality [82] and Schwarz's result in inner product spaces due to Dragomir [118], we provided some refinements and improvements of some known results, which are related to the celebrated Cauchy-Bunyakovsky-Schwarz inequality for functions defined by the power series. Besides Buzano's and Schwarz's results, the other crucial tools that have been used for our investigations are Young's, Hölder's and Jensen type inequalities, as well as their refinements, reverses and counterparts. Particular examples that are related to some fundamental real or complex functions such as exponential, logarithm, trigonometric and hyperbolic functions are presented. Some applications for special functions such as polylogarithm, hypergeometric, Bessel and modified Bessel functions of the first kind are presented as well in the first part of this dissertation.

Geometric Function Theory is the branch of Complex Analysis, which deals with the geometric properties of analytic functions, founded around the turn of the 20th century. The cornerstone of this theory is the Theory of Univalent Functions, which is mainly concerned with the analytic and univalent functions $f(z)$ in a unit disk, and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. It is a fact that all functions in the class S, which are analytic and univalent function in a unit disk can be represented by convergent power series with real or complex coefficients.

Conjecture Bieberbach [67] is one of the major problems in the field of Geometric Function Theory, which asserts the modulus of the k-th coefficient in the power series expansion of functions in the class S is less than or equal to k for all $k \geq 2$. In spite of this famous coefficient estimate that was completely solved by de Brange [75] in 1984, it suggests various approaches and directions for the study of Univalent Function Theory, and Geometric Function Theory as a whole. Hence, various subclasses of analytic and univalent functions such as starlike, spirallike, convex, close-to-convex, etc., have been introduced by numerous researchers in the last few decades and developed many interesting properties of functions in these classes.

In the other direction, the new subclasses of analytic functions, which are defined involving operators were introduced. The Sălăgean differential operator D^n, $n \in \mathbb{N}_0$, is one of the famous operators, which was established by Sălăgean in [383]. In [343], Opoola introduced and studied the new class $T_n^{\alpha}(\beta)$, $\alpha > 0$, $0 \leq \beta < 1$, $n \in \mathbb{N} \cup \{0\}$ of analytic functions, which satisfies certain conditions involving the Sălăgean differential operator. Regarding this class, we investigate in second part of this dissertation, some properties of functions in this class. In order to stimulate further investigation in this field and the related topics, we introduce the class $\widetilde{T}_n^{\alpha}(\beta)$ and $T_{n,\lambda}^{\alpha}(\beta)$, $\lambda \geq 1$, and provide new properties and coefficient inequalities of functions in these classes. The properties of the generalized Sălăgean operators $D^n f^{\alpha}$ and $D_{\lambda}^n f^{\alpha}$ are also established. We conclude the second part of this dissertation with some results on the Fekete-Szegö theorem concerning analytic and univalent functions.

Functions in the class of analytic and univalent functions S and its subclasses, which map the unit disk or half-plane conformally onto another simply connected domain in a complex plane, have found many applications in engineering, physics, electronics, medicines and other branches of applied mathematics (see [153], [238]).

1.5 Thesis Outline

This dissertation is partitioned into two main parts, where each part is further divided into several chapters that have a number of sections and subsections. Overall, this dissertation contains eight chapters including Chapter 1 as above.

In the first chapter, we provide a brief introduction to the power series, Inequalities Theory and Univalent Function Theory. The power series that are the most important elements throughout this dissertation, are also discussed in this chapter. Some examples of analytic functions with their corresponding power series expansions are given as well for useful references in the subsequent chapters.

Part I consists of Chapter 2 - Chapter 5, and mainly deals with Analytic Inequalities in Real and Complex Analysis. The main purpose of this part is to establish some inequalities concerning the power series with real or nonnegative coefficients. Chapter 2 provides some basic concepts and introductory materials in Inequalities Theory including the triangle, means and convexity inequalities. Some of the most important and classical inequalities such as the Cauchy-Bunyakovsky-Schwarz, Young's, Hölder's and Jensen type inequalities are emphasized in order to provide an essential background for the subsequent chapters.

In chapter 3, we derive new and better inequalities for functions defined by the power series, which are related to the celebrated Cauchy-Bunyakovsky-Schwarz inequality. In particular, we provide some refinements and improvements of the Cauchy-Bunyakovsky-Schwarz inequality by utilising Buzano's and Schwarz's results in inner product spaces, and the technique based on the continuity properties of modulus. Particular inequalities that are related to some fundamental functions of interest such as the exponential, logarithm, trigonometric and hyperbolic functions are obtained.

More inequalities for functions defined by the power series are established in Chapter 4. To develop some inequalities for the power series by utilising the standard of Young's inequality, the refinement and the reverse of Young's inequality, the Jensen type inequality and one of its reverses due to Dragomir and Ionescu [139], is the main purpose of this chapter. Applications for some fundamental functions of interest are also presented.

Chapter 5 is devoted to some inequalities for special functions. In this chapter, we employ some results established in Chapter 3 and Chapter 4 to obtain particular inequalities involving special functions such as polylogarithm, hypergeometric, Bessel and modified Bessel functions of the first kinds.

Part II consists of Chapter 6 - Chapter 7, devotes to the study of Univalent Function Theory. In this part, we mainly concern with the class of functions, which are analytic and univalent in a unit disk, and normalized by a standard conditions. To investigate some properties of certain subclasses of analytic and univalent functions is the main objective throughout this part.

Chapter 6 is written as a reference point for the subsequent chapter. Some fundamental facts and basic concepts regarding analytic and univalent functions are provided. We emphasize the class A of analytic functions and the class S of analytic, univalent and normalized functions in the unit disk D. Some basic properties and coefficient inequalities of functions in the class S and its subclasses are presented. We also present the well-known Sălăgean differential operators and their related subclasses at the end of this chapter.

Chapter 7 presents some new properties and coefficients inequalities for functions in certain subclasses S. The properties of the generalized Sălăgean differential operator [383] are also given. Some results of the Fekete-Szegö functional concerning analytic and univalent functions are included at the end of this chapter.

Finally, we summarize the work of this dissertation in Chapter 8, which contains the summary and main achievements of the work.

The various theorems, corollaries, lemmas, propositions, remarks and examples are numbered in order, consecutively throughout the thesis, whereas, the equations are numbered consecutively within each chapter. The end of the proof of a theorem, corollary or lemma is indicated by a solid square '■'.

Part I

ANALYTIC INEQUALITIES IN REAL AND COMPLEX ANALYSIS

Chapter 2

Elementary and Some Classical Inequalities

Chapter 2 provides some essential background for the subsequent chapters of the first part of this dissertation. This includes the introductory materials and basic inequalities in Real and Complex Analysis. The elementary inequalities such as triangle, means and convexity inequalities are discussed in Section 2.2. The basic properties of normed, linear and inner product spaces are also addressed.

We emphasize in Section 2.3 the basic inequalities of the classical Cauchy-Bunyakovsky-Schwarz type in real and complex numbers, and in inner product spaces. The corresponding version of the Cauchy-Bunyakovsky-Schwarz inequality for functions defined by the power series is also given. Some results related to thsese types of inequalities such as the de Bruijn inequality, the Buzano's and Schwarz's results are mentioned as a foundation for the next chapters. Young's inequality and its variants, and Hölder's inequality and its generalizations to the power series functions are briefly discussed in Section 2.4 and Section 2.5, respectively. Inequalities of the Jensen type and their reverses are given in Section 2.6. Some other well-known inequalities in analysis are provided at the end of this chapter.

All the results are given without proof. Some of their proof can be found in the literature, see for instance, the books of Beckenbach and Bellman [55], Bullen [79], Hardy, Littlewood and Pólya [185], Mitrinović [316], Mitrinović, Pečarić and

Fink [318] and the numerous textbooks devoted to the Inequalities Theory and their related topics.

2.1 Symbols and Notations

We start this chapter by introducing some of the symbols and notations that we shall use in this dissertation. First, let z, z_0 be two points in a complex plane $\mathbb{C}$ and R is a positive real number. Then, the set of all points that satisfies the inequality $|z - z_0| < R$ is called the *open disk* (or simply the *disk*) with center z_0 and radius R [44, p. 13]. It is denoted by $D(z_0, R)$, i.e.,

$$D(z_0, R) = \{z \in \mathbb{C} : |z - z_0| < R\}. \tag{2.1}$$

The boundary is denoted by $\partial D(z_0, R)$, where $|z - z_0| = R$, while, the closure $\overline{D}(z_0, R)$ contains the set of all points given by (2.1), including its boundary points. If the center of the open disk is zero (i.e., $z_0 = 0$), then from (2.1) we have

$$D(0, R) = \{z \in \mathbb{C} : |z| < R\}. \tag{2.2}$$

The *open unit disk* (or simply *unit disk*) is the disk centered at the origin and the radius is 1 (see [357, p. 10]), i.e.,

$$D = D(0, 1) = \{z \in \mathbb{C} : |z| < 1\}. \tag{2.3}$$

Let n be a nonnegative integer and $a \in \mathbb{C}$ such that $a \neq \{\ldots, -2, -1\}$. Then, the *Pochhammer symbol* $(a)_n$ is defined by (see [4, p. 256], [404])

$$(a)_n = \begin{cases} 1, & \text{for } n = 0; \\ a(a+1)\cdots(a+n-1), & \text{for } n \geq 1. \end{cases} \tag{2.4}$$

This symbol, which is also known as the *rising factorial* or the *shifted factorial function,* can also be expressed in the following form (see [26, p. 86]):

$$(a)_n = \prod_{j=1}^{n} (a + j - 1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n = 1, 2, \ldots, \tag{2.5}$$

for a, $a+n \in \mathbb{C}\backslash\{\ldots,-2,-1\}$, where $\Gamma(z)$ is the *gamma function* defined by $\Gamma(z) = \int_0^\infty t^{z-1}e^t dt$, for any $z \in \mathbb{C}$ such that $\operatorname{Re}(z) > 0$. Other notations such as $a^{\overline{n}}$, $a^{(n)}$ and (a,n) may be found in the literature, which represent the Pochhammer symbol as well.

The $(a)_n$ has simple values at the arguments 0 and 1, i.e.,

$$(0)_0 = 1, \quad \text{and} \quad (1)_n = n! \tag{2.6}$$

Some other useful formulas associated with the Pochhammer symbol are given in the following:

$$\begin{aligned} (a)_{-n} &= \frac{(-1)^n}{(1-a)_n}, \qquad & (a)_n &= \frac{n!}{(a-n)!} = \frac{(a+n-1)!}{(a-1)!}, \\ (a+1)_n &= \frac{(a+n)!}{n!}, \qquad & (a)_{n+k} &= (a+k)_k\,(a)_n\,, \end{aligned} \tag{2.7}$$

where n and k are nonnegative integers. The Pochhammer symbol is widely used in the Theory of Special Functions, particularly to represent the coefficients of hypergeometric functions.

A *binomial series* is an infinite series defined by

$$(1+z)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k, \tag{2.8}$$

where α is an arbitrary complex number. The series in the right hand-side of (2.8) converges absolutely for all $z \in D$, with the generalized *binomial coefficients* defined by

$$\binom{\alpha}{k} = \frac{\alpha!}{(\alpha-k)!k!}, \tag{2.9}$$

for any $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}_0$. Some well-known formulas of binomial coefficients are given as follows:

$$\begin{aligned} \binom{\alpha}{0} &= \binom{\alpha}{\alpha} = 1, \qquad & \binom{\alpha}{1} &= \binom{\alpha}{\alpha-1} = \alpha, \\ \binom{\alpha}{k} &= \binom{\alpha}{\alpha-k}, \qquad & \binom{-\alpha}{k} &= (-1)^k \binom{\alpha+k-1}{k}, \end{aligned} \tag{2.10}$$

which holding for all $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}_0$.

A *real signum* or *real sign function* is a function denoted by $y = sgn\,(x)$ (read 'y is equal to sign of x'), and it is defined by the equations

$$sgn\,(x) = \begin{cases} -1, & \text{for } x < 0; \\ 0, & \text{for } x = 0; \\ 1, & \text{for } x > 0. \end{cases} \tag{2.11}$$

In the other interactive way, the function $y = sgn(x)$ is related to the function $y = |x|$, where its moment's reflection shows that for each real number a,

$$|a| = a\ sgn\,(a) \quad \text{or equivalent to} \quad a = |a|\, sgn\,(a)\,. \tag{2.12}$$

On the other hand, the equation in (2.12) provides another characterization of the absolute value of a real number a. In the complex plane, a *complex sign function* is defined by

$$sgn\,(z) = \begin{cases} \dfrac{z}{|z|}, & \text{for } z \in \mathbb{C}\setminus\{0\}\,; \\ 0, & \text{for } z = 0. \end{cases} \tag{2.13}$$

The two positive real numbers p, q such that

$$\frac{1}{p} + \frac{1}{q} = 1, \tag{2.14}$$

are called the *conjugate exponent.* For $p = 1$, the conjugate exponents of p is $q = \infty$. If p and q are integers, then the only pair of the conjugate exponents is 2, 2.

2.2 Elementary Inequalities in Analysis

We shall begin this section with some basic inequalities in Real and Complex Analysis. These include the triangle inequality, mean inequalities and the inequalities involving convex functions.

2.2.1 Basic Inequalities

There are many basic facts, which are fundamental in the Theory of Inequalities. First, we state the two simple propositions called *axioms* (see [55, p. 7], [239, p. 2]):

Axiom 10 *A given real number x satisfies precisely one of the following: $x = 0$, $x > 0$ or $x < 0$.*

Axiom 11 *If x and y are positive numbers, then the sum $x + y$ and the product xy are positive numbers.*

Consequently, there are obvious variations and extensions of these axioms, for example,

Axiom 12 *If $x,\ y \in \mathbb{R}$, then one and only one of the following relationships holds:*

$$x = y, \qquad x > y \qquad \textit{or} \qquad x < y, \tag{2.15}$$

where the symbols "$>$" and "$<$" represent the strict inequalities.

The following fundamental rules of algebra also hold for all the real numbers $x,\ y,\ z \in \mathbb{R}$ (see [55, p. 7-21]):

(a) If $x \geq y$ and $y \geq z$, then $x \geq z$.

(b) If $x \leq y$, then $x + a \leq y + a$ for any $a \in \mathbb{R}$.

(c) If $x \geq y$ and $a > 0$, then $ax \geq ay$ and $\dfrac{x}{a} \geq \dfrac{y}{a}$. If $a < 0$, then $ax \leq ay$ and $\dfrac{x}{a} \leq \dfrac{y}{a}$.

(d) If $x > y$ and $a > b$, then $x + a > y + b$ for any $a,\ b \in \mathbb{R}$.

(e) If $x > y > 0$ and a, b are positive integers, then $x^{a/b} > y^{a/b}$.

The next result is of fundamental importance and frequently used in dealing with inequalities.

Theorem 13 ([55, p. 10]) *The product xy of a positive number x and negative number y is a negative number, while, the product of two negative numbers x and y is a positive number.*

We observe that the square of any real number can never be negative. Thus, we obtain one of the simplest and most useful results in the Theory of Inequalities as follows.

Theorem 14 ([55, p. 11]) *Any real number x satisfies the inequality $x^2 \geq 0$, with the equality holds if and only if $x = 0$.*

Accordingly, for any two real numbers a and b, we have the quantity that $(a-b)^2 \geq 0$, which implies

$$\frac{a^2+b^2}{2} \geq ab, \tag{2.16}$$

with equality occurring in (2.16) if and only if $a = b$. If $a > 0$ and $b = 1$ in (2.16), then we have

$$a + \frac{1}{a} \geq 2. \tag{2.17}$$

The inequality (2.16) can be written in the form [55, p. 48]

$$\frac{x+y}{2} \geq \sqrt{xy}, \tag{2.18}$$

by replacing a^2 by x and b^2 by y, with x and y being positive real numbers. Clearly, the equality in (2.18) occurs if and only if $x = y$. This inequality is often called the *arithmetic mean-geometric mean inequality* for the two positive real numbers (we shall discuss further this inequality in Section 2.2.3).

Analogously, for complex numbers, $|z-w|^2 \geq 0$, we have

$$\frac{|z|^2+|w|^2}{2} \geq \operatorname{Re}(z\overline{w}), \tag{2.19}$$

for any z, $w \in \mathbb{C}$. The equality in (2.19) is achieved if and only if $z = w$.

The inequalities given by (2.16) and (2.19) are of fundamental importance in the study of Analytic Inequalities in Real and Complex Analysis. They have a much richer interpretation in the Theory of Inequalities as well. In fact, these bounds are the sources and the methods of proof of many important classical inequalities in analysis.

2.2.2 Triangle Inequality

If z and w are the two points in the complex plane, then we can associate the vectors from the origin to the each point. Now, we consider the triangle with the vertices are the origin, z and w, with the lengths of its sides are $|z|$, $|w|$ and $|z-w|$ (or $|w-z|$). Thus, using the geometrical fact that the sum of the lengths of two sides of a triangle is greater than or equal to the third side, implies the inequality

$$|z| + |w| \geq |z - w|, \tag{2.20}$$

with the sign of equality holding in (2.20) if and only if $\operatorname{Re}(z\overline{w}) = |zw|$. Since w represents an arbitrary complex number, then the inequality (2.20) still holds if w is replaced by $-w$, thus we obtain

$$|z| + |w| \geq |z + w|, \tag{2.21}$$

for all z, $w \in \mathbb{C}$, in which, the equality occurs again in (2.21) if and only if $\operatorname{Re}(z\overline{w}) = |zw|$. The inequality (2.21) is popularly known as the *triangle inequality* for two complex numbers. On the same basis, we also have the inequality

$$|z - w| \geq ||z| - |w||, \tag{2.22}$$

which is concerned with the length of the difference of the two complex numbers. We note that, if w is replaced by $-w$ in (2.22), it then yields

$$|z + w| \geq ||z| - |w||. \tag{2.23}$$

The inequality (2.23) is called the *reverse triangle inequality*, which provides the lower bounds instead of the upper bounds of $|z+w|$. Thus, for any z, $w \in \mathbb{C}$ we have

$$||z| - |w|| \leq |z \pm w| \leq |z| + |w|, \tag{2.24}$$

which obviously follows from the inequalities (2.20) - (2.23).

The triangle inequality in (2.21) can be generalized for n complex numbers $z_1, z_2, \ldots, z_n \in \mathbb{C}$ as follows:

$$\left|\sum_{j=1}^{n} z_j\right| \leq \sum_{j=1}^{n} |z_j|, \tag{2.25}$$

with the equality occurring in (2.25) if and only if the ratio of any nonzero terms is positive [4, p. 11]. In terms of their weighted version, the following inequality holds:

$$\left|\sum_{j=1}^{n} \alpha_j z_j\right| \leq \sum_{j=1}^{n} \alpha_j |z_j|, \tag{2.26}$$

where $\alpha_j > 0$ for $j \in \{1, 2, \ldots, n\}$ such that $\sum_{j=1}^{n} \alpha_j = 1$. The last two types of the triangle inequalities are the most important and frequently used in mathematical analysis.

2.2.3 Mean Inequalities

A *mean* is a notion of *average* for collection of numbers, data, etc. For instance, the *arithmetic mean* is the simplest type of means that is commonly used in Probability and Statistics Theory to measure the central tendency of numbers or data. Besides this mean, there are several types of means occurring in various contexts of measuring the central tendency of numbers, such as geometric mean and harmonic mean. The values of these means have an important relationship, that we shall emphasize in this section. Before that, we shall recall the basic definitions of the arithmetic, geometric and harmonic means for n positive real numbers.

Let $a = (a_1, a_2, \ldots, a_n)$ be a sequence of positive real numbers. Then, we denote the *arithmetic mean*, *geometric mean* and *harmonic mean* for n numbers by AM, GM and HM respectively, which are defined by the following equations (see [77, p. 19-20], [239, p. 16]):

$$\mathrm{AM}\,(a) = \frac{1}{n}\sum_{j=1}^{n} a_j, \quad \mathrm{GM}\,(a) = \left(\prod_{j=1}^{n} a_j\right)^{1/n}, \quad \mathrm{HM}\,(a) = n\left(\sum_{j=1}^{n} \tfrac{1}{a_j}\right)^{-1}. \tag{2.27}$$

The value of these means always lies between the lowest and the highest of its numbers. In the same manner, we can generalize these means into their weighted versions. Let us fix a set of positive real numbers $w_j > 0$ for $j \in \{1, 2, \ldots, n\}$. Then, the weighted version of the AM, GM and HM are defined by the following expressions:

$$\mathrm{AM}\,(w,a) = \sum_{j=1}^{n} w_j a_j \left(\sum_{j=1}^{n} w_j\right)^{-1}, \qquad \mathrm{GM}\,(w,a) = \left(\prod_{j=1}^{n} a_j^{w_j}\right)^{1/\sum_{j=1}^{n} w_j} \tag{2.28}$$

and

$$\mathrm{HM}\,(w,a) = \sum_{j=1}^{n} w_j \left(\sum_{j=1}^{n} \frac{w_j}{a_j}\right)^{-1}, \tag{2.29}$$

where the w_j's, $j \in \{1, 2, \ldots, n\}$ are called the *weights.* The ordinary means mentioned in (2.27) are the special case of these weighted means, when the weights w_j are equal for all $j \in \{1, 2, \ldots, n\}$.

The most important in the Theory of Inequalities is the relations between these means. As mentioned in the book of Kapur [237, p. 1] (see also [55, p. 54]), the relation HM $\leq$ AM $\leq$ GM holds for n positive real numbers. More precisely, we have the following result:

Theorem 15 ([55, p. 54]) *For any n nonnegative real numbers $a_1, a_n, ..., a_n$,*

$$\frac{1}{n}\sum_{j=1}^{n} a_j \geq \left(\prod_{j=1}^{n} a_j\right)^{\frac{1}{n}}. \tag{2.30}$$

The sign of equality holds in (2.30) if and only if all the numbers $a_1, a_2, \ldots, a_n$ are equal.

Since there is the obvious relation between the arithmetic mean and the harmonic mean, that is,

$$\mathrm{HM}(a) = \left[\mathrm{AM}\left(\frac{1}{a}\right)\right]^{-1}, \tag{2.31}$$

where $1/a = (1/a_1, 1/a_2, \ldots, 1/a_n)$ and a_j, $j \in \{1, 2, \ldots, n\}$ are positive real numbers, we can derive a similar inequality between the harmonic mean and geometric mean as follows:

$$\left(\prod_{j=1}^{n} a_j\right)^{1/n} \geq n\left(\sum_{j=1}^{n} \frac{1}{a_j}\right)^{-1}, \tag{2.32}$$

for $a_j > 0$, $j \in \{1, 2, \ldots, n\}$. Again, the case of equality occurs in (2.32) if and only if $a_1 = a_2 = \cdots = a_n$. The inequalities given by (2.30) and (2.32) are well-known in the literature as the *arithmetic mean-geometric mean inequality* and the *geometric mean-harmonic mean inequality* for n numbers respectively, or they are simply denoted by (AM-GM)-inequality and (GM-HM)-inequality. These inequalities are widely used in Optimization Theory, for instance, in solving of the optimization problems for two or more dimensional spaces, where the maximum or minimum achieved at the point of the equality of a given inequality occurs (see [55, Chapt. 5], [237, Chapt. 2]).

It can be clearly seen from (2.32) that the harmonic mean is bounded by the geometric mean, which in turn is bounded by the arithmetic mean (2.30). Hence, this string of the fundamental inequalities can be expressed as follows (see [4, p. 10], [237, p. 1]):

$$\frac{1}{n}\sum_{j=1}^{n} a_j \geq \left(\prod_{j=1}^{n} a_j\right)^{1/n} \geq n\left(\sum_{j=1}^{n} \frac{1}{a_j}\right)^{-1}, \tag{2.33}$$

for any positive real numbers $a_1, a_2, \ldots, a_n$. The equalities occur in (2.33) if and only if $a_1 = a_2 = \cdots = a_n$. The double inequality in (2.33) is often called the *arithmetic mean-geometric mean-harmonic mean inequality*, or simply denoted as the (AM-GM-HM)-inequality. The weighted means of the (AM-GM-HM)-inequality also holds [237, p. 225], namely

$$\sum_{j=1}^{n} w_j a_j \geq \prod_{j=1}^{n} a_j^{w_j} \geq \left(\sum_{j=1}^{n} \frac{w_j}{a_j}\right)^{-1}, \tag{2.34}$$

for $w_j > 0$, $j \in \{1, 2, \ldots, n\}$ such that $\sum_{j=1}^{n} w_j > 0$. The simplest case of (2.33) for two positive real numbers is that

$$\frac{x+y}{2} \geq \sqrt{xy} \geq \frac{2}{\frac{1}{x}+\frac{1}{y}}, \tag{2.35}$$

for any x, $y > 0$, where the first inequality in (2.35) has been mentioned in the previous section as (2.18). Hence, we also have from (2.34) that [91, p. 4]

$$\frac{\alpha a + \beta b}{\alpha + \beta} \leq a^{\frac{\alpha}{\alpha+\beta}} b^{\frac{\beta}{\alpha+\beta}} \leq \frac{\alpha+\beta}{\frac{\alpha}{a}+\frac{\beta}{b}}, \tag{2.36}$$

for the weighted version of the (AM-GM-HM)-inequality, with a, $b > 0$ and α, $\beta \geq 0$ such that $\alpha + \beta > 0$. The equality holds in (2.36) if and only if $a = b$. We note that by letting $p = (\alpha + \beta)/\alpha$, $q = (\alpha + \beta)/\beta$, $x = a^{\alpha/(\alpha+\beta)}$ and $b = y^{\beta/(\alpha+\beta)}$ in the first inequality of (2.36), it reduces to

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}, \tag{2.37}$$

for any x, $y \geq 0$, with p, $q > 1$ such that $1/p + 1/q = 1$. This inequality (2.37) is one the most important results in Analytic Inequalities, which provides a principal tool for deriving some classical inequalities in analysis; such as the Cauchy-Bunyakovsky-Schwarz's, Hölder, Minkowski inequalities, etc. It also happens to be a special case of the classical Young's inequality for the product of two real numbers. In the other sense, the result in (2.37) becomes one of the most important tools of our investigations in Chapter 4.

2.2.4 Convexity Inequalities

Convexity is a simple and natural notion of functions, which can be traced back to Archimedes (c. 250 B.C), in connection with his famous estimate of the value of π. It has a central role in the study of Convex Function Theory. Convex functions, which are extensively treated in various textbooks on calculus (see for instance [333], [352, Chapt. 2], [360], [375], [377]), play an extremely important role in many branches of mathematics as well as their applications in various fields.

The subject of convex functions was studied by Danish mathematician J. Jensen ([222], [223]) in the 1900s, although this class of functions was already treated by Ch. Hermite [195], O. Hölder [201], J. Hadamard [176] and O. Stolz [410]. In the third chapter of Hardy, Littlewood and Pólya's famous work [185] and the sixth chapter of Steela's work [407], they revealed that the convexity and its generalizations, play a crucial rule in the Theory of Inequalities. Many classical inequalities such as the triangle, Cauchy-Bunyakovsky-Schwarz, Hölder, Minkowski inequalities, etc., are closely related to the *convexity* of certain underlying functions.

Roughly speaking, there are several basic properties of convex functions that made them so widely used in theoretical and applied mathematics. As noted in [334, p. 11], the basic ideas of convex functions begins in the context of real-valued functions of a real variable. So, we have the following property of convex functions defined on an interval (see also [318, p. 1], [375, p. 2]).

Definition 16 *Let I be an interval in $\mathbb{R}$. Then, the real-valued function $f : I \to \mathbb{R}$ is said to be convex if for all x, $y \in I$ and $\lambda \in [0, 1]$,*

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y). \tag{2.38}$$

It is called strictly convex provided that the inequality in (2.38) is strict for $x \neq y$ and $\lambda \in (0, 1)$. If the sign of inequality in (2.38) is reversed, then the function f is *concave* (strictly concave). In other words, we say f is concave (strictly concave) if and only if $-f$ is convex (strictly convex). If f is both convex and concave, then f is said to be an affine function.

Geometrically, the property in (2.38) on the other hand, requires that the entire secant line joining the points $(x, f(x))$ and $(y, f(y))$ on the graph of a convex function f lie above (or on) the graph of function between x_1 and x_2, if λ assumes all real values in $[0, 1]$.

Remark 17 *The definition of convexity of functions (Definition 16) has a natural generalization to the real-valued functions defined on arbitrary linear space $\mathbf{X}$, in which, it requires that the domain of $f(x)$ be a convex and open set $\mathbf{U}$. In other words, we may define the function $f(x)$ to be convex on $\mathbf{U} \subseteq \mathbf{X}$, then the inequality (2.38) holds true for all x, $y \in \mathbf{U}$ and $\lambda \in [0, 1]$.*

Many convex functions are well-known in analysis. For instance, $f(x) = x^2$, $f(x) = \exp(x)$ and $f(x) = |x|$, are convex functions on any open interval $I \subset \mathbb{R}$. In elementary calculus, the following simple criteria are commonly used to verify the convexity of a given function (see [104, p. 70], [318, p. 2], [375, p. 10-11]).

Theorem 18 ([375, p. 10-11]) *(i) Suppose $f(x)$ is differentiable on the interval I. Then, the function $f(x)$ is convex (strictly convex) on I if and only if $f'(x)$ is increasing (strictly increasing). (ii) Suppose the second derivative $f''(x)$ exists on I. Then, $f(x)$ is convex on I if and only if $f''(x) \geq 0$ for each $x \in I$, and if $f''(x) > 0$ on I, then $f(x)$ is strictly convex on the interval.*

These criteria give sufficient conditions that functions be convex on an interval I, however such functions need to be continuous everywhere on this interval. Checking that a given function is convex or not, is not an easy task, but there are several useful criteria of convexity properties available in the literature. The simplest one of the basic properties of convex functions in one dimensional case is due to Jensen [223], see also ([79, p. 48], [318, p.3], [334, p. 14]).

Theorem 19 ([223]) *Let $f : I \to \mathbb{R}$ be a continuous real-valued function. Then, f is convex if and only if f is midpoint convex (or simply midconvex), that is, for all points x, $y \in I$, the inequality*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \tag{2.39}$$

is valid.

The function f is said to be strictly midconvex, if for all pairs of points (x, y), $x \neq y$, then the strict inequality holds in (2.39). If the sign of inequality in (2.39) is reversed, then we shall say the function f is concave. The functions, which fulfil the *Jensen's functional inequality* (2.39), are also known as the *Jensen-convex* or simply *J-convex* functions. The inequality in (2.39) is obviously the special case of (2.38) for $\lambda = 1/2$. In addition, Hardy, Littlewood and Pölya [185, p. 96] showed that every convex function is a Jensen-convex.

One of the most important aspects in the Theory of Inequalities is that, the convexity inequalities have been generalized, extended and refined in several ways. For instance, the following refinement of the inequality (2.39) was proved by Hadamard in [176].

Theorem 20 ([176]) *Let $f : I \to \mathbb{R}$ be a convex function, where $I = [a, b]$. Then, the inequalities*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{2.40}$$

are valid.

The inequality (2.40) is called the *Hadamard's inequality.* Besides the mid-convex, the other forms of convexity of functions are log-convex and quasi-convex. We shall say that a positive function $f : I \to \mathbb{R}^+$ is *logarithmic convex,* or for short, *log-convex,* if $\log f$ is a convex function, or equivalently, if x, $y \in I$ and $\lambda \in [0, 1]$, one has the inequality (see [79, p. 48], [318, p. 3], [352, p. 7])

$$f\left(\lambda x + (1-\lambda)\, y\right) \le f^{\lambda}(x) f^{1-\lambda}(y). \tag{2.41}$$

It is said to be a positive log-concave if the sign of inequality in (2.41) is reversed. Note that, if f and g are convex functions and g is increasing, then the convolution of f and g, i.e., $f \circ g$ is convex. Moreover, since $f = \exp(\log f)$, it follows that a log-convex function is convex, but the converse may not necessarily be true [352, p. 7].

The function $f : I \to \mathbb{R}$ is said to be *quasi-convex* if for all x, $y \in I$ and all $\lambda \in [0, 1]$, then (see [318, p. 3])

$$f\left(\lambda x + (1-\lambda)\, y\right) \le \max\left[f(x), f(y)\right]. \tag{2.42}$$

Further, the concept of *m-convexity* was introduced by Toader [421] to provide a generalization of the convexity of functions (2.38). It states that, a function $f : [0, b] \to \mathbb{R}$ is said to be m-convex if it satisfies the following condition:

$$f\left(\lambda x + m\,(1-\lambda)\, y\right) \le \lambda f(x) + m(1-\lambda) f(y), \tag{2.43}$$

where $m \in [0, 1]$, $x, y \in [0, b]$ and $\lambda \in [0, 1]$.

Another extension of the property in (2.39) was discovered by Jensen in his work ([222], [223]), which is known in the literature as *Jensen's inequality.* We shall discuss this inequality in detail in Section 2.6).

2.2.5 Normed, Linear and Inner Product Spaces

Suppose that **X** is a *linear space* over the real or complex number field $\mathbb{K}$. The inner product on **X** is a real-valued function $\langle \cdot, \cdot \rangle$ [1] $: \mathbf{X} \times \mathbf{X} \to \mathbb{K}$, which satisfies the following conditions (see [161, p. 14], [439, p. 3]). For all $x, y, z \in \mathbf{X}$ and $\alpha, \beta \in \mathbb{K}$,

$$
\begin{aligned}
&\text{(i)} \quad \text{Positive definiteness: } \langle x, x \rangle \geq 0 \text{ and } \langle x, x \rangle = 0 \text{ if and only if } x = 0. \\
&\text{(ii)} \quad \text{Hermitian condition: } \langle x, y \rangle = \overline{\langle y, x \rangle}. \\
&\text{(iii)} \quad \text{Linearity: } \langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle \quad \text{and} \\
&\qquad\qquad \langle x, \alpha y + \beta z \rangle = \overline{\alpha} \langle x, y \rangle + \overline{\beta} \langle x, z \rangle .
\end{aligned}
\tag{2.44}
$$

The notation $\langle x, y \rangle$ is called the *inner product* of x and y. The linear space **X** equipped with the inner product $\langle \cdot, \cdot \rangle$ defined on **X**, is called an *inner product space* or *pre-Hilbert space.* The $(\mathbf{X}, \langle \cdot, \cdot \rangle)$ is the standard notation for the inner product spaces **X**.

One example of the inner product spaces is $(\mathbb{C}^n, \langle \cdot, \cdot \rangle)$, with its usual inner product defined by

$$\langle z, w \rangle = \sum_{k=1}^{n} z_k \overline{w}_k, \tag{2.45}$$

for any $z, w \in \mathbb{C}^n$ and $\overline{w}_k$, $k \in \{1, 2, \ldots, n\}$ are the complex conjugate of w_k. For the real inner product space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$, the inner product corresponds to the standard *dot product* of the two vectors x, $y \in \mathbb{R}^n$, namely

$$\langle x, y \rangle = \sum_{k=1}^{n} x_k y_k. \tag{2.46}$$

Consider the complex-valued functions $f, g : [a, b] \to \mathcal{C} \subset \mathbb{C}$, which are continuous on the bounded interval $[a, b]$. Then, the expression

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx, \tag{2.47}$$

[1] In other disciplines, the notation $\langle \cdot \mid \cdot \rangle$ may be used for an inner product.

defines the inner product on the complex function space $\mathcal{C}[a,b]$.

There are many examples of useful inner products that occur in a great variety of mathematical contexts. For instance, if we fix a set of positive real numbers $q_k > 0$, $k \in \{1, 2, \ldots, n\}$, then we can easily define an inner product on $\mathbb{C}^n$ with the weighted sums, i.e.,

$$\langle z, w\rangle = \sum_{k=1}^{n} q_k z_k \overline{w}_k. \tag{2.48}$$

Also, if $w : [a,b] \to \mathbb{R}$ is a real-valued continuous function such that $w(x) > 0$ for all $x \in [a,b]$, then we might define the weighted inner product on $C[a,b]$ as well by setting

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}w(x)\,dx. \tag{2.49}$$

The definitions of the weighted inner products given by (2.48) and (2.49), satisfy all of the properties (i) - (iii) as mentioned in (2.44) that one requires for the inner product.

Another family of the linear spaces, which is of great important in mathematical analysis, is the family of *normed space.* Every inner product induces a *norm* on $\mathbf{X}$ by the following identity:

$$\|z\| = \langle z, z\rangle^{1/2}, \tag{2.50}$$

for all $z \in \mathbf{X}$. We denote the above norm by $\|\cdot\|$. In general, a norm on a linear space $\mathbf{X}$ is a function $\|\cdot\| : \mathbf{X} \to \mathbb{K}$ satisfying the following properties:

$$\begin{aligned} &\text{(i)} && \|v\| > 0 \text{ for all nonzero } v \in \mathbf{X}. \\ &\text{(ii)} && \|cv\| = |c|\,\|v\| \text{ for all } c \in \mathbb{C} \text{ and all } v \in \mathbf{X}. \\ &\text{(iii)} && \|v + w\| \le \|v\| + \|w\| \text{ for all } v, w \in \mathbf{X} \text{ (the triangle inequality).} \end{aligned} \tag{2.51}$$

Consequently, from the above examples (2.45) - (2.49), thus we have

$$\|x\|^2 = \sum_{k=1}^{n} |x_k|^2, \quad \text{or} \quad \|x\|^2 = \sum_{k=1}^{n} q_k |x_k|^2, \; q_k > 0 \tag{2.52}$$

on the complex inner product space, and

$$\|f\|^2 = \int_a^b |f(x)|^2 \, dx, \quad \text{or} \quad \|f\|^2 = \int_a^b |f(x)|^2 \, w(x) \, dx \tag{2.53}$$

on the function space $\mathcal{C}[a,b]$.

It is called a *normed linear space* $(\mathbf{X}, \|\cdot\|)$ for a linear space $\mathbf{X}$ equipped with the norm $\|\cdot\|$ and a *normed space* $(V, \|\cdot\|)$ is a vector space with a norm, where V is a real or complex vector space and $\|\cdot\|$ is a norm defined on V.

Let $a = (a_1, a_2, \ldots, a_n)$ be an n-tuple of real numbers and $1 \leq p < \infty$. Then, we can write

$$\|a\|_p = \left(\sum_{k=1}^{n} |a_k|^p \right)^{1/p}, \tag{2.54}$$

where for $p = \infty$ we simply set $\|a\|_\infty = \max\limits_{1 \leq k \leq n} |a_k|$. The quantity $\|a\|_p$ is called the p-norm or the *l^p-norm* of the n-tuple of real numbers.

The function $a \longmapsto \|a\|_p$, where $\|a\|_p$ is defined by (2.54) indeed satisfies the properties required by the definition of the norm. Specifically, the function $\|a\|_p$, $p \geq 1$ needs to satisfy the following properties:

$$\begin{aligned} &\text{(i)} \quad \|a\|_p = 0 \text{ if and only if } a = 0, \\ &\text{(ii)} \quad \|\alpha a\|_p = |\alpha| \, \|a\|_p \text{ for all } \alpha \in \mathbb{R}, \\ &\text{(iii)} \quad \|a + b\|_p \leq \|a\|_p + \|b\|_p \text{ for all real } n\text{-tuple } a \text{ and } b, \end{aligned} \tag{2.55}$$

where the third property in (2.55) appears as *Minkowski's inequality.*

Remark 21 *The function (2.54) reduces to simple norm (2.52) for the case $p = 2$.*

The following section discusses the classical Cauchy-Bunyakovsky-Schwarz inequality that is one of the famous and important inequalities in analysis.

2.3 The (CBS)-Type Inequalities

The (CBS)-inequality is a short-term of the *Cauchy-Bunyakovsky-Schwarz inequality,* is also known in the literature as *Cauchy's*, *Schwarz's* or *Cauchy-Schwarz's inequality.* For simplicity, we shall refer to it throughout this dissertation as the (CBS)-inequality. This inequality plays an important role in different branches of Modern Mathematics including Hilbert Spaces Theory, Probability and Statistics Theory, Classical Real and Complex Analysis, Numerical Analysis, Qualitative Theory of Differential Equations and their applications (see [63], [66], [125], [135], [246], [270], [300], [302], [355], [380]).

Several versions of the classical (CBS)-inequality have been found in the literature, starting from Cauchy [88], who published the (CBS)-inequality in elementary form. Then, Bunyakovsky [81] derived the corresponding inequality for integrals, and later, it was rediscovered by Schwarz [388]. Schwarz [389] also discovered the corresponding version of the (CBS)-inequality in inner product spaces. For more information about the name and history of these type of inequalities, see ([387], [407]). In any case, the discrete version of the (CBS)-inequality is stated in the following section.

2.3.1 (CBS)-Inequalities for Real and Complex Numbers

In the following, we state the elementary inequality of the (CBS)-inequality for sequences of real numbers, which it is also known in the literature as *Cauchy's inequality* [318, p. 83] (see also [55, p. 66], [91, p. 13], [117, p. 1]).

Theorem 22 ([318, p. 83]) *If $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ are sequences of real numbers, then*

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right), \tag{2.56}$$

with equality holding in (2.56) if and only if the sequences a and b are proportional, i.e., there is a real number $r \in \mathbb{R}$ such that $a_k = rb_k$ for each $k \in \{1, 2, \ldots, n\}$.

The following version of the (CBS)-inequality for complex numbers also holds (see [117, p. 2], [318, p. 84]).

Theorem 23 *If* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ *and* $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ *are sequences of complex numbers, then*

$$\left|\sum_{k=1}^{n} a_k b_k\right|^2 \leq \left(\sum_{k=1}^{n} |a_k|^2\right)\left(\sum_{k=1}^{n} |b_k|^2\right), \tag{2.57}$$

with equality holding in (2.57) if and only if the sequences a *and* $\bar{b}$ *are proportional, i.e., there is a complex number* $c \in \mathbb{C}$ *such that* $a_k = c\bar{b}_k$ *for any* $k \in \{1, 2, \ldots, n\}$.

These types of the (CBS)-inequality have been proved in several ways, see for instance the works of Mitrinovic et al. [318, p. 83], Dragomir [117, p. 1-2], Cerone and Dragomir [91, p. 13-14], Bullen [79, p. 183] and Steele [407, p. 4-6].

An analogous statement of the (CBS)-inequality is the Cauchy-Schwarz inequality for convergent infinite series and integrals [77, p. 31], i.e.,

$$\left(\sum_{k=1}^{\infty} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{\infty} a_k^2\right)\left(\sum_{k=1}^{\infty} b_k^2\right) \tag{2.58}$$

and

$$\left[\int_a^b f(x)\, g(x)\, dx\right]^2 \leq \int_a^b [f(x)]^2\, dx \int_a^b [g(x)]^2\, dx, \tag{2.59}$$

for f and g are square integrable functions on (a, b).

Let $(p_1, p_2, \ldots, p_n)$ be a sequence of nonnegative real numbers. Then, we can state the weighted version of the (CBS)$-$inequality as follows:

$$\left|\sum_{k=1}^{n} p_k a_k b_k\right|^2 \leq \sum_{k=1}^{n} p_k |a_k|^2 \sum_{k=1}^{n} p_k |b_k|^2, \tag{2.60}$$

for any a_k, $b_k \in \mathbb{C}$, $p_k \geq 0$, $k \in \{1, 2, \ldots, n\}$. Hence, if in (2.60) we choose $p_k = 1$ for all $k \in \{1, 2, \ldots, n\}$, then we recapture the inequality (2.57).

Remark 24 *By the (CBS)-inequality for real numbers (2.56) and the generalized triangle inequality for complex numbers (2.25), we also have*

$$\left|\sum_{k=1}^{n} a_k b_k\right|^2 \leq \left(\sum_{k=1}^{n} |a_k b_k|\right)^2 \leq \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 , \tag{2.61}$$

for $a_k, b_k \in \mathbb{C},\ k \in \{1, 2, \ldots, n\}$.

2.3.2 (CBS)-Inequalities in Inner Product Spaces

In 1885, Schwarz [389] gave another proof of the (CBS)-inequality for sums by extending it to an inner product space. The result states that if x and y are vectors in an inner product space $(H, \langle\cdot,\cdot\rangle)$ over a real or complex number $\mathbb{K}$, then

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle \langle y, y\rangle , \tag{2.62}$$

for any $x, y \in H$, with the equality holding in (2.62) if and only if the vectors x and y are *linearly dependent*, i.e., there exists a nonzero constant $\lambda \in \mathbb{K}$ such that $x = \lambda y$. The inequality (2.62) is mainly known in the literature as *Schwarz*'s or *Cauchy-Schwarz's inequality* (see [161, p. 15]).

We note that, if we consider the norm of $x, y \in H$ as defined by (2.50), then the inequality (2.62) can be written in the form

$$|\langle x, y\rangle|^2 \leq \|x\|^2 \|y\|^2 \tag{2.63}$$

or, equivalently,

$$|\langle x, y\rangle| \leq \|x\| \|y\| , \tag{2.64}$$

with the case of equality holding in (2.63) or (2.64) if and only if there exists a scalar $\lambda \in \mathbb{C}$ such that $y = \lambda x$. Also, we note that the inequality (2.62) reduces to the elementary (CBS)-inequality (2.56) by considering the real inner product space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$, with its usual inner product as defined by (2.46).

2.3.3 (CBS)-Inequalities for Power Series

We now generalize the situation of the celebrated (CBS)-inequality for real or complex numbers and sequences into the following case. Let $f(z)$ be an analytic function defined by the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{2.65}$$

which is convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If the coefficients a_n, $n \in \{1, 2, \ldots\}$ in (2.65) are complex numbers and applying the well-known (CBS)-inequality for complex numbers (2.57), then we can deduce that

$$|f(z)|^2 = \left| \sum_{n=0}^{\infty} a_n z^n \right|^2 \leq \sum_{n=0}^{\infty} |a_n|^2 \sum_{n=0}^{\infty} |z|^{2n} = \frac{1}{1 - |z|^2} \sum_{n=0}^{\infty} |a_n|^2, \tag{2.66}$$

for any $z \in D(0,R) \cap D(0,1)$, where R is the radius of convergence of $f(z)$. The above inequality (2.66) gives some information about the magnitude of the function $f(z)$ provided the numerical series $\sum_{n=0}^{\infty} |a_n|^2$ is convergent and z is not too close to the boundary of the open disk $D(0,1)$.

In practice, many usual fundamental complex functions can be represented by the power series with real or nonnegative coefficients (see Section 1.3). Thus, if we assume the coefficients in the power series (2.65) are nonnegative and utilizing the weighted version of the (CBS)-inequality for complex numbers (2.60), then we can state the following inequality:

$$|f(zw)|^2 = \left| \sum_{n=0}^{\infty} a_n z^n w^n \right|^2 \leq \sum_{n=0}^{\infty} a_n |z|^{2n} \sum_{n=0}^{\infty} a_n |w|^{2n} = f\left(|z|^2\right) f\left(|w|^2\right), \tag{2.67}$$

for any z, $w \in \mathbb{C}$ with zw, $|z|^2$, $|w|^2 \in D(0,R)$. In particular, if $w = a \in \mathbb{R}$, then we get from (2.67)

$$|f(az)|^2 \leq f\left(a^2\right) f\left(|z|^2\right), \tag{2.68}$$

for any $z \in \mathbb{C}$ with az, a^2, $|z|^2 \in D(0,R)$.

Specifically, for functions, which are defined by the power series (2.65) with real coefficients a_n, $n \in \{0, 1, 2, \ldots\}$, we can naturally construct another power series, where the coefficients of the new series are given by the absolute values of the coefficients of the original series, namely

$$f_A(z) = \sum_{n=0}^{\infty} |a_n| z^n, \tag{2.69}$$

for all $z \in D(0, R)$, where $a_n = |a_n| \, sgn(a_n)$, $n \in \{0, 1, 2, \ldots\}$. The $sgn(x)$ is the real signum function defined by (2.11). It is obvious that the new power series $f_A(z)$ given by (2.69) has the same radius of convergence as the original power series $f(z)$ given by (2.65).

As some natural examples of functions and their transformations that may be useful for later applications, we can point out that if the functions $g(z)$, $h(z)$, $k(z)$ and $l(z)$ are respectively given by (1.20), (1.30), (1.32) and (1.33), then the corresponding functions constructed by the use of the absolute values of the coefficients (2.69) are obviously

$$\begin{aligned} g_A(z) &= \frac{1}{1-z}, \qquad & h_A(z) &= \ln\left(\frac{1}{1-z}\right), \\ k_A(z) &= \sinh(z), \qquad & l_A(z) &= \cosh(z). \end{aligned} \tag{2.70}$$

These functions are defined on the same domain as their generating functions respectively.

In a similar way, the following inequality holds for any $z, w \in \mathbb{C}$ such that zw, $|z|^2$, $|w|^2 \in D(0, R)$, by utilising the weighted version of the (CBS)-inequality for complex numbers (2.60):

$$\begin{aligned} |f(zw)|^2 &= \left| \sum_{n=0}^{\infty} |a_n| \, sgn(a_n) \, z^n w^n \right|^2 \le \sum_{n=0}^{\infty} |a_n| \, |z|^{2n} \sum_{n=0}^{\infty} |a_n| \, |w|^{2n} \\ &= f_A\left(|z|^2\right) f_A\left(|w|^2\right). \end{aligned} \tag{2.71}$$

In particular, if $w = a \in \mathbb{R}$, then we get from (2.71)

$$|f(az)|^2 \leq f_A\left(a^2\right) f_A\left(|z|^2\right) \tag{2.72}$$

for all $z \in \mathbb{C}$ with $az, a^2, |z|^2 \in D(0, R)$.

2.3.4 Some Results Related to (CBS)-Type

It is well-known that the classical (CBS)-type inequalities have been improved, generalized, refined and applied by a remarkable large number of researchers in the last few decades. For the details, see particularly the survey paper of Dragomir [127], the relevant chapters in the books of Dragomir ([117], [120, Chapt. 1], [121, Chapt. 2]) and numerous references, which are cited therein.

In this section, we provide some of the results related to the (CBS)-type inequalities as important tools for our investigation in the next chapters. First, in 1960, N. G. de Bruijn [78] established the following refinement of the classical (CBS)-inequality for a sequence of real numbers and the second sequence of complex numbers (see also [117, p. 48], [121, p. 43], [318, p. 89]).

Theorem 25 (de Bruijn [78]) *Let $(a_1, a_2, \ldots, a_n)$ be $n-$tuple of real numbers and $(z_1, z_2, \ldots, z_n)$ an $n-$tuple of complex numbers. Then,*

$$\left|\sum_{k=1}^{n} a_k z_k\right|^2 \leq \frac{1}{2}\sum_{k=1}^{n} a_k^2 \left(\sum_{k=1}^{n} |z_k|^2 + \left|\sum_{k=1}^{n} z_k^2\right|\right) \tag{2.73}$$
$$\left(\leq \sum_{k=1}^{n} a_k^2 \cdot \sum_{k=1}^{n} |z_k|^2\right).$$

Equality holds in (2.73) if and only if, $a_k = \operatorname{Re}(\lambda z_k)$ for $k \in \{1, 2, \ldots, n\}$, where λ is a complex number such that the quantity $\lambda^2 \sum_{k=1}^{n} z_k^2$ is a nonnegative real number.

This inequality (2.73) is popularly known as the *de Bruijn inequality.* The proof of Theorem 25 can also be found in the work of Mitrinović, Pečarić and Fink [318, p. 89] and Dragomir [117, p. 48]. The weighted version of the de Bruijn inequality also holds, namely

$$\left|\sum_{k=1}^{n} p_k a_k z_k\right|^2 \le \frac{1}{2}\sum_{k=1}^{n} p_k a_k^2 \left(\sum_{k=1}^{n} p_k |z_k|^2 + \left|\sum_{k=1}^{n} p_k z_k^2\right|\right), \tag{2.74}$$

where $p_k \ge 0$, $a_k \in \mathbb{R}$, $z_k \in \mathbb{C}$, $k \in \{1, 2, \ldots, n\}$.

In [82], M. L. Buzano obtained the extension of the celebrated (CBS)-inequality (2.63) for a real or complex inner product space $(H, \langle \cdot, \cdot \rangle)$. The result is stated as follows:

Theorem 26 (Buzano [82]) *Let H be a complete inner product space and let x, a, b be elements of H with $x \neq 0$. Then,*

$$|\langle x, a\rangle \langle x, b\rangle| \le \frac{1}{2}(\|a\| \|b\| + |\langle a, b\rangle|) \|x\|^2, \tag{2.75}$$

with equality holds in (2.75) if and only if there exists a scalar $\lambda \in \mathbb{K}$ such that $x = \lambda a$.

It is clear that for $a = b$, the above inequality (2.75) becomes the (CBS)-inequality (2.62) in inner product spaces. A similar inequality to (2.75) has been independently obtained by Richard [373] for H is a real inner product space (see also [361]), namely

$$\frac{1}{2}\|a\|^2 (|\langle a, b\rangle| - \|a\| \|b\|) \le \langle x, a\rangle \langle x, b\rangle \le \frac{1}{2}\|x\|^2 (|\langle a, b\rangle| - \|a\| \|b\|). \tag{2.76}$$

Further, in 1985, Dragomir [118] proved the following refinement of the (CBS)-inequality (2.63) in inner product space $(H; \langle \cdot, \cdot \rangle)$ over the real and complex number field $\mathbb{K}$.

Theorem 27 (Dragomir [118]) *For any x, $y \in H$ and $e \in H$ with $\|e\| = 1$, the following refinement of the (CBS)-inequality holds:*

$$\|x\| \|y\| \ge |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| + |\langle x, e\rangle \langle e, y\rangle| \ge |\langle x, y\rangle|. \tag{2.77}$$

Remark 28 *If in the first inequality of (2.77), we choose $e = z/\|z\|$, $z \in H \backslash \{0\}$, then we get*

$$\|x\| \|y\| \|z\|^2 - |\langle x, z\rangle \langle z, y\rangle| \ge \left|\langle x, y\rangle \|z\|^2 - \langle x, z\rangle \langle z, y\rangle\right|, \tag{2.78}$$

for any x, y, $z \in H$.

2.4 Young's Inequality and Its Variants

In 1912, W. H. Young [440] presented the following integral inequality:

Theorem 29 ([440]) *Let $f : [0, c] \to R$ be a real-valued, continuous and strictly increasing function on $[0, c]$ with $c > 0$. If $f(0) = 0$, $x \in [0, c]$ and $y \in [0, f(c)]$, then*

$$xy \leq \int_0^x f(t)\,dt + \int_0^y f^{-1}(t)\,dt, \tag{2.79}$$

where $f^{-1}(t)$ is the inverse function of $f(t)$. Equality occurs in (2.79) if and if $y = f(x)$ or equivalently, $x = f^{-1}(y)$.

The above inequality (2.79) has an easy geometric interpretation shown in [422], so that some monographs simply refer to it and omit the proof. The above result also can be found in the work of Hardy, Littlewood and Pólya [185, p. 111-113], but there was no analytic proof until Diaz and Metcalf [114] supplied it in 1970. The complete proof is also given in [316, p. 48-50], [432] and for more information on Young's inequality, we may refer to [318, p. 379-389] and the references which are cited therein.

The inequality (2.79) provides an important tool in deriving other classical inequalities. For instance, we shall obtain the (AM-GM)-inequality given in (2.18) by choosing the identity function $f(x) = x$ in (2.79). The most useful consequence of the Theorem 29 is the inequality

$$xy \leq \frac{x^q}{q} + \frac{y^p}{p}, \tag{2.80}$$

for x, y are nonnegative numbers and p, $q > 1$ with $1/p + 1/q = 1$, where it can be easily derived from the inequality (2.79) by choosing the exponential function $f(t) = t^{p-1}$ and $q = p/(p-1)$. The equality holds in (2.80) if and only if $x^q = y^p$. This classical inequality (2.80) is called *Young's inequality* for the scalar products of real numbers. It appears as Theorem 61 in the remarkable work of Hardy, Littlewood and Pòlya [185, p. 61], but it is not credited to any individual.

This form of Young's inequality (2.80) can be used to establish some important inequalities such as the (CBS), Hölder and Minkowski inequalities [422]. The inequality (2.80) is itself, a special case of the (AM-GM)-inequality (2.18) when $p = q = 2$.

Now, let $a = x^q$, $b = y^p$, $v = 1/q$ and $1 - v = 1/p$. Then, the inequality (2.80) can be written in the form

$$a^v b^{1-v} \leq va + (1-v)\, b, \tag{2.81}$$

where $a, b \geq 0$ and $v \in [0,1]$, with the equality holds in (2.81) if and only if $a = b$. This inequality gives us another useful form of the classical Young's inequality for two scalar products. We note that, if $v = 1/2$, then we obtain from (2.81) the basic result of the arithmetic-geometric mean inequality (2.18). This shows that, the inequality (2.81) is the generalizations of the weighted (AM-GM)-inequality (2.36), as it is mentioned earlier in Section 2.2.3.

In [250], Kittaneh and Manasrah defined the Heinz means as

$$H_v\,(a,b) = \frac{a^v b^{1-v} + a^{1-v} b^v}{2}, \tag{2.82}$$

for a, $b \geq 0$, $v \in [0,1]$ and showed from the inequalities (2.80), (2.81) and (2.18) that the Heinz means interpolates between the arithmetic mean and geometric mean as follows:

$$\sqrt{ab} \leq H_v\,(a,b) \leq \frac{a+b}{2}, \tag{2.83}$$

for a, $b \geq 0$ and $v \in [0,1]$. Again, the equality holds in (2.81) if and only if $a = b$.

The famous Young's inequality (2.81) has been studied by several authors, see [10], [11], [157], [158], [198], [250] and the references cited therein. For instance, Hirzallah and Kittaneh [198] obtained the refinement of the inequality (2.81) as follows:

$$[va + (1-v)\,b]^2 - \left(a^v b^{1-v}\right)^2 \geq r^2\,(a-b)^2\,, \tag{2.84}$$

for $a, b \geq 0$, $v \in [0,1]$ and $r = \min\{v, 1-v\}$. Kittaneh and Manasrah [250] provided the refinement of Young's inequality (2.81) in the following form:

$$va + (1 - v)\, b - a^v b^{1-v} \geq r \left(\sqrt{a} - \sqrt{b}\right)^2, \tag{2.85}$$

for all $a, b \geq 0$, $v \in [0, 1]$ and $r = \min\{v, 1 - v\}$. The following inequality has been proved by Aldaz [10],

$$\begin{aligned} 2v\left(\frac{a+b}{2} - \sqrt{ab}\right) &\leq va + (1 - v)\, b - \left(a^v b^{1-v}\right)^2 \\ &\leq 2(1 - v)\left(\frac{a+b}{2} - \sqrt{ab}\right), \end{aligned} \tag{2.86}$$

for any $a, b \geq 0$ and $v \in [0, 1/2]$, which provided a refinement and a reverse of Young's inequality as well. This inequality (2.86) has been further studied by the same author in ([11], [12]) and Furuichi in [158].

A generalization of Young's inequality (2.81) was given by Furuichi in [159], namely

$$\sum_{j=1}^{n} p_j a_j - \prod_{j=1}^{n} a_j^{p_j} \geq n p_{\min} \left(\frac{1}{n}\sum_{j=1}^{n} a_j - \prod_{j=1}^{n} a_j^{1/n}\right), \tag{2.87}$$

for $a_j, p_j \geq 0$, $j \in \{1, 2, \ldots, n\}$ with $\sum_{j=1}^{n} p_j = 1$ and $p_{\min} = \min\{p_1, p_2, \ldots, p_n\}$. This inequality (2.87) becomes an equality if and only if $a_1 = a_2 = \cdots = a_n$. Note that, for $n = 2$, the inequality (2.87) reduces to Kittaneh and Manasrah's result (2.85).

Other generalizations of Young's inequality can be found in [13] and [14]. See also [21], [72, p. 205], [119], [160], [204], [259], [310], [335], [369], [423], [441] and the references cited therein, for some improvements of Young's inequality and their recent advances.

2.5 Hölder's Inequality

The classical Hölder's inequality for sequences of real numbers was first derived in 1888 by L. J. Rogers [378]. Then a year later, it was rediscovered in another way by O. L. Hölder [201] and named after him.

Hölder's inequality, which is also known is the literature as *Rogers's inequality,* was built around the two real numbers p and q that satisfy the conditions $p > 1$ and $1/p + 1/q = 1$. This inequality asserts that for all nonnegative numbers a_k, $b_k \in \mathbb{R}$, $k \in \{1, 2, \ldots, n\}$ and $p > 1$ with $1/p + 1/q = 1$, one has the bound [55, p. 68]

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q}, \tag{2.88}$$

with the equality holding in (2.88) if and only if the sequences (a_k^p) and (b_k^q) for $k \in \{1, 2, \ldots, n\}$ are proportional, i.e., there exists a real number $r \in \mathbb{R}$ such that $b_k^{1/p} = r a_k^{1/q}$ for all $k \in \{1, 2, \ldots, n\}$. The inequality (2.88) is reversed if $p < 1$ $(p \neq 0)$.

Hölder's inequality holds for complex numbers as well [318, p. 105], namely

$$\left|\sum_{k=1}^{n} a_k b_k\right| \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |b_k|^q\right)^{1/q}, \tag{2.89}$$

for all a_k, $b_k \in \mathbb{C}$, $k \in \{1, 2, \ldots, n\}$, $p > 1$ and q satisfying $1/p + 1/q = 1$. The equality occurs in (2.89) if and only if the sequences $\{|a_k|^p\}$ and $\{|b_k|^q\}$, $k \in \{1, 2, \ldots, n\}$ are proportional and the $\arg(a_k b_k)$ is independent of k.

The following inequality also holds, which is called the weighted version of Hölder's inequality,

$$\left|\sum_{k=1}^{n} m_k a_k b_k\right| \leq \left(\sum_{k=1}^{n} m_k |a_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} m_k |b_k|^q\right)^{1/q}, \tag{2.90}$$

where $m_k \geq 0$, $a_k, b_k \in \mathbb{C}$, $k \in \{1, 2, \ldots, n\}$ and $p > 1$ such that $1/p + 1/q = 1$. Again, the equality occurs in (2.90) if and only if the sequences $\{|a_k|^p\}$ and $\{|b_k|^q\}$ for $k \in \{1, 2, \ldots, n\}$ are proportional the $\arg(a_k b_k)$ is independent of k. We note that, if all the weights m_k for $k \in \{1, 2, \ldots, n\}$ are equal, then (2.90) reduces to the inequality (2.89).

Several proofs of Hölder's inequality (2.88) can be found in the literature. For instance, Tolsted in [422] (see also [301, p. 457], [381, p. 63-64]) showed that

Hölder's inequality (2.88) can be easily proved by using the standard Young's inequality (2.80).

The above inequality (2.88) can be written in the simpler form as follows:

$$\left|\sum_{k=1}^{n} a_k b_k\right| \leq \|a\|_p \|b\|_q\,, \tag{2.91}$$

where for $1 < p < \infty$, the pair (p, q) are the usual conjugate exponents and the $\|x\|_p$ is the p-norm of n-tuple of real numbers x as defined by (2.54).

The inequality (2.88) also appears in ([4, p. 11], [55, p. 19-21], [185, p. 24, 26], [239, p. 67, 71], [318, p. 99]. One of the standard proofs of Hölder's inequality is based on Young's inequality (2.80) for the scalar products (see [422]). Alternative proof is by utilising the Jensen type inequality for the convex function $f(x) = x^p$, $p > 1$. We note that, the (CBS)-inequality (2.56), which is established in the previous section, is the special case of Hölder's inequality (2.88) with $p = q = 2$. In other words, we have to remark that Hölder's inequality is one of the remarkable extensions of the (CBS)-inequality.

The inequality (2.89) is also valid for countable infinite pairs of numbers, where from the convergence of the series on the right-hand side $\sum_{k=1}^{\infty} |a_k b_k|$, the convergent of the left-hand side follows.

Remark 30 *Hölder's inequality (2.89) is also valid for countable infinite pairs of numbers, i.e.,*

$$\left|\sum_{k=1}^{\infty} a_k b_k\right| \leq \left(\sum_{k=1}^{\infty} |a_k|^p\right)^{1/p} \left(\sum_{k=1}^{\infty} |b_k|^q\right)^{1/q}, \tag{2.92}$$

where from the convergence of the series on the right-hand side, the convergence of the left-hand side follows [77, p. 32].

If now, we consider an analytic function $f(z)$ defined by the real power series (2.65) with real or nonnegative coefficients and convergent on the interval $(-R, R)$, $R > 0$, and applying the weighted version of the Hölder inequality for real numbers, then we can state that

$$
\begin{aligned}
f(xy) &= \sum_{n=0}^{\infty} a_n x^n y^n \le \left(\sum_{n=0}^{\infty} a_n x^{pn}\right)^{1/p} \left(\sum_{n=0}^{\infty} a_n y^{qn}\right)^{1/q} \\
&= f^{1/p}(x^p) f^{1/q}(y^q) \qquad (2.93)
\end{aligned}
$$

for any $x,\ y \in \mathbb{R}$ such as $xy,\ x^p,\ y^q < R$. A similar result can be obtained by applying the complex power series (2.65) with real coefficients for the weighted version of the Hölder inequality (2.90), namely

$$
\begin{aligned}
|f(xy)| &= \left|\sum_{n=0}^{\infty} a_n x^n y^n\right| \le \left(\sum_{n=0}^{\infty} |a_n|\, |x|^{pn}\right)^{1/p} \left(\sum_{n=0}^{\infty} |a_n|\, |y|^{qn}\right)^{1/q} \\
&= f_A^{1/p}(|x|^p) f_A^{1/q}(|y|^q), \qquad (2.94)
\end{aligned}
$$

for any $x,\ y \in \mathbb{C}$ with $xy,\ |x|^p,\ |y|^q \in D(0,R)$ and $f_A(z)$ is the power series defined by (2.69).

Various extensions, generalizations, refinements, etc. of the Hölder inequality have been obtained by many authors, see for instance [3], [87], [112], [192], [248], [271], [317], [365], [436], [437], [438] and the numerous references which are cited therein.

2.6 Jensen Types Inequalities and Their Reverses

Many important classical inequalities depend upon convexity. One of them is *Jensen's inequality*. Jensen's inequality, which was proved by Jensen ([222], [223]), concerned with the bounding of convex functions of sums or integrals (see also [79, p. 31], [91, p. 43], [317, p.], [334, p. 12]). It is one of the most important inequalities and has had a tremendous impact on many different fields such as Probability and Statistics Theory, Information Theory, Control and Systems Theory, etc. Since its discovery in 1906 [223], Jensen's inequality has been proven to be one of the most useful inequalities in mathematical analysis, because it implies many of other classical results in Inequalities Theory. The triangle, (AM-GM), (CBS), Young, Hölder, Minkowski, Schnnon, Ky Fan inequalities, etc., can be obtained as particular cases of Jensen's inequality. For classical and contemporary developments related to Jensen's inequality, we may refer to the works of ([56], [101], [119], [140], [318, Chapt. I]) and the references, which are cited therein.

There are several types of Jensen's inequality that appear in the literature. The classical form of Jensen's inequality involving real numbers and weights is stated as follows (see [79, p. 31], [317, p. 6]). Let f be a convex function on an interval $I \subset \mathbb{R}$ and p_j, $j \in \{1, 2, \ldots, n\}$ be nonnegative scalars with $P_n := \sum_{j=1}^{n} p_j > 0$, $n \geq 2$. Then,

$$f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right) \leq \frac{1}{P_n}\sum_{j=1}^{n} p_j f(x_j), \tag{2.95}$$

for $x_j \in I$ such that $x_j \geq 0$ for each $j \in \{1, 2, \ldots, n\}$, with equality holds in (2.95) if and only if $x_j = x_k$ for all $j, k \in \{1, 2, \ldots, n\}$. If $f(x)$ is strictly convex for all the distinct points x_j and all weights p_j, $j \in \{1, 2, \ldots, n\}$ are positive, then the strict inequality holds in (2.95). The above inequality (2.95) is known as the weighted version of Jensen inequality with p_j, $j \in \{1, 2, \ldots, n\}$ are called the weights.

The reverse of the Jensen inequality holds, namely (see [79, p. 43], [318, p. 6], [427])

$$f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right) \geq \frac{1}{P_n}\sum_{j=1}^{n} p_j f(x_j), \tag{2.96}$$

for a convex function f on I, $x_j \in I$, $j \in \{1, 2, \ldots, n\}$, $p_1 > 0$, $p_j < 0$ for $2 \leq j \leq n$ and $P = \frac{1}{P_n}\sum_{j=1}^{n} p_j x_j \in I$. On the other hand, the inequality in (2.96), which provides the lower bound instead of the upper bound, particularly holds for concave functions $f(x)$ on I.

We note that, if $n = 2$, then we have from (2.95) the inequality

$$f\left(\frac{px + qy}{p + q}\right) \leq \frac{pf(x) + qf(y)}{p + q}, \tag{2.97}$$

for any $x,\ y \in I$, where $p,\ q \geq 0$ such that $p+q>0$. Thus, the inequality in (2.39) as shown in the previous section, is the special case of (2.95) when $p=q=1$. Another remarkable particular case is obtained when $p_j = 1/n$ for all $j \in \{1,2,\ldots,n\}$. That is, the inequality (2.95) becomes [79, p. 32]

$$f\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right) \leq \frac{1}{n}\sum_{j=1}^{n} f\left(x_j\right), \tag{2.98}$$

holding for a convex function f on the interval I and $x_j \in I$, $j \in \{1,2,\ldots,n\}$. Again, the sign of inequality reverses in (2.98) for concave functions. Obviously, the inequality (2.98) holds for every Jensen-convex function f on I. For an analytic proof of the Jensen type inequalities, we may refer to the works of ([185, p. 71], [352, p. 53]).

Further, we have the following inequality

$$f\left(\sum_{j=1}^{n} q_j x_j\right) \leq \sum_{j=1}^{n} q_j f\left(x_j\right), \tag{2.99}$$

which holds for a convex function $f: I \to \mathbb{R}$, $x_j \in I$, $j \in \{1,2,\ldots,n\}$ and q_j are probabilities, namely $q_j \geq 0$ for $j \in \{1,2,\ldots,n\}$ such that $\sum_{j=1}^{n} q_j = 1$. The equality holds in (2.99) if and only if all the x_j, $j \in \{1,2,\ldots,n\}$ are equal, or if $f(x)$ reduces to a linear function. In a real sense, the Jensen type inequality given by (2.99) is the extended property of the inequality in (2.38), in which it defines a convex function for the case $n=2$.

The forms of Jensen's discrete inequality given by (2.98), (2.95) and (2.99) as well as their reverses (2.96) allow us to derive some other important classical results in Inequalities Theory. Through judicious choice of convex (or concave) functions f, we can derive, for instance, the generalized triangle inequality (2.25) for real numbers, by choosing in (2.99) the convex function $f(x) = |x|$, $x \in \mathbb{R}$. Jensen's inequality (2.96) reduces to the celebrated (CBS)-inequality (2.56) for the convex function $f(x) = x^2$, and noting that $x_j = b_j/a_j$ and $p_j = a_j^2$ for $j \in \{1,2,\ldots,n\}$. Taking the function $f(x) = x^p$, $q_j = b_j^q$ and $x_j = \left(a_j^p/b_j^q\right)^{1/p}$ with $a_j,\ b_j > 0$ for $j \in \{1,2,\ldots,n\}$ and $p,q>0$ such that $1/p + 1/q = 1$, we can prove the famous Hölder's inequality.

Some studies related to Jensen's inequality concerning generalizations, extensions, refinements, reverses, counterparts, etc., can be found in (see for example [27], [56], [57], [80, p. 139-142], [101], [126], [129], [130], [139], [273], [318, p. 1-20]). Also, see [286], [322], [351] for further references.

2.7 Other Inequalities

In this section, we state some other important classical inequalities in analysis, which are Minkowski's and Čebyšev's inequalities. *Minkowski's inequality* was established in 1896 by Hermann Minkowski [313] in his book "*Geometrie der Zahlen*" (Geometry of Numbers). It states that for $p > 1$, one has the bound

$$\left(\sum_{k=t1}^{n} |a_k + b_k|^p\right)^{1/p} \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |b_k|^p\right)^{1/p}, \tag{2.100}$$

for any $a_k, b_k \in \mathbb{C}$, $k \in \{1, 2, \ldots, n\}$. Equality holds in (2.100) if and only if there exist a constant $\lambda \in \mathbb{R}$ such that $|b_k| = \lambda |a_k|$ for all $k \in \{1, 2, \ldots, n\}$. If $p < 1$ ($p \neq 0$), then the inequality sign in (2.100) is reversed. This inequality arises as Theorem 25 in the work of Hardy, Littlewood and Pòlya [185, p. 31], see also [4, p. 11], [55, p. 69], [185, p. 31], [239, p. 69-70] and the references cited therein. We note that the special case of (2.100) with $p = 2$ is the triangle inequality for arbitrary complex numbers.

The inequality (2.100) can be generalized for their weighted version as follows,

$$\left(\sum_{k=1}^{n} m_k |a_k + b_k|\right)^{1/p} \leq \left(\sum_{k=1}^{n} m_k |a_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} m_k |b_k|^p\right)^{1/p}, \tag{2.101}$$

for any $a_k, b_k \in \mathbb{C}, m_k \geq 0$ for $k \in \{1, 2, \ldots, n\}$, $p > 1$ and again the equality holds in (2.101) if and only if the sequences $\{a_k\}$ and $\{b_k\}$ are proportional.

Čebyšev's inequality provides the lower bound of $\sum_{k=1}^{n} a_k b_k$, which says that if $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be real numbers, then

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) \le n \sum_{k=1}^{n} a_k b_k, \tag{2.102}$$

with equality occurring if and only if, either all the a_k, $k \in \{1, 2, \ldots, n\}$ are equal or all the b_k are equal. It is important to order, either the $a_k s$ and $b_k s$ in both descending and ascending order in the inequality (2.102).

Remark 31 *The inequality (2.102) is also true in the case when $a_1 \ge a_2 \ge \cdots \ge a_n$ and $b_1 \ge b_2 \ge \cdots \ge b_n$, but if $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \ge b_2 \ge \cdots \ge b_n$ (or the reverse), then the sign in inequality (2.102) is reversed.*

In terms of its weighted version, Čebyšev's inequality can be written as follow:

$$\left(\sum_{k=1}^{n} a_k m_k\right)\left(\sum_{k=1}^{n} b_k m_k\right) \le \sum_{k=1}^{n} a_k b_k m_k, \tag{2.103}$$

for $a_1 \le a_2 \le \cdots \le a_n$, $b_1 \le b_2 \le \cdots \le b_n$ be real numbers and $m_1, m_2, \ldots, m_n$ be nonnegative real numbers such that $m_1 + m_2 + \ldots + m_n = 1$. The equality occurs if and only if $a_1 = a_2 = \ldots = a_n$ or $b_1 = b_2 = \ldots = b_n$. Note that, if in (2.103) we choose $m_1 = m_2 = \ldots = m_n = 1/n$, then we get the inequality (2.102).

Chapter 3

Power Series and the Cauchy-Bunyakovsky-Schwarz Inequality

Chapter 3 is devoted to some inequalities concerning the power series that are related to the (CBS)-type inequalities. It is well-known that the classical (CBS)-type inequalities have been generalized, extended, refined and applied by a remarkable large number of researchers for different and various motivations (see for instance [117], [120, Chapt. 1], [121, Chapt. 2], [127]). In this chapter, we provide some generalizations, improvements and refinements of the (CBS)-inequality for functions defined by the power series with real or nonnegative coefficients and convergent on an open disk.

We start this chapter with some known results on the power series inequalities proved by Dragomir [122], Cerone and Dragomir [89]. In Section 3.2, we derive new and better inequalities for functions defined by the power series with real coefficients, by utilizing Buzano's result in inner product spaces. Particular inequalities are obtained by applying the results for some fundamental functions of interest, such as the exponential, logarithm, trigonometric and hyperbolic functions.

In Section 3.3, we utilize a refinement of the celebrated (CBS)-inequality in inner product spaces established by Dragomir in [118], to develop some other inequalities for functions defined by the power series with real and nonnegative coefficients. Some applications for elementary complex functions of interest are also presented.

We end this chapter with some more inequalities involving the power series functions that have been developed by utilising a different technique based on the continuity properties of modulus. Those results have natural applications for fundamental functions as well, which is a subsequent aim of this section.

All the results contained in this chapter, are mainly taken from the author's research papers in collaboration with Dragomir and Darus (see [208], [209], [214]). The same results can also be found in the survey research paper published by the author and Dragomir (see [217]).

3.1 Introduction and Preliminary Results

The (CBS)-inequality is one of the most important inequalities in analysis. Nowadays, a large number of results concerning new proofs, noteworthy extensions, generalizations, refinements, etc., of the classical (CBS)-inequality have been published in the literature. Most of the results are discrete and involve finite sums. A few attempts to extend those results to the infinite series have been considered by several authors (see [49], [51], [89], [122], [123], [173]). For instance, in [122], Dragomir provided the generalization of the (CBS)-inequality for functions defined by the real power series (2.65) with nonnegative coefficients and convergent on the interval $(-R, R)$, $R > 0$. The result is stated as follows (see also [117, p. 19]):

Theorem 32 (Dragomir [122]) *Let $f : (-R, R) \to \mathbb{R}$, $f(x) = \sum_{k=0}^{\infty} \alpha_k x^k$ with $\alpha_k \geq 0$, $k \in \mathbb{N}$. If $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ are sequences of real numbers such that $a_k b_k$, a_k^2, $b_k^2 \in (-R, R)$ for any $j \in \{1, 2, \ldots, n\}$, then one has the inequality*

$$\left(\sum_{k=1}^{n} f(a_k b_k) \right)^2 \leq \sum_{k=1}^{n} f\left(a_k^2\right) \sum_{k=1}^{n} f\left(b_k^2\right). \tag{3.1}$$

We note that for $f(x) = x$, $x \in \mathbb{R}$ in (3.1), we simply recapture the classical (CBS)-inequality for real numbers (2.56).

Particular inequalities of (3.1) for some fundamental functions are also given in [122]. For instance, if a and b are sequences of real numbers, then one has the inequality

$$\begin{aligned}
\left(\sum_{j=1}^{n} \exp(a_j b_j)\right)^2 &\leq \sum_{j=1}^{n} \exp\left(a_j^2\right) \sum_{j=1}^{n} \exp\left(b_j^2\right), \\
\left(\sum_{j=1}^{n} \sinh(a_j b_j)\right)^2 &\leq \sum_{j=1}^{n} \sinh\left(a_j^2\right) \sum_{j=1}^{n} \sinh\left(b_j^2\right), \\
\left(\sum_{j=1}^{n} \cosh(a_j b_j)\right)^2 &\leq \sum_{j=1}^{n} \cosh\left(a_j^2\right) \sum_{j=1}^{n} \cosh\left(b_j^2\right).
\end{aligned} \tag{3.2}$$

Other inequalities concerning the power series (2.65) with real or nonnegative coefficients, have been established by Cerone and Dragomir in [89]. On utilising the known result that has been available in the literature, which is called the de Bruijn inequality (2.73), they obtained some refinements of the (CBS)-inequality for functions defined by the power series (2.68) and (2.72). The results are summarized as follows:

Theorem 33 (Cerone and Dragomir [89]) *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an analytic function defined by a power series with nonnegative coefficients a_k, $k \geq 0$ and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. If a is a real number and z a complex number such that az, a^2, z^2, $|z|^2 \in D(0, R)$, then*

$$|f(az)|^2 \leq \frac{1}{2} f\left(a^2\right) \left[f\left(|z|^2\right) + \left|f\left(z^2\right)\right|\right]. \tag{3.3}$$

Cerone and Dragomir [89] have also proved an analogous inequality of (3.3) for functions defined by the complex power series with real coefficients.

Theorem 34 (Cerone and Dragomir [89]) *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be a function defined by a power series with real coefficients and convergent on the open* *disk* $D(0,R) \subset \mathbb{C}$, $R > 0$. *If* $a \in \mathbb{R}$ *and* $z \in \mathbb{C}$ *are such that* az, a^2, z^2, $|z|^2 \in D(0,R)$, *then*

$$|f(az)|^2 \le \frac{1}{2} f_A(a^2)\left[f_A\left(|z|^2\right) + \left|f_A\left(z^2\right)\right|\right]. \tag{3.4}$$

Particularly, the results given by (3.3) and (3.4) provide the improvements of the de Bruijn inequality (2.73) as well as the refinements of the inequalities (2.68) and (2.72) of the (CBS)-type for functions defined by the complex power series with real or nonnegative coefficients and convergent on the open disk.

Particular inequalities for various examples of fundamental functions such as the exponential, logarithm, trigonometric and hyperbolic functions are also pointed out in [89]. For instance,

$$\left[1 - |z|^2 + |1 - z|\right] |1 - az|^2 \ge 2\left(1 - a^2\right)\left(1 - |z|^2\right)\left|1 - z^2\right|,$$

$$|\ln(1 - az)|^2 \le \frac{1}{2}\ln\left(\frac{1}{1 - a^2}\right)\left[\ln\left(\frac{1}{1 - |z|^2}\right) + \left|\ln\left(1 - z^2\right)\right|\right],$$

$$\left|\sin^{-1}(az)\right|^2 \le \frac{1}{2}\sin^{-1}\left(a^2\right)\left[\sin^{-1}\left(|z|^2\right) + \left|\sin^{-1}\left(z^2\right)\right|\right], \tag{3.5}$$

$$\left|\tanh^{-1}(az)\right|^2 \le \frac{1}{2}\tanh^{-1}\left(a^2\right)\left[\tanh^{-1}\left(|z|^2\right) + \left|\tanh^{-1}\left(z^2\right)\right|\right],$$

for any $a \in (-1,1)$ and $z \in D(0,1)$. Applications for special function such as polylogarithm, are given as well.

Motivated by the results given by (3.3), (3.4) and some other results established in [89], we derive new and better inequalities for functions defined by the power series with real or nonnegative coefficients, by utilising Buzano's and Schwarz's result in complex inner product spaces, and a different technique based on the continuity properties of modulus. Natural applications for fundamental functions of interest such as the exponential, logarithm, trigonometric and hyperbolic are also provided.

3.2 Power Series Inequalities Via Buzano's Results

In [124], Dragomir has observed that from [78], on utilizing Buzano's inequality (2.75) in the complex inner product space $(H; \langle\cdot,\cdot\rangle)$, where the inner product is defined by (2.48), one can obtain the following discrete inequality:

$$\begin{aligned} &\left|\sum_{j=1}^{n} p_j c_j \overline{x_j} \sum_{j=1}^{n} p_j x_j \overline{b_j}\right| \\ &\leq \frac{1}{2}\left[\left(\sum_{j=1}^{n} p_j |c_j|^2 \sum_{j=1}^{n} p_j |b_j|^2\right)^{1/2} + \left|\sum_{j=1}^{n} p_j c_j \overline{b_j}\right|\right] \sum_{j=1}^{n} p_j |x_j|^2, \end{aligned} \tag{3.6}$$

where $p_j \geq 0$, $x_j, b_j, c_j \in \mathbb{C}$, $j \in \{1, \dots, n\}$. If we take in (3.6) $b_j = \overline{c_j}$ for $j \in \{1, 2, ..., n\}$, then we get

$$\left|\sum_{j=1}^{n} p_j c_j \overline{x_j} \sum_{j=1}^{n} p_j c_j x_j\right| \leq \frac{1}{2}\left[\sum_{j=1}^{n} p_j |c_j|^2 + \left|\sum_{j=1}^{n} p_j c_j^2\right|\right] \sum_{j=1}^{n} p_j |x_j|^2, \tag{3.7}$$

for any $p_j \geq 0$, x_j, $c_j \in \mathbb{C}$, $j \in \{1, 2, ..., n\}$.

As pointed out in [124] and also in [120, p. 49], if x_j, $j \in \{1, 2, ..., n\}$, are real numbers, then (3.6) generates the de Bruijn refinement of the celebrated weighted (CBS)-inequality, namely

$$\left|\sum_{j=1}^{n} p_j x_j z_j\right|^2 \leq \frac{1}{2}\sum_{j=1}^{n} p_j x_j^2 \left[\sum_{j=1}^{n} p_j |z_j|^2 + \left|\sum_{j=1}^{n} p_j z_j^2\right|\right], \tag{3.8}$$

where $p_j \geq 0$, $x_j \in \mathbb{R}$, $z_j \in \mathbb{C}$, $j \in \{1, 2, ..., n\}$. Therefore, the Buzano's result (3.6) may be regarded as a generalization of the de Bruijn inequality (2.73) as well.

First, we prove the following result [208] that has been obtained on utilising Buzano's result (3.6), see also [217].

Theorem 35 (Ibrahim and Dragomir [208]) *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients a_n and convergent on the open disk $D(0,R)$. If x, α, $\beta \in \mathbb{C}$ so that $\alpha\overline{x}$, $\overline{\beta}x$, $|\alpha|^2$, β^2, $a\overline{\beta}$, $|x|^2 \in D(0,R)$, then*

$$\left|f(\alpha\overline{x}) f(\overline{\beta}x)\right| \leq \frac{1}{2}\left[\left[f\left(|\alpha|^2\right) f\left(|\beta|^2\right)\right]^{1/2} + \left|f\left(\alpha\overline{\beta}\right)\right|\right] f\left(|x|^2\right). \tag{3.9}$$

Proof. On utilizing the inequality (3.6) for the choices $p_n = a_n$, $c_n = \alpha^n$, $x_n = x^n$ and $b_n = \beta^n$, $n \geq 0$, we have

$$\begin{aligned}&\left|\sum_{n=0}^{m} a_n \alpha^n (\overline{x})^n \sum_{n=0}^{m} a_n \left(\overline{\beta}\right)^n x^n\right| \\ &\leq \frac{1}{2}\left[\left(\sum_{n=0}^{m} a_n |\alpha|^{2n} \sum_{n=0}^{m} a_n |\beta|^{2n}\right)^{1/2} + \left|\sum_{n=0}^{m} a_n \alpha^n (\overline{\beta})^n\right|\right] \sum_{n=0}^{m} a_n |x|^{2n},\end{aligned} \tag{3.10}$$

for any $m \geq 0$.

Since $\alpha\overline{x}$, $\overline{\beta}x$, $|\alpha|^2$, $|\beta|^2$, $\alpha\overline{\beta}$, $|x|^2$ belong to the convergence disk $D(0,R)$, hence the series in (3.10) are convergent and letting $m \to \infty$, we deduce the desired inequality (3.9). ■

A particular case of interest is as follows:

Corollary 36 *Let $f(z)$ be as in Theorem 35 and $z, x \in \mathbb{C}$ with $z\overline{x}$, zx, $|z|^2$, z^2, $|x|^2 \in D(0,R)$. Then,*

$$|f(z\overline{x}) f(zx)| \leq \frac{1}{2}\left[f\left(|z|^2\right) + \left|f\left(z^2\right)\right|\right] f\left(|x|^2\right). \tag{3.11}$$

Proof. This follows from Theorem 35 by choosing $\alpha = z$ and $\beta = \overline{z}$. ■

Remark 37 *In particular, if $x = a \in \mathbb{R}$, then from (3.11) we deduce the inequality (3.3) [89].*

The above result (3.9) has some natural applications for particular complex functions of interest as follows:

(1) If we apply the inequality (3.9) for the function $f(z)$ given by (1.19), then we get

$$\begin{aligned}&\left|\frac{1}{1-\alpha\overline{x}}\cdot\frac{1}{1-\overline{\beta}x}\right|\\&\leq\frac{1}{2}\left[\left(\frac{1}{1-|\alpha|^{2}}\cdot\frac{1}{1-|\beta|^{2}}\right)^{1/2}+\left|\frac{1}{1-\alpha\overline{\beta}}\right|\right]\frac{1}{1-|x|^{2}},\end{aligned}\tag{3.12}$$

for any x, α, $\beta\in D(0,1)$. This is equivalent with

$$\begin{aligned}&2\left(1-|x|^{2}\right)\left|1-\alpha\overline{\beta}\right|\sqrt{\left(1-|\alpha|^{2}\right)\left(1-|\beta|\right)^{2}}\\&\leq|1-\alpha\overline{x}|\left|1-\overline{\beta}x\right|\left[\left|1-\alpha\overline{\beta}\right|+\sqrt{\left(1-|\alpha|^{2}\right)\left(1-|\beta|\right)^{2}}\right],\end{aligned}\tag{3.13}$$

for x, α, $\beta\in D(0,1)$. In particular, if $\beta=\overline{\alpha}$, then we get from (3.13) that

$$\begin{aligned}&2\left(1-|x|\right)\left(1-|\alpha|^{2}\right)\left|1-\alpha^{2}\right|\\&\leq|1-\alpha\overline{x}|\,|1-\alpha x|\left[\left|1-\alpha^{2}\right|+1-|\alpha|^{2}\right],\end{aligned}\tag{3.14}$$

for any x, $\alpha\in D(0,1)$.

(2) If we apply (3.9) for the exponential function $f(z)$ given by (1.25), then we get the inequality

$$\begin{aligned}&\left|\exp\left(\alpha\overline{x}+\overline{\beta}x\right)\right|\\&\leq\frac{1}{2}\left[\left(\exp\left(|\alpha|^{2}+|\beta|^{2}\right)\right)^{1/2}+\left|\exp\left(\alpha\overline{\beta}\right)\right|\right]\exp\left(|x|^{2}\right),\end{aligned}\tag{3.15}$$

for any α, β, $x\in\mathbb{C}$. In particular, if $\alpha=\overline{\beta}$, then we get from (3.15) that

$$|\exp\left(2\alpha\operatorname{Re}(x)\right)|\leq\frac{1}{2}\left[\left(\exp(2|\alpha|^{2})\right)^{1/2}+\left|\exp\left(\alpha^{2}\right)\right|\right]\exp\left(|x|^{2}\right),\tag{3.16}$$

for any α, $x\in\mathbb{C}$.

(3) If we apply (3.9) for the Koebe function $f(z)$ defined by (1.23), then we get

$$\begin{aligned}&\left|\frac{\alpha\overline{\beta}\,|x|^2}{(1-\alpha\overline{x})^2\left(1-\overline{\beta}x\right)^2}\right|\\&\leq\frac{1}{2}\left(\frac{|\alpha\beta|}{\left(1-|\alpha|^2\right)\left(1-|\beta|^2\right)}+\left|\frac{\alpha\overline{\beta}}{\left(1-\alpha\overline{\beta}\right)^2}\right|\right)\frac{|x|^2}{\left(1-|x|^2\right)^2},\end{aligned}\tag{3.17}$$

for any $x,\ \alpha,\ \beta\in D(0,1)$. If we simplify the above inequality (3.17), then we have

$$\begin{aligned}&\frac{1-|x|^2}{\left|(1-\alpha\overline{x})\left(1-\overline{\beta}x\right)\right|}\\&\leq\left(\frac{1}{2\left(1-|\alpha|^2\right)\left(1-|\beta|^2\right)}+\frac{1}{2\left|1-\alpha\overline{\beta}\right|^2}\right)^{1/2},\end{aligned}\tag{3.18}$$

for any $\alpha,\ \beta,\ x\in D(0,1)$. In particular if $\beta=\overline{\alpha}$, then we get from (3.18) that

$$\frac{1-|x|^2}{|(1-\alpha\overline{x})(1-\alpha x)|}\leq\left(\frac{1}{2\left(1-|\alpha|^2\right)^2}+\frac{1}{2\left|1-\alpha^2\right|^2}\right)^{1/2},\tag{3.19}$$

for any $\alpha,\ x\in D(0,1)$.

(4) If we apply the same inequality (3.9) for the hyperbolic function $f(z)$ given by (1.39), then we obtain

$$\begin{aligned}&\left|\cosh\left(\alpha\overline{x}+\overline{\beta}x\right)+\cosh\left(\alpha\overline{x}-\overline{\beta}x\right)\right|\\&\leq\left(\left[\frac{1}{2}\left(\cosh\left(|\alpha|^2+|\beta|^2\right)+\cosh\left(|\alpha|^2-|\beta|^2\right)\right)\right]^{1/2}\right.\\&\quad\left.+\left|\cosh\left(\alpha\overline{\beta}\right)\right|\right)\cosh\left(|x|^2\right),\end{aligned}\tag{3.20}$$

for any $x,\ \alpha,\ \beta\in\mathbb{C}$. In particular, for $\beta=\overline{\alpha}$, we get that from (3.20)

$$\begin{aligned}&\left|\cosh\left(2\alpha\operatorname{Re}(x)\right)+\cosh\left(2i\alpha\operatorname{Im}(x)\right)\right|\\&\leq\left[\cosh\left(|\alpha|^2\right)+\left|\cosh\left(\alpha^2\right)\right|\right]\cosh\left(|x|^2\right),\end{aligned}\tag{3.21}$$

that holds for any α, $x\in\mathbb{C}$.

We also obtain the analogous result to (3.9), which connects the power series $f(z)$ with its transform f_A [208] (see also [217]).

Theorem 38 (Ibrahim and Dragomir [208]) *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a function defined by a power series with real coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. If α, β, x are complex numbers such that $\alpha\overline{x}$, $\overline{\beta}x$, $\alpha\overline{\beta}$, $|\alpha|^2$, $|\beta|^2$, $|x^2|\in D(0,R)$, then*

$$\left|f(\alpha\overline{x})f\left(\overline{\beta}x\right)\right|\leq\frac{1}{2}\left(\left[f_A\left(|\alpha|^2\right)f_A\left(|\beta|^2\right)\right]^{1/2}+\left|f_A\left(\alpha\overline{\beta}\right)\right|\right)f_A\left(|x|^2\right).\tag{3.22}$$

Proof. By choosing $p_n=|a_n|$, $c_n=\alpha^n$, $b_n=\beta^n$ and $x_n=sgn(a_n)x^n$, $n\geq 0$ in (3.6), where $sgn(t)$ is a real sign function defined by (2.11), we have

$$\begin{aligned}&\left|\sum_{n=0}^{m}a_n(\alpha\overline{x})^n\sum_{n=0}^{m}a_n\left(\overline{\beta}x\right)^n\right|\\&=\left|\sum_{n=0}^{m}|a_n|\,sgn(a_n)\,\alpha^n(\overline{x})^n\sum_{n=0}^{m}|a_n|\,sgn(a_n)\,x^n\left(\overline{\beta}\right)^n\right|,\\&\leq\frac{1}{2}\left(\left[\sum_{n=0}^{m}|a_n|\left(|\alpha|^2\right)^n\sum_{n=0}^{m}|a_n|\left(|\beta|^2\right)^n\right]^{1/2}+\left|\sum_{n=0}^{m}|a_n|\left(\alpha\overline{\beta}\right)^n\right|\right)\\&\times\sum_{n=0}^{m}|a_n|\left(|x|^2\right)^n,\end{aligned}\tag{3.23}$$

for any α, β, $x\in\mathbb{C}$ with $\alpha\overline{x}$, $\overline{\beta}x$, $\alpha\overline{\beta}$, $|\alpha|^2$, $|\beta|^2$, $|x|^2\in D(0,R)$. Now, taking the limit as $m\to\infty$ in (3.23) and noticing that all the involved series in (3.23) are convergent, then we deduce the desired inequality (3.22). ■

Remark 39 *Taking in (3.22) $\alpha=z\in\mathbb{C}$, $\beta=\overline{z}$ and $x=a\in R$ will produce the inequality (3.4) [89].*

In what follows, we provide some applications of the inequality (3.22) for particular functions of interest.

(1) If we take the function $f(z)$ given by (1.20), then we have the transform $f_A(z)$ given by (1.19).Applying Theorem 38, we get the following inequality

$$2\left|1-\alpha\overline{\beta}\right|\left(1-|x|^2\right)\left[\left(1-|\alpha|^2\right)\left(1-|\beta|^2\right)\right]^{1/2} \\ \leq |1+\alpha\overline{x}|\left|1+\overline{\beta}x\right|\left(\left|1-\alpha\overline{\beta}\right|+\left[\left(1-|\alpha|^2\right)\left(1-|\beta|^2\right)\right]^{1/2}\right), \tag{3.24}$$

for any $\alpha, \beta, x \in D(0,1)$. In particular, if $\alpha = \overline{\beta}$, then from (3.24) we obtain

$$2\left|1-\alpha^2\right|\left(1-|\alpha|^2\right)\left(1-|x|^2\right) \\ \leq |1+\alpha x|\,|1+\alpha\overline{x}|\left(1-|\alpha|^2+\left|1-\alpha^2\right|\right), \tag{3.25}$$

for any $\alpha, x \in D(0,1)$.

(2) For the exponential function $f(z)$ given by (1.26), we have the transform

$$f_A(z) = \sum_{n=0}^{\infty}\frac{z^n}{n!} = e^z, \;\; z \in \mathbb{C}. \tag{3.26}$$

Utilising the inequality (3.22) we obtain

$$\frac{1}{\left|\exp\left(\alpha\overline{x}+\overline{\beta}x\right)\right|} \\ \leq \frac{1}{2}\left(\exp\left[\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)+|x|^2\right]+\exp\left(|x|^2\right)\left|\exp\left(\alpha\overline{\beta}\right)\right|\right), \tag{3.27}$$

for any $\alpha, \beta, x \in \mathbb{C}$. In particular, if $\alpha = \overline{\beta}$ in (3.27), then we get

$$\frac{1}{|\exp\left(2\alpha\operatorname{Re}(x)\right)|} \leq \frac{1}{2}\left[\exp\left(|\alpha|^2+|\beta|^2\right)+\exp\left(|x|^2\right)\left|\exp\left(\alpha^2\right)\right|\right], \tag{3.28}$$

for any $\alpha, x \in \mathbb{C}$.

(3) If in (3.22) we choose the trigonometric function $f(z)$ given by (1.33), then the transform $f_A(z)$ is given by (1.39). Applying the inequality (3.22) will produce the following result:

$$\begin{aligned}&\left|\cos\left(\alpha\overline{x}\right)\cos\left(\overline{\beta}x\right)\right| \\ &\leq \frac{1}{2}\left(\left[\cosh\left(|\alpha|^2\right)\cosh\left(|\beta|^2\right)\right]^{1/2} + \left|\cosh\left(\alpha\overline{\beta}\right)\right|\right)\cosh\left(|x|^2\right), \qquad (3.29)\end{aligned}$$

for any α, β, $x \in \mathbb{C}$. In particular, if we choose $\alpha = \overline{\beta}$ in (3.29), then we obtain the inequality

$$\left|\cos\left(\alpha\overline{x}\right)\cos\left(\alpha x\right)\right| \leq \frac{1}{2}\left[\cosh\left(|\alpha|^2\right) + \left|\cosh\left(\alpha^2\right)\right|\right]\cosh\left(|x|^2\right), \qquad (3.30)$$

for any α, $x \in \mathbb{C}$.

Next, we have proved the following result [208], which connects the two power series, one having positive coefficients (see also [217]).

Theorem 40 (Ibrahim and Dragomir [208]) *Let $g(z) = \sum_{n=0}^{\infty} g_n z^n$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be two power series with $g_n \in \mathbb{C}$ and $a_n > 0$ for $n \geq 0$. If f and g are convergent on $D(0, R_1)$ and $D(0, R_2)$ respectively, and the numerical series $\sum_{n=0}^{\infty} \frac{|g_n|^2}{a_n}$ is convergent, then we have the inequality:*

$$|g(z)\,g(\overline{z})| \leq \frac{1}{2}\sum_{n=0}^{\infty}\frac{|g_n|^2}{a_n}\left[f\left(|z|^2\right) + \left|f\left(z^2\right)\right|\right], \qquad (3.31)$$

for any $z \in \mathbb{C}$ with z, z^2, $|z|^2 \in D(0, R_1) \cap D(0, R_2)$.

Proof. On utilizing the inequality (3.7) for the choices $p_n = a_n$, $c_n = z^n$, $x_n = g_n/a_n$, $n \geq 0$, we have

$$\left|\sum_{n=0}^{m} g_n z^n \sum_{n=0}^{m} \overline{g_n} z^n\right| \leq \frac{1}{2}\left[\sum_{n=0}^{m} a_n\left(|z|^2\right)^n + \left|\sum_{n=0}^{m} a_n\left(z^2\right)^n\right|\right]\sum_{n=0}^{m}\frac{|g_n|^2}{a_n}, \qquad (3.32)$$

for any $m \geq 0$. We observe that

$$\sum_{n=0}^{m}\overline{g_n}z^n=\overline{\sum_{n=0}^{m}g_n\left(\overline{z}\right)^n}\quad\text{and then}\quad\left|\sum_{n=0}^{m}\overline{g_n}z^n\right|=\left|\sum_{n=0}^{m}g_n\left(\overline{z}\right)^n\right|. \tag{3.33}$$

Replacing (3.33) into the inequality (3.32) we get

$$\left|\sum_{n=0}^{m}g_nz^n\sum_{n=0n}^{m}g_n\left(\overline{z}\right)^n\right|\leq\frac{1}{2}\sum_{n=0}^{m}\frac{|g_n|^2}{a_n}\left[\sum_{n=0}^{m}a_n\left(|z|^2\right)^n+\left|\sum_{n=0}^{m}a_n\left(z^2\right)^n\right|\right]. \tag{3.34}$$

Since z, z^2, $|z|^2\in D(0,R_1)\cap D(0,R_2)$, hence the series in (3.34) are convergent and letting $m\to\infty$, we deduce the desired inequality (3.31). ■

Remark 41 *If the coefficients g_n, $n\geq 0$ are real, then by (3.31) we recapture the result of Cerone and Dragomir in [89], namely*

$$|g(z)|^2\leq\frac{1}{2}\sum_{n=0}^{\infty}\frac{g_n^2}{a_n}\left[f\left(|z|^2\right)+\left|f\left(z^2\right)\right|\right], \tag{3.35}$$

for any z, z^2, $|z|^2\in D(0,R_1)\cap D(0,R_2)$.

As natural consequence of Theorem 40, the following corollaries hold for particular choices of functions $f(z)$. First, we have

Corollary 42 *Let $g(z)=\sum_{n=0}^{\infty}g_nz^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$. If the numerical series $\sum_{n=0}^{\infty}|g_n|^2$ is convergent, then*

$$|g(z)g(\overline{z})|\leq\frac{1}{2}\sum_{n=0}^{\infty}|g_n|^2\left[\frac{1-|z|^2+|1-z^2|}{\left(1-|z|^2\right)|1-z^2|}\right], \tag{3.36}$$

for any $z\in D(0,1)\cap D(0,R)$.

Proof. The proof simply follows from (3.31) for the function $f(z)$ given by (1.19). ■

If we consider the series expansion

$$\frac{1}{iz}\ln\left(\frac{1}{1-iz}\right)=\sum_{n=0}^{\infty}\frac{i^n}{n+1}z^n;\ z\in D(0,1)\setminus\{0\}, \tag{3.37}$$

then, on utilising the inequality (3.36) for the choice $g_n = i^n / (n+1)$ and taking into account the identity (d) in (1.45), we can state the following inequality

$$\left| \ln\left(\frac{1}{1-iz}\right) \ln\left(\frac{1}{1-i\overline{z}}\right) \right| \le \frac{\pi^2}{12} \left(\frac{|z|^2}{1-|z|^2}\right) \left(\frac{1-|z|^2+|1-z^2|}{|1-z^2|}\right), \tag{3.38}$$

for all $z \in D(0,1)$.

Corollary 43 *Let $g(z) = \sum_{n=0}^{\infty} g_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$. If the numerical series $\sum_{n=0}^{\infty} n! \, |g_n|^2$ is convergent, then*

$$|g(z)g(\overline{z})| \le \frac{1}{2} \sum_{n=0}^{\infty} n! \, |g_n|^2 \left[\exp\left(|z|^2\right) + \left|\exp\left(z^2\right)\right|\right], \tag{3.39}$$

for any $z \in D(0,R)$.

Proof. The proof follows from Theorem 40 by choosing the exponential function $f(z)$ given by (1.25). ■

Some applications of the inequality (3.39) are as follows.

(1) If we apply the inequality (3.39) for the function

$$\sin(iz) = \sum_{n=0}^{\infty} \frac{i}{(2n+1)!} z^{2n+1}, \quad z \in \mathbb{C}, \tag{3.40}$$

then, we obtain the inequality

$$|\sin(iz)\sin(i\overline{z})| \le \frac{1}{2} \sum_{n=0}^{\infty} \frac{n!}{[(2n+1)!]^2} \left[\exp\left(|z|^2\right) + \left|\exp\left(z^2\right)\right|\right], \tag{3.41}$$

for any $z \in \mathbb{C}$.

(2) If we apply the inequality (3.39) for the function

$$\sinh(iz) = \sum_{n=0}^{\infty} \frac{(-1)^n i}{(2n+1)!} z^{2n+1}, \quad z \in \mathbb{C}, \tag{3.42}$$

then, we obtain the inequality

$$|\sin(z)|^2 \le \frac{1}{2}\sum_{n=0}^{\infty} \frac{n!}{[(2n+1)!]^2}\left[\exp\left(|z|^2\right) + \left|\exp\left(z^2\right)\right|\right], \tag{3.43}$$

for any $z \in \mathbb{C}$. Indeed, observing that

$$\begin{aligned} |\sinh(iz)\sinh(i\overline{z})| &= |i\sin(z) \cdot i\sin(\overline{z})| = |\sin(z)\sin(\overline{z})| \\ &= |\sin(z)|^2, \end{aligned} \tag{3.44}$$

and by (3.39) we have

$$|\sinh(iz)\sinh(i\overline{z})| \le \frac{1}{2}\sum_{n=0}^{\infty} \frac{n!}{[(2n+1)!]^2}\left[\exp\left(|z|^2\right) + \left|\exp\left(z^2\right)\right|\right], \tag{3.45}$$

for any $z \in \mathbb{C}$, then we deduce the desired inequality (3.43).

Finally, we obtain the following result [208], which provides the connection between three functions defined by the power series, one having positive coefficients, while the others have complex coefficients (see also [217]).

Theorem 44 (Ibrahim and Dragomir [208]) *Let $g(z) = \sum_{n=0}^{\infty} g_n z^n$, $h(z) = \sum_{n=0}^{\infty} h_n z^n$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be three power series with g_n, $h_n \in \mathbb{C}$ and $a_n > 0$ for $n \ge 0$. If f, g and h are convergent on $D(0,R_1)$, $D(0,R_2)$ and $D(0,R_3)$ respectively, and the numerical series $\sum_{n=0}^{\infty} \frac{|g_n|^2}{a_n}$, $\sum_{n=0}^{\infty} \frac{|h_n|^2}{a_n}$ and $\sum_{n=0}^{\infty} \frac{g_n \overline{h_n}}{a_n}$ are convergent, then we have the inequality*

$$|g(z)h(z)| \le \frac{1}{2}\left(\left[\sum_{n=0}^{\infty}\frac{|g_n|^2}{a_n}\sum_{n=0}^{\infty}\frac{|h_n|^2}{a_n}\right]^{1/2} + \left|\sum_{n=0}^{\infty}\frac{g_n\overline{h_n}}{a_n}\right|\right) f\left(|z|^2\right), \tag{3.46}$$

for any $z \in \mathbb{C}$ with z, $|z|^2 \in D(0,R_1) \cap D(0,R_2) \cap D(0,R_3)$.

Proof. Again, on utilising the Buzano inequality (3.6) for the choices $p_n = a_n$, $c_n = g_n/a_n$, $b_n = h_n/a_n$, $x_n = z^n$, $n \ge 0$, we can state that

$$|g(z)h(z)| = \left|\sum_{n=0}^{\infty} a_n\left(\frac{g_n}{a_n}\right) z^n \sum_{n=0}^{\infty} a_n\left(\frac{h_n}{a_n}\right) z^n\right|,$$

$$\le \frac{1}{2}\left(\left[\sum_{n=0}^{\infty} a_n \left|\frac{g_n}{a_n}\right|^2 \sum_{n=0}^{\infty} a_n \left|\frac{h_n}{a_n}\right|^2\right]^{1/2}\right.$$
$$\left. + \left|\sum_{n=0}^{\infty} a_n \left(\frac{g_n}{a_n}\right)\overline{\left(\frac{h_n}{a_n}\right)}\right|\right) \sum_{n=0}^{\infty} a_n \left(|z|^2\right)^n,$$
$$= \frac{1}{2}\left(\left[\sum_{n=0}^{\infty} \frac{|g_n|^2}{a_n} \sum_{n=0}^{\infty} \frac{|h_n|^2}{a_n}\right]^{1/2} + \left|\sum_{n=0}^{\infty} \frac{g_n \overline{h_n}}{a_n}\right|\right) f\left(|z|^2\right), \tag{3.47}$$

for any $z \in \mathbb{C}$ with $z, |z|^2 \in D(0, R_1) \cap D(0, R_2) \cap D(0, R_3)$. ■

Remark 45 *In particular, if $g_n = h_n$, then from (3.46) we have*

$$|g(z)|^2 \le f\left(|z|^2\right) \sum_{n=0}^{\infty} \frac{|g_n|^2}{a_n}, \tag{3.48}$$

for any z, $|z|^2 \in D(0, R_1) \cap D(0, R_2)$.

Remark 46 *Also, if $h_n = \overline{g_n}$, then from (3.46) we get the following inequality*

$$|g(z)g(\overline{z})| \le \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{|g_n|^2}{a_n} + \left|\sum_{n=0}^{\infty} g_n^2\right|\right) f\left(|z|^2\right), \tag{3.49}$$

for any $z, |z|^2 \in D(0, R_1) \cap D(0, R_2)$.

For particular choices of function $f(z)$ in (3.46), the following result holds.

Corollary 47 *Let $g(z)$ and $h(z)$ be power series as in Theorem 44. If the numerical series $\sum_{n=0}^{\infty} |g_n|^2$, $\sum_{n=0}^{\infty} |h_n|^2$ and $\sum_{n=0}^{\infty} \left|g_n \overline{h_n}\right|$ are convergent, then*

$$|g(z)h(z)| \le \frac{1}{2\left(1 - |z|^2\right)}\left(\left[\sum_{n=0}^{\infty} |g_n|^2 \sum_{n=0}^{\infty} |h_n|^2\right]^{1/2} + \left|\sum_{n=0}^{\infty} g_n \overline{h_n}\right|\right), \tag{3.50}$$

for any $z \in D(0, 1) \cap D(0, R_2) \cap D(0, R_3)$.

Proof. This follows from (3.46) for the function $f(z)$ given by (1.19). ■

If we consider the series expansion given by (3.37) and the series

$$\ln\left(\frac{1}{1+iz}\right)=\sum_{n=1}^{\infty}\frac{(-i)^n}{n}z^n,\ z\in D(0,1), \tag{3.51}$$

then on utilizing the inequality (3.50) for the choices $g_0=h_0=0$, $g_n=i^n/(n+1)$, $h_n=(-i)^n/n$, $n\geq 1$, and taking into account that

$$\sum_{n=0}^{\infty}g_n\overline{h_n}=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1, \tag{3.52}$$

and the identity (d) as mentioned in (1.45), we obtain the following inequality

$$\left|\ln\left(\frac{1}{1-iz}\right)\ln\left(\frac{1}{1+iz}\right)\right|\leq\frac{\pi^2+6}{12}\left(\frac{|z|}{1-|z|^2}\right), \tag{3.53}$$

for any $z\in D(0,1)$.

In the next section, we establish some other inequalities for functions defined by the power series with nonnegative coefficients and convergent on an open disk. A refinement of the (CBS)-inequality in inner product spaces due to Dragomir [118], is being an important tool for this investigations. Natural applications for some fundamental functions of interest such as the exponential, logarithm, trigonometric and hyperbolic functions are also pointed out.

3.3 Power Series Inequalities Via a Refinement of the Schwarz Inequality

If we write the inequality (2.78) for the particular inner product space $(\mathbb{K}^n;\langle\cdot,\cdot\rangle)$, where the weighted inner product is defined by (2.48), for $x=(x_1,x_2,...,x_n)$, $y=(y_1,y_2,...,y_n)\in\mathbb{K}^n$ and $p=(p_1,p_2,...,p_n)$ with $p_j\geq 0$, $j\in\{1,2,...,n\}$, then we get the following discrete inequality

$$\left(\sum_{j=1}^{n} p_j |x_j|^2\right)^{1/2} \left(\sum_{j=1}^{n} p_j |y_j|^2\right)^{1/2} \sum_{j=1}^{n} p_j |z_j|^2 - \left|\sum_{j=1}^{n} p_j x_j \overline{z_j} \sum_{j=1}^{n} p_j z_j \overline{y_j}\right|$$
$$\geq \left|\sum_{j=1}^{n} p_j x_j \overline{y_j} \sum_{j=1}^{n} p_j |z_j|^2 - \sum_{j=1}^{n} p_j x_j \overline{z_j} \sum_{j=1}^{n} p_j z_j \overline{y_j}\right| \tag{3.54}$$

where $p_j \geq 0,\ x_j, y_j, z_j \in \mathbb{K},\ \ j \in \{1, 2, ..., n\}$. In particular, if we take in (3.54) $y_j = \overline{x_j}$ for $j \in \{1, 2, ..., n\}$, then we obtain

$$\sum_{j=1}^{n} p_j |x_j|^2 \sum_{j=1}^{n} p_j |z_j|^2 - \left|\sum_{j=1}^{n} p_j x_j \overline{z_j} \sum_{j=1}^{n} p_j x_j z_j\right|$$
$$\geq \left|\sum_{j=1}^{n} p_j x_j^2 \sum_{j=1}^{n} p_j |z_j|^2 - \sum_{j=1}^{n} p_j x_j \overline{z_j} \sum_{j=1}^{n} p_j x_j z_j\right|, \tag{3.55}$$

for $p_j \geq 0,\ x_j, z_j \in \mathbb{K},\ j \in \{1, 2, ..., n\}$.

On applying the inequality (3.54) for functions defined by the power series with nonnegative coefficients, we establish the following result [209] (see also [217]).

Theorem 48 (Ibrahim and Dragomir [209]) *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients a_n and convergent on the open disk $D(0, R)$. If $x,\ y,\ z \in \mathbb{C}$, so that $|x|^2,\ |y|^2,\ |z|^2,\ x\overline{z},\ z\overline{y},\ x\overline{y} \in D(0, R)$, then*

$$\left[f\left(|x|^2\right) f\left(|y|^2\right)\right]^{1/2} f\left(|z|^2\right) - \left|f(x\overline{z}) f(z\overline{y})\right|$$
$$\geq \left|f(x\overline{y}) f\left(|z|^2\right) - f(x\overline{z}) f(z\overline{y})\right|. \tag{3.56}$$

Proof. If we choose $p_n = a_n,\ x_n = x^n,\ y_n = y^n$ and $z_n = z^n,\ n \in \{0, 1, 2, ..., m\}$ in (3.54), then we have

$$\left[\sum_{n=0}^{m} a_n \left(|x|^2\right)^n\right]^{1/2} \left[\sum_{n=0}^{m} a_n \left(|y|^2\right)^n\right]^{1/2} \sum_{n=0}^{m} a_n \left(|z|^2\right)^n$$
$$- \left|\sum_{n=0}^{m} a_n (x\overline{z})^n \sum_{n=0}^{m} a_n (z\overline{y})^n\right|$$

$$\geq \left| \sum_{n=0}^{m} a_n (x\overline{y})^n \sum_{n=0}^{m} a_n \left(|z|^2\right)^n - \sum_{n=0}^{m} a_n (x\overline{z})^n \sum_{n=0}^{m} a_n (x\overline{y})^n \right|. \tag{3.57}$$

Since $|x|^2$, $|y|^2$, $|z|^2$, $x\overline{z}$, $z\overline{y}$, $x\overline{y}$ belong to the convergence disk $D(0,R)$ and taking the limit as $m \to \infty$ in (3.57), we deduce the desired inequality (3.56). ■

Some examples for particular functions that are generated by the power series with nonnegative coefficients, are as follows:

(1) If we choose in the above inequality (3.56) for the function $f(z)$ given by (1.19), then we have

$$\frac{|(1 - x\overline{z})(1 - z\overline{y})|}{\left[\left(1 - |x|^2\right)\left(1 - |y|^2\right)\right]^{1/2}\left(1 - |z|^2\right)} - 1$$

$$\geq \left| \frac{(1 - x\overline{z})(1 - z\overline{y})}{(1 - x\overline{y})\left(1 - |z|^2\right)} - 1 \right|, \tag{3.58}$$

for any x, y, $z \in D(0,1)$. In particular, for $z = \overline{x}$ in (3.58) we get

$$\frac{|(1 - x^2)(1 - \overline{xy})|}{\left(1 - |x|^2\right)^{3/2}\left(1 - |y|^2\right)^{1/2}} - 1 \geq \left| \frac{(1 - x^2)(1 - \overline{xy})}{(1 - x\overline{y})\left(1 - |x|^2\right)} - 1 \right|, \tag{3.59}$$

for any x, $y \in D(0,1)$. Also, if $z = a \in \mathbb{R}$ and x, $y \in \mathbb{C}$, then from (3.58), we obtain the following result:

$$\left| \frac{(1 - ax)(1 - a\overline{y})}{(1 - x\overline{y})(1 - a^2)} - 1 \right|$$

$$\leq \frac{|(1 - ax)(1 - a\overline{y})|}{\left[\left(1 - |x|^2\right)\left(1 - |y|^2\right)\right]^{1/2}(1 - a^2)} - 1, \tag{3.60}$$

for any x, $y \in D(0,1)$ and $a \in (-1,1)$.

(2) If we apply (3.56) for the exponential function $f(z)$ given by (1.25), then we get

$$\exp\left(\frac{|x|^2+|y|^2}{2}+|z|^2\right)-|\exp(x\overline{z}+z\overline{y})|$$
$$\geq\left|\exp\left(x\overline{y}+|z|^2\right)-\exp(x\overline{z}+z\overline{y})\right|, \tag{3.61}$$

for any x, y, $z\in\mathbb{C}$. In particular, for $z=\overline{x}$ in (3.61), we obtain

$$\left|\exp\left(x\overline{y}+|x|^2\right)-\exp\left(x^2+\overline{xy}\right)\right|$$
$$\leq\exp\left(\frac{3|x|^2+|y|^2}{2}\right)-\left|\exp\left(x^2+\overline{xy}\right)\right|, \tag{3.62}$$

for any x, $y\in\mathbb{C}$. Also, if $z=a\in\mathbb{R}$ and x, $y\in\mathbb{C}$, then from (3.61) we get

$$\left|\exp\left(x\overline{y}+a^2\right)-\exp\left[a\left(x+\overline{y}\right)\right]\right|$$
$$\leq\exp\left(\frac{|x|^2+|y|^2}{2}+a^2\right)-\left|\exp\left[a\left(x+\overline{y}\right)\right]\right|, \tag{3.63}$$

for any x, $y\in\mathbb{C}$ and $a\in\mathbb{R}$.

(3) For the *Koebe function* defined by (1.23), we get from (3.56) the following inequality:

$$\frac{1}{\left(1-|x|^2\right)\left(1-|y|^2\right)\left(1-|z|^2\right)^2}-\frac{1}{\left|\left[(1-x\overline{z})\left(1-z\overline{y}\right)\right]^2\right|}$$
$$\geq\left|\frac{1}{\left[\left(1-x\overline{y}\right)\left(1-|z|^2\right)\right]^2}-\frac{1}{\left[(1-x\overline{z})\left(1-z\overline{y}\right)\right]^2}\right|, \tag{3.64}$$

for any x, y, $z\in D(0,1)$. In particular, for $y=\overline{x}$ in (3.64) we get

$$\frac{1}{\left[\left(1-|x|^2\right)\left(1-|z|^2\right)\right]^2}-\frac{1}{\left|\left[(1-x\overline{z})(1-xz)\right]^2\right|}$$
$$\geq\left|\frac{1}{\left[\left(1-x^2\right)\left(1-|z|^2\right)\right]^2}-\frac{1}{\left[(1-x\overline{z})(1-xz)\right]^2}\right|, \tag{3.65}$$

for any x, $z\in D(0,1)$. Also, for $z=\overline{x}$, we have from (3.64)

$$\frac{1}{\left(1-|x|^{2}\right)^{3}\left(1-|y|^{2}\right)}-\frac{1}{\left|\left[\left(1-x^{2}\right)\left(1-\overline{xy}\right)\right]^{2}\right|}$$

$$\geq\left|\frac{1}{\left[\left(1-x\overline{y}\right)\left(1-|x|^{2}\right)\right]^{2}}-\frac{1}{\left[\left(1-x^{2}\right)\left(1-\overline{xy}\right)\right]^{2}}\right|, \tag{3.66}$$

for any $x,\ y\in D\left(0,1\right)$. If $z=a\in\mathbb{R}$ and $x,\ y\in\mathbb{C}$, then from (3.64)

$$\frac{1}{\left(1-|x|^{2}\right)\left(1-|y|^{2}\right)\left(1-a^{2}\right)^{2}}-\frac{1}{\left|\left[\left(1-ax\right)\left(1-a\overline{y}\right)\right]^{2}\right|}$$

$$\geq\left|\frac{1}{\left[\left(1-x\overline{y}\right)\left(1-a^{2}\right)\right]^{2}}-\frac{1}{\left[\left(1-ax\right)\left(1-a\overline{y}\right)\right]^{2}}\right|, \tag{3.67}$$

for any $x,\ y\in D\left(0,1\right)$ and $a\in\left(-1,1\right)$.

Remark 49 *If $z=0$, then from (3.56) we obtain*

$$\left[f\left(|x|^{2}\right)f\left(|y|^{2}\right)\right]^{1/2}-\left|f\left(0\right)\right|\geq\left|f\left(x\overline{y}\right)-f\left(0\right)\right|, \tag{3.68}$$

where $f(0)=a_{0}>0,\ |x|^{2},\ |y|^{2},\ x\overline{y}\in D\left(0,R\right)$.

The above result (3.68) also has natural applications for particular functions of interest, which are pointed out as follows:

(1) If we apply the inequality (3.68) for the exponential function $f\left(z\right)$ given by (1.25), then we obtain the inequality

$$\exp\left(\frac{|x|^{2}+|y|^{2}}{2}\right)-1\geq\left|\exp\left(x\overline{y}\right)-1\right|, \tag{3.69}$$

for any $x,\ y\in\mathbb{C}$. Moreover, if $y=\overline{x}$, then from (3.69) we get

$$\exp\left(|x|^{2}\right)-1\geq\left|\exp\left(x^{2}\right)-1\right|,$$

for any $x\in\mathbb{C}$.

(2) If we apply the same inequality (3.68) for the trigonometric function $f(z)$ given by (1.33), then we get the following inequality

$$\left[\cos\left(|x|^2\right)\cos\left(|y|^2\right)\right]^{1/2} - 1 \geq |\cos(x\overline{y}) - 1|, \tag{3.70}$$

for any $x,\ y \in \mathbb{C}$. Also, if $y = \overline{x}$, then from (3.70) we get

$$\cos\left(|x|^2\right) - 1 \geq \left|\cos\left(x^2\right) - 1\right|,$$

for any $x \in \mathbb{C}$.

(3) For the function $f(x)$ given by (1.19) and applying the inequality (3.68) we obtain

$$\frac{1}{\left[\left(1-|x|^2\right)\left(1-|y|^2\right)\right]^{1/2}} - 1 \geq \left|\frac{x\overline{y}}{1-x\overline{y}}\right|, \tag{3.71}$$

for any $x,\ y \in \mathbb{C}$ with $|x|^2,\ |y|^2,\ x\overline{y} \in D(0,1)$.

The following result also holds [209].

Remark 50 *If $y = \overline{x}$ in (3.56), then we get*

$$f\left(|x|^2\right) f\left(|z|^2\right) - |f(xz)f(x\overline{z})| \geq \left|f\left(x^2\right) f\left(|z|^2\right) - f(xz)f(x\overline{z})\right|, \tag{3.72}$$

for $x,\ z \in \mathbb{C}$ with $|x|^2,\ |z|^2,\ x\overline{z},\ zx \in D(0,R)$. Moreover, for $z = a \in \mathbb{R}$, from (3.72) we deduce

$$f\left(|x|^2\right) f\left(a^2\right) - |f(ax)|^2 \geq \left|f\left(x^2\right) f\left(a^2\right) - f^2(ax)\right|, \tag{3.73}$$

for any $x \in \mathbb{C}$, $a \in \mathbb{R}$. If we choose in (3.73) $a = 1$, then we have the inequality

$$f\left(|x|^2\right) f(1) - |f(x)|^2 \geq \left|f\left(x^2\right) f(1) - f^2(x)\right|, \tag{3.74}$$

for any $x \in \mathbb{C}$.

Remark 51 *If $z = \overline{x}$ in (3.56) then we get*

$$\begin{aligned} &\left[f\left(|x|^2\right) f\left(|y|^2\right)\right]^{1/2} f\left(|x|^2\right) - \left|f\left(x^2\right) f(xy)\right| \\ &\geq \left|f(x\overline{y}) f\left(|x|^2\right) - f\left(x^2\right) f\left(\overline{xy}\right)\right|, \end{aligned} \tag{3.75}$$

for $x,\ y \in \mathbb{C}$ with $x^2,\ xy,\ |x|^2,\ |y|^2 \in D(0,R)$.

For some application, we apply the inequality (3.74) for the exponential function $f(z)$ given by (1.25), then we have

$$\exp\left(|x|^2+1\right)-|\exp(2x)|\geq\left|\exp\left(x^2+1\right)-\exp(2x)\right|, \tag{3.76}$$

for any $x\in\mathbb{C}$. Since $|\exp(2x)|\neq 0$, then the inequality (3.76) is equivalent with

$$\frac{\exp\left(|x|^2+1\right)}{|\exp(2x)|}-1\geq\left|\exp(x-1)^2-1\right|, \tag{3.77}$$

for any $x\in\mathbb{C}$.

Also, if we apply the inequality (3.75) for the exponential function $f(z)$ given by (1.25), then we get

$$\begin{aligned}&\exp\left(\frac{3|x|^2+|y|^2}{2}\right)-\left|\exp\left[x^2+\overline{xy}\right]\right|\\&\geq\left|\exp\left(x\overline{y}+|x|^2\right)-\exp\left(x^2+\overline{xy}\right)\right|,\end{aligned} \tag{3.78}$$

for any x, $y\in\mathbb{C}$. Moreover if $x=a\in\mathbb{R}$, then from (3.78) we obtain

$$\exp\left(\frac{3a^2+|y|^2}{2}\right)\geq|\exp\left[a(a+y)\right]|, \tag{3.79}$$

for any $y\in\mathbb{C}$ and $a\in\mathbb{R}$.

The second result on the power series inequalities via a refinement of the (CBS)-inequality in inner product spaces (3.54), is incorporated in the following theorem [209] (see also [217]).

Theorem 52 (Ibrahim and Dragomir [209]) *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series with real coefficients a_n and convergent on $D(0,R)\subset\mathbb{C}$, $R>0$. If x, y, $z\in\mathbb{C}$, so that $|x|^2$, $|y|^2$, $|z|^2$, $x\overline{z}$, $z\overline{y}$, $x\overline{y}\in D(0,R)$, then*

$$\begin{aligned}&\left[f_A\left(|x|^2\right)f_A\left(|y|^2\right)\right]^{1/2}f_A\left(|z|^2\right)-|f(x\overline{z})f(z\overline{y})|\\&\geq\left|f_A(x\overline{y})f_A\left(|z|^2\right)-f(x\overline{z})f(z\overline{y})\right|.\end{aligned} \tag{3.80}$$

Proof. By choosing $p_n = |a_n| \geq 0$, $x_n = x^n$, $y_n = y^n$ and $z_n = sgn(a_n) z^n$, $n \geq 0$ in (3.54), we get

$$\begin{aligned}
&\left|\sum_{n=0}^{m} a_n (x\overline{z})^n \sum_{n=0}^{m} a_n (z\overline{y})^n\right| \\
&= \left|\sum_{n=0}^{m} |a_n|\, sgn(a_n)\, x^n (\overline{z})^n \sum_{n=0}^{m} |a_n|\, sgn(a_n)\, z^n (\overline{y})^n\right|, \\
&\leq \left(\sum_{n=0}^{m} |a_n| |x|^{2n}\right)^{1/2} \left(\sum_{n=0}^{m} |a_n| |y|^{2n}\right)^{1/2} \sum_{n=0}^{m} |a_n| |sgn(a_n) z^n|^2 \\
&- \left|\sum_{n=0}^{m} |a_n|\, x^n (\overline{y})^n \sum_{n=0}^{m} |a_n| |sgn(a_n) z^n|^2\right. \\
&\left.- \sum_{n=0}^{m} |a_n|\, x^n [sgn(a_n) (\overline{z})^n] \times \sum_{n=0}^{m} |a_n| [sgn(a_n) z^n] (\overline{y})^n\right|, \\
&= \left(\sum_{n=0}^{m} |a_n| \left(|x|^2\right)^n\right)^{1/2} \left(\sum_{n=0}^{m} |a_n| \left(|y|^2\right)^n\right)^{1/2} \sum_{n=0}^{m} |a_n| \left(|z|^2\right)^n
\end{aligned}$$

$$- \left|\sum_{n=0}^{m} |a_n| (x\overline{y})^n \sum_{n=0}^{m} |a_n| \left(|z|^2\right)^n - \sum_{n=0}^{m} a_n (x\overline{z})^n \sum_{n=0}^{m} a_n (z\overline{y})^n\right|, \tag{3.81}$$

for any $x, y, z \in \mathbb{C}$ with $x\overline{y}, x\overline{z}, z\overline{y}, |x|^2, |y|^2, |z|^2 \in D(0, R)$. Since all the series involved in (3.81) are convergent, by taking the limit as $m \to \infty$ in (3.81), we deduce the desired inequality (3.80). ∎

In what follows, we provide some applications of the inequality (3.80) for particular functions of interest.

(1) If we take the function $f(z)$ given by (1.20), then the transform $f_A(z)$ is given by (1.19). Applying the inequality (3.80), we can state that

$$\begin{aligned}
&\left[\frac{1}{1-|x|^2} \cdot \frac{1}{1-|y|^2}\right]^{1/2} \left(\frac{1}{1-|z|^2}\right) - \left|\frac{1}{1+x\overline{z}} \cdot \frac{1}{1+z\overline{y}}\right| f_A(z) \\
&= \frac{1}{\left[\left(1-|x|^2\right)\left(1-|y|^2\right)\right]^{1/2}\left(1-|z|^2\right)} - \frac{1}{|(1+x\overline{z})(1+z\overline{y})|}
\end{aligned}$$

$$= \frac{|(1+x\overline{z})(1+z\overline{y})| - \left[\left(1-|x|^2\right)\left(1-|y|^2\right)\right]^{1/2}\left(1-|z|^2\right)}{\left[\left(1-|x|^2\right)\left(1-|y|^2\right)\right]^{1/2}\left(1-|z|^2\right)|(1+x\overline{z})(1+z\overline{y})|},$$

$$\geq \left|\frac{1}{1-x\overline{y}}\cdot\frac{1}{1-|z|^2} - \frac{1}{1+x\overline{z}}\cdot\frac{1}{1+z\overline{y}}\right|$$

$$= \frac{\left|(1+x\overline{z})(1+z\overline{y}) - (1-x\overline{y})\left(1-|z|^2\right)\right|}{\left|(1-x\overline{y})\left(1-|z|^2\right)\right||(1+x\overline{z})(1+z\overline{y})|}. \tag{3.82}$$

Hence we have

$$\frac{|(1+x\overline{z})(1+z\overline{y})|}{\left[\left(1-|x|^2\right)\left(1-|y|^2\right)\right]^{1/2}\left(1-|z|^2\right)} - 1$$
$$\geq \left|\frac{(1+x\overline{z})(1+z\overline{y})}{(1-x\overline{y})\left(1-|z|^2\right)} - 1\right|, \tag{3.83}$$

for any $x,\ y,\ z \in D(0,1)$. In particular, if $y = \overline{x}$, $z = a \in \mathbb{R}$, then from (3.83) we get

$$\frac{|1+ax|^2}{\left(1-|x|^2\right)(1-a^2)} - 1 \geq \left|\frac{(1+ax)^2}{(1-x^2)(1-a^2)} - 1\right|, \tag{3.84}$$

for any $x \in D(0,1)\,, a \in \mathbb{R}$.

(2) For the exponential function $f(z)$ is given by (1.26), we have the transform $f_A(z)$ given by (3.26). Utilising the inequality (3.80), we obtain

$$\exp\left(\frac{|x|^2+|y|^2}{2} + |z|^2\right)|\exp(x\overline{z}+z\overline{y})| - 1$$
$$\geq \left|\exp\left(x\overline{y} + |z|^2 + x\overline{z} + z\overline{y}\right) - 1\right|, \tag{3.85}$$

for any $x,\ y,\ z \in \mathbb{C}$. In particular, if $y = \overline{x}$, $z = a \in \mathbb{R}$, then from (3.85) we get

$$\exp\left(|x|^2 + a^2\right)|\exp(2ax)| - 1 \geq \left|\exp\left(|x|^2 + a + 2ax\right) - 1\right|, \tag{3.86}$$

for any $x \in \mathbb{C}$, $a \in \mathbb{R}$.

Similar result to (3.31) is also obtained that connects the two convergent power series, one having positive coefficients, while the other has complex coefficients [209] (see also [217]).

Theorem 53 (Ibrahim and Dragomir [209]) *Let* $g(z) = \sum_{n=0}^{\infty} g_n z^n$ *and* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be two power series with* $g_n \in \mathbb{C}$ *and* $a_n > 0,\ n \geq 0$. *If* f *and* g *are convergent on* $D(0, R_1)$ *and* $D(0, R_2)$ *respectively, and the numerical series* $\sum_{n=0}^{\infty} \frac{|g_n|^2}{a_n}$ *is convergent, then we have the inequality*

$$\sum_{n=0}^{\infty} \frac{|g_n|^2}{a_n} f\left(|z|^2\right) - |g(z)g(\overline{z})| \geq \left| \sum_{n=0}^{\infty} \frac{|g_n|^2}{a_n} f\left(z^2\right) - g(z)\overline{g\left(\overline{z}\right)} \right|, \tag{3.87}$$

for any $z \in \mathbb{C}$ *with* $z,\ z^2,\ |z|^2 \in D(0, R_1) \cap D(0, R_2)$.

Proof. On utilizing the inequality (3.55) for the choices $p_n = a_n$, $x_n = z^n$ and $z_n = g_n/a_n$, $n \in \{0, 1, 2, ..., m\}$, we have

$$\begin{aligned}
&\sum_{n=0}^{m} a_n |z|^{2n} \sum_{n=0}^{m} a_n \left|\frac{g_n}{a_n}\right|^2 - \left| \sum_{n=0}^{m} a_n z^n \overline{\left(\frac{g_n}{a_n}\right)} \sum_{n=0}^{n} a_n z^n \left(\frac{g_n}{a_n}\right) \right| \\
&= \sum_{n=0}^{m} a_n \left(|z|^2\right)^n \sum_{n=0}^{m} \frac{|g_n|^2}{a_n} - \left| \sum_{n=0}^{m} \overline{g_n} z^n \sum_{n=0}^{n} g_n z^n \right| \\
&\geq \left| \sum_{n=0}^{m} a_n z^{2n} \sum_{n=0}^{m} a_n \left|\frac{g_n}{a_n}\right|^2 - \sum_{n=0}^{m} a_n z^n \overline{\left(\frac{g_n}{a_n}\right)} \sum_{n=0}^{n} a_n z^n \left(\frac{g_n}{a_n}\right) \right| \\
&= \left| \sum_{n=0}^{m} a_n \left(z^2\right)^n \sum_{n=0}^{m} \frac{|g_n|^2}{a_n} - \sum_{n=0}^{m} \overline{g_n} z^n \sum_{n=0}^{n} g_n z^n \right|,
\end{aligned} \tag{3.88}$$

for any $m \geq 0$. Replacing the equation (3.33) in (3.88), we get

$$\begin{aligned}
&\sum_{n=0}^{m} \frac{|g_n|^2}{a_n} \sum_{n=0}^{m} a_n \left(|z|^2\right)^n - \left| \sum_{n=0}^{m} g_n \left(\overline{z}\right)^n \sum_{n=0}^{n} g_n z^n \right| \\
&\geq \left| \sum_{n=0}^{m} a_n \left(z^2\right)^n \sum_{n=0}^{m} \frac{|g_n|^2}{a_n} - \sum_{n=0}^{m} \overline{g_n} z^n \sum_{n=0}^{n} g_n z^n \right|.
\end{aligned} \tag{3.89}$$

Since z, z^2, $|z|^2 \in D(0,R_1) \cap D(0,R_2)$, the series in (3.89) are convergent, by letting $m \to \infty$, we deduce the desired inequality (3.87). ■

Remark 54 *If the coefficients g_n, $n \geq 0$ are real, then we have the inequality*

$$\sum_{n=0}^{\infty} \frac{g_n^2}{a_n} f\left(|z|^2\right) - |g(z)g(\overline{z})| \geq \left| \sum_{n=0}^{\infty} \frac{g_n^2}{a_n} f\left(z^2\right) - g^2(z) \right|, \tag{3.90}$$

for any $z \in \mathbb{C}$ with z, z^2, $|z|^2 \in D(0,R_1) \cap D(0,R_2)$.

The following corollary is the natural consequence of Theorem 53 (see [209], [217]).

Corollary 55 *Let $g(z) = \sum_{n=0}^{\infty} g_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$. If the numerical series $\sum_{n=0}^{\infty} |g_n|^2$ is convergent, then*

$$\left(\frac{1}{1-|z|^2}\right) \sum_{n=0}^{\infty} |g_n|^2 - |g(z)g(\overline{z})| \geq \left| \left(\frac{1}{1-z^2}\right) \sum_{n=0}^{\infty} |g_n|^2 - g(z)\overline{g(\overline{z})} \right|, \tag{3.91}$$

for any $z \in D(0,1) \cap D(0,R)$.

Proof. It follows from (3.87) for the function $f(z)$ given by (1.19). ■

If we consider the series expansion given by (3.37), then on utilizing the inequality (3.91) for the choice $g_n = i^n/(n+1)$, and taking into account the equality (d) in (1.45), we can state the following inequality

$$\begin{aligned} &\frac{\pi^2}{6}\left(\frac{|z|^2}{1-|z|^2}\right) - \left| \ln\left(\frac{1}{1-iz}\right) \ln\left(\frac{1}{1-i\overline{z}}\right) \right| \\ &\geq \left| \frac{\pi^2}{6}\left(\frac{z^2}{1-z^2}\right) + \ln\left(\frac{1}{1-iz}\right) \ln\left(\frac{1}{1+iz}\right) \right|, \end{aligned} \tag{3.92}$$

for any $z \in D(0,1)$.

Corollary 56 *Let $g(z) = \sum_{n=0}^{\infty} g_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0, R)$. If the numerical series $\sum_{n=0}^{\infty} n! \left|g_n\right|^2$ is convergent, then*

$$\sum_{n=0}^{\infty} n! \left|g_n\right|^2 \exp\left(|z|^2\right) - |g(z)g(\overline{z})| \geq \left|\sum_{n=0}^{\infty} n! \left|g_n\right|^2 \exp\left(z^2\right) - g(z)\overline{g\left(\overline{z}\right)}\right|, \quad (3.93)$$

for any $z \in D(0, R)$.

Proof. It follows from Theorem 53 by choosing the exponential function $f(z)$ given by (1.25). ■

If we apply the inequality (3.93) for the trigonometric function $f(z)$ given by (3.40), then we obtain the following inequality

$$\sum_{n=0}^{\infty} \frac{n!}{(2n+1)!} \exp\left(|z|^2\right) - |\sin(iz)\sin(i\overline{z})|$$
$$\geq \left|\sum_{n=0}^{\infty} \frac{n!}{(2n+1)!} \exp\left(z^2\right) - \sin^2(iz)\right|, \quad (3.94)$$

for any $z \in \mathbb{C}$.

More results on the celebrated (CBS)-type inequalities for functions defined by the power series with real coefficients, are discussed in the following section.

3.4 Other Refinements of the (CBS)-type

Utilising a different technique based on the continuity properties of modulus (2.22), in this section, we develop some more inequalities for the power series functions that are related to the (CBS)-type inequality (see [214], [217]). Applications for some fundamental functions are included as well.

We begin with the following result.

Theorem 57 (Ibrahim, Dragomir and Darus [214]) *Assume that the power series $f(z) = \sum_{n=0}^{\infty} p_n z^n$ with real coefficients is convergent on the disk $D(0,R)$, $R > 0$. If x, $z \in \mathbb{C}$ are such that x, xz, $|x|\,|z|^2 \in D(0,R)$, then we have the inequality*

$$\begin{aligned} & f_A\left(|x|\,|z|^2\right) f_A(|x|) - |f_A(|x|\,z)|^2 \\ & \geq |f(x)f(x\,|z|\,z) - f(xz)f(x\,|z|)| \geq 0. \end{aligned} \tag{3.95}$$

Proof. For arbitrary complex number z with $z \in D(0,R)$, we have for n, $j \in \mathbb{N}$,

$$\left|z^n - z^j\right|^2 = \left|z^n - z^j\right| \left|z^n - z^j\right| \geq \left|z^n - z^j\right| \left||z|^n - |z|^j\right|, \tag{3.96}$$

by utilising the continuity properties of modulus (2.22). We also have

$$\left|z^n - z^j\right|^2 = |z^n|^2 - 2\operatorname{Re}\left(z^n \overline{z}^j\right) + \left|z^j\right|^2 = |z|^{2n} - 2\operatorname{Re}\left(z^n \overline{z}^j\right) + |z|^{2j}, \tag{3.97}$$

and

$$\left|z^n - z^j\right| \cdot \left||z|^n - |z|^j\right| = \left|z^n\,|z|^n + z^j\,|z|^j - z^n\,|z|^j - z^j\,|z|^n\right|, \tag{3.98}$$

for any n, $j \in \mathbb{N}$. Utilizing (3.96) we get the inequality

$$|z|^{2n} - 2\operatorname{Re}\left(z^n \overline{z}^j\right) + |z|^{2j} \geq \left|z^n\,|z|^n + z^j\,|z|^j - |z|^j\,z^n - |z|^n\,z^j\right|, \tag{3.99}$$

for any n, $j \in \mathbb{N}$. If we multiply the inequality (3.99) by nonnegative quantity, i.e., $|p_n|\,|x|^n\,|p_j|\,|x|^j \geq 0$, where $x \in D(0,R)$ and n, $j \in \mathbb{N}$, then we have

$$\begin{aligned} & |p_n|\,|x|^n\,|z|^{2n}\,p_j\,|x|^j + |p_n|\,|x|^n\,p_j\,|x|^j\,|z|^{2j} \\ & - 2\operatorname{Re}\left(|p_n|\,|x|^n\,z^n\,|p_j|\,|x|^j\,\overline{z}^j\right) \\ & \geq \Big|p_n x^n\,|z|^n\,z^n p_j x^j + p_n x^n p_j x^j\,|z|^j\,z^j - p_n x^n z^n p_j x^j\,|z|^j \\ & - p_n x^n\,|z|^n\,p_j x^j z^j\Big|, \end{aligned} \tag{3.100}$$

for any n, $j \in \mathbb{N}$. Summing over n and j from 0 to k, and utilizing the triangle inequality for the modulus (2.25), we obtain from (3.100) that

$$\begin{aligned}&\sum_{n=0}^{k}|p_n||x|^n|z|^{2n}\sum_{j=0}^{k}|p_j||x|^j+\sum_{n=0}^{k}|p_n||x|^n\sum_{j=0}^{k}|p_j||x|^j|z|^{2j}\\&-2\operatorname{Re}\left(\sum_{n=0}^{k}|p_n||x|^n z^n\sum_{j=0}^{k}|p_j||x|^j(\overline{z})^j\right)\\&\geq\left|\sum_{n=0}^{k}p_nx^n|z|^n z^n\sum_{j=0}^{k}p_jx^j+\sum_{n=0}^{k}p_nx^n\sum_{j=0}^{k}p_jx^j|z|^j z^j\right.\\&\left.-\sum_{n=0}^{k}p_nx^nz^n\sum_{j=0}^{k}p_jx^j|z|^j-\sum_{n=0}^{k}p_nx^n|z|^n\sum_{j=0}^{k}p_jx^jz^j\right|.\end{aligned}\tag{3.101}$$

Since

$$\sum_{j=0}^{k}|p_j||x|^j(\overline{z})^j=\overline{\sum_{n=0}^{k}|p_n||x|^n z^n},\tag{3.102}$$

then

$$\operatorname{Re}\left(\sum_{n=0}^{k}|p_n||x|^n z^n\sum_{j=0}^{k}|p_j||x|^j(\overline{z})^j\right)=\left|\sum_{n=0}^{k}|p_n||x|^n z^n\right|^2.\tag{3.103}$$

Hence, we get the following inequality, by replacing the equality (3.103) into (3.101),

$$\begin{aligned}&\sum_{n=0}^{k}|p_n||x|^n|z|^{2n}\sum_{n=0}^{k}|p_n||x|^n-\left|\sum_{n=0}^{k}|p_n||x|^n z^n\right|^2\\&\geq\left|\sum_{n=0}^{k}p_nx^n\sum_{n=0}^{k}p_nx^n|z|^n z^n-\sum_{n=0}^{k}p_nx^nz^n\sum_{n=0}^{k}p_nx^n|z|^n\right|.\end{aligned}\tag{3.104}$$

Since all the series whose partial sums are involved in (3.104), are convergent, then by taking the limit as $k\to\infty$ in (3.104), we deduce the desired inequality (3.95). ■

The particular cases are as follows:

Corollary 58 *If* $\sum_{n=0}^{\infty}|p_n|<\infty$, *i.e.,* $f_A(1)<\infty$, *then for any* $\zeta\in\mathbb{C}$ *with* $|\zeta|=1$, *we have*

$$f_A\left(|z|^2\right)f_A(1)-|f_A(z)|^2\geq|f(\zeta)f(\zeta|z|z)-f(\zeta z)f(\zeta|z|)|\geq 0.\tag{3.105}$$

In particular, for $\zeta = 1$*, we have*

$$f_A\left(|z|^2\right) f_A(1) - |f_A(z)|^2 \geq |f(1)f(|z|\, z) - f(z)f(|z|)| \geq 0, \tag{3.106}$$

for any $z, |z|^2 \in D(0, R)$.

Some applications of the inequalities (3.95) and (3.106) for particular functions of interest are pointed out as follows:

(1) If we apply the inequality (3.95) for the function $f(z)$ given by (1.19), then we get

$$\begin{aligned} &\frac{|1-z|}{(1-|x|)\left(1-|x|\,|z|^2\right)|1-|x|\,z|^2} \\ &\geq \left|\frac{1-|z|}{(1-x)(1-xz)(1-x\,|z|)(1-x\,|z|\,z)}\right| \end{aligned} \tag{3.107}$$

for any $x,\ z \in \mathbb{C}$ with $x,\ |x|\,|z|^2 \in D(0,1)$.

(2) If we apply the inequality (3.95) for the function $f(z)$ given by (1.20), then we get the inequality

$$\begin{aligned} &\frac{|1-z|}{(1-|x|)\left(1-|x|\,|z|^2\right)|1-|x|\,z|^2} \\ &\geq \left|\frac{1-|z|}{(1+x)(1+xz)(1+x\,|z|)(1+x\,|z|\,z)}\right| \end{aligned} \tag{3.108}$$

for any $x,\ z \in \mathbb{C}$ with $x,\ xz,\ |x|\,|z|^2 \in D(0,1)$.

(3) If we apply the inequality (3.106) for the exponential function $f(z)$ given by (1.25), then we get the inequality

$$\exp\left(|z|^2+1\right) - |\exp(z)|^2 \geq |\exp(z\,|z|+1) - \exp(z+|z|)| \tag{3.109}$$

for any $z \in \mathbb{C}$.

Remark 59 *The inequality (3.95) can be written in the form*

$$\det\begin{bmatrix} f_A\left(|x|\,|z|^2\right) & f_A\left(|x|\,z\right) \\ f_A\left(|x|\,\overline{z}\right) & f_A\left(|x|\right) \end{bmatrix} \geq \left|\det\begin{bmatrix} f(x) & f(xz) \\ f(x\,|z|) & f\left(x\,|z|\,z\right) \end{bmatrix}\right| \tag{3.110}$$

for any $x,\ z \in \mathbb{C}$ with $x,\ xz,\ |x|\,|z|^2 \in D(0,R)$.

The following result also holds.

Theorem 60 (Ibrahim, Dragomir and Darus [214]) *Assume that the power series $f(z)=\sum_{n=0}^{\infty} p_n z^n$ with real coefficients is convergent on the disk $D(0,R)$, $R>0$. If $x,\ z\in\mathbb{C}$ are such that $x,\ xz,\ |x|\,|z|^2 \in D(0,R)$, then we have the inequality*

$$\begin{aligned} & f_A(|x|)\, f_A\left(|x|\,|z|^2\right) - \operatorname{Re}\left[f_A^2\left(|x|\,z\right)\right] \\ & \geq \frac{1}{2}\left|f(x)f\left(x\,|z|\,z\right)+f(x)f\left(x\,|z|\,\overline{z}\right)-f(xz)f(x\,|z|)\right. \\ & \left.-f\left(x\,|z|\right)f\left(x\overline{z}\right)\right|. \end{aligned} \tag{3.111}$$

Proof. If $z\in D(0,R)$, then

$$\begin{aligned} \left|z^n-(\overline{z})^j\right|^2 &= \left|z^n-(\overline{z})^j\right|\left|z^n-(\overline{z})^j\right| \geq \left|z^n-(\overline{z})^j\right|\left||z|^n-|z|^j\right| \\ &= \left||z|^n z^n+(\overline{z})^j|z|^j-|z|^j z^n-|z|^n(\overline{z})^j\right|, \end{aligned} \tag{3.112}$$

for any $n,\ j\in\mathbb{N}$. We also have

$$\begin{aligned} \left|z^n-(\overline{z})^j\right|^2 &= |z^n|^2-2\operatorname{Re}\left(z^n z^j\right)+\left|z^j\right|^2 \\ &= |z|^{2n}-2\operatorname{Re}\left(z^n z^j\right)+|z|^{2j}, \end{aligned} \tag{3.113}$$

for any $n,\ j\in\mathbb{N}$. Utilizing (3.112) we have the inequality

$$|z|^{2n}-2\operatorname{Re}\left(z^n z^j\right)+|z|^{2j} \geq \left||z|^n z^n+(\overline{z})^j z^n-|z|^j z^n-|z|^n(\overline{z})^j\right| \tag{3.114}$$

for any $n,\ j\in\mathbb{N}$. Now, on utilizing the similar argument to the one in the proof of Theorem 57 above, we deduce the desired result (3.111). The details are omitted. ■

Corollary 61 *If $z=\overline{x}$ in (3.111), then we have*

$$\begin{aligned}
&f_A(|x|)\, f_A\left(|x|^3\right)-\operatorname{Re}\left[f_A^2\left(|x|\,\overline{x}\right)\right]\\
&\geq \frac{1}{2}\left|f(x)f\left(|x|^3\right)+f(x)f\left(|x|\,x^2\right)-f(|x|^2)f(|x|\,x)\right.\\
&\left.-f\left(|x|\,x\right)f\left(x^2\right)\right| \qquad (3.115)
\end{aligned}$$

for any $x\in\mathbb{C}$ such that $x,\ |x|\,x,\ |x|\,x^2\in D(0,R)$.

In the following, we give some applications of above inequality (3.115) for particular complex functions of interest.

(1) If we take the function $f(z)$ given by (1.20), the we have the transform $f_A(z)$ given by (1.19). Applying the inequality (3.115), we get the following result

$$\begin{aligned}
&\frac{1}{(1-|x|)\left(1-|x|^3\right)}-\operatorname{Re}\left(\frac{1}{1-|x|\,\overline{x}}\right)^2\\
&\geq \frac{1}{2}\left|\frac{2+|x|\,x^2+|x|^3}{(1+x)\left(1+|x|^3\right)\left(1+|x|\,x^2\right)}\right.\\
&\left.-\frac{2+x^2+|x|^2}{(1+x^2)\left(1+|x|\,x\right)\left(1+|x|^2\right)}\right|, \qquad (3.116)
\end{aligned}$$

for any $x,\ |x|\,x,\ |x|\,x^2\in D(0,1)$.

(2) If we apply the inequality (3.115) for the function $f(z)$ given by (1.25), then we get

$$\begin{aligned}
&\exp\left(|x|+|x|^3\right)-\operatorname{Re}\left[\exp\left(2\,|x|\,\overline{x}\right)\right]\\
&\geq \frac{1}{2}\left|\exp(x+|x|^3)+\exp(x+|x|\,x^2)-\exp(|x|^2+|x|\,x)\right.\\
&\left.-\exp(|x|\,x+x^2)\right| \qquad (3.117)
\end{aligned}$$

for any $x\in\mathbb{C}$.

Theorem 62 (Ibrahim, Dragomir and Darus [214]) *Assume that the power series $f(z) = \sum_{n=0}^{\infty} p_n z^n$ with real coefficients is convergent on the disk $D(0,R)$, $R > 0$. If $x, y \in \mathbb{C}$ are such that $|x|^2, |y|^2 < R$, then we have the inequality*

$$\begin{aligned} & f_A\left(|x|^2\right) f_A\left(|y|^2\right) - \left|f_A(xy)\right|^2 \\ & \geq \left| f(|x|\,x) f(|y|\,\overline{y}) - f(|y|\,x) f(|x|\,\overline{y}) \right| . \end{aligned} \tag{3.118}$$

Proof. If $x, y \in \mathbb{C}$, then we have

$$\begin{aligned} \left| x^n (\overline{y})^j - x^j (\overline{y})^n \right|^2 &= \left| x^n (\overline{y})^j - x^j (\overline{y})^n \right| \left| x^n (\overline{y})^j - x^j (\overline{y})^n \right| \\ &\geq \left| x^n (\overline{y})^j - x^j (\overline{y})^n \right| \left| |x|^n |y|^j - |x|^j |y|^n \right|, \end{aligned} \tag{3.119}$$

for any $n, j \in \mathbb{N}$. We have upon simple calculations that

$$\begin{aligned} & |x|^{2n} |y|^{2j} - 2\operatorname{Re}\left(x^n y^n (\overline{x})^j (\overline{y})^j \right) + |y|^{2n} |x|^{2j} \\ & \geq \Big| |x|^n x^n |y|^j (\overline{y})^j + |y|^n (\overline{y})^n |x|^j x^j - |y|^n x^n |x|^j (\overline{y})^j \\ & - |x|^n (\overline{y})^n |y|^j x^j \Big| , \end{aligned} \tag{3.120}$$

for any $n, j \in \mathbb{N}$.

If we multiply the inequality (3.120) with the positive quantity, i.e., $|p_n|\,|p_j| \geq 0$, and summing over n and j from 0 to k, then we have

$$\begin{aligned} & \sum_{n=0}^{k} |p_n| \, |x|^{2n} \sum_{j=0}^{\infty} |p_j| \, |y|^{2j} + \sum_{n=0}^{k} |p_n| \, |y|^{2n} \sum_{j=0}^{k} |p_j| \, |x|^{2j} \\ & - 2\operatorname{Re}\left(\sum_{n=0}^{k} |p_n| \, x^n y^n \sum_{j=0}^{k} |p_j| \, (\overline{x})^j (\overline{y})^j \right) \\ & \geq \Bigg| \sum_{n=0}^{k} p_n |x|^n x^n \sum_{j=0}^{k} p_j |y|^j (\overline{y})^j + \sum_{n=0}^{k} p_n |y|^n (\overline{y})^n \sum_{j=0}^{k} p_j |x|^j x^j \end{aligned}$$

$$-\sum_{n=0}^{k} p_n |y|^n x^n \sum_{j=0}^{k} p_j |x|^j (\overline{y})^j - \sum_{n=0}^{k} p_n |x|^n (\overline{y})^n \sum_{j=0}^{k} p_j |y|^j x^j \Bigg| . \qquad (3.121)$$

Due to the fact that

$$\sum_{n=0}^{k} |p_n| x^n y^n \sum_{j=0}^{\infty} |p_j| (\overline{x})^j (\overline{y})^j = \left| \sum_{n=0}^{k} |p_n| x^n y^n \right|^2 , \qquad (3.122)$$

then the inequality (3.121) is equivalent with

$$\sum_{n=0}^{k} |p_n| |x|^{2n} \sum_{n=0}^{k} |p_n| |y|^{2n} - \left| \sum_{n=0}^{k} |p_n| x^n y^n \right|^2$$

$$\geq \left| \sum_{n=0}^{k} p_n |x|^n x^n \sum_{n=0}^{k} p_n |y|^n (\overline{y})^n - \sum_{n=0}^{k} p_n |y|^n x^n \sum_{n=0}^{k} p_n |x|^n (\overline{y})^n \right| . \qquad (3.123)$$

Since all the series with the partial sums are involved in (3.123), are convergent, then by taking the limit over $k \to \infty$ in (3.123), we deduce the desired result (3.118). ■

Remark 63 *The inequality (3.118) is also equivalent to*

$$\det \begin{bmatrix} f_A\left(|x|^2\right) & f_A(xy) \\ f_A(\overline{xy}) & f_A\left(|y|^2\right) \end{bmatrix} \geq \left| \det \begin{bmatrix} f(|x|\, x) & f(|y|\, x) \\ f(|x|\, \overline{y}) & f(|y|\, \overline{y}) \end{bmatrix} \right| \qquad (3.124)$$

for any x, $y \in \mathbb{C}$ with $|x|^2$, $|y|^2 < R$.

The inequality (3.118) has some applications for particular complex functions of interest, which will be pointed out as follows.

(1) If we apply the inequality (3.118) for the function $f(z)$ given by (1.19), then we get

$$\frac{1}{\left(1-|x|^2\right)\left(1-|y|^2\right)} - \frac{1}{|1-xy|^2}$$

$$\geq \left| \frac{1}{(1-x|x|)(1-|y|\,\overline{y})} - \frac{1}{(1-|y|\,x)(1-|x|\,\overline{y})} \right| \qquad (3.125)$$

for any $x,\ y \in \mathbb{C}$. In particular, if in (3.125) we choose $y = 0$, then we obtain the simpler inequality

$$\frac{1}{1-|x|^2} - 1 \geq \left| \frac{1}{1-x|x|} - 1 \right| \tag{3.126}$$

for any $x \in \mathbb{C}$.

(2) If we apply the inequality (3.118) for the function $f(z)$ given by (1.25), then we have

$$\begin{aligned} &\exp\left(|x|^2 + |y|^2\right) - |\exp(xy)|^2 \\ &\geq |\exp(x|x| + |y|\overline{y}) - \exp(|y|x + |x|\overline{y})| \end{aligned} \tag{3.127}$$

for any $x,\ y \in \mathbb{C}$. In particular, if in (3.127) we choose $y = 0$, then we get

$$\exp\left(|x|^2\right) - 1 \geq |\exp(x|x|) - 1| \tag{3.128}$$

for any $x \in \mathbb{C}$.

(3) If we take the trigonometric function $f(z)$ given by (1.33) with its transform $f_A(z)$ is given by (1.39), then utilizing the inequality (3.118) for $f(z)$ as above gives

$$\begin{aligned} &\cosh\left(|x|^2\right)\cosh\left(|y|^2\right) - |\cosh(xy)|^2 \\ &\geq |\cos(x|x|)\cos(|y|\overline{y}) - \cos(|y|x)\cos(|x|\overline{y})| \end{aligned} \tag{3.129}$$

for any $x,\ y \in \mathbb{C}$. In particular, we have, with $y = 0$ in (3.129),

$$\cosh\left(|x|^2\right) - 1 \geq |\cos(x|x|) - 1| \tag{3.130}$$

for any $x \in \mathbb{C}$.

We end this section by proving the following results.

Theorem 64 (Ibrahim, Dragomir and Darus [214]) *Assume that the power series $f(z) = \sum_{n=0}^{\infty} p_n z^n$ with real coefficients is convergent on the disk*

$D(0,R)$, $R>0$. If x, $y\in\mathbb{C}$ are such that $|x|^2$, $|y|^2<R$, then we have the inequality

$$\begin{aligned}&f_A\left(|x|^2\right)f_A\left(|y|^2\right)-\operatorname{Re}\left[f_A^2\left(x\overline{y}\right)\right]\\&\geq\frac{1}{2}\left|f(|x|\,x)f\left(|y|\,\overline{y}\right)+f(|x|\,\overline{x})f\left(|y|\,y\right)-f(|x|\,y)f(|y|\,\overline{x})\right.\\&\left.-f(|y|\,x)f(|x|\,\overline{y})\right|.\end{aligned}\tag{3.131}$$

Proof. If x, $y\in D(0,R)$, then we have

$$\begin{aligned}\left|x^n\left(\overline{y}\right)^j-\left(\overline{x}\right)^j y^n\right|^2&=\left|x^n\left(\overline{y}\right)^j-\left(\overline{x}\right)^j y^n\right|\left|x^n\left(\overline{y}\right)^j-\left(\overline{x}\right)^j y^n\right|\\&\geq\left|x^n\left(\overline{y}\right)^j-\left(\overline{x}\right)^j y^n\right|\left||x|^n|y|^j-|x|^j|y|^n\right|\end{aligned}\tag{3.132}$$

for any n, $j\in\mathbb{N}$. Doing simple calculations we get that

$$\begin{aligned}&|x|^{2n}|y|^{2j}-2\operatorname{Re}\left[x^n\left(\overline{y}\right)^j x^j\left(\overline{y}\right)^n\right]+|x|^{2j}|y|^{2n}\\&\geq\left||x|^n x^n|y|^j\left(\overline{y}\right)^j+|x|^j\left(\overline{x}\right)^j|y|^n y^n-|x|^n y^n|y|^j\left(\overline{x}\right)^j\right.\\&\left.-|y|^n x^n|x|^j\left(\overline{y}\right)^j\right|\end{aligned}\tag{3.133}$$

for any n, $j\in\mathbb{N}$. If we multiply the inequality (3.133) with $|p_n|\,|p_j|\geq 0$, and summing over n and j from 0 to k, then we get

$$\begin{aligned}&\sum_{n=0}^k|p_n|\,|x|^{2n}\sum_{j=0}^k|p_j|\,|y|^{2j}-2\operatorname{Re}\left(\sum_{n=0}^k|p_n|\,x^n\left(\overline{y}\right)^n\sum_{j=0}^k|p_j|\,x^j\left(\overline{y}\right)^j\right)\\&+\sum_{n=0}^k|p_n|\,|y|^{2n}\sum_{j=0}^k|p_j|\,|x|^{2j}\\&\geq\left|\sum_{n=0}^k p_n|x|^n x^n\sum_{j=0}^k p_j|y|^j\left(\overline{y}\right)^j+\sum_{n=0}^k p_n y^n|y|^n\sum_{j=0}^k p_j|x|^j\left(\overline{x}\right)^j\right.\\&\left.-\sum_{n=0}^k p_n|x|^n y^n\sum_{j=0}^k p_j|y|^j\left(\overline{x}\right)^j-\sum_{n=0}^k p_n|y|^n x^n\sum_{j=0}^k p_j|x|^j\left(\overline{y}\right)^j\right|,\end{aligned}\tag{3.134}$$

for any n, $j\in\mathbb{N}$.

Since all the series whose partial sums are involved in (3.134), are convergent, then by taking the limit over $k\to\infty$ in (3.134), we deduce the desired result (3.131). ■

The inequality (3.131) is also a valuable source of particular inequalities for complex functions of interest, that will be outlined in the following.

(1) In (3.131), we take the function $f(z)$ given by (1.25), then we can state that

$$\begin{aligned}
&\exp\left(|x|^2+|y|^2\right)-\operatorname{Re}\left[\exp\left(2x\overline{y}\right)\right]\\
&\geq \frac{1}{2}\left|\exp\left(|x|\,x+|y|\,\overline{y}\right)+\exp\left(|x|\,\overline{x}\right)+|y|\,y\right)\\
&-\exp\left(|x|\,y\right)+|y|\,\overline{x}\right)-\exp\left(|y|\,x+|x|\,\overline{y}\right)\right|,
\end{aligned}\tag{3.135}$$

for any $x, y \in \mathbb{C}$. If in (3.135) we choose $y=0$, then we obtain the simpler result

$$\exp\left(|x|^2\right)-1\geq \frac{1}{2}\left|\exp\left(|x|\,x\right)+\exp\left(|x|\,\overline{x}\right)-2\right|,\tag{3.136}$$

for any $x \in \mathbb{C}$.

(2) If we apply the inequality (3.131) for the trigonometric function $f(z)$ given by (1.33) with its transform $f_A(z)$ is given by (1.39), then we get

$$\begin{aligned}
&\cosh\left(|x|^2\right)\cosh\left(|y|^2\right)-\operatorname{Re}\left[\cosh^2\left(x\overline{y}\right)\right]\\
&\geq \frac{1}{2}\left|\cos(|x|\,x)\cos\left(|y|\,\overline{y}\right)+\cos(|x|\,\overline{x})\cos\left(|y|\,y\right)\right.\\
&\left.-\cos(|x|\,y)\cos(|y|\,\overline{x})-\cos(|y|\,x)\cos(|x|\,\overline{y})\right|,
\end{aligned}\tag{3.137}$$

for any $x, y \in \mathbb{C}$. In particular, if in (3.137) we choose $y=0$, then we obtain that

$$\cosh\left(|x|^2\right)-1\geq \frac{1}{2}\left|\cos(|x|\,x)+\cos(|x|\,\overline{x})-2\right|,\tag{3.138}$$

for any $x \in \mathbb{C}$.

Chapter 4

More Inequalities on Power Series with Real Coefficients

In the previous chapter, some inequalities concerning the power series have been established by utilizing a different technique based on the continuity properties of modulus, Buzano's inequality and Schwarz's result in inner product spaces. In this chapter, we employ the similar method to develop some more inequalities for functions defined by the power series in a real and complex variable, with the crucial tools for these investigations being Young's, Hölder's and Jensen's type inequalities, as well as their reverses and counterparts.

The main purpose of Chapter 4 is two-fold. The first aim is to derive some inequalities for the power series that are related to the classical Young's inequality. Motivated by the results established by Dragomir and Sàndor [136], we derive new and better inequalities for functions defined by the power series with real coefficients via Young's inequality for the product of two scalars. All the results and their applications for some fundamental functions are presented in Section 4.1.2. More inequalities on the power series functions are obtained by utilising a refinement and a reverse of Young's inequality, see Section 4.1.3. Natural applications for some fundamental functions of interest are also included.

The second purpose of this chapter is discussed in Section 4.2: that is, to develop some inequalities on the power series functions by making use the convexity properties of certain underlying functions. The celebrated Jensen type inequalities and one of their reverses due to Dragomir and Ionescu [139], are the useful tools that have been used in Section 4.2.3, in order to derive new inequalities for functions defined by the real power series with positive coefficients. Applications for some fundamental functions such as the exponential, logarithm, trigonometric and hyperbolic functions are provided as well.

All the results contained in this chapter are mainly taken from the several research papers published by the author in collaboration with Dragomir, Darus and Cerone (see [213], [215], [216]).

4.1 Some Results Related to Young's Inequality

4.1.1 Introduction and Preliminary Results

It has been shown in the literature that the famous (CBS)-inequality and its generalization, such as Hölder's inequality, can be derived by utilising Young's inequality for the product of two real numbers (see [422], [301, p. 457], [381, p. 63-64]). Consequently, some results concerning the extensions, generalizations, refinements, etc., of the (CBS) or Hölder's inequality, are closely related to Young's inequality as well.

In [122], Dragomir provided the generalization of the (CBS)-inequality by utilising Young's inequality (2.80) as follows (see also [117, p. 11], [136]):

$$\left(\sum_{k=1}^{n}|x_k y_k|\right)^2 \le \frac{1}{p}\sum_{k=1}^{n}|x_k|^p\sum_{k=1}^{n}|y_k|^p + \frac{1}{q}\sum_{k=1}^{n}|x_k|^q\sum_{k=1}^{n}|y_k|^q\,, \tag{4.1}$$

for $x_k, y_k \in \mathbb{C}$, $k \in \{1, 2, \ldots, n\}$ and $p,\ q > 1$ with $1/p + 1/q = 1$. It is clearly seen that this result reduces to the (CBS)-inequality for complex numbers when $p = q = 2$. Further generalizations of the (CBS)-inequality via Young's inequality (2.80) have been obtained in [136] by Dragomir and Sàndor (see also [117, p. 10]), namely

$$
\begin{aligned}
&\sum_{k=1}^{n} p_k \left|x_k y_k\right| \sum_{k=1}^{n} q_k \left|x_k y_k\right| \\
&\leq \frac{1}{p} \sum_{k=1}^{n} p_k \left|x_k\right|^p \sum_{k=1}^{n} q_k \left|y_k\right|^p + \frac{1}{q} \sum_{k=1}^{n} q_k \left|x_k\right|^q \sum_{k=1}^{n} p_k \left|y_k\right|^q, \qquad (4.2)
\end{aligned}
$$

for any x_k, $y_k \in \mathbb{C}$, p_k, $q_k \geq 0$, $k \in \{1, 2, \ldots, n\}$ and $p, q > 1$ with $1/p + 1/q = 1$. If we assume in (4.2) that $p_k = q_k = 1$ for all $k \in \{1, 2, \ldots, n\}$, then it reduces to the inequality (4.1).

Motivated by the result (4.2) and other results from [136], we derive some inequalities related to the Hölder's type (2.94) for functions defined by the real and complex power series. The crucial tool that has been used for these investigations is Young's inequality (2.80), as well as a refinement and its reverse. Particular inequalities are also obtained by applying the results for some fundamental real and complex functions such as the exponential, logarithm, trigonometric and hyperbolic functions.

4.1.2 Power Series Inequalities Via Young's Inequality

On utilising Young's inequality (2.80) for the power series, we establish the following result, which improves the Hölder's type (2.94), as well as generalizes the (CBS)-inequality (2.71) for the power series with real coefficients (see [215]).

Theorem 65 (Ibrahim, Dragomir and Darus [215]) *Let $f(z) = \sum_{n=0}^{\infty} p_n z^n$ and $g(z) = \sum_{n=0}^{\infty} q_n z^n$ be two power series with real coefficients and convergent on the open disk $D(0, R)$, $R > 0$. If $p > 1$ with $1/p + 1/q = 1$ and x, $y \in \mathbb{C}$, x, $y \neq 0$ so that xy, $|x|^p$, $|x|^q$, $|y|^p$, $|y|^q \in D(0, R)$, then*

$$
\frac{1}{p} g_A\left(|x|^p\right) f_A\left(|y|^p\right) + \frac{1}{q} f_A\left(|x|^q\right) g_A\left(|y|^q\right) \geq \left|f\left(xy\right) g\left(xy\right)\right| \qquad (4.3)
$$

and

$$
\frac{1}{p} g_A\left(|x|^p\right) f_A\left(|y|^q\right) + \frac{1}{q} f_A\left(|x|^q\right) g_A\left(|y|^p\right) \geq \left|f\left(x\left|y\right|^{q-1}\right) g\left(x\left|y\right|^{p-1}\right)\right|. \qquad (4.4)
$$

Proof. If we choose $x = |x|^j |y|^k$ and $y = |x|^k |y|^j$, $j,k \in \{0,1,2,\ldots,n\}$ in the inequality (2.80), then we have

$$p\,|x|^{qj}\,|y|^{qk} + q\,|x|^{pk}\,|y|^{pj} \geq pq\,|xy|^j\,|xy|^k\,, \tag{4.5}$$

for any $j,\ k \in \{0,1,2,\ldots,n\}$. Now, if we multiply the above inequality (4.5) with the positive quantities $|p_j|\,|q_k| > 0$, for $j,\ k \in \{0,1,2,\ldots,n\}$ and sum over j and k from 0 to n, then we derive

$$\begin{aligned} & p\sum_{j=0}^{n}|p_j|\,|x|^{qj}\sum_{k=0}^{n}|q_k|\,|y|^{qk} + q\sum_{k=0}^{n}|q_k|\,|x|^{pk}\sum_{j=0}^{n}|p_j|\,|y|^{pj} \\ & \geq pq\left|\sum_{j=0}^{n}p_j\,(xy)^j\sum_{k=0}^{n}q_k\,(xy)^k\right|. \end{aligned} \tag{4.6}$$

Since all the series, whose the partial sums are involved in the inequality (4.6), are convergent on the disk $D(0,R)$ by taking the limit as $n \to \infty$ in (4.6), we deduce the desired result (4.3).

Further, if we choose in (2.80) $x = |x|^j / |y|^j$ and $y = |x|^k / |y|^k$, then we get the inequality

$$p\left(\frac{|x|^j}{|y|^j}\right)^q + q\left(\frac{|x|^k}{|y|^k}\right)^p \geq pq\frac{|x|^j}{|y|^j}\frac{|x|^k}{|y|^k}, \tag{4.7}$$

for any $|y|^j,\ |y|^k \neq 0,\ j,k \in \{0,1,2,\ldots,n\}$. Simplifying the above inequality (4.7), then we obtain that

$$\begin{aligned} p\,|x|^{qj}\,|y|^{pk} + q\,|x|^{pk}\,|y|^{qj} &\geq pq\,|x|^j\,|y|^{(q-1)j}\,|x|^k\,|y|^{(p-1)k} \\ &= pq\left|\left(x\,|y|^{q-1}\right)^j\left(x\,|y|^{p-1}\right)^k\right|, \end{aligned} \tag{4.8}$$

for any $j,k \in \{0,1,2,\ldots,n\}$. Multiplying the inequality (4.8) by $|p_j|\,|q_k| \geq 0$, for $j,k \in \{0,1,2,\ldots,n\}$ and summing over j and k from 0 to n, we get

$$\begin{aligned} & p\sum_{j=0}^{n}|p_j|\,|x|^{qj}\sum_{k=0}^{n}|q_k|\,|y|^{pk} + q\sum_{k=0}^{n}|q_k|\,|x|^{pk}\sum_{j=0}^{n}|p_j|\,|y|^{qj} \\ & \geq pq\left|\sum_{j=0}^{n}p_j\left(x\,|y|^{q-1}\right)^j\sum_{k=0}^{n}q_k\left(x\,|y|^{p-1}\right)^k\right|. \end{aligned} \tag{4.9}$$

Since all the series, whose the partial sums are involved in the inequality (4.9), are convergent on the disk $D(0,R)$ and letting $n \to \infty$ in (4.9), we deduce the desired result (4.4). ■

The following result is the particular case of interest (see [215]).

Corollary 66 *If $g(z) = f(z)$ in (4.3) and (4.4), then*

$$\frac{1}{p} f_A\left(|x|^p\right) f_A\left(|y|^p\right) + \frac{1}{q} f_A\left(|x|^q\right) f_A\left(|y|^q\right) \geq |f(xy)|^2 \tag{4.10}$$

and

$$\frac{1}{p} f_A\left(|x|^p\right) f_A\left(|y|^q\right) + \frac{1}{q} f_A\left(|x|^q\right) f_A\left(|y|^p\right) \geq \left|f\left(x\,|y|^{q-1}\right) f\left(x\,|y|^{p-1}\right)\right| \tag{4.11}$$

respectively, where $p > 1$, $1/p + 1/q = 1$ and x, $y \neq 0$ with xy, $|x|^p$, $|x|^q$, $|y|^p$, $|y|^q \in D(0,R)$. In particular, if $y = x$ in (4.10) and (4.11), then we have

$$\frac{1}{p} f_A^2\left(|x|^p\right) + \frac{1}{q} f_A^2\left(|x|^q\right) \geq \left|f\left(x^2\right)\right|^2 \tag{4.12}$$

and

$$f_A\left(|x|^p\right) f_A\left(|x|^q\right) \geq \left|f\left(sgn\left(x\right)|x|^q\right) f\left(sgn\left(x\right)|x|^p\right)\right| \tag{4.13}$$

respectively, for any $x \in \mathbb{C}$, $x \neq 0$ with x^2, $|x|^p$, $|x|^q \in D(0,R)$ and $sgn(x)$ is the complex signum function defined by (2.13).

Remark 67 *In the particular case $p = q = 2$ in (4.10), we recapture the (CBS)-type inequality (2.71) for the power series, and we also have from (4.11) that*

$$f_A\left(|x|^2\right) f_A\left(|y|^2\right) \geq |f(x\,|y|)|^2,$$

for any $x, y \in \mathbb{C}$ with $x\,|y|$, $|x|^2$, $|y|^2 \in D(0,R)$.

Some applications of the inequalities (4.10) and (4.11) for particular functions of interest are as follows:

(1) If we apply the inequalities (4.10) and (4.11) for the function $f(z)$ given by (1.19), then we get

$$\frac{1}{|1-xy|^{2}} \le \frac{1}{p\,(1-|x|^{p})\,(1-|y|^{p})} + \frac{1}{q\,(1-|x|^{q})\,(1-|y|^{q})} \tag{4.14}$$

and

$$\begin{aligned} &\frac{1}{\left|1-x\,|y|^{q-1}\right|\left|1-x\,|y|^{p-1}\right|} \\ &\le \frac{1}{p\,(1-|x|^{p})\,(1-|y|^{q})} + \frac{1}{q\,(1-|x|^{q})\,(1-|y|^{p})} \end{aligned} \tag{4.15}$$

respectively, for any $x, y \in \mathbb{C}$ such that $x,\ y \neq 0,\ xy,\ |x|^{p},\ |x|^{q},\ |y|^{p},\ |y|^{q} \in D(0,1)$ and $p>1$ with $1/p+1/q=1$.

(2) If we apply the inequalities (4.10) and (4.11) for the exponential function $f(z)$ given by (1.25), then we can state that

$$|\exp(xy)|^{2} \le \frac{1}{p}\exp(|x|^{p}+|y|^{p}) + \frac{1}{q}\exp(|x|^{q}+|y|^{q}) \tag{4.16}$$

and

$$\begin{aligned} &\left|\exp\left(x\,|y|^{q-1}+x\,|y|^{p-1}\right)\right| \\ &\le \frac{1}{p}\exp(|x|^{p}+|y|^{q}) + \frac{1}{q}\exp(|x|^{q}+|y|^{p}) \end{aligned} \tag{4.17}$$

respectively, for any $x,\ y \in \mathbb{C}$ and $p>1$ with $1/p+1/q=1$.

(3) If we apply the logarithmic function $f(z)$ given by (1.31), then from (4.10) and (4.11) we have the inequalities

$$\begin{aligned} &|\ln(1-xy)|^{2} \\ &\le \frac{1}{p}\ln(1-|x|^{p})\ln(1-|y|^{p}) + \frac{1}{q}\ln(1-|x|^{q})\ln(1-|y|^{q}) \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} &\left|\ln\left(1-x\,|y|^{q-1}\right)\ln\left(1-x\,|y|^{p-1}\right)\right| \\ &\le \frac{1}{p}\ln(1-|x|^{p})\ln(1-|y|^{q}) + \frac{1}{q}\ln(1-|x|^{q})\ln(1-|y|^{p}) \end{aligned} \tag{4.19}$$

respectively, for any $x,\ y \in \mathbb{C}$ with $x,\ y \neq 0,\ |x|^{p},\ |x|^{q},\ |y|^{p},\ |y|^{q} \in D(0,1)$ and $p>1,\ 1/p+1/q=1$.

(4) If we consider the trigonometric function $f(z)$ given by (1.32), then we obviously have the transform $f_A(z)$ given by (1.38). Applying the inequalities (4.10) and (4.11) for these functions, we get

$$|\sin(xy)|^2 \leq \frac{1}{p}\sinh(|x|^p)\sinh(|y|^p) + \frac{1}{q}\sinh(|x|^q)\sinh(|y|^q) \tag{4.20}$$

and

$$\begin{aligned}&\left|\sin\left(x|y|^{q-1}\right)\sin\left(x|y|^{p-1}\right)\right| \\ &\leq \frac{1}{p}\sinh(|x|^p)\sinh(|y|^q) + \frac{1}{q}\sinh(|x|^q)\sinh(|y|^p)\end{aligned} \tag{4.21}$$

respectively, for $x,\ y \in \mathbb{C}$ and $p > 1$ with $1/p + 1/q = 1$.

(5) Similar results can be obtained for the cosine function $f(z)$ given by (1.33) as well, that is, from (4.10) and (4.11) we get

$$|\cos(xy)|^2 \leq \frac{1}{p}\cosh(|x|^p)\cosh(|y|^p) + \frac{1}{q}\cosh(|x|^q)\cosh(|y|^q) \tag{4.22}$$

and

$$\begin{aligned}&\left|\cos\left(x|y|^{q-1}\right)\cos\left(x|y|^{p-1}\right)\right| \\ &\leq \frac{1}{p}\cosh(|x|^p)\cosh(|y|^q) + \frac{1}{q}\cosh(|x|^q)\cosh(|y|^p)\end{aligned} \tag{4.23}$$

respectively, for $x,\ y \in \mathbb{C}$ and $p > 1$ with $1/p + 1/q = 1$.

The second improvement of Hölder's inequality (2.94) for the power series via Young's inequality is incorporated in the following theorem [215].

Theorem 68 (Ibrahim, Dragomir and Darus [215]) *Let $f(z)$ and $g(z)$ be as in Theorem 65. Then one has the inequality*

$$\frac{1}{p}g_A(|x|^p)\,f_A(|y|^q) + \frac{1}{q}f_A(|x|^p)\,g_A(|y|^q) \geq \left|f\left(|x|^{p-1}|y|^{q-1}\right)g(xy)\right| \tag{4.24}$$

and

$$\frac{1}{p}f_A(|x|^p)\,g_A\left(|y|^2\right) + \frac{1}{q}g_A\left(|x|^2\right)f_A(|y|^q) \geq \left|f(xy)\,g\left(|x|^{2/q}|y|^{2/p}\right)\right|. \tag{4.25}$$

Proof. If we choose in (2.80) $x = |y|^k / |y|^j$, $y = |x|^k / |x|^j$, $|x|^j$, $|y|^j \neq 0$, j, $k \in \{0, 1, 2, \cdots, n\}$, then we have

$$\begin{aligned} p\,|y|^{qk}\,|x|^{pj} + q\,|x|^{pk}\,|y|^{qj} &\geq pq\,|x|^{(p-1)j}\,|y|^{(q-1)j}\,|xy|^k \\ &= pq\left|\left(|x|^{p-1}\,|y|^{q-1}\right)^j (xy)^k\right| \qquad (4.26)\end{aligned}$$

for any j, $k \in \{0, 1, 2, \cdots, n\}$. Multiplying the inequality (4.26) with $|p_j|\,|q_k| \geq 0$, and summing over j and k from 0 to n, we obtain that

$$\begin{aligned} & p\sum_{k=0}^{n}|q_k|\,|y|^{qk}\sum_{j=0}^{n}|p_j|\,|x|^{pj} + q\sum_{k=0}^{n}|q_k|\,|x|^{pk}\sum_{j=0}^{n}|p_j|\,|y|^{qj} \\ & \geq pq\left|\sum_{j=0}^{n}p_j\left(|x|^{p-1}\,|y|^{q-1}\right)^j\sum_{k=0}^{n}q_k\,(xy)^k\right|. \qquad (4.27)\end{aligned}$$

From (2.80), we also have the inequality

$$\begin{aligned} & p\sum_{k=1}^{n}|q_k|\,|x|^{2k}\sum_{j=1}^{n}|p_j|\,|y|^{qj} + q\sum_{j=1}^{n}|p_j|\,|x|^{pj}\sum_{k=1}^{n}|q_k|\,|y|^{2k} \\ & \geq pq\left|\sum_{j=1}^{n}p_j\,(xy)^j\sum_{k=1}^{n}q_k\left(|x|^{2/q}\,|y|^{2/p}\right)^k\right|, \qquad (4.28)\end{aligned}$$

for any x, $y \in \mathbb{C}$, $p > 1$ with $1/p + 1/q = 1$, which was obtained from (2.80) by choosing $x = |x|^{(2/q)k}\,|y|^j$, $y = |x|^j\,|y|^{(2/p)k}$, and repeating the same method as above.

Now, since all the series, whose the partial sums are involved in the inequalities (4.27) and (4.28), are convergent on the disk $D(0, R)$ and letting $n \to \infty$ in the both inequalities (4.27) and (4.28), we deduce the desired results (4.24) and (4.25). ■

The following corollary holds for the particular cases of interest (see [215]).

Corollary 69 *If $g(z) = f(z)$ in (4.24) and (4.25), then we get*

$$f_A\left(|x|^p\right) f_A\left(|y|^q\right) \geq \left|f\,(xy)\,f\left(|x|^{p-1}\,|y|^{q-1}\right)\right| \qquad (4.29)$$

and

$$\frac{1}{p}f_A\left(|x|^p\right)f_A\left(|y|^2\right)+\frac{1}{q}f_A\left(|x|^2\right)f_A\left(|y|^q\right)\geq\left|f\left(xy\right)f\left(|x|^{2/q}|y|^{2/p}\right)\right| \tag{4.30}$$

respectively, where $p>1$, $1/p+1/q=1$ *and* $x,y\neq 0$ *with* xy, $|x|^2$, $|x|^p$, $|x|^{2/q}$, $|y|^2$, $|y|^q$, $|y|^{2/p}\in D\left(0,R\right)$*. In particular, if* $y=x$ *in (4.29) and (4.30), then we have*

$$f_A\left(|x|^p\right)f_A\left(|x|^q\right)\geq\left|f\left(x^2\right)f\left(|x|^{pq-2}\right)\right| \tag{4.31}$$

and

$$f_A\left(|x|^2\right)\left[\frac{1}{p}f_A\left(|x|^p\right)+\frac{1}{q}f_A\left(|x|^q\right)\right]\geq\left|f\left(x^2\right)f\left(|x|^2\right)\right|, \tag{4.32}$$

for $x\neq 0$ *with* x^2, $|x|^2$, $|x|^p$, $|x|^q$, $\in D\left(0,R\right)$*.*

Remark 70 *In particular case* $p=q=2$ *in (4.29) or (4.30), we get the inequality*

$$f_A\left(|x|^2\right)f_A\left(|y|^2\right)\geq\left|f\left(xy\right)f\left(|xy|\right)\right|, \tag{4.33}$$

for any $x,y\in\mathbb{C}$ *with* xy, $|xy|$, $|x|^2$, $|y|^2\in D\left(0,R\right)$*.*

In what follows, we provide some applications of the inequalities (4.29) and (4.30) for particular functions of interest:

(1) If we apply the inequalities (4.29) and (4.30) for the function $f\left(z\right)$ given by (1.19), then we get

$$\left(1-|x|^p\right)\left(1-|y|^q\right)\leq\left|1-xy\right|\left|1-|x|^{p-1}|y|^{q-1}\right| \tag{4.34}$$

and

$$\begin{aligned}&\frac{1}{\left|1-xy\right|\left|1-|x|^{2/q}|y|^{2/p}\right|}\\&\leq\frac{1}{p\left(1-|x|^p\right)\left(1-|y|^2\right)}+\frac{1}{q\left(1-|x|^2\right)\left(1-|y|^q\right)}\end{aligned} \tag{4.35}$$

respectively, where $p>1$, $1/p+1/q=1$ and $x,y\neq 0$ with xy, $|x|^2$, $|x|^p$, $|x|^{1/q}$, $|y|^2$, $|y|^q$, $|y|^{1/p}\in D\left(0,1\right)$.

(2) If we apply the inequalities (4.29) and (4.30) for the exponential function $f(z)$ given by (1.25), then we can state that

$$\left|\exp\left(xy+|x|^{p-1}|y|^{q-1}\right)\right| \le \exp\left(|x|^p+|y|^q\right) \tag{4.36}$$

and

$$\left|\exp\left(xy+|x|^{2/q}|y|^{2/p}\right)\right| \le \frac{1}{p}\exp\left(|x|^p+|y|^2\right)+\frac{1}{q}\exp\left(|x|^2+|y|^q\right) \tag{4.37}$$

respectively, for any $x,\ y\in\mathbb{C}$ with $x,\ y\neq 0$.

(3) If we take the logarithmic function $f(z)$ given by (1.31), then from (4.29) and (4.30) we have

$$\left|\ln(1-xy)\ln\left(1-|x|^{p-1}|y|^{q-1}\right)\right| \le \ln\left(1-|x|^p\right)\ln\left(1-|y|^q\right) \tag{4.38}$$

and

$$\begin{aligned}&\left|\ln(1-xy)\ln\left(1-|x|^{2/q}|y|^{2/p}\right)\right|\\&\le\frac{1}{p}\ln\left(1-|x|^p\right)\ln\left(1-|y|^2\right)+\frac{1}{q}\ln\left(1-|x|^2\right)\ln\left(1-|y|^q\right)\end{aligned} \tag{4.39}$$

respectively, where $p>1,\ 1/p+1/q=1$ and $x,\ y\neq 0$ with $xy,\ |x|^2,\ |x|^p,\ |x|^{1/q},\ |y|^2,\ |y|^q,\ |y|^{1/p}\in D(0,1)$.

(4) If we consider the trigonometric function $f(z)$ given by (1.32), then the transform $f_A(z)$ is given by (1.38). Applying the inequalities (4.29) and (4.30) for these functions, we get

$$\left|\sin(xy)\sin\left(|x|^{p-1}|y|^{q-1}\right)\right| \le \sinh\left(|x|^p\right)\sinh\left(|y|^q\right) \tag{4.40}$$

and

$$\begin{aligned}&\left|\sin(xy)\sin\left(|x|^{2/q}|y|^{2/p}\right)\right|\\&\le\frac{1}{p}\sinh\left(|x|^p\right)\sinh\left(|y|^2\right)+\frac{1}{q}\sinh\left(|x|^2\right)\sinh\left(|y|^q\right)\end{aligned} \tag{4.41}$$

respectively, where $p>1,\ 1/p+1/q=1$ and $x,\ y\in\mathbb{C}$ with $x,\ y\neq 0$.

(5) A similar result can be obtained for the cosine function $f(z)$ given by (1.33) as well, namely

$$\left|\cos(xy)\cos\left(|x|^{p-1}|y|^{q-1}\right)\right| \le \cosh(|x|^p)\cosh(|y|^q) \tag{4.42}$$

and

$$\begin{aligned}&\left|\cos(xy)\cos\left(|x|^{2/q}|y|^{2/p}\right)\right|\\ &\le \frac{1}{p}\cosh(|x|^p)\cosh\left(|y|^2\right)+\frac{1}{q}\cosh\left(|x|^2\right)\cosh(|y|^q)\end{aligned} \tag{4.43}$$

respectively, where $p>1$, $1/p+1/q=1$ and $x,\ y\in\mathbb{C}$ with $x,\ y\neq 0$.

Further, the following result provides another improvement of Hölder's inequality via Young's inequality [215].

Theorem 71 (Ibrahim, Dragomir and Darus [215]) *Let $f(z)$ and $g(z)$ be as in Theorem 65. Then one has the inequality*

$$\begin{aligned}&\left|f\left(|x|^{p-1}|y|^{q-1}\right)g\left(|x|^{2/p}|y|^{2/q}\right)\right|\\ &\le \frac{1}{p}g_A\left(|x|^2\right)f_A(|y|^q)+\frac{1}{q}f_A(|x|^p)\,g_A\left(|y|^2\right)\end{aligned} \tag{4.44}$$

and

$$\left|f\left(|x|^{2/q}y\right)g\left(|x|^{2/p}y\right)\right| \le \frac{1}{p}g_A\left(|x|^2\right)f_A(|y|^p)+\frac{1}{q}f_A\left(|x|^2\right)g_A(|y|^q). \tag{4.45}$$

Proof. This follows from the inequality (2.80) by choosing $x=|y|^{(2/q)k}/|y|^j$, $y=|x|^{(2/p)k}/|x|^j$ and $x=|x|^{(2/q)j}|y|^k$, $y=|x|^{(2/p)k}|y|^j$. That is, for any $j,k\in\{0,1,2,\dots n\}$, we have the following inequalities

$$\begin{aligned}p|x|^{pj}|y|^{2k}+q|x|^{2k}|y|^{qj} &\ge pq|x|^{(p-1)j}|y|^{(q-1)j}|x|^{(2/p)k}|y|^{(2/q)k}\\ &= pq\left|\left(|x|^{(p-1)}|y|^{(q-1)}\right)^j\left(|x|^{2/p}|y|^{2/q}\right)^k\right|\end{aligned} \tag{4.46}$$

and

$$\begin{aligned}p|x|^{2j}|y|^{qk}+q|x|^{2k}|y|^{pj} &\ge pq|x|^{(2/q)j}|y|^j|x|^{(2/p)k}|y|^k\\ &= pq\left|\left(|x|^{2/q}y\right)^j\left(|x|^{2/p}y\right)^k\right|\end{aligned} \tag{4.47}$$

respectively. Repeating the same method as in Theorem 65 for the both inequalities (4.46) and (4.47), we deduce the desired results (4.44) and (4.45). ■

As a particular case of interest, we can state the following corollary:

Corollary 72 *If $g(z) = f(z)$ in (4.44) and (4.45), then*

$$\begin{aligned}&\left| f\left(|x|^{p-1}|y|^{q-1}\right) f\left(|x|^{2/p}|y|^{2/q}\right)\right| \\ &\le \frac{1}{p} f_A\left(|x|^2\right) f_A\left(|y|^q\right) + \frac{1}{q} f_A\left(|x|^p\right) f_A\left(|y|^2\right)\end{aligned} \tag{4.48}$$

and

$$\left| f\left(|x|^{2/q} y\right) f\left(|x|^{2/p} y\right)\right| \le f_A\left(|x|^2\right)\left[\frac{1}{p} f_A\left(|y|^p\right) + \frac{1}{q} f_A\left(|y|^q\right)\right], \tag{4.49}$$

where $p > 1$, $1/p + 1/q = 1$ and $x,\ y \neq 0$ with $|x|^2,\ |x|^p,\ |x|^q,\ |y|^2,\ |y|^p,\ |y|^q \in D(0,R)$. In particular, if $y = x$ in (4.48) and (4.49), then we have

$$\left| f\left(|x|^2\right) f\left(|x|^{pq-2}\right)\right| \le f_A\left(|x|^2\right)\left[\frac{1}{p} f_A\left(|x|^q\right) + \frac{1}{q} f_A\left(|x|^p\right)\right] \tag{4.50}$$

and

$$\left| f\left(|x|^{2/q} x\right) f\left(|x|^{2/p} x\right)\right| \le f_A\left(|x|^2\right)\left[\frac{1}{p} f_A\left(|x|^p\right) + \frac{1}{q} f_A\left(|x|^q\right)\right] \tag{4.51}$$

for $x \neq 0$, $|x|^2$, $|x|^p$, $|x|^q \in D(0,R)$.

The inequalities (4.48) and (4.49) are also the valuable sources of particular inequalities for some fundamental functions as will be outlined in the following:

(1) If we apply the inequalities (4.48) and (4.49) for the function $f(z)$ given by (1.19), then we get

$$\begin{aligned}&\frac{1}{\left|\left(1-|x|^{p-1}|y|^{q-1}\right)\left(1-|x|^{2/p}|y|^{2/q}\right)\right|} \\ &\le \frac{1}{p\left(1-|x|^2\right)\left(1-|y|^q\right)} + \frac{1}{q\left(1-|x|^p\right)\left(1-|y|^2\right)}\end{aligned} \tag{4.52}$$

and

$$\frac{1}{\left|\left(1-|x|^{2/q}y\right)\left(1-|x|^{2/p}y\right)\right|} \\ \leq \frac{1}{1-|x|^2}\left(\frac{1}{p\left(1-|y|^p\right)}+\frac{1}{q\left(1-|y|^q\right)}\right) \tag{4.53}$$

respectively, for any $x,\ y \in \mathbb{C},\ x,\ y \neq 0$ with $|x|^2,\ |y|^2,\ |x|^p,\ |y|^q,\ |x|^{1/p}$, $|x|^{1/q},\ |y|^{1/p},\ |y|^{1/q} \in D(0,1)$.

(2) If we apply the inequalities (4.48) and (4.49) for the exponential function $f(z)$ given by (1.25), then we can state that

$$\left|\exp\left(|x|^{p-1}|y|^{q-1}+|x|^{2/p}|y|^{2/q}\right)\right| \\ \leq \frac{1}{p}\exp\left(|x|^2+|y|^q\right)+\frac{1}{q}\exp\left(|x|^p+|y|^2\right) \tag{4.54}$$

and

$$\left|\exp\left[\left(|x|^{2/q}+|x|^{2/p}\right)y\right]\right| \\ \leq \exp\left(|x|^2\right)\left[\frac{1}{p}\exp\left(|y|^p\right)+\frac{1}{q}\exp\left(|y|^q\right)\right] \tag{4.55}$$

respectively, for $x,\ y \in \mathbb{C},\ x,\ y \neq 0$.

(3) If we apply the logarithmic function $f(z)$ given by (1.31), then from (4.48) and (4.49) we have

$$\left|\ln\left(1-|x|^{p-1}|y|^{q-1}\right)\ln\left(1-|x|^{2/p}|y|^{2/q}\right)\right| \\ \leq \frac{1}{p}\ln\left(1-|x|^2\right)\ln\left(1-|y|^q\right)+\frac{1}{q}\ln\left(1-|x|^p\right)\ln\left(1-|y|^2\right) \tag{4.56}$$

and

$$\left|\ln\left(1-|x|^{2/q}y\right)\ln\left(1-|x|^{2/p}y\right)\right| \\ \leq \ln\left(1-|x|^2\right)\left[\frac{1}{p}\ln\left(1-|y|^p\right)+\frac{1}{q}\ln\left(1-|y|^q\right)\right] \tag{4.57}$$

respectively, for any $x,\ y \in \mathbb{C},\ x,\ y \neq 0$ with $|x|^2,\ |y|^2,\ |x|^p,\ |y|^q,\ |x|^{1/p}$, $|x|^{1/q},\ |y|^{1/p},\ |y|^{1/q} \in D(0,1)$.

(4) If we consider the trigonometric function $f(z)$ given by (1.32), then we have $f_A(z)$ as given by (1.38). Applying the inequalities (4.48) and (4.49) for these functions, we get

$$\begin{aligned}&\left|\sin\left(|x|^{p-1}|y|^{q-1}\right)\sin\left(|x|^{2/p}|y|^{2/q}\right)\right| \\ &\leq \frac{1}{p}\sinh\left(|x|^2\right)\sinh\left(|y|^q\right)+\frac{1}{q}\sinh\left(|x|^p\right)\sinh\left(|y|^2\right)\end{aligned} \tag{4.58}$$

and

$$\begin{aligned}&\left|\sin\left(|x|^{2/q}y\right)\sin\left(|x|^{2/p}y\right)\right| \\ &\leq \sinh\left(|x|^2\right)\left[\frac{1}{p}\sinh\left(|y|^p\right)+\frac{1}{q}\sinh\left(|y|^q\right)\right]\end{aligned} \tag{4.59}$$

respectively, for any $x,\ y\in\mathbb{C},\ x,\ y\neq 0$.

(5) A similar result can be obtained for the hyperbolic function $f(z)$ given by (1.38) as well, namely

$$\begin{aligned}&\left|\sinh\left(|x|^{p-1}|y|^{q-1}\right)\sinh\left(|x|^{2/p}|y|^{2/q}\right)\right| \\ &\leq \frac{1}{p}\sinh\left(|x|^2\right)\sinh\left(|y|^q\right)+\frac{1}{q}\sinh\left(|x|^p\right)\sinh\left(|y|^2\right)\end{aligned} \tag{4.60}$$

and

$$\begin{aligned}&\left|\sinh\left(|x|^{2/q}y\right)\sinh\left(|x|^{2/p}y\right)\right| \\ &\leq \sinh\left(|x|^2\right)\left[\frac{1}{p}\sinh\left(|y|^p\right)+\frac{1}{q}\sinh\left(|y|^q\right)\right]\end{aligned} \tag{4.61}$$

respectively, for any $x,\ y\in\mathbb{C},\ x,\ y\neq 0$.

More recent studies for different types of Young's inequality can be found in the literature (see for instance [24], [90], [157], [160], [203], [220], [250], [259], [310], [412], [432], [433], [448], [449]).

In the next section, we refine Young's inequality of the form (2.81) and (2.85) of Kittaneh and Manasrah's result [250], then utilising these results, we obtain further improvements of the Hölder's type inequality (2.94) for function defined by the real power series with real coefficients. In particular, we refine the Hölder's type inequalities from Section 4.1.2. Natural examples for some fundamental real functions are also presented.

4.1.3 Further Improvements of Hölder's Inequality for Power Series

Before we state our results, we first prove the following lemma, which provides the refinement and a reverse of the Young's inequality (2.81) for real numbers as follows (see [216]).

Lemma 73 *For any a, $b \geq 0$ and $v \in [0,1]$, we have*

$$2\min\{v,1-v\}\left(\frac{a+b}{2}-\sqrt{ab}\right) \leq va+(1-v)\,b-a^{v}b^{1-v} \leq 2\max\{v,1-v\}\left(\frac{a+b}{2}-\sqrt{ab}\right). \quad (4.62)$$

Proof. We recall the following result obtained by Dragomir in [119] that provides a refinement and a reverse of the weighted Jensen's discrete inequality (2.95), namely

$$\begin{aligned} & n\min_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j)-\Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right] \\ & \leq \frac{1}{P_n}\sum_{j=1}^{n}p_j\Phi(x_j)-\Phi\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j\right) \\ & \leq n\max_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j)-\Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right], \end{aligned} \quad (4.63)$$

where $\Phi : \mathbf{C} \to \mathbb{R}$ is a convex function defined on a convex subset $\mathbf{C}$ of the linear space $\mathbf{X}$, $\{x_j\}$, $j \in \{1,2,\dots,n\}$ are vectors in $\mathbf{C}$ and $\{p_j\}$, $j \in \{1,2,\dots,n\}$ are nonnegative numbers with $P_n = \sum_{j=1}^{n} p_j > 0$.

We notice that Furuichi's result (2.87) is the particular case of (4.63), which applied for the convex function $f(t) = \exp(t)$, and denoting the function $\exp(x_j)$ as a_j for $j \in \{1, \ldots, n\}$. Also, for $n = 2$, we deduce from (4.63) that

$$\begin{aligned}
&2\min\{v, 1-v\}\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right] \\
&\le v\Phi(x) + (1-v)\Phi(y) - \Phi[vx + (1-v)y] \\
&\le 2\max\{v, 1-v\}\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right], \qquad (4.64)
\end{aligned}$$

for any $x, y \in \mathbb{R}$ and $v \in [0, 1]$. If we take the function $\Phi(x) = \exp(x)$, then we get from (4.64)

$$\begin{aligned}
&2\min\{v, 1-v\}\left[\frac{\exp(x)+\exp(y)}{2} - \exp\left(\frac{x+y}{2}\right)\right] \\
&\le v\exp(x) + (1-v)\exp(y) - \exp[vx + (1-v)y] \\
&\le 2\max\{v, 1-v\}\left[\frac{\exp(x)+\exp(y)}{2} - \exp\left(\frac{x+y}{2}\right)\right], \qquad (4.65)
\end{aligned}$$

for any $x, y \in \mathbb{R}$ and $v \in [0, 1]$. Further, we denote $\exp(x) = a$, $\exp(y) = b$ with $a, b > 0$, then from the (4.65) we obtain the desired result (4.62). ■

From the refinement and its reverse of Young's inequality (4.62), we have the following corollary (see [216]).

Corollary 74 *For any x, $y \ge 0$ and $p > 1$ with $1/p + 1/q = 1$, we have*

$$\begin{aligned}
&2\min\left\{\frac{1}{q}, \frac{1}{p}\right\}\left(\frac{x^q + y^p}{2} - x^{q/2}y^{p/2}\right) \\
&\le \frac{x^q}{q} + \frac{y^p}{p} - xy \\
&\le 2\max\left\{\frac{1}{q}, \frac{1}{p}\right\}\left(\frac{x^q + y^p}{2} - x^{q/2}y^{p/2}\right). \qquad (4.66)
\end{aligned}$$

Proof. The proof follows by choosing $a = x^q$, $b = y^p$, $v = 1/q$, $1 - v = q/p$ in Lemma 73. ■

Remark 75 *The first inequality in (4.62) provides Kittaneh and Manasrah's result (2.85) in [250] as well as a reverse of that result. One can also see that it is equivalent with Aldaz's result (2.86) [10].*

First, utilizing the inequality (2.81) for functions defined by the real power series with positive coefficients, we obtain the following result (see [216]).

Theorem 76 *Let $f(x) = \sum_{n=0}^{\infty} p_n x^n$ be a power series with positive coefficients and convergent on $(0, R)$. Then for $v \in [0,1]$, x, $y \geq 0$ such that y, xy, $x^v y$, $x^{1-v} y \in (0, R)$, we have*

$$f(x^v y) f\left(x^{1-v} y\right) \leq f(xy) f(y). \tag{4.67}$$

Proof. The proof follows by choosing in (2.81) $a = x^j$ and $b = x^k$ for $j, k \in \{0, 1, \ldots, n\}$. Thus, we have

$$x^{vj} x^{(1-v)k} \leq v x^j + (1 - v) x^k, \tag{4.68}$$

for any $x \geq 0$ and $v \in [0,1]$. If now we multiply the inequality (4.68) with the positive quantity, i.e., $p_j y^j p_k y^k \geq 0$, j, $k \in \{0, 1, \ldots, n\}$, $y \in (0, R)$, and sum over j and k from 0 to n, then we get

$$\begin{aligned} &\sum_{j=0}^{n} p_j (x^v y)^j \sum_{k=0}^{n} p_k \left(x^{1-v} y\right)^k \\ &\leq v \sum_{j=0}^{n} p_j (xy)^j \sum_{k=0}^{n} p_k y^k + (1 - v) \sum_{j=0}^{n} p_j y^j \sum_{k=0}^{n} p_k (xy)^k . \end{aligned} \tag{4.69}$$

Since all the series whose partial sums are involved in the inequality (4.69), are convergent on the interval $(0, R)$, by taking the limit as $n \to \infty$ in (4.69), we deduce the desired result (4.67). ■

Remark 77 *(i) If $xy = z$ in (4.67), then we have*

$$f\left(y^v z^{1-v}\right) f\left(y^{1-v} z^v\right) \leq f(y)f(z), \tag{4.70}$$

for y, z, $y^v z^{1-v}$, $y^{1-v}z^v \in (0, R)$ and $v \in [0, 1]$.

(ii) If $y = x$ in (4.67), then we also have

$$f\left(x^{1+v}\right) f\left(x^{2-v}\right) \leq f(x^2)f(x), \tag{4.71}$$

for x, x^2, x^{1+v}, $x^{2-v} \in (0, R)$ and $v \in [0, 1]$.

Some applications of the inequality (4.70) for particular functions of interest are as follows:

(1) If we apply the inequality (4.70) for the function $f(x) = 1/(1-x)$, $x \in (0, 1)$, then we get

$$(1-y)(1-z) \leq \left(1 - y^v z^{1-v}\right)\left(1 - y^{1-v}z^v\right), \tag{4.72}$$

for y, z, $y^v z^{1-v}$, $y^{1-v}z^v \in (0, 1)$ and $v \in [0, 1]$.

(2) If we consider the logarithm function $f(x) = \ln[1/(1-x)]$, $x \in (0, 1)$, and apply the inequality (4.70), then we get

$$\ln\left(1 - y^v z^{1-v}\right)\ln\left(1 - y^{1-v}z^v\right) \leq \ln(1-y)\ln(1-z), \tag{4.73}$$

for y, z, $y^v z^{1-v}$, $y^{1-v}z^v \in (0, 1)$ and $v \in [0, 1]$.

Next, we prove the following inequality based on the refinement and the reverse of Young's inequality (4.62) (see [216]).

Theorem 78 (Ibrahim, Dragomir and Darus [216]) *Let $f(x)$ be as in Theorem 76. Then, one has the inequality*

$$\begin{aligned} &2\min\{v, 1-v\}\left[f(xy)f(y) - f^2\left(x^{1/2}y\right)\right] \\ &\leq f(xy)f(y) - f(x^v y) f\left(x^{1-v}y\right) \\ &\leq 2\max\{v, 1-v\}\left[f(xy)f(y) - f^2\left(x^{1/2}y\right)\right] \end{aligned} \tag{4.74}$$

for x, $y \geq 0$ such that xy, y, $x^{1/2}y$, $x^v y$, $x^{1-v}y \in (0, R)$ and $v \in [0, 1]$.

Proof. We use the inequality (4.62) for $a = x^j$ and $b = x^k$, $j, k \in \{0, 1, \ldots, n\}$ to get

$$\begin{aligned}
&2\min\{v, 1-v\}\left(\frac{x^j + x^k}{2} - x^{j/2}x^{k/2}\right) \\
&\le vx^j + (1-v)x^k - x^{vj}x^{(1-v)k} \\
&\le 2\max\{v, 1-v\}\left(\frac{x^j + x^k}{2} - x^{j/2}x^{k/2}\right) \qquad (4.75)
\end{aligned}$$

for any $x, y \ge 0$ and $v \in [0, 1]$. Then, multiplying the inequality (4.75) with $p_j y^j p_k y^k \ge 0$, $j, k \in \{0, 1, \ldots, n\}$ and summing over j and k from 0 to n, we have

$$\begin{aligned}
&2t\left[\frac{1}{2}\left(\sum_{j=0}^{n} p_j x^j y^j \sum_{k=0}^{n} p_k y^k + \sum_{j=0}^{n} p_j y^j \sum_{k=0}^{n} p_k x^k y^k\right)\right. \\
&\left. - \sum_{j=0}^{n} p_j x^{j/2} y^j \sum_{k=0}^{n} p_k x^{k/2} y^k\right] \\
&\le v\sum_{j=0}^{n} p_j x^j y^j \sum_{k=0}^{n} p_k y^k + (1-v)\sum_{j=0}^{n} p_j y^j \sum_{k=0}^{n} p_k x^k y^k \\
&- \sum_{j=0}^{n} p_j x^{vj} y^j \sum_{k=0}^{n} p_k x^{(1-v)k} y^k \\
&\le 2T\left[\frac{1}{2}\left(\sum_{j=0}^{n} p_j x^j y^j \sum_{k=0}^{n} p_k y^k + \sum_{j=0}^{n} p_j y^j \sum_{k=0}^{n} p_k x^k y^k\right)\right. \\
&\left. - \sum_{j=0}^{n} p_j x^{j/2} y^j \sum_{k=0}^{n} p_k x^{k/2} y^k\right], \qquad (4.76)
\end{aligned}$$

where $t = \min\{v, 1-v\}$ and $T = \max\{v, 1-v\}$.

Since all the series whose partial sums are involved in the inequality (4.76) are convergent on the interval $(0, R)$, by taking the limit as $n \to \infty$ in (4.76), we deduce the desired result (4.74). ■

Remark 79 *(a) If $xy = z$ in (4.74), then we have*

$$\begin{aligned}
&2\min\{v,1-v\}\left[f(y)f(z)-f^2\left(\sqrt{yz}\right)\right]\\
&\le f(y)f(z)-f\left(y^v z^{1-v}\right)f\left(y^{1-v}z^v\right)\\
&\le 2\max\{v,1-v\}\left[f(y)f(z)-f^2\left(\sqrt{yz}\right)\right] \qquad (4.77)
\end{aligned}$$

for $y,\ z,\ z^v y^{1-v},\ z^{1-v}y^v \in (0,R)$ *and* $v\in[0,1]$. *This result provides somehow a symmetric form for the inequality (4.74) and has some nice applications as well, see (4.79).*

(b) If $y=x$ *in (4.74), then we also have*

$$\begin{aligned}
&2\min\{v,1-v\}\left[f(x)f\left(x^2\right)-f^2\left(x^{3/2}\right)\right]\\
&\le f(x)f(x^2)-f\left(x^{1+v}\right)f\left(x^{2-v}\right)\\
&\le 2\max\{\nu,1-\nu\}\left[f(x)f\left(x^2\right)-f^2\left(x^{3/2}\right)\right] \qquad (4.78)
\end{aligned}$$

for $x,\ x^2,\ x^{3/2},\ x^{1+v},\ x^{2-v}\in(0,R)$ *and* $v\in[0,1]$.

Now, if we consider the exponential function $f(x)=\exp(x)$, $x\in\mathbb{R}$ and apply the inequality (4.77), then we get

$$\begin{aligned}
&2\min\{v,1-v\}\left[\exp(y+z)-\exp\left(2\sqrt{yz}\right)\right]\\
&\le \exp(y+z)-\exp\left(y^v z^{1-v}+y^{1-v}z^v\right)\\
&\le 2\max\{v,1-v\}\left[\exp(y+z)-\exp\left(2\sqrt{yz}\right)\right] \qquad (4.79)
\end{aligned}$$

for any $y,z\ge 0$ and $v\in[0,1]$.

The second improvement of Hölder's inequality (2.94) via the refinement and the reverse of Young's inequality (4.66) is incorporated in the following theorem (see [216]).

Theorem 80 (Ibrahim, Dragomir and Darus [216]) *Let* $f(x)$ *be as in Theorem 76. If* $p>1$, $1/p+1/q=1$ *and* $x,y\ge 0$ *such that* $xy,\ x^q,\ y^p,\ x^{q/2}y^{p/2}\in(0,R)$, *then*

$$2t\left[\frac{1}{2}\left[f\left(x^q\right)+f\left(y^p\right)\right]-f\left(x^{q/2}y^{p/2}\right)\right]$$

$$\leq \frac{1}{q} f\left(x^q\right) + \frac{1}{p} f\left(y^p\right) - f\left(xy\right)$$
$$\leq 2T\left[\frac{1}{2}\left[f\left(x^q\right) + f\left(y^p\right)\right] - f\left(x^{q/2}y^{p/2}\right)\right] \tag{4.80}$$

where $t = \min\{1/q, 1/p\}$ *and* $T = \max\{1/q, 1/p\}$.

Proof. If we choose $x = x^j$ and $y = y^j$, $j \in \{0, 1, 2, \ldots, n\}$, then we have from (4.66) that

$$\begin{aligned} 2t\left(\frac{x^{qj} + y^{pj}}{2} - x^{(q/2)j}y^{(p/2)j}\right) &\leq \frac{x^{qj}}{q} + \frac{y^{pj}}{p} - (xy)^j \\ &\leq 2T\left(\frac{x^{qj} + y^{pj}}{2} - x^{(q/2)j}y^{(p/2)j}\right) \end{aligned} \tag{4.81}$$

for any x, $y \geq 0$ and $p > 1$ with $1/p + 1/q = 1$. If we multiply this inequality (4.81) with the positive quantities $p_j > 0$, $j \in \{0, 1, 2, \ldots, n\}$, and sum over j from 0 to n, then we get

$$\begin{aligned} &2t\left(\frac{1}{2}\left[\sum_{j=0}^{n} p_j x^{qj} + \sum_{j=0}^{n} p_j y^{pj}\right] - \sum_{j=0}^{n} p_j \left(x^{q/2}y^{p/2}\right)^j\right) \\ &\leq \frac{1}{q}\sum_{j=0}^{n} p_j x^{qj} + \frac{1}{p}\sum_{j=0}^{n} p_j y^{pj} - \sum_{j=0}^{n} p_j (xy)^j \\ &\leq 2T\left(\frac{1}{2}\left[\sum_{j=0}^{n} p_j x^{qj} + \sum_{j=0}^{n} p_j y^{pj}\right] - \sum_{j=0}^{n} p_j \left(x^{q/2}y^{p/2}\right)^j\right). \end{aligned} \tag{4.82}$$

Since all the series whose partial sums are involved in the inequality (4.82) are convergent on the interval $(0, R)$, by taking the limit as $n \to \infty$ in (4.82), we deduce the desired inequality (4.80). ■

Corollary 81 *If* $y = x$ *in (4.80), then for any* x^2, x^q, x^p, $x^{pq/2} \in (0, R)$ *and* $p > 1$ *with* $1/p + 1/q = 1$ *we have*

$$2t\left[\frac{1}{2}\left[f\left(x^q\right) + f\left(x^p\right)\right] - f\left(x^{pq/2}\right)\right]$$
$$\leq \frac{1}{q} f\left(x^q\right) + \frac{1}{p} f\left(x^p\right) - f\left(x^2\right)$$

$$\leq 2T\left[\frac{1}{2}\left[f\left(x^q\right)+f\left(x^p\right)\right]-f\left(x^{pq/2}\right)\right]. \tag{4.83}$$

In the following, we give some applications of the inequality (4.83) for particular functions of interest:

(1) If we apply the inequality (4.83) for the function $f(x) = 1/(1-x)$, $x \in (0,1)$, then we get

$$\begin{aligned} & t\left[\frac{1}{1-x^q}+\frac{1}{1-x^p}-\frac{2}{1-x^{pq/2}}\right] \\ & \leq \frac{1}{q\left(1-x^q\right)}+\frac{1}{p\left(1-x^p\right)}-\frac{1}{1-x^2} \\ & \leq T\left[\frac{1}{1-x^q}+\frac{1}{1-x^p}-\frac{2}{1-x^{pq/2}}\right] \end{aligned} \tag{4.84}$$

for x^2, x^q, x^p, $x^{pq/2} \in (0,1)$, $p > 1$ with $1/p + 1/q = 1$.

(2) If we apply the inequality (4.83) for the function $f(x) = \ln\left[1/(1-x)\right]$, $x \in (0,1)$, then we get

$$\begin{aligned} \left(\frac{\left(1-x^{pq/2}\right)^2}{\left(1-x^q\right)\left(1-x^p\right)}\right)^t & \leq \frac{1-x^2}{\left(1-x^q\right)^{1/q}\left(1-x^p\right)^{1/p}} \\ & \leq \left(\frac{\left(1-x^{pq/2}\right)^2}{\left(1-x^q\right)\left(1-x^p\right)}\right)^T, \end{aligned} \tag{4.85}$$

for x^2, x^q, x^p, $x^{pq/2} \in (0,1)$, $p > 1$ with $1/p + 1/q = 1$.

(3) If we consider the hyperbolic function $f(x) = \sinh(x)$, $x \in \mathbb{R}$ and apply the inequality (4.83), then we get

$$\begin{aligned} & t\left[\sinh\left(x^q\right)+\sinh\left(x^p\right)-2\sinh\left(x^{pq/2}\right)\right] \\ & \leq \frac{1}{q}\sinh\left(x^q\right)+\frac{1}{p}\sinh\left(x^p\right)-\sinh\left(x^2\right) \\ & \leq T\left[\sinh\left(x^q\right)+\sinh\left(x^p\right)-2\sinh\left(x^{pq/2}\right)\right] \end{aligned} \tag{4.86}$$

for any $x \geq 0$, $p > 1$ with $1/p + 1/q = 1$.

(4) A similar result can be obtained for the function $f(x) = \cosh(x)$, namely

$$\begin{aligned} & t\left[\cosh(x^q) + \cosh(x^p) - 2\cosh\left(x^{pq/2}\right)\right] \\ & \leq \frac{1}{q}\cosh(x^q) + \frac{1}{p}\cosh(x^p) - \cosh\left(x^2\right) \\ & \leq T\left[\cosh f(x^q) + \cosh(x^p) - 2\cosh\left(x^{pq/2}\right)\right] \end{aligned} \tag{4.87}$$

for any $x \geq 0$, $p > 1$ with $1/p + 1/q = 1$.

Further, we utilize the inequality (4.66) to improve the results from Section 4.1.2, giving the refinements and the reverses of Hölder's inequality for two functions defined by the real power series with positive coefficients. First, we obtain the following result (see [216]).

Theorem 82 (Ibrahim, Dragomir and Darus [216]) *Let $f(x) = \sum_{n=0}^{\infty} p_n x^n$ and $g(x) = \sum_{n=0}^{\infty} q_n x^n$ be two power series with positive coefficients and convergent on $(0, R)$. If $p > 1$, $1/p + 1/q = 1$ and $x, y \geq 0$ such that xy, x^p, x^q, y^p, y^q, $x^{q/2}y^{p/2}$, $x^{p/2}y^{q/2}$, $(xy)^{q/2}$, $(xy)^{p/2}$, xy^{q-1}, $xy^{p-1} \in (0, R)$, then*

$$\begin{aligned} & t\left[f(x^q)g(y^q) + g(x^p)f(y^p) - 2f\left(x^{q/2}y^{p/2}\right)g\left(x^{p/2}y^{q/2}\right)\right] \\ & \leq \frac{1}{q}f(x^q)g(y^q) + \frac{1}{p}g(x^p)f(y^p) - f(xy)\,g(xy) \\ & \leq T\left[f(x^q)g(y^q) + g(x^p)f(y^p) - 2f\left(x^{q/2}y^{p/2}\right)g\left(x^{p/2}y^{q/2}\right)\right] \end{aligned} \tag{4.88}$$

and

$$\begin{aligned} & t\left[f(x^q)g(y^p) + g(x^p)f(y^q) - 2f(x^{q/2}y^{q/2})g(x^{p/2}y^{p/2})\right] \\ & \leq \frac{1}{q}f(x^q)g(y^p) + \frac{1}{p}g(x^p)f(y^q) - f(xy^{q-1})g(xy^{p-1}) \\ & \leq T\left[f(x^q)g(y^p) + g(x^p)f(y^q) - 2f(x^{q/2}y^{q/2})g(x^{p/2}y^{p/2})\right]. \end{aligned} \tag{4.89}$$

Proof. If we choose in (4.66) $x = x^j y^k$ and $y = x^k y^j$, $j, k \in \{0, 1, 2 \ldots, n\}$, then we have

$$2t\left(\frac{x^{qj}y^{qk} + x^{pk}y^{pj}}{2} - x^{(q/2)j}y^{(p/2)j}x^{(p/2)k}y^{(q/2)k}\right)$$

$$\leq \frac{1}{q}\left(x^{qj}y^{qk}\right)+\frac{1}{p}\left(x^{pk}y^{pj}\right)-(xy)^j(xy)^k$$
$$\leq 2T\left(\frac{x^{qj}y^{qk}+x^{pk}y^{pj}}{2}-x^{(q/2)j}y^{(p/2)j}x^{(p/2)k}y^{(q/2)k}\right) \tag{4.90}$$

for any x, $y\geq 0$ and $p>1$ with $1/p+1/q=1$. Multiplying this inequality (4.90) with $p_jq_k\geq 0$, j, $k\in\{0,1,2\ldots,n\}$ and summing over j and k from 0 to n, we get

$$2t\left(\frac{1}{2}\left[\sum_{j=0}^{n}p_jx^{qj}\sum_{k=0}^{n}q_ky^{qk}+\sum_{k=0}^{n}q_kx^{pk}\sum_{j=0}^{n}p_jy^{pj}\right]\right.$$
$$\left.-\sum_{j=0}^{n}p_jx^{(q/2)j}y^{(p/2)j}\sum_{k=0}^{n}q_kx^{(p/2)k}y^{(q/2)k}\right)$$
$$\leq\frac{1}{q}\left(\sum_{j=0}^{n}p_jx^{qj}\sum_{k=0}^{n}q_ky^{qk}\right)+\frac{1}{p}\left(\sum_{k=0}^{n}q_kx^{pk}\sum_{j=0}^{n}p_jy^{pj}\right)$$
$$-\sum_{j=0}^{n}p_j(xy)^j\sum_{k=0}^{n}q_k(xy)^k$$
$$\leq 2T\left(\frac{1}{2}\left[\sum_{j=0}^{n}p_jx^{qj}\sum_{k=0}^{n}q_ky^{qk}+\sum_{k=0}^{n}q_kx^{pk}\sum_{j=0}^{n}p_jy^{pj}\right]\right.$$
$$\left.-\sum_{j=0}^{n}p_jx^{(q/2)j}y^{(p/2)j}\sum_{k=0}^{n}q_kx^{(p/2)k}y^{(q/2)k}\right), \tag{4.91}$$

where $p>1$, $1/p+1/q=1$.

Further, if we choose in (4.66) $x=x^j/y^j$ and $y=x^k/y^k$, $y\neq 0$, $j,k\in\{0,1,2,\ldots,n\}$ and repeat the same method as above, then we get

$$2t\left(\frac{1}{2}\left[\sum_{j=0}^{n}p_jx^{qj}\sum_{k=0}^{n}q_ky^{pk}+\sum_{k=0}^{n}q_kx^{pk}\sum_{j=0}^{n}p_jy^{qj}\right]\right.$$
$$\left.-\sum_{j=0}^{n}p_jx^{(q/2)j}y^{(q/2)j}\sum_{k=0}^{n}q_kx^{(p/2)k}y^{(p/2)k}\right)$$
$$\leq\frac{1}{q}\sum_{j=0}^{n}p_jx^{qj}\sum_{k=0}^{n}q_ky^{pk}+\frac{1}{p}\sum_{k=0}^{n}q_kx^{pk}\sum_{j=0}^{n}p_jy^{qj}$$

$$
\begin{aligned}
&-\sum_{j=0}^{n} p_j x^j y^{(q-1)j} \sum_{k=0}^{n} q_k x^k y^{(p-1)k} \\
&\leq 2T\left(\frac{1}{2}\left[\sum_{j=0}^{n} p_j x^{qj} \sum_{k=0}^{n} q_k y^{pk} + \sum_{k=0}^{n} q_k x^{pk} \sum_{j=0}^{n} p_j y^{qj}\right]\right. \\
&\left.-\sum_{j=0}^{n} p_j x^{(q/2)j} y^{(q/2)j} \sum_{k=0}^{n} q_k x^{(p/2)k} y^{(p/2)k}\right) \qquad (4.92)
\end{aligned}
$$

where $p > 1$, $1/p + 1/q = 1$.

Since all the series whose partial sums are involved in the inequalities (4.91) and (4.92) are convergent on the interval $(0, R)$, by taking the limit as $n \to \infty$ in the both inequalities (4.91) and (4.92), we deduce the desired results (4.88) and (4.89). ■

The natural consequence of Theorem 82 is as follows:

Corollary 83 *If $g(x) = f(x)$ in (4.88) and (4.89), then we have*

$$
\begin{aligned}
&t\left[f(x^q)f(y^q) + f(x^p)f(y^p) - 2f\left(x^{q/2}y^{p/2}\right) f\left(x^{p/2}y^{q/2}\right)\right] \\
&\leq \frac{1}{q} f(x^q)f(y^q) + \frac{1}{p} f(x^p)f(y^p) - f^2(xy) \\
&\leq T\left[f(x^q)f(y^q) + f(x^p)f(y^p) - 2f\left(x^{q/2}y^{p/2}\right) f\left(x^{p/2}y^{q/2}\right)\right] \qquad (4.93)
\end{aligned}
$$

and

$$
\begin{aligned}
&t\left[f(x^q)f(y^p) + f(x^p)f(y^q) - 2f(x^{q/2}y^{q/2})f(x^{p/2}y^{p/2})\right] \\
&\leq \frac{1}{q} f(x^q)f(y^p) + \frac{1}{p} f(x^p)f(y^q) - f(xy^{q-1})f(xy^{p-1}) \\
&\leq T\left[f(x^q)f(y^p) + f(x^p)f(y^q) - 2f(x^{q/2}y^{q/2})f(x^{p/2}y^{p/2})\right] \qquad (4.94)
\end{aligned}
$$

respectively, for any x, $y \geq 0$ such that xy, x^p, x^q, y^p, $y^q \in (0, R)$ and $p > 1$ with $1/p + 1/q = 1$.

The above results (4.93) and (4.94) have some natural applications for some particular functions of interest. For example, if we apply the inequalities (4.93) and (4.94) for the exponential function $f(x)$ given by (1.25), $x \in \mathbb{R}$, then we get the following inequalities:

$$\begin{aligned}
&t\left[\exp(x^q+y^q)+\exp(x^p+y^p)-2\exp\left(x^{q/2}y^{p/2}+x^{p/2}y^{q/2}\right)\right] \\
&\le \frac{1}{q}\exp(x^q+y^q)+\frac{1}{p}\exp(x^p+y^p)-\exp(2xy) \\
&\le T\left[\exp(x^q+y^q)+\exp(x^p+y^p)-2\exp\left(x^{q/2}y^{p/2}+x^{p/2}y^{q/2}\right)\right] \qquad (4.95)
\end{aligned}$$

and

$$\begin{aligned}
&t\left[\exp(x^q+y^p)+\exp(x^p+y^q)-2\exp(x^{q/2}y^{q/2}+x^{p/2}y^{p/2})\right] \\
&\le \frac{1}{q}\exp(x^q+y^p)+\frac{1}{p}\exp(x^p+y^q)-\exp(xy^{q-1}+xy^{p-1}) \\
&\le T\left[\exp(x^q+y^p)+\exp(x^p+y^q)-2\exp(x^{q/2}y^{q/2}+x^{p/2}y^{p/2})\right], \qquad (4.96)
\end{aligned}$$

respectively, for any $x \ge 0$, $y > 0$ and $p > 1$ with $1/p + 1/q = 1$.

We also obtain the following result (see [216]).

Theorem 84 (Ibrahim, Dragomir and Darus [216]) *Let $f(x)$ and $g(x)$ be as in Theorem 82. Then, one has the inequalities*

$$\begin{aligned}
&t\left[f(x^p)g(y^q)+g(x^p)f(y^q)-2g\left(x^{p/2}y^{q/2}\right)f\left(x^{p/2}y^{q/2}\right)\right] \\
&\le \frac{1}{q}f(x^p)g(y^q)+\frac{1}{p}g(x^p)f(y^q)-f\left(x^{p-1}y^{q-1}\right)g(xy) \\
&\le T\left[f(x^p)g(y^q)+g(x^p)f(y^q)-2g\left(x^{p/2}y^{q/2}\right)f\left(x^{p/2}y^{q/2}\right)\right] \qquad (4.97)
\end{aligned}$$

and

$$\begin{aligned}
&t\left[g\left(x^2\right)f\left(y^q\right)+f\left(x^p\right)g\left(y^2\right)-2f\left(x^{p/2}y^{q/2}\right)g(xy)\right] \\
&\le \frac{1}{q}g\left(x^2\right)f\left(y^q\right)+\frac{1}{p}f\left(x^p\right)g\left(y^2\right)-f(xy)\,g\left(x^{2/q}y^{2/p}\right) \\
&\le T\left[g\left(x^2\right)f\left(y^q\right)+f\left(x^p\right)g\left(y^2\right)-2f\left(x^{p/2}y^{q/2}\right)g(xy)\right]. \qquad (4.98)
\end{aligned}$$

Proof. If we choose in (4.66) $x = y^k/y^j$, $y = x^k/x^j$, $x,\ y \ne 0$ and $x = x^{(2/q)k}y^j$, $y = x^jy^{(2/p)k}$, $j,k \in \{0,1,2,\ldots,n\}$, then we have the following inequalities:

$$t\left(x^{pj}y^{qk}+x^{pk}y^{qj}-2x^{(p/2)k}y^{(q/2)k}x^{(p/2)j}y^{(q/2)j}\right)$$

$$\leq \frac{1}{q}x^{pj}y^{qk} + \frac{1}{p}x^{pk}y^{qj} - x^{(p-1)j}y^{(q-1)j}x^ky^k$$
$$\leq T\left(x^{pj}y^{qk} + x^{pk}y^{qj} - 2x^{(p/2)k}y^{(q/2)k}x^{(p/2)j}y^{(q/2)j}\right) \tag{4.99}$$

and

$$t\left(x^{2k}y^{qj} + x^{pj}y^{2k} - 2x^{(p/2)j}y^{(q/2)j}x^ky^k\right)$$
$$\leq \frac{1}{q}x^{2k}y^{qj} + \frac{1}{p}x^{pj}y^{2k} - x^jy^jx^{(2/q)k}y^{(2/p)k}$$
$$\leq T\left(x^{2k}y^{qj} + x^{pj}y^{2k} - 2x^{(p/2)j}y^{(q/2)j}x^ky^k\right), \tag{4.100}$$

respectively, for any x, $y \geq 0$ and $p > 1$ with $1/p + 1/q = 1$. Now, repeating the same method as in Theorem 82, we obtain the desired inequalities (4.97) and (4.98). ■

Corollary 85 *If $g(x) = f(x)$ in (4.97) and (4.98), then we have*

$$2t\left[f(x^p)f(y^q) - f^2\left(x^{p/2}y^{q/2}\right)\right] \leq f(x^p)f(y^q) - f\left(x^{p-1}y^{q-1}\right)f(xy)$$
$$\leq 2T\left[f(x^p)f(y^q) - f^2\left(x^{p/2}y^{q/2}\right)\right] \tag{4.101}$$

and

$$t\left[f\left(x^2\right)f\left(y^q\right) + f\left(x^p\right)f\left(y^2\right) - 2f\left(xy\right)f\left(x^{p/2}y^{q/2}\right)\right]$$
$$\leq \frac{1}{q}f\left(x^2\right)f\left(y^q\right) + \frac{1}{p}f\left(x^p\right)f\left(y^2\right) - f\left(xy\right)f\left(x^{2/q}y^{2/p}\right)$$
$$\leq T\left[f\left(x^2\right)f\left(y^q\right) + f\left(x^p\right)g\left(y^2\right) - 2f\left(xy\right)f\left(x^{p/2}y^{q/2}\right)\right]. \tag{4.102}$$

The above inequalities also provide some natural applications for particular functions of interest. We give some examples here as follows:

(1) If we apply the inequality (4.97) for the hyperbolic functions $f(x) = \sinh(x)$ and $g(x) = \cosh(x)$, $x \in \mathbb{R}$, then we get

$$t\left[\sinh(x^p + y^q) - \sinh\left(2x^{p/2}y^{q/2}\right)\right]$$
$$\leq \frac{1}{q}\sinh(x^p)\cosh(y^q) + \frac{1}{p}\cosh(x^p)\sinh(y^q)$$
$$-\sinh\left(x^{p-1}y^{q-1}\right)\cosh\left(xy\right)$$
$$\leq T\left[\sinh(x^p + y^q) - \sinh\left(2x^{p/2}y^{q/2}\right)\right], \tag{4.103}$$

for any $x, y \geq 0$ and $p > 1$ with $1/p + 1/q = 1$.

(2) Further, if we consider the exponential function $f(x) = \exp(x)$, $x \in \mathbb{R}$ and apply the inequality (4.101), then we get

$$\begin{aligned}
&2t\left[\exp(x^p + y^q) - \exp\left(2x^{p/2}y^{q/2}\right)\right] \\
&\leq \exp(x^p + y^q) - \exp\left(x^{p-1}y^{q-1} + xy\right) \\
&\leq 2T\left[\exp(x^p + y^q) - \exp\left(2x^{p/2}y^{q/2}\right)\right], \qquad (4.104)
\end{aligned}$$

for any $x, y \geq 0$ and $p > 1$ with $1/p + 1/q = 1$.

We end this section with the following result (see [216]).

Theorem 86 (Ibrahim, Dragomir and Darus [216]) *Let $f(x)$ and $g(x)$ be as in Theorem 82. Then one has the inequality*

$$\begin{aligned}
&t\left[f(x^p)\, g\left(y^2\right) + g\left(x^2\right) f(y^q) - 2f\left(x^{p/2}y^{q/2}\right) g(xy)\right] \\
&\leq \frac{1}{q} f(x^p)\, g\left(y^2\right) + \frac{1}{p} g\left(x^2\right) f(y^q) - f\left(x^{p-1}y^{q-1}\right) g\left(x^{2/p}y^{2/q}\right) \\
&\leq T\left[f(x^p)\, g\left(y^2\right) + g\left(x^2\right) f(y^q) - 2f\left(x^{p/2}y^{q/2}\right) g(xy)\right] \qquad (4.105)
\end{aligned}$$

and

$$\begin{aligned}
&t\left[f\left(x^2\right) g(y^q) + g\left(x^2\right) f(y^p) - 2f\left(xy^{p/2}\right) g\left(xy^{q/2}\right)\right] \\
&\leq \frac{1}{q} f\left(x^2\right) g(y^q) + \frac{1}{p} g\left(x^2\right) f(y^p) - f\left(x^{2/q}y\right) g\left(x^{2/p}y\right) \\
&\leq T\left[f\left(x^2\right) g(y^q) + g\left(x^2\right) f(y^p) - 2f\left(xy^{p/2}\right) g\left(xy^{q/2}\right)\right]. \qquad (4.106)
\end{aligned}$$

Proof. Again, the proof follows by using the same method as in Theorem 82, on choosing in (4.66) as $x = y^{(2/q)k}/y^j$, $y = x^{(2/p)k}/x^j$, $x, y \neq 0$ and $x = x^{(2/q)j}y^k$, $y = x^{(2/p)k}y^j$, $j, k \in \{0, 1, 2, \ldots, n\}$ respectively. The details are omitted. ■

Corollary 87 *If $g(x) = f(x)$ in (4.105) and (4.106), then we have*

$$t\left[f(x^p)\, f\left(y^2\right) + f\left(x^2\right) f(y^q) - 2f(xy)\, f\left(x^{p/2}y^{q/2}\right)\right]$$

$$\leq \frac{1}{q} f\left(x^p\right) f\left(y^2\right) + \frac{1}{p} f\left(x^2\right) f\left(y^q\right) - f\left(x^{p-1} y^{q-1}\right) f\left(x^{2/p} y^{2/q}\right)$$
$$\leq T\left[f\left(x^p\right) f\left(y^2\right) + f\left(x^2\right) f\left(y^q\right) - 2 f(xy) f\left(x^{p/2} y^{q/2}\right)\right] \tag{4.107}$$

and

$$t\left[f\left(x^2\right)\left[f\left(y^q\right) + f\left(y^p\right)\right] - 2 f\left(x y^{p/2}\right) f\left(x y^{q/2}\right)\right]$$
$$\leq f\left(x^2\right)\left[\frac{1}{q} f\left(y^q\right) + \frac{1}{p} f\left(y^p\right)\right] - f\left(x^{2/q} y\right) f\left(x^{2/p} y\right)$$
$$\leq T\left[f\left(x^2\right)\left[f\left(y^q\right) + f\left(y^p\right)\right] - 2 f\left(x y^{p/2}\right) f\left(x y^{q/2}\right)\right]. \tag{4.108}$$

4.2 Some Results Via Convexity and Jensen's Type Inequalities

4.2.1 Introduction and Preliminary Results

In 1989, Pečarić and Dragomir [351] established a refinement of Jensen's inequality (2.95) as follows:

$$\begin{aligned} f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right) &\leq \frac{1}{P_n^{m+1}} \sum_{j_1, j_2, \ldots, j_{m+1}} p_{j_1} \ldots p_{j_m} f\left(\frac{x_{j_1} + \ldots + x_{j_{m+1}}}{m+1}\right) \\ &\leq \frac{1}{P_n^{m}} \sum_{j_1, j_2, \ldots, j_m = 1} p_{j_1} \ldots p_{j_m} f\left(\frac{x_{j_1} + \ldots + x_{j_m}}{m}\right) \\ &\leq \ldots \leq \frac{1}{p_n} \sum_{j=1}^{n} p_j f\left(x_j\right), \end{aligned} \tag{4.109}$$

for a convex function $f : I \subseteq \mathbb{R} \to \mathbb{R}$, $x_j \in I$, $j \in \{1, 2, \ldots, n\}$ and $p_j \geq 0$ with $P_n = \sum_{j=1}^{n} p_j > 0$, where $m \geq 1$, $m \in \mathbb{N}$. The following refinement of the weighted Jensen's inequality was obtained by Dragomir and Scarmozzino [137] based on the properties of modulus of differentiable convex functions. It states that (see also [117, p. 204])

$$\begin{aligned}
&\frac{1}{p_n}\sum_{j=1}^{n} p_j f(x_j) - f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right) \\
&\geq \left|\frac{1}{p_n}\sum_{j=1}^{n} p_j \left|f(x_j) - f\left(\frac{1}{p_n}\sum_{j=1}^{n} p_j x_j\right)\right|\right. \\
&\left.-\left|f'\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right)\right| \cdot \frac{1}{P_n}\sum_{j=1}^{n} p_j \left|x_j - \frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right|\right| \geq 0, \qquad (4.110)
\end{aligned}$$

for any $x_j \in I$, $j \in \{1, 2, \ldots, n\}$ and $p_j \geq 0$ with $P_n = \sum_{j=1}^{n} p_j > 0$. In particular, we have from (4.110) the following result for unweighted means:

$$\begin{aligned}
&\frac{f(x_1) + \cdots + f(x_n)}{n} - f\left(\frac{x_1 + \cdots + x_n}{n}\right) \\
&\geq \left|\frac{1}{n}\sum_{j=1}^{n} \left|f(x_j) - f\left(\frac{x_1 + \cdots + x_n}{n}\right)\right|\right. \\
&\left.-\left|f'\left(\frac{x_1 + \cdots + x_n}{n}\right)\right| \cdot \frac{1}{n}\sum_{j=1}^{n} \left|x_j - \frac{1}{n}\sum_{k=1}^{n} x_k\right|\right| \quad (\geq 0), \qquad (4.111)
\end{aligned}$$

for any $x_j \in I$, $j \in \{1, 2, \ldots, n\}$. If we apply the inequality (4.110) for the convex function $f(x) = x^2$, $x \in \mathbb{R}$, and choose $x_j = a_j / b_j$, $p_j = b_j^2$, $j \in \{1, 2, \ldots, n\}$, then we get a refinement of the classical (CBS)-inequality for real numbers.

Meanwhile, in 1994, Dragomir and Ionescu [139] proved the following reverse of the Jensen's type inequality for a differentiable convex function,

$$\begin{aligned}
0 &\leq \sum_{j=1}^{n} q_j f(x_j) - f\left(\sum_{j=1}^{n} q_j x_j\right) \\
&\leq \sum_{j=1}^{n} q_j x_j f'(x_j) - \sum_{j=1}^{n} q_j x_j \sum_{j=1}^{n} q_j f'(x_j),
\end{aligned}$$

where $q_j \geq 0$, $j \in \{1, 2, \ldots, n\}$ such that $\sum_{j=1}^{n} q_j = 0$, or in more general case, they have

$$0 \leq \frac{\sum_{j=1}^{n} p_j f(x_j)}{\sum_{j=1}^{n} p_j} - f\left(\frac{\sum_{j=1}^{n} p_j x_j}{\sum_{j=1}^{n} p_j}\right)$$

$$\leq \frac{\sum_{j=1}^{n} p_j x_j f'(x_j)}{\sum_{j=1}^{n} p_j} - \frac{\sum_{j=1}^{n} p_j x_j}{\sum_{j=1}^{n} p_j} \cdot \frac{\sum_{j=1}^{n} p_j f'(x_j)}{\sum_{j=1}^{n} wp_j}, \tag{4.112}$$

provided that $f : I \to \mathbb{R}$ is a differentiable convex function on the interval $I \subset \mathbb{R}$, $x_j \in I$, $p_j \geq 0$ for $j \in \{1, 2, \ldots, n\}$ such that $\sum_{j=1}^{n} p_j > 0$. If f is strictly convex on $\overset{o}{I}$ (i.e., the interior of I), then the equality holds in (4.112) if and only if $x_1 = x_2 = \cdots = x_n$.

For other refinements, generalizations and applications of the Jensen's type inequalities, see for instance ([117, p. 190-213], [126], [129], [130], [132], [141]) and the references which are cited therein.

Utilising the celebrated Jensen's discrete inequality (2.95) and its reverse (4.112) for particular real convex functions, we establish in this section, some new and interesting inequalities for functions defined by the real power series (2.65) with real coefficients and convergent on the interval $(-R, R)$, $R > 0$. Applications for some fundamental functions of interest are also presented.

4.2.2 Power Series Inequalities Via Convexity

First, we state the following result that has been obtained on utilising the convexity properties of certain underlying functions (see [213]).

Theorem 88 (Ibrahim, Dragomir, Cerone and Darus [213]) *Assume that $f(x) = \sum_{n=0}^{\infty} p_n x^n$ is a function with nonnegative coefficients defined on $(-R, R)$, $R > 0$. If $a, b, c > 0$ are such that $ac, bc \in (0, R)$, then*

$$\begin{aligned}(\ln a - \ln b)\, acf'(ac) &\geq f(ac) - f(bc) \\ &\geq (\ln a - \ln b)\, bcf'(bc).\end{aligned} \tag{4.113}$$

Proof. It is well known that if $f : I \subset \mathbb{R} \to \mathbb{R}$ is a differentiable convex function on $\mathring{I}$, then for any $x, y \in \mathring{I}$ we have

$$f'(y)(y - x) \geq f(y) - f(x). \tag{4.114}$$

Now, if we apply the property (4.114) for the function $f(t) = -\ln t$, $t > 0$, then we get

$$\frac{x}{y} - 1 \geq \ln x - \ln y, \tag{4.115}$$

for any x, $y > 0$. If in (4.115) we choose $x = a^n$, $y = b^n$, $n \geq 0$, then we obtain

$$a^n - b^n \geq nb^n \ln a - nb^n \ln b, \tag{4.116}$$

for any $a, b > 0$, $n \geq 0$. If we multiply (4.116) by $p_n c^n \geq 0$, $n \geq 0$ and sum over n from 0 to k, then we derive

$$\sum_{n=0}^{k} p_n c^n a^n - \sum_{n=0}^{k} p_n b^n c^n \geq \ln a \sum_{n=0}^{k} np_n b^n c^n - \ln b \sum_{n=0}^{k} np_n b^n c^n. \tag{4.117}$$

Since

$$\sum_{n=0}^{\infty} p_n c^n a^n = f(ac), \qquad \sum_{n=0}^{\infty} p_n c^n b^n = f(cb) \tag{4.118}$$

and

$$\sum_{n=0}^{\infty} np_n b^n c^n = bc \sum_{n=1}^{\infty} np_n b^{n-1} c^{n-1} = bcf'(bc), \tag{4.119}$$

then by letting $k \to \infty$ in (4.117), we deduce

$$f(ac) - f(bc) \geq (\ln a - \ln b)\, bcf'(bc). \tag{4.120}$$

Now, replacing a with b in (4.120), we have

$$f(bc) - f(ac) \geq (\ln b - \ln a)\, acf'(ac), \tag{4.121}$$

which is equivalent with

$$f(ac) - f(bc) \leq (\ln a - \ln b)\, acf'(ac). \tag{4.122}$$

Thus, from the inequalities (4.120) and (4.122), we derive the desired result (4.113). ■

Corollary 89 *With the assumptions of Theorem 88 and if $a, b \in (0, R)$, then*

$$(\ln a - \ln b)\, af'(a) \geq f(a) - f(b) \geq (\ln a - \ln b)\, bf'(b). \tag{4.123}$$

Corollary 90 *With the assumptions of Theorem 88 and if $a, c > 0$ and $a, ac \in (0, R)$, then*

$$ac \ln a \; f'(ac) \geq f(ac) - f(c) \geq c \ln a \; f'(c). \tag{4.124}$$

The above results (4.123) and (4.124) have some natural applications for particular real functions of interest as follows:

(1) If we apply the inequality (4.123) for the function $f(x) = 1/(1-x)$, $x \in (-1, 1)$, then for any $a, b \in (0, 1)$ we get

$$\begin{aligned}(\ln a - \ln b) \cdot \frac{a}{(1-a)^2} &\geq \frac{1}{1-a} - \frac{1}{1-b} \\ &\geq (\ln a - \ln b) \cdot \frac{b}{(1-b)^2}.\end{aligned} \tag{4.125}$$

Hence

$$\frac{(1-b)(a-b)}{b(1-a)} \geq \ln a - \ln b \geq \frac{(1-a)(a-b)}{a(1-b)}, \tag{4.126}$$

for any $a, b \in (0, 1)$.

(2) If we apply the inequality (4.123) for the function $f(x) = x/(1-x)^2$, $x \in (-1, 1)$, then for any $a, b \in (0, 1)$ we obtain

$$\begin{aligned}(\ln a - \ln b) \cdot \frac{a(1+a)}{(1-a)^3} &\geq \frac{a}{(1-a)^2} - \frac{b}{(1-b)^2} \\ &\geq (\ln a - \ln b) \cdot \frac{b(1+b)}{(1-b)^3},\end{aligned} \tag{4.127}$$

which implies that

$$\begin{aligned}\frac{(1-b)(1-ab)(a-b)}{b(1+b)(1-a)^2} &\geq \ln a - \ln b \\ &\geq \frac{(1-a)(1-ab)(a-b)}{a(1+a)(1-b)^2},\end{aligned} \tag{4.128}$$

for any $a, b \in (0, 1)$.

(3) If in (4.123), we choose the logarithm function $f(x) = -\ln(1-x)$, $x \in (-1,1)$, then for any $a, b \in (0,1)$ we get the inequality

$$\ln\left(\frac{a}{b}\right)^{a/(1-a)} \geq \ln\left(\frac{1-b}{1-a}\right) \geq \ln\left(\frac{a}{b}\right)^{b/(1-b)}. \tag{4.129}$$

Hence

$$\left(\frac{a}{b}\right)^{a/(1-a)} \geq \frac{1-b}{1-a} \geq \left(\frac{a}{b}\right)^{b/(1-b)}, \tag{4.130}$$

for any $a, b \in (0,1)$. In particular, if in (4.130) we choose $b = 1-a$, then we obtain the following result:

$$\left(\frac{a}{1-a}\right)^{a/(1-a)} \geq \frac{a}{1-a} \geq \left(\frac{a}{1-a}\right)^{(1-a)/a}, \tag{4.131}$$

for any $a \in (0,1)$.

(4) If in (4.123) we choose the function $f(x) = \frac{1}{2}\ln\left[(1+x)/(1-x)\right]$, $x \in (-1,1)$, then for any $a, b \in (0,1)$ we get the following inequality:

$$\begin{aligned}(\ln a - \ln b) \cdot \frac{a}{1-a^2} &\geq \frac{1}{2}\left[\ln\left(\frac{1+a}{1-a}\right) - \ln\left(\frac{1+b}{1-b}\right)\right] \\ &\geq (\ln a - \ln b)\frac{b}{1-b^2},\end{aligned} \tag{4.132}$$

which is equivalent to

$$\ln\left(\frac{a}{b}\right)^{a/(1-a^2)} \geq \ln\left(\frac{(1+a)(1-b)}{(1-a)(1+b)}\right)^{1/2} \geq \ln\left(\frac{a}{b}\right)^{b/(1-b)}. \tag{4.133}$$

Hence

$$\left(\frac{a}{b}\right)^{2a/(1-a^2)} \geq \frac{(1+a)(1-b)}{(1-a)(1+b)} \geq \left(\frac{a}{b}\right)^{2b/(1-b^2)}, \tag{4.134}$$

for any $a, b \in (0,1)$.

(5) If we apply the same inequality (4.123) for the hyperbolic function $f(x) = \cosh(x)$, $x \in \mathbb{R}$,then we obtain that

$$\ln\left(\frac{a}{b}\right)^{a\sinh(a)} \geq \cosh(a) - \cosh(b) \geq \ln\left(\frac{a}{b}\right)^{b\sinh(b)}, \tag{4.135}$$

for any $a, b \in (0, R), R > 0$.

(6) Further, if we apply the inequality (4.124) for the function $f(x) = \cosh(x)$, $x \in \mathbb{R}$, then for any $a, b \in \mathbb{R}$ we have

$$\ln a^{ab\sinh(ab)} \geq \cosh(ab) - \cosh(b) \geq \ln a^{b\sinh(b)}. \tag{4.136}$$

In particular, if in (4.136) we choose $b = 1$, then we obtain that

$$\ln a^{2ea\sinh(a)} \geq 2e\cosh(a) - e^2 - 1 \geq \ln a^{e^2-1}, \tag{4.137}$$

for any $a \in \mathbb{R}$.

Now, we prove the following reverse of the Jensen type inequality that has been obtained by utilising the inequality (4.123) for a differentiable convex function (see [213]).

Theorem 91 (Ibrahim, Dragomir, Cerone and Darus [213]) *Assume that f is as in Theorem 88 and $R = 1$ or $R = +\infty$. If $a_i \in (0, R)$ and $p_k \geq 0$, $k \in \{1, \ldots, n\}$ with $\sum_{k=1}^{n} p_k = 1$, then we have the inequalities*

$$\ln\left[\frac{\prod_{k=1}^{n} a_k^{p_k a_k f'(a_k)}}{\left(\prod_{j=1}^{n} a_j^{p_j}\right)^{\sum_{k=1}^{n} p_k a_k f'(a_k)}}\right] \geq \sum_{k=1}^{n} p_k f(a_k) - f\left(\prod_{j=1}^{n} a_j^{p_j}\right) \geq 0. \tag{4.138}$$

Proof. We use the inequality (4.123) for the choices $a = a_i$ and $b = \prod_{j=1}^{n} a_j^{p_j}$ to get

$$\begin{aligned}
&\left[\ln a_k - \ln\left(\prod_{j=1}^{n} a_j^{p_j}\right)\right] a_k f'(a_k) \\
&\geq f(a_k) - f\left(\prod_{j=1}^{n} a_j^{p_j}\right) \\
&\geq \left[\ln a_k - \ln\left(\prod_{j=1}^{n} a_j^{p_j}\right)\right]\left(\prod_{j=1}^{n} a_j^{p_j}\right) f'\left(\prod_{j=1}^{n} a_j^{p_j}\right),
\end{aligned} \tag{4.139}$$

for any $k \in \{1, \ldots, n\}$. Now, if we multiply (4.139) by $p_k \geq 0$ and sum over k from 1 to n, then we get

$$\begin{aligned} &\ln\left(\prod_{k=1}^{n} a_k^{p_k a_k f'(a_k)}\right) - \sum_{k=1}^{n} p_k a_k f'(a_k) \cdot \ln\left(\prod_{j=1}^{n} a_j^{p_j}\right) \\ &\geq \sum_{k=1}^{n} p_k f(a_k) - f\left(\prod_{j=1}^{n} a_j^{p_j}\right) \geq 0, \end{aligned} \tag{4.140}$$

which is clearly equivalent with the desired result (4.138). ■

Remark 92 *The second inequality in (4.138) shows that*

$$\lambda f(x) + (1-\lambda) f(y) \geq f\left(x^{\lambda} y^{1-\lambda}\right) \tag{4.141}$$

for any $\lambda \in [0,1]$ *and* $x, y \in (0, R)$ $(R = 1 \quad or \quad R = \infty)$, *i.e.,* f *is an* $GA-$*convex function in the sense of terminology introduced by Anderson et al. in [22]. For other similar results, see [23], [47], [79, p.], [304], [333].*

The following result is also established, which provides a refinement as well as a reverse of the Jensen's type for the power series.

Theorem 93 (Ibrahim, Dragomir, Cerone and Darus [213]) *Assume that* f, a_k *and* p_k *are as in Theorem 91. Then, we have the inequalities:*

$$\begin{aligned} &\ln\left[\left(\prod_{k=1}^{n} a_k^{p_k a_k f'(a_k)}\right) \cdot \left(\sum_{j=1}^{n} \frac{p_j}{a_j}\right)^{\sum_{k=1}^{n} p_k a_k f'(a_k)}\right] \\ &\geq \sum_{k=1}^{n} p_k f(a_k) - f\left[\left(\sum_{k=1}^{n} \frac{p_k}{a_k}\right)^{-1}\right] \\ &\geq \ln\left[\left(\prod_{k=1}^{n} a_k^{p_k}\right) \cdot \left(\sum_{j=1}^{n} \frac{p_j}{a_j}\right)\right]^{f'\left[\left(\sum_{k=1}^{n} \frac{p_k}{a_k}\right)^{-1}\right]\left(\sum_{j=1}^{n} \frac{p_j}{a_j}\right)^{-1}} \geq 0. \end{aligned} \tag{4.142}$$

Proof. From the inequality (4.123), we have for the choices $a = a_k$, $b = \left(\sum_{j=1}^{n} \frac{p_j}{a_j}\right)^{-1}$ that

$$\begin{aligned}
&\left[\ln a_k - \ln\left(\frac{1}{\sum_{j=1}^{n}\frac{p_j}{a_j}}\right)\right] a_k f'(a_k) \\
&\geq f(a_k) - f\left(\frac{1}{\sum_{j=1}^{n}\frac{p_j}{a_j}}\right) \\
&\geq \left[\ln a_k - \ln\left(\frac{1}{\sum_{j=1}^{n}\frac{p_j}{a_j}}\right)\right] \frac{1}{\sum_{j=1}^{n}\frac{p_j}{a_j}} f'\left(\frac{1}{\sum_{j=1}^{n}\frac{p_j}{a_j}}\right),
\end{aligned} \tag{4.143}$$

for any $k \in \{1, \ldots, n\}$. Now, if we multiply (4.143) by $p_k \geq 0$ and sum over k from 1 to n, then we deduce

$$\begin{aligned}
&\ln\left[\frac{\prod_{k=1}^{n} a_k^{p_k a_k f'(a_k)}}{\left(\prod_{j=1}^{n} a_j^{p_j}\right)^{-\sum_{k=1}^{n} p_k a_k f'(a_k)}}\right] \\
&\geq \sum_{k=1}^{n} p_k f(a_k) - f\left(\frac{1}{\sum_{j=1}^{n}\frac{p_j}{a_j}}\right) \\
&\geq \ln\left[\frac{\prod_{k=1}^{n} a_k^{p_k}}{\left(\sum_{j=1}^{n}\frac{p_j}{a_j}\right)^{-1}}\right]^{f'\left(1\big/\sum_{j=1}^{n}\frac{p_j}{a_j}\right)\left(\sum_{j=1}^{n}\frac{p_j}{a_j}\right)^{-1}} \geq 0
\end{aligned} \tag{4.144}$$

giving the desired result (4.142). ■

Finally, we establish the following result.

Theorem 94 (Ibrahim, Dragomir, Cerone and Darus [213]) *With the assumptions in Theorem 88, we have the inequality:*

$$f\left(\left(\prod_{k=1}^{n} a_k^{p_k a_k f'(a_k)}\right)^{\frac{1}{\sum_{k=1}^{n} p_k a_k f'(a_k)}}\right) \geq \sum_{k=1}^{n} p_k f(a_k). \tag{4.145}$$

Proof. From the inequality (4.123) we have

$$f(a) - f(a_k) \geq (\ln a - \ln a_k)\, a_k f'(a_k), \tag{4.146}$$

for any $k \in \{1, \ldots, n\}$. Now, if we multiply (4.146) by $p_k \geq 0$ and sum over k from 1 to n, then we get

$$f(a) - \sum_{k=1}^{n} p_k f(a_k) \geq \ln a \cdot \sum_{k=1}^{n} p_k a_k f'(a_k) - \sum_{k=1}^{n} p_k a_k f'(a_k) \ln a_k. \tag{4.147}$$

Now, if we choose a so that

$$\ln a = \frac{\sum_{k=1}^{n} p_k a_k f'(a_k) \ln a_k}{\sum_{k=1}^{n} p_k a_k f'(a_k)} = \ln \left(\prod_{k=1}^{n} a_k^{p_k a_k f'(a_k)} \right)^{\frac{1}{\sum_{k=1}^{n} p_k a_k f'(a_k)}}, \tag{4.148}$$

then we get the desired result (4.145). ■

4.2.3 Power Series Inequalities Via Jensen Type

In this section, we employ the inequality (2.95) and also the reverse of the Jensen inequality (4.112) due to Dragomir and Ionescu [139], to derive some new and interesting inequalities for functions defined by the real power series with positive coefficients.

First, we get the following result by utilising Jensen's inequality (2.95) for a particular convex function.

Theorem 95 (Ibrahim, Dragomir, Cerone and Darus [213]) *Assume that the function $f(x) = \sum_{n=0}^{\infty} p_n x^n$ is defined on the interval $(-R, R)$, $R > 0$ and $p_0 > 0$, $p_n \geq 0$ for $n \geq 1$. If $a, b > 0$ are such that $a, ab \in (0, R)$, then*

$$b^{af'(a)/f(a)} \leq \frac{f(ab)}{f(a)} \leq b^{abf'(ab)/f(ab)}. \tag{4.149}$$

Proof. If we apply Jensen's inequality (2.95) for the convex function $g(t) = -\ln t$, $t > 0$, then we have

$$\ln \left(\frac{\sum_{n=0}^{k} p_n a^n b^n}{\sum_{n=0}^{k} p_n a^n} \right)$$

$$\geq \frac{\sum_{n=0}^{k} p_n a^n \ln b^n}{\sum_{n=0}^{k} p_n a^n} = \frac{\ln b \sum_{n=0}^{k} n p_n a^n}{\sum_{n=0}^{k} p_n a^n} = \frac{a \ln b \sum_{n=1}^{k} n p_n a^{n-1}}{\sum_{n=0}^{k} p_n a^n}, \tag{4.150}$$

for any $a, b > 0$ with $a, ab \in (0, R)$. Taking the limit over $k \to \infty$ and since

$$\sum_{n=0}^{\infty} p_n a^n b^n = f(ab), \quad \sum_{n=1}^{\infty} n p_n a^{n-1} = f'(a), \quad \sum_{n=0}^{\infty} p_n a^n = f(a), \tag{4.151}$$

then from (4.150) we have

$$\ln\left(\frac{f(ab)}{f(a)}\right) \geq \frac{a \ln b f'(a)}{f(a)}, \tag{4.152}$$

for $a, b > 0$ with $a, ab \in (0, R)$, which is equivalent with the first inequality in (4.149).

Now, if we apply Jensen's inequality (2.95) for the convex function $g(t) = t \ln t$, $t > 0$, then we can state that

$$\begin{aligned} &\left(\frac{\sum_{n=0}^{k} p_n a^n b^n}{\sum_{n=0}^{k} p_n a^n}\right) \cdot \ln\left(\frac{\sum_{n=0}^{k} p_n a^n b^n}{\sum_{n=0}^{k} p_n a^n}\right) \\ &\leq \frac{\sum_{n=0}^{k} p_n a^n b^n \ln b^n}{\sum_{n=0}^{k} p_n a^n} = \frac{\ln b \sum_{n=0}^{k} n p_n a^n b^n}{\sum_{n=0}^{k} p_n a^n} \\ &= \frac{ab \ln b \sum_{n=1}^{k} n p_n a^{n-1} b^{n-1}}{\sum_{n=0}^{k} p_n a^n}, \end{aligned} \tag{4.153}$$

for $a, b > 0$ with $a, ab \in (0, R)$. Taking the limit over $k \to \infty$ and since

$$\sum_{n=1}^{\infty} n p_n a^{n-1} b^{n-1} = f'(ab), \tag{4.154}$$

then from (4.153) we have

$$\frac{f(ab)}{f(a)} \ln\left(\frac{f(ab)}{f(a)}\right) \leq \frac{ab \ln b f'(ab)}{f(a)}, \tag{4.155}$$

which is clearly equivalent with the second inequality in (4.149). ∎

Corollary 96 *With the assumptions in Theorem 95, and if $a, c \in (0, R)$, then*

$$\left(\frac{c}{a}\right)^{af'(a)/f(a)} \le \frac{f\left(c\right)}{f\left(a\right)} \le \left(\frac{c}{a}\right)^{cf'(c)/f(c)}. \tag{4.156}$$

Proof. It follows from Theorem 95 on choosing $c = ab$. ∎

Some applications of the inequality (4.156) for particular functions of interest are given in the following:

(1) If we apply the inequality (4.156) for the function $f(x) = -\ln(1-x)$, $x \in (-1, 1)$, then we get

$$\left(\frac{c}{a}\right)^{-a/(1-a)\ln(1-a)} \le \frac{\ln(1-c)}{\ln(1-a)} \le \left(\frac{c}{a}\right)^{-c/(1-c)\ln(1-c)}, \tag{4.157}$$

for any $a, c \in (0, 1)$.

(2) If in (4.156) we choose the function $f(x) = \exp(x)$, $x \in \mathbb{R}$, then we can state that

$$\left(\frac{c}{a}\right)^{a} \le \exp\left(c-a\right) \le \left(\frac{c}{a}\right)^{c}, \tag{4.158}$$

for any $a, c \in (0, R)$.

(3) If we apply the same inequality (4.156) for the function $f(x) = \sinh(x)$, $x \in \mathbb{R}$, then we obtain

$$\left(\frac{c}{a}\right)^{a\coth(a)} \le \frac{\sinh\left(c\right)}{\sinh\left(a\right)} \le \left(\frac{c}{a}\right)^{c\coth(c)} \tag{4.159}$$

for any $a, c \in (0, R)$.

Next, we utilize the reverse of the Jensen type inequality (4.112) for a particular convex function to obtain the following result.

Theorem 97 (Ibrahim, Dragomir, Cerone and Darus [213]) *Let f be as in Theorem 95. If a, $b > 0$ are such that a, ab, $ab \in (0, R)$, then*

$$\frac{f\left(ab\right)}{f\left(a\right)} \le b^{af'(a)/f(a)} \cdot \exp\left[\frac{f\left(ab\right)f\left(\frac{a}{b}\right)}{f^{2}\left(a\right)} - 1\right]. \tag{4.160}$$

Proof. Utilizing the inequality (4.112), for the convex function $g(t) = -\ln t$, $t > 0$, we can write that

$$\ln\left(\frac{\sum_{n=0}^{k} p_n a^n b^n}{\sum_{n=0}^{k} p_n a^n}\right) - \frac{\sum_{n=0}^{k} p_n a^n \ln b^n}{\sum_{n=0}^{k} p_n a^n}$$

$$\leq \frac{\sum_{n=0}^{k} p_n a^n b^n}{\sum_{n=0}^{k} p_n a^n} \cdot \frac{\sum_{n=0}^{k} \frac{p_n a^n}{b^n}}{\sum_{n=0}^{k} p_n a^n} - 1, \tag{4.161}$$

for $a, b > 0$ with $a, ab, ..ab \in (0, R)$. Taking the limit over $k \to \infty$ and since

$$\sum_{n=0}^{\infty} p_n \left(\frac{a}{b}\right)^n = f\left(\frac{a}{b}\right), \tag{4.162}$$

then by (4.161) we deduce

$$\ln\left(\frac{f(ab)}{f(a)}\right) - \frac{a \ln b f'(a)}{f(a)} \leq \frac{f(ab) f\left(\frac{a}{b}\right)}{f^2(a)} - 1, \tag{4.163}$$

which is clearly equivalent with the desired result (4.160). ■

The particular case of Theorem 97 can be stated as well.

Corollary 98 *With the assumptions in Theorem 97 and if $a, c \in (0, R)$ with $a^2/c \in (0, R)$, then we have the inequality*

$$\frac{f(c)}{f(a)} \leq \left(\frac{c}{a}\right)^{af'(a)/f(a)} \exp\left[\frac{f(c) f\left(\frac{a^2}{c}\right)}{f^2(a)} - 1\right]. \tag{4.164}$$

If $c^2/a \in (0, R)$, then we also have

$$\left(\frac{c}{a}\right)^{cf'(c)/f(c)} \exp\left[1 - \frac{f(a) f\left(\frac{c^2}{a}\right)}{f^2(c)}\right] \leq \frac{f(c)}{f(a)}. \tag{4.165}$$

Some examples for particular inequalities that are generated by the real power series with positive coefficients are as follows.

(1) If we apply the inequality (4.164) for the function $f(x) = 1/(1-x)$, $x \in (-1,1)$, then for any $a, c \in (0,1)$ we get

$$\frac{1-a}{1-c} \leq \left(\frac{c}{a}\right)^{a/(1-a)} \exp\left[\frac{c(1-a)^2}{(1-c)(c-a^2)} - 1\right]. \tag{4.166}$$

(2) If in (4.164) we choose the function $f(x) = \exp(x)$, $x \in \mathbb{R}$, then for any $a, c \in (0,R)$ we can state that

$$\exp(c-a) \leq \left(\frac{c}{a}\right)^{a} \exp\left[\exp\left(\frac{(c-a)^2}{c}\right) - 1\right]. \tag{4.167}$$

The following result is also established.

Theorem 99 (Ibrahim, Dragomir, Cerone and Darus [213]) *With the assumptions of Theorem 4.149 for the function f and if $x_k \in (0,R)$ and $p_k \geq 0$ with $\sum_{k=1}^{n} p_k = 1$, then we have the inequality*

$$\begin{aligned}
&\sum_{k=1}^{n} p_k \left(\frac{x_k}{\sum_{j=1}^{n} p_j x_j}\right)^{\frac{\sum_{j=1}^{n} p_j x_j f'\left(\sum_{j=1}^{n} p_j x_j\right)}{f\left(\sum_{j=1}^{n} p_j x_j\right)}} \\
&\leq \frac{\sum_{k=1}^{n} p_k f(x_k)}{f\left(\sum_{j=1}^{n} p_j x_j\right)} \leq \sum_{k=1}^{n} p_k \left(\frac{x_k}{\sum_{j=1}^{n} p_j x_j}\right)^{\frac{x_k f'(x_k)}{f(x_k)}}.
\end{aligned} \tag{4.168}$$

Proof. It is obvious by the inequality (4.156) on noticing that for $c = x_k$ and $a = \sum_{j=1}^{n} p_j x_j$. Thus, we have

$$\begin{aligned}
&\left(\frac{x_k}{\sum_{j=1}^{n} p_j x_j}\right)^{\frac{\sum_{j=1}^{n} p_j x_j f'\left(\sum_{j=1}^{n} p_j x_j\right)}{f\left(\sum_{j=1}^{n} p_j x_j\right)}} \\
&\leq \frac{f(x_k)}{f\left(\sum_{j=1}^{n} p_j x_j\right)} \leq \left(\frac{x_k}{\sum_{j=1}^{n} p_j x_j}\right)^{\frac{x_k f'(x_k)}{f(x_k)}},
\end{aligned} \tag{4.169}$$

for all $k \in \{1, \ldots, n\}$. On multiplying the inequality (4.169) by $p_k \geq 0$ and summing over k from 1 to n, we get the desired result. ■

A more natural result of the Jensen type inequality is incorporated in the following theorem.

Theorem 100 (Ibrahim, Dragomir, Cerone and Darus [213]) *With the assumptions of Theorem 4.168, we have*

$$1 \leq \frac{G_p(f(x))}{f(G_p(x))} \leq \frac{G_p\left(x^{\frac{xf'(x)}{f(x)}}\right)}{[G_p(x)]\, A_p\left(\frac{xf'(x)}{f(x)}\right)}, \tag{4.170}$$

where

$$A_p(h(x)) := \sum p_j h(x_j) \tag{4.171}$$

and

$$G_p(h(x)) := \prod_{j=1}^{n} h^{p_j}(x_j),\, h : (0, \infty) \to (0, \infty). \tag{4.172}$$

Proof. If we choose in (4.156) $c = x_j$, $a = G_p(x) = \prod_{j=1}^{n} x_j^{p_j}$, then we have

$$\left(\frac{x_j}{G_p(x)}\right)^{\frac{G_p(x) f'(G_p(x))}{f(G_p(x))}} \leq \frac{f(x_j)}{f(G_p(x))} \leq \left(\frac{x_j}{G_p(x)}\right)^{\frac{x_j f'(x_j)}{f(x_j)}}, \tag{4.173}$$

which by taking the power p_j gives us that

$$\left(\frac{x_j}{G_p(x)}\right)^{p_j A} \leq \left[\frac{f(x_j)}{f(G_p(x))}\right]^{p_j} \leq \left(\frac{x_j}{G_p(x)}\right)^{\frac{p_j x_j f'(x_j)}{f(x_j)}}, \tag{4.174}$$

with

$$A := \frac{G_p(x) f'(G_p(x))}{f(G_p(x))}. \tag{4.175}$$

Now, multiplying the inequality (4.174) over j from 1 to n, we get

$$1 \le \frac{\prod_{j=1}^{n} f^{p_j}(x_j)}{f(G_p(x))} \le \frac{\prod_{j=1}^{n} x_j^{\frac{p_j x_j f'(x_j)}{f(x_j)}}}{[G_p(x)]^{\sum_{j=1}^{n} \frac{p_j x_j f'(x_j)}{f(x_j)}}}, \tag{4.176}$$

and the proof is completed. ■

For some other results related to the Jensen's type inequalities and their applications, see for instance [27], [57], [129], [130], [131], [132], [133], [134], [139], [273], [303], [351], [421] and the references which are cited therein.

Chapter 5

Applications to Special Functions

Special functions are another important type of functions that are defined by the power series. They arise in almost all areas of Modern Mathematics, with physics, engineering, chemistry, statistics and computer science as the most well-known application areas of these functions. In this chapter, we provide some inequalities involving the special functions, such as polylogarithm, hypergeometric, Bessel and modified Bessel functions of the first kind, that have been obtained on utilising the Buzano, Schwarz, Young, Hölder and Jensen type inequalities.

Section 5.1 briefly introduces some general facts and basic concepts of the special functions. We address some of their important relations to the fundamental functions. Some inequalities involving the polylogarithm, hypergeometric, Bessel and modified Bessel functions of the first kind are presented subsequently in Section 5.2, Section 5.3 and Section 5.4.

The author's research papers in collaboration with Dragomir, Darus and Cerone (see [208], [209], [213], [214], [215], [216]) are the main contributions to this chapter. The survey research paper published by the author and Dragomir [217], is also incorporated with some of the results.

5.1 Definitions and Basic Concepts

Before we state our results for the polylogarithm, hypergeometric, Bessel and modified Bessel functions, we recall in this section, the definitions and some basic concepts of these functions. The details about these types of special functions can be found in the work of (see for instance [4], [25], [102], [296], [342, Chapt. 60]), and numerous classical and modern texts devoted to the special functions.

We start by introducing the classical gamma, zeta and their related functions. The gamma function is useful in defining some other types of special functions.

5.1.1 Gamma, Zeta and Related Functions

The *gamma function,* denoted by $\Gamma(z)$, is a function defined by the *Euler's integral* [102, p. 221], i.e.,

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \tag{5.1}$$

for any $z \in \mathbb{C}$ such that $\operatorname{Re}(z) > 0$. It is clear that from (5.1) we have

$$\Gamma(1) = 1, \tag{5.2}$$

and it also satisfies the following recurrent formulas:

$$\text{(i)}\quad \Gamma(1+z) = z\Gamma(z), \tag{5.3}$$

$$\text{(ii)}\quad \Gamma(1-z) = -z\Gamma(-z), \tag{5.4}$$

for all $z \in \mathbb{C}$ with $\operatorname{Re}(z) > 1$. The gamma function (5.1) reduces to the factorial function when its argument is a positive real integer, i.e.,

$$\Gamma(n) = (n-1)! \tag{5.5}$$

for $n \in \mathbb{N}$. For positive half-integer arguments, the gamma function has the special form as follows:

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n-1)!!\sqrt{\pi}}{2^n}, \tag{5.6}$$

where $n!!$ denotes the *double factorial function*[1] of integer n, $n \geq -1$. It can be seen that the relations in (5.4), (5.5) and (5.6), define the gamma functions in whole real numbers.

[1]For integers $k \geq -1$, the double factorial $k!!$ are defined by $(-1)!! = 0!! = 1$, $(2k)!! = \prod_{j=1}^{k} 2j$, $(2k+1)!! = \prod_{j=0}^{k} (2j+1)$ and it is undefined for $k \leq -2$.

Some common values of the gamma function are given in the following [4, Chapt. 6]:

$$\begin{aligned}
&\Gamma\left(-\frac{3}{2}\right)=\frac{4\sqrt{\pi}}{3}, && \Gamma\left(-\frac{1}{2}\right)=-2\sqrt{\pi},\\
&\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}, && \Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{2},\\
&\Gamma(2)=1, && \Gamma\left(\frac{5}{2}\right)=\frac{3\sqrt{\pi}}{4},\\
&\Gamma(3)=2, && \Gamma(4)=6.
\end{aligned} \tag{5.7}$$

The derivative of the gamma function is

$$\Gamma'(z)=\int_0^\infty e^{-t}t^{z-1}\log(t)\,dt, \tag{5.8}$$

and the n-th derivative is given by

$$\Gamma^{(n)}(z)=\int_0^\infty e^{-t}t^{z-1}\log^n(t)\,dt. \tag{5.9}$$

These can be derived by differentiating the integral form of the gamma function (5.1) with respect to z, and using the technique of differentiation under the integral sign. Closely related to the derivative of the gamma function is the *digamma* or *psi function* $\Psi(z)$, which is defined by

$$\Psi(z)=\frac{d}{dz}[\ln\Gamma(z)]=\frac{\Gamma'(z)}{\Gamma(z)},\ z\neq 0,1,2,\dots\ . \tag{5.10}$$

Further differentiations of (5.8) lead to the family of the *polygamma functions* $\Psi_n(z)$,

$$\Psi_n(z)=\Gamma^{(n+1)}[\log\Gamma(z)], \tag{5.11}$$

where $\Psi_0(z)=\Psi(z)$. The *incomplete gamma function* $\gamma(\alpha,z)$ and the *complementary incomplete gamma function* $\Gamma(\alpha,z)$, which are defined by

$$\gamma(\alpha,z)=\int_0^z t^{\alpha-1}e^{-t}dt,\quad \mathrm{Re}(\alpha)>0 \tag{5.12}$$

and

$$\Gamma(\alpha, z) = \int_{z}^{\infty} t^{\alpha-1} e^{-t} dt, \quad \mathrm{Re}(\alpha) > 0 \tag{5.13}$$

respectively, have a relation to the gamma function as follows:

$$\Gamma(\alpha) = \gamma(\alpha, z) + \Gamma(\alpha, z). \tag{5.14}$$

Although, the gamma function does not have a power series representation, the incomplete gamma functions can be represented by the following power series:

$$\gamma(\alpha, z) = z^{\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+\alpha)} \frac{z^n}{n!}. \tag{5.15}$$

Hence,

$$\Gamma(\alpha, z) = \Gamma(\alpha) - z^{\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+\alpha)} \frac{z^n}{n!}, \tag{5.16}$$

for all α, $z \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$.

Another important function related to the gamma function is the *beta function* $B(x, y)$, where

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{5.17}$$

for all x, $y \in \mathbb{R}$ such that x, $y > 0$. The beta function is a real function of two variables, and it is defined by the formula

$$B(x, y) = \int_{0}^{1} e^{x-1} (1-t)^{y-1} dt, \tag{5.18}$$

for x, $y > 0$.

The gamma function (5.5) with the natural extension of the factorial of an integer was first introduced by Euler in 1930 (see [111]). It is of fundamental importance to many areas of science including Probability Theory, Mathematical Physics, Number Theory and Special Functions Theory. For more on its properties and some historical remarks, we may refer to [25, Chapt. 1], [34], [102, Chapt. 12], [167], [296, Chapt. 2], [395] and the references which are cited therein.

The *Riemann zeta function* or *Euler-Riemann zeta function* $\zeta(z)$, is a function defined by [187, p. 60]

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t - 1} dt, \tag{5.19}$$

for $z \in \mathbb{C}$ with $\operatorname{Re}(z) > 1$, where $\Gamma(z)$ is the gamma function defined by (5.1). Another useful definition of the Riemann zeta function is given by the infinite series [4, p. 807], namely

$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}, \tag{5.20}$$

for any $z \in \mathbb{C}$ such that $\operatorname{Re}(z) > 1$. For even and negative integer arguments, the Riemann zeta function can be expressed in terms of the Bernoulli numbers, that is,

$$\zeta(2k) = (-1)^k \frac{|B_{2k}| (2\pi)^{2k}}{2(2k)!} \quad \text{and} \quad \zeta(-k) = -\frac{B_{k+1}}{k+1} \tag{5.21}$$

for $k \in \mathbb{N}$.

Some common values of the Riemann zeta function are as follows:

$$\begin{aligned}
&\zeta(-3) = \frac{1}{120}, && \zeta(-2) = 0, && \zeta(-1) = -\frac{1}{12}, \\
&\zeta(0) = -\frac{1}{2}, && \zeta(1) = \infty && \zeta(2) = \frac{\pi^2}{6} \\
&\zeta(3) = Z, && \zeta(4) = \frac{\pi^4}{90}, && \zeta(\infty) = 1,
\end{aligned} \tag{5.22}$$

where Z is known as the *Apéry's constant*[2] (cf. Apéry [30]).

The *Hurwitz zeta function* $\zeta(s, q)$ is the generalization of the Riemann zeta function, that is,

$$\zeta(s, q) = \sum_{k=0}^{\infty} \frac{1}{(k+q)^s}, \tag{5.23}$$

which coincides with the Riemann zeta function (5.20) when $q = 1$ (see [4, p. 807], [149, p. 24-27]). Some useful functions related to the Riemann zeta function are the *Dirichlet beta* $\beta(s)$, *Dirichlet eta* $\eta(s)$ and *lambda* $\lambda(s)$ functions, where

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[2] The Apéry's constant $Z := \sum_{k=0}^{\infty} \frac{1}{k^3} = 1.20205690\ldots$. It appears in a number of physical problems, for instance, in the electron's gyromagnetic ratio, computed using quantum electrodynamics, Debye model, Stefen-Boltzman law, etc.

$$\frac{\zeta(s)}{2^s} = \frac{\eta(s)}{2^s - 1} = \frac{\lambda(s)}{2^s - 2}, \qquad 2\lambda(s) = \zeta(s) + \eta(s), \tag{5.24}$$

and

$$\beta(s) = 2^{-2s}\left[\zeta\left(s, \frac{1}{4}\right) - \zeta\left(s, \frac{3}{4}\right)\right], \tag{5.25}$$

in which $\beta(s)$, $\eta(s)$ and $\lambda(s)$ are defined by the following series:

$$\beta(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^s}, \ \eta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s}, \ \lambda(s) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^s}, \tag{5.26}$$

respectively, for $\mathrm{Re}(s) > 0$. In the following, we give some special values of the $\beta(s)$, $\eta(s)$ and $\lambda(s)$ functions.

$$\begin{aligned}
&\beta(0) = \frac{1}{2}, && \beta(1) = \frac{\pi}{4}, && \beta(2) = G, \\
&\beta(3) = \frac{\pi^3}{32}, && \eta(-1) = \frac{1}{4}, && \eta(0) = \frac{1}{2}, \\
&\eta(1) = \ln 2, && \eta(2) = \frac{\pi^2}{12}, && \eta(3) = \frac{3}{4}Z, \\
&\eta(4) = \frac{7\pi^4}{720}, && \lambda(2) = \frac{\pi^2}{8}, && \lambda(4) = \frac{\pi^4}{96},
\end{aligned} \tag{5.27}$$

where Z and G, respectively represent the Apèry's constant and *Catalan's constant*[3] (see [59, p. 153]).

Similarly, the Riemann zeta function is also one of the most important classical functions in analysis. It is widely used in various areas such as asymptotic series, definite integration, Number Theory, hypergeometric series, etc., and with applications even in physics, Probability Theory and Applied Statistic. Further reading on this subject can be found in the various textbooks devoted to the Theory of Special Functions, see for instance [4, p. 807-808], [146], [218], [224, Chapt. 9], [356] and the references which are cited therein.

[3] $G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.915965594\ldots$.

5.1.2 Polylogarithm Functions

The classical s-th *polylogarithm function*, or for short, the *polylogarithms* $Li_s(z)$ is a function defined by the power series (see [281, p. 239])

$$Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}. \tag{5.28}$$

This series (5.28) converges absolutely for all complex values of the order s and the argument z, where $\mathrm{Re}(s) > 0$ and $z \in D(0,1)$. The polylogarithm function, which is also known in the literature as *Jonquiére's function* (cf. Jonquiere [225]), appears in many different branches of mathematics such as combinatorics, Algebraic Number Theory and Knot Theory (see [1], [71], [272]). It also plays an important role in physics and engineering including Quantum Field Theory, dynamical system, Feynman integrals, thermodynamic, network analysis, etc., (see [48], [154], [311], [443, p. 497-512]).

The alternative definition of the polylogarithm function is given by the following integral representation [364, p. 762]:

$$Li_s(z) = \frac{z}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - z} dt, \tag{5.29}$$

for all s, $z \in \mathbb{C}$ such that $\mathrm{Re}(s) > 0$ and $|z| < 1$, where $\Gamma(s)$ is the gamma function defined by (5.1).

The polylogarithm functions (5.28) reduce to the elementary functions for the orders $s = 0$ and 1, that is,

$$Li_0(z) = \frac{z}{1-z} \tag{5.30}$$

and

$$Li_1(z) = \ln\left(\frac{1}{1-z}\right), \tag{5.31}$$

respectively, for $z \in D(0,1)$, while for $n = 2$, we have

$$Li_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} = -\int_0^z \frac{\ln(1-t)}{t} dt, \quad |z| < 1, \tag{5.32}$$

which is called the *dilogarithm* or *Spence's function* (cf. Spence [405, p. 24-36]). The function (5.32) was first introduced by Leibniz (cf. Leibniz [280, p. 336-339]) in 1696, and it has bee extensively studied in its own right, see for instance [71, p. 103-104], [142], [196], [202], [267], [281, Chapt. 1], [306] and [442]. For the cases of the integer orders $s = 3, 4, \ldots$, they are known, respectively as the *trilogarithm*, *quadrilogarithm*, and so on. However, for the higher of integer orders $s \geq 2$, the functions $Li_s(z)$ cannot be expressed in terms of the simple elementary functions.

The polylogarithm functions can also be defined through successive steps of their integration by

$$Li_s(z) = \int_0^z \frac{Li_{s-1}(t)}{t} dt, \tag{5.33}$$

for $s = 1, 2, 3 \ldots$, with $Li_0(z)$ is the well-known function as defined in (5.30). Then, the differential relation follows from (5.33) that

$$\frac{d\left[Li_s(z)\right]}{ds} = \frac{Li_{s-1}(z)}{z} \tag{5.34}$$

or, in general,

$$\frac{z^m d^{(m)}\left[Li_s(z)\right]}{ds^m} = Li_{s-m}(z) \tag{5.35}$$

for $m = 1, 2, \ldots$ and $d^{(m)}$ is denoted as the m-th of the derivative. Thus, by applying the recurrence formulas given by (5.34) and the equation (5.30) for $s = \ldots, -3, -2, -1, 0$, we have the following relations [277]:

$$\begin{aligned} Li_{-1}(z) &= \frac{z}{(1-z)^2}, & Li_{-2}(z) &= \frac{z(z+1)}{(1-z)^3}, \\ Li_{-3}(z) &= \frac{z(1+4z+z^2)}{(1-z)^4}, & Li_{-4}(z) &= \frac{z(1+z)(1+10z+z^2)}{(1-z)^5} \end{aligned} \tag{5.36}$$

and so on, for all $z \in \mathbb{C}$ such that $|z| < 1$. It is clearly seen that for all nonnegative integer order n, $Li_{-n}(z)$ is a rational function in z, whose the denominator is $(1-z)^{n+1}$. Hence, for any desired function of the lower orders $n = 1, 2, 3, \ldots$ and $|z| < 1$, we have the following expression [99, p. 245]:

$$Li_{-n}(z) = \frac{1}{(1-z)^{n+1}} \sum_{k=0}^{n} E_{n,k} z^{n-k}, \tag{5.37}$$

where the coefficients $E_{n,k}$, $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ are *Eulerian number*[4], namely, we recall that these numbers can be obtained by the following series (see [85], [99, p. 243]):

$$E_{n,k} = \sum_{j=0}^{k+1} (-1)^j \binom{n+1}{j} (k+1-j)^n \tag{5.38}$$

for integers $n \geq 1$. Polylogarithms also arise in the sum of *generalized harmonic numbers* $H_{n,r}$ as

$$\sum_{n=1}^{\infty} H_{n,r} z^n = \frac{Li_r(z)}{1-z}, \quad |z| < 1 \tag{5.39}$$

for $z \in D(0,1)$, where

$$H_{n,r} := \sum_{k=1}^{\infty} \frac{1}{k^r} \qquad \text{and} \qquad H_n := H_{n,1} = \sum_{k=1}^{\infty} \frac{1}{k}. \tag{5.40}$$

There are some special cases of polylogarithms, with first remarkable values (see [146, p. 11], [443, p. 497-512])

$$Li_s(1) = \zeta(s), \tag{5.41}$$

for $\mathrm{Re}(s) > 1$ and

$$Li_s(-1) = \left(2^{1-s} - 1\right) \zeta(s), \tag{5.42}$$

for $s \neq 1$, where $\zeta(s)$ is the Riemann zeta function defined by (5.20). Clearly, from the equation (5.41), the polylogarithm function (5.28) can be viewed as a generalization of the Riemann zeta function as well. Instead of (5.41), the polylogarithm also has relationships to other functions for some special cases of argument z. For instance:

$$Li_n(-1) = -\eta(n), \tag{5.43}$$

$$Li_n(\pm i) = \frac{1}{2^s}\eta(n) \pm i\beta(n), \tag{5.44}$$

where $\beta(n)$ and $\eta(n)$ are the Dirichlet beta and Dirichlet eta functions, which

[4]Some particular cases of Eulerian numbers are $E_{1,0} = E_{2,0} = E_{3,0} = E_{4,0} = 1$, $E_{2,1} = 1, E_{3,1} = 4$, $E_{3,2} = 1, E_{4,1} = 11$, etc. Eulerian numbers can also be obtained by the recurrence equation: $E_{m,k} = (m+1-k)\,E_{m-1,k-1} + kE_{m-1,k}$.

were introduced in the previous section in (5.26).

There are some numerical values of $Li_2(z)$ and $Li_3(z)$ for the special arguments z, for instance [282, Chapt. 1, 13],

$$Li_1\left(\frac{1}{2}\right) = \ln 2, \qquad Li_2(-1) = -\frac{\pi^2}{12}, \qquad Li_2(0) = 0,$$

$$Li_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\ln^2(2), \qquad Li_2(1) = \frac{\pi^2}{6}, \qquad Li_2(2) = \frac{\pi^2}{4},$$

$$Li_2(i) = -\frac{\pi^2}{48} + iG, \qquad Li_2\left(\tfrac{\sqrt{5}-1}{2}\right) = \frac{\pi^2}{10} - \frac{1}{2}\ln^2\left(\tfrac{\sqrt{5}+1}{2}\right),$$

$$Li_3(i) = -\frac{3}{48}Z + i\pi^3, \qquad Li_3\left(\frac{1}{2}\right) = \frac{1}{24}\left[4\ln^3(2) - 2\pi^2\ln(2) + 21Z\right],$$

where Z and G are the Apéry and Catalan constant, respectively. For similar equations containing the sum of dilogarithms of other arguments, see [249, p. 69-74], and see also [143], [179], [281, Chapt. 7], [363, p. 762-763] for some other formulae involving $Li_s(z)$.

Next, we shall introduce another important class of the special functions called hypergeometric functions.

5.1.3 Hypergeometric Functions

There are varieties of the hypergeometric-type functions ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$, where p and q are nonnegative integers. However, the most important are the functions of the form ${}_2F_1(a, b; c; z)$ and ${}_1F_1(a; c; z)$, which are called the *Gauss hypergeometric function* and the *confluent hypergeometric function* of the first kind. In this section, we shall discuss some basic properties of these types of hypergeometric functions. For the details on this topic, we may refer to the work of Erdélyi et al. [149], and some other well-known textbooks such as Abramowitz and Stegun [4, p. 556-565], Andrews [26, p. 85-101] and Oldham et al. [342]. See also [25], [46], [186], [368] for some historical overview of hypergeometric functions.

First, we give the definition of the hypergeometric series with any positive integer of parameters p and q, which is called the *generalized hypergeometric series* ${}_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;z\right)$. This series is defined by (see [368, p. 73])

$$ {}_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;z\right)=\sum_{n=0}^{\infty}\frac{\left(a_1\right)_n\cdots\left(a_p\right)_n}{\left(b_1\right)_n\cdots\left(b_q\right)_n}\frac{z^n}{n!} \tag{5.45}$$

for arbitrary $z\in\mathbb{C}$, a_j, $b_j\in\mathbb{R}\backslash\mathbb{Z}_0^-$, $1\le i\le p$, $1\le j\le q$, and $(t)_n$, $n\in\mathbb{N}_0$ is the Pochhammer symbol, which is defined by (2.4). The subscripts p and q indicate the number of parameters in numerator and denominator of the coefficients in its power series expansion. The series (5.45) converges absolutely for all $z\in\mathbb{C}$ when $p<q+1$, and it converges only for $z=0$ when $p>q+1$. If $p=q+1$, then the series (5.45) converges absolutely for $|z|<1$, and it diverges for $|z|>1$, while, for $z=1$, it converges absolutely when the $\mathrm{Re}\left(\sum_{k=1}^{q}b_k-\sum_{k=1}^{p}a_k\right)>0$. If the hypergeometric series converges, then it corresponds to the hypergeometric function.

The *Gauss hypergeometric function* or simply *hypergeometric function*, is the special case of the series (5.45) in which $p=2$ and $q=1$, that is,

$$ {}_2F_1\left(a,b;c;z\right)=\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n}\frac{z^n}{n!} \tag{5.46}$$

for any $a,b,c\in\mathbb{R}$, $z\in\mathbb{C}$ such that $|z|<1$ and $c\notin\{0,-1,-2,\ldots\}$, $n\in\mathbb{N}$ (see [53], [162]). This series arises in many areas of mathematics, physics and engineering with several notations, for instance $F\left(a,b;c;z\right)$, $F\left({a,\ b \atop c}\mid z\right)$, ${}_2F_1\left({a,\ b \atop c}\mid z\right)$ and $\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\left[z\frac{a-1,b-1}{0,c-1}\right]$.

In general, the hypergeometric function defined by (5.46), is the solution of the *Euler's hypergeometric differential equation*, which is

$$ z\left(1-z\right)w''\left(z\right)+\left[c-\left(a+b+1\right)z\right]w'\left(z\right)-abw=0 \tag{5.47}$$

with the initial conditions $w\left(0\right)=1$ and $w'\left(0\right)=ab/c$.

It is obvious that ${}_2F_1\left(a,b;c;z\right)={}_2F_1\left(b,a;c;z\right)$. That is, the hypergeometric function (5.46) is symmetrical with respect to the interchange of its two parameter of the numerators a and b. The function (5.46) is also an analytic and *univalent* in the unit disk D. For the fixed b, c and z, it is an entire function of a, while, if a, c and z are fixed, then it is an entire function of b. If a or b are zero or negative integers, then the series in (5.46) terminates after a finite number of its terms, thus the power series reduces to a polynomial in z. For example, if $a = -m$, $m \in \{0, 1, 2, \ldots\}$ and $c \notin \{\ldots, -2, -1, 0\}$, then the functions

$$ {}_2F_1\left(-m, b; c; z\right) = \sum_{n=0}^{m} \frac{(-m)_n\,(b)_n}{(c)_n}\frac{z^n}{n!} = \sum_{n=0}^{\infty}(-1)\binom{m}{n}\frac{(b)_n}{(c)_n}z^n \tag{5.48}$$

are the polynomials in z.

Some values of (5.46) for the special arguments z are given as follows:

$$\begin{aligned} &{}_2F_1\left(a, b; c; 0\right) = 1, \\ &{}_2F_1\left(a, b; c; -\infty\right) = 1, \quad \text{for } a \text{ or } b = 0; \\ &{}_2F_1\left(a, b; c; -\infty\right) = 0, \quad \text{for } a\ < 0 \text{ and } b < 0; \\ &{}_2F_1\left(a, b; c; -\infty\right) = \infty, \quad \text{for } a \text{ or } b > 0. \end{aligned} \tag{5.49}$$

Many fundamental functions in a real or complex variable can be expressed in terms of the hypergeometric function as well, and some of their typical examples are given in the following [4, p. 556]:

$$\begin{aligned} (1+z)^p &= {}_2F_1\left(-p, 1; 1; -z\right), p \notin \mathbb{Z}_0^- \ ; \\ (1-z)^{-1} &= {}_2F_1\left(1, p; p; z\right); \\ (1-z)^{-2} &= {}_2F_1\left(1, 2; 1; z\right); \\ \frac{1}{(1-z)^a} &= {}_2F_1\left(a, b; b; z\right),\ a \in \mathbb{Z}^+; \\ z\ln\left(\frac{1}{1-z}\right) &= {}_2F_1\left(1, 1; 2; z\right), \\ \frac{1}{z}\ln\left(1+z\right) &= {}_2F_1\left(1, 1; 2; -z\right), \\ \frac{1}{2z}\ln\left(\frac{1+z}{1-z}\right) &= {}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; z^2\right), \\ e^z &= \lim_{b\to\infty} {}_2F_1\left(1, b; 1; \frac{z}{b}\right), \\ \cos\left(z\right) &= {}_2F_1\left(\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}; \sin^2\left(z\right)\right), \end{aligned} \tag{5.50}$$

$$\begin{aligned}
\frac{1}{z}\arcsin(z) &= {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^2\right), \\
\arccos(z) &= {}_2F_1\left(\frac{1}{2},1;\frac{3}{2};-z^2\right), \qquad (5.51) \\
\frac{1}{z}\arctan(z) &= {}_2F_1\left(\frac{1}{2},1;\frac{3}{2};-z^2\right),
\end{aligned}$$

The derivative of the hypergeometric function (5.46) is given by

$$\frac{d}{dz}\left[{}_2F_1(a,b;c;z)\right] = \frac{ab}{c}\,{}_2F_1(a+1,b+1;c+1;z) \qquad (5.52)$$

or

$$\frac{d^k}{dz^k}\left[{}_2F_1(a,b;c;z)\right] = \frac{(a)_k(b)_k}{(c)_k}\,{}_2F_1(a+k,b+k;c+k;z), \qquad (5.53)$$

for all $k \in \{1,2,\ldots,n\}$, $a,b,c,z \in \mathbb{C}$ such as $|z|<1$, $c \notin \mathbb{Z}_0^-$, while the integration is defined by

$$\int_0^z \left[{}_2F_1(a,b;c;t)\right]dt = \frac{c-1}{(a-1)(b-1)}\left[{}_2F_1(a-1,b-1;c-1;z)-1\right], \qquad (5.54)$$

for all $a,b,c,z \in \mathbb{C}$ with $|z|<1$, $c \notin \mathbb{Z}_0^-$ and $\mathrm{Re}(a)$, $\mathrm{Re}(b)$, $\mathrm{Re}(c) \neq 1$.

Like the polylogarithms, the hypergeometric functions (5.46) can also be represented in terms of integrals by

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt, \qquad (5.55)$$

where $z \in \mathbb{C}\backslash[1,\infty)$, $a \in \mathbb{C}, b, c-b \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $0 < \mathrm{Re}(b) < \mathrm{Re}(c)$, as proved by Euler in 1748 (see [46, p. 4-5]). Other integral representations and similar formulas defining analytic functions can be found in (see for instance [25, p. 65], [149, p.], [268]). If the argument $z=1$ and the $\mathrm{Re}(c-a-b)>0$, then we have from (5.55) that

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-a)}. \qquad (5.56)$$

The functions of the form ${}_1F_1(a;b;z)$ are called the *confluent hypergeometric function of the first kind,* which is corresponding to $p=q=1$, i.e.,

$$
{}_1F_1(a;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}, \tag{5.57}
$$

where $a, z \in \mathbb{C}$, $b \in \mathbb{C}\backslash\mathbb{Z}_0^-$. It can be derived from the Gauss hypergeometric function (5.46) by taking the limit as follows:

$$
\lim_{a\to\infty} {}_2F_1\left(a, b; c; \frac{z}{a}\right) = {}_1F_1(b; c; z). \tag{5.58}
$$

This function ${}_1F_1(a;b;z)$ or $F\left(\begin{smallmatrix} a \\ b \end{smallmatrix} \mid z\right)$ is sometimes written as $M(a,b,z)$ or $\Phi(a;b;z)$ with its corresponding second-order differential equation called the *confluent hypergeometric differential equation* or *Kummer's differential equation* [4, p. 504], namely,

$$
z\frac{d^2w}{dz^2} + (b-z)\frac{dw}{dz} - aw = 0, \tag{5.59}
$$

for any $a \in \mathbb{C}$, $b \in \mathbb{C}\backslash Z_0^-$ with $w(0) = 0$ and $w'(0) = a/b$. The derivative of the function (5.57) is given by [4, p. 507]

$$
\frac{d}{dz}\, {}_1F_1(a;b;z) = \frac{a}{b}\, {}_1F_1(a+1; b+1; z). \tag{5.60}
$$

There are many elementary functions connects to (5.57), for example,

$$
{}_1F_1(a;a;z) = e^z, \qquad {}_1F_1(a;2;2z) = \tfrac{e^z}{z} \sinh(z). \tag{5.61}
$$

The *Kummer's function* $U(a,b,z)$ or $\Psi(a;b;z)$ is also related to the hypergeometric functions of type ${}_2F_0(a, b; \ ; z)$, where

$$
{}_2F_0\left(a, 1+a-b; \ ; -\frac{1}{z}\right) = z^a U(a,b,z). \tag{5.62}
$$

Many relations of more complicated hypergeometric functions can be found in (see for example [4, Chapt. 15], [149, p. 686-706], [298, Chapt. II]).

5.1.4 Bessel and Modified Bessel Functions

The *Bessel functions of the first kind* $J_\alpha(z)$, are defined as the solutions to the *Bessel's differential equation*, i.e.,

$$z^2\frac{d^2y}{dz^2}+z\frac{dy}{dz}+\left(z^2-\alpha^2\right)y=0,\ \ \mathrm{Re}\left(\alpha\right)\geq 0, \tag{5.63}$$

for an arbitrary real or complex order α. The solution of (5.63) is an analytic function of z in $\mathbb{C}$, except for a point $z=0$ when α is not an integer. The solutions, denoted by $J_\alpha\left(z\right)$ are defined by Taylor series expansion around the origin (see [4, p. 360], [26, p. 57]), i.e.,

$$J_\alpha\left(z\right)=\sum_{k=0}^{\infty}\frac{\left(-1\right)^k}{\Gamma\left(k+\alpha+1\right)k!}\left(\frac{z}{2}\right)^{2k+\alpha}, \tag{5.64}$$

for any α, $z\in\mathbb{C}$ with $\mathrm{Re}\left(\alpha\right)\geq 0$ and $\Gamma\left(z\right)$ is the gamma function.

The most common of the Bessel functions that occur in practice are those of the integer orders $\alpha=n\in\mathbb{N}$. For such functions, we have

$$J_n\left(z\right)=\sum_{k=0}^{\infty}\frac{\left(-1\right)^k}{(k+n)!k!}\left(\frac{z}{2}\right)^{2k+n}. \tag{5.65}$$

For a non-integer order α, $J_\alpha\left(z\right)$ and $J_{-\alpha}\left(z\right)$ are linearly independent, and therefore the two solutions of the differential equation (5.63). On the other hand, for an integer order α, the $J_\alpha\left(z\right)$ and $J_{-\alpha}\left(z\right)$ are linearly dependent solutions of (5.63), hence the following relationship is valid [4, p. 358]:

$$J_{-\alpha}\left(z\right)=\left(-1\right)^\alpha J_\alpha\left(z\right). \tag{5.66}$$

The Bessel functions satisfy a large number of identities and integral relations, some of which are provided below [26, p. 58]:

$$J_0\left(0\right) = 1, \tag{5.67}$$

$$J_\alpha\left(0\right) = 0,\quad \alpha>0, \tag{5.68}$$

$$\frac{2\alpha}{x}J_\alpha\left(x\right) = J_{\alpha-1}\left(x\right)+J_{\alpha+1}\left(x\right), \tag{5.69}$$

$$2J_\alpha^{'}\left(x\right) = J_{\alpha-1}\left(x\right)-J_{\alpha+1}\left(x\right), \tag{5.70}$$

$$J_0\left(x\right) = \frac{1}{2\pi}\int_0^{2\pi}e^{\pm ix\cos\theta}d\theta, \tag{5.71}$$

$$J_n\left(x\right) = \frac{1}{\pi}\int_0^{2\pi}\cos\left(n\theta-x\sin\theta\right)d\theta,\ \ n\in\mathbb{N}_0, \tag{5.72}$$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin(x), \quad x > 0, \tag{5.73}$$

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos(x), \quad x > 0. \tag{5.74}$$

If the argument z in (5.63) is replaced by $\pm iz$, then the solutions of the second order differential equations are called the *modified Bessel functions of the first kind.* We denoted these functions by $I_{\alpha}(z)$. It is easy to verify from (5.64) that the modified Bessel functions $I_{\alpha}(z)$ are defined by the following power series [4, p. 375]:

$$I_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+\alpha+1)\, k!} \left(\frac{z}{2}\right)^{2k+\alpha}, \tag{5.75}$$

for all $\alpha, z \in \mathbb{C}$ such as $\mathrm{Re}(\alpha) \geq 0$. We observe that the function $I_{\alpha}(z)$ has all the nonnegative coefficients. If $\alpha = n \in \mathbb{N}_0$, then we have

$$I_n(z) = \sum_{k=0}^{\infty} \frac{1}{(k+n)!k!} \left(\frac{z}{2}\right)^{2k+\alpha}.$$

Similar to the Bessel functions, the modified Bessel functions of the first kind (5.75) also satisfy the following relations,

$$I_{\alpha}(-z) = (-1)^{\alpha} I_{\alpha}(z) \quad \text{and} \quad I_{-\alpha}(z) = I_{\alpha}(z) \tag{5.76}$$

holding for all $\alpha \in \mathbb{Z}$, $z \in \mathbb{C}$ such as $|z| < 1$. The modified Bessel functions of the first kind of order α, $I_{\alpha}(z)$, can be expressed by Bessel functions of the first kind as follows:

$$J_{\alpha}(iz) = i^{\alpha} I_{\alpha}(z). \tag{5.77}$$

Most of the properties of the modified Bessel functions are analogous to the ordinary Bessel functions, some of which are provided below [26, p. 62]:

$$I_0(0) = 1, \quad I_{\alpha}(0) = 0, \quad \alpha > 0, \tag{5.78}$$

$$\frac{2\alpha}{x} I_{\alpha}(x) = I_{\alpha-1}(x) - I_{\alpha+1}(x), \tag{5.79}$$

$$2I_{\alpha}'(x) = I_{\alpha-1}(x) + I_{\alpha+1}(x), \tag{5.80}$$

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{\pm x\cos\theta}d\theta, \tag{5.81}$$

$$I_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cosh(x), \quad x>0, \tag{5.82}$$

$$I_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sinh(x), \quad x>0. \tag{5.83}$$

In the following section, we provide some results involving the special functions. We start by presenting some inequalities for the polylogarithm functions.

5.2 Inequalities for Polylogarithm Functions

It is clearly seen that the polylogarithm function (5.28) is the power series with nonnegative coefficients and convergent on the open disk $D(0,1)$, so that all the results established in Section 3.2 - Section 3.4 of Chapter 3, and Section 4.1.2, Section 4.1.3, Section 4.2.2, Section 4.2.3 of Chapter 4 hold true. Therefore, for the examples, and for the main purpose of this section, we derive some inequalities involving the polylogarithm functions by considering the inequalities (3.9), (3.56), (3.95), (3.118), (4.29), (4.70) and (4.156).

First, we give the following result proved by Cerone and Dragomir [89].

Corollary 101 *If $Li_n(z)$ is the polylogarithm function, then we have*

$$|Li_n(az)|^2 \le \frac{1}{2}Li_n\left(a^2\right)\left[Li_n\left(|z|^2\right)+\left|Li_n(z^2)\right|\right], \tag{5.84}$$

for $a\in(-1,1)$, $z\in\mathbb{C}$ with z, az, a^2, z^2, $|z|^2\in D(0,1)$ and n is a negative or positive integer.

Utilising the inequalities (3.9) and (3.56) for the polylogarithm functions (5.28), we obtain the following results (see [208], [209], [217]).

Corollary 102 *If $Li_n(z)$ is the polylogarithm function, then we have*

$$\begin{aligned}&\left|Li_n\left(\alpha\overline{x}\right)Li_n\left(\overline{\beta}x\right)\right|\\&\leq\frac{1}{2}\left[\left[Li_n\left(|\alpha|^2\right)Li_n\left(|\beta|^2\right)\right]^{1/2}+\left|Li_n\left(\alpha\overline{\beta}\right)\right|\right]Li_n\left(|x|^2\right),\end{aligned}\tag{5.85}$$

for any $\alpha,\beta,x\in\mathbb{C}$ with $|\alpha|^2$, $|\beta|^2$, $|x|^2$, $\alpha\overline{\beta}$, $\alpha\overline{x}$, $\overline{\beta}x\in D(0,1)$ and n is a negative or positive integer.

Corollary 103 *If $Li_n(z)$ is the polylogarithm function, then we have*

$$\begin{aligned}&\left[Li_n\left(|x|^2\right)Li_n\left(|y|^2\right)\right]^{1/2}Li_n\left(|z|^2\right)-\left|Li_n\left(x\overline{z}\right)Li_n\left(z\overline{y}\right)\right|\\&\geq\left|Li_n\left(x\overline{y}\right)Li_n\left(|z|^2\right)-Li_n\left(x\overline{z}\right)Li_n\left(z\overline{y}\right)\right|,\end{aligned}\tag{5.86}$$

for any x, y, $z\in\mathbb{C}$ such that $|x|^2$, $|y|^2$, $|z|^2$, $x\overline{z}$, $z\overline{y}$, $x\overline{y}\in D(0,1)$ and n is a negative or a positive integer.

We also obtain the following results from (3.95) and (3.118), see [214], [217].

Corollary 104 *If $Li_n(z)$ is the polylogarithm function, then we have*

$$\begin{aligned}&Li_n\left(|x|\,|z|^2\right)Li_n\left(|x|\right)-\left|Li_n\left(|x|\,z\right)\right|^2\\&\geq\left|Li_n(x)Li_n\left(x\,|z|\,z\right)-Li_n(xz)Li_n(x\,|z|)\right|,\end{aligned}\tag{5.87}$$

for any x, $z\in\mathbb{C}$ such that x, xz, $|x|\,|z|^2\in D(0,1)$ and n is a negative or positive integer.

Corollary 105 *If $Li_n(z)$ is the polylogarithm function, then we have*

$$\begin{aligned}&Li_n\left(|x|^2\right)Li_n\left(|y|^2\right)-\left|Li_n\left(xy\right)\right|^2\\&\geq\left|Li_n(|x|\,x)Li_n\left(|y|\,\overline{y}\right)-Li_n(|y|\,x)Li_n(|x|\,\overline{y})\right|,\end{aligned}\tag{5.88}$$

for any x, $y\in\mathbb{C}$ with $|x|^2$, $|y|^2<R$ and n is a negative or positive integer.

As a consequence of the above results, we get the following inequalities, which incorporate the Riemann zeta function $\zeta(z)$, see ([214],

Corollary 106 *(i) If we choose $x = 1$ in (5.87), then we get the following inequality, which incorporates the zeta function, i.e.,*

$$\begin{aligned}
&\zeta(n)\,Li_n\left(|z|^2\right) - |Li_n(z)|^2 \\
&\geq |\zeta(n)\,Li_n(|z|\,z) - Li_n(z)\,Li_n(|z|)|\,,
\end{aligned} \tag{5.89}$$

for any $z \in D(0,1)$, where n is a positive or negative integer. In particular, if $n = 2$, then by the inequality (5.89) we get

$$\frac{\pi^2}{6}Li_2\left(|z|^2\right) - |Li_2(z)|^2 \geq \left|\frac{\pi^2}{6}Li_2(|z|\,z) - Li_2(z)\,Li_2(|z|)\right|, \tag{5.90}$$

for any $z \in D(0,1)$ and the $Li_2(z)$ is the dilogarithm or Spence's function which is defined in (5.32).

(ii) If we choose $x = i$ in (5.87), then we have

$$\begin{aligned}
&\zeta(n)Li_n\left(|z|^2\right) - |Li_n(z)|^2 \\
&\geq \left|\left(\frac{1}{2^n}\eta(n) + i\beta(n)\right)Li_n(i\,|z|\,z) - Li_n(iz)\,Li_n(i\,|z|)\right|,
\end{aligned} \tag{5.91}$$

for any $z \in D(0,1)$ and n is a positive or negative integer. In particular, for $n = 2$, we have from (5.91) that

$$\begin{aligned}
&\frac{\pi^2}{6}Li_2\left(|z|^2\right) - |Li_2(z)|^2 \\
&\geq \left|\left(\frac{\pi^2}{48} + iG\right)Li_2(i\,|z|\,z) - Li_2(iz)\,Li_2(i\,|z|)\right|,
\end{aligned} \tag{5.92}$$

for any $z \in D(0,1)$, where G is the Catalan's constant.

Corollary 107 *(i) If we choose $x = 1$ in (5.88), then we have*

$$\zeta(n)\,Li_n\left(|y|^2\right) - |Li_n(y)|^2 \geq |\zeta(n)\,Li_n(|y|\,\overline{y}) - Li_n(|y|)Li_n(\overline{y})| \tag{5.93}$$

for any $z \in D(0,1)$, where n is a positive or negative integer. In particular, for $n = 2$, we get from the inequality (5.93)

$$\frac{\pi^2}{6}Li_2\left(|y|^2\right) - |Li_2(y)|^2 \geq \left|\frac{\pi^2}{6}Li_2(|y|\,\overline{y}) - Li_2(|y|)Li_2(\overline{y})\right| \tag{5.94}$$

for any $z \in D(0,1)$.

(ii) If we choose $x = i$ *in (5.88), then we have*

$$\zeta(n) Li_n\left(|y|^2\right) - |Li_n(iy)|^2$$
$$\geq \left| \left(\frac{1}{2^n} \eta(n) + i\beta(n) \right) Li_n(|y|\,\overline{y}) - Li_n(i\,|y|) Li_n(\overline{y}) \right|, \tag{5.95}$$

for any $y \in D(0,1)$*, where* n *is a positive or negative integer. In particular, we get the following inequality by choosing* $n = 2$ *in (5.95),*

$$\frac{\pi^2}{6} Li_2\left(|y|^2\right) - |Li_2(iy)|^2$$
$$\geq \left| \left(\frac{\pi^2}{48} + iG \right) Li_2(|y|\,\overline{y}) - Li_2(i\,|y|) Li_2(\overline{y}) \right|, \tag{5.96}$$

for any $z \in D(0,1)$.

Similarly, if we apply the inequalities (4.29), (4.70) and (4.156) for the polylogarithm functions (5.28) for, then we get the following results (see [213], [215], [216], [217]).

Corollary 108 *Let* $Li_n(z)$ *be the polylogarithm function. Then, we have*

$$Li_n(|x|^p) Li_n(|y|^q) \geq \left| Li_n(xy) Li_n\left(|x|^{p-1} |y|^{q-1}\right) \right|, \tag{5.97}$$

for any x*,* $y \in \mathbb{C}$*,* $x, y \neq 0$ *with* xy*,* $|x|^p$*,* $|y|^q \in D(0,1)$ *and* p*,* $q > 1$*,* $1/p + 1/q = 1$*. In particular, if* $n = 0$ *in (5.97), then we get the following inequality:*

$$|1 - xy| \left|1 - |x|^{p-1} |y|^{q-1}\right| \geq (1 - |x|^p)(1 - |y|^q), \tag{5.98}$$

for any x*,* $y \neq 0$*,* xy*,* $|x|^p$*,* $|y|^q \in D(0,1)$ *and* p*,* $q > 1$ *with* $1/p + 1/q = 1$*. If we take* $n = 1$ *in (5.97), then we get the inequality (4.38) for all* x*,* $y \neq 0$ *with* xy*,* $|x|^p$*,* $|y|^q \in D(0,1)$ *and* p*,* $q > 1$*,* $1/p + 1/q = 1$*. Also, if we choose in (5.97)* $n = 2$*, then we obtain*

$$Li_2(|x|^p) Li_2(|y|^q) \geq \left| Li_2(xy) Li_2\left(|x|^{p-1} |y|^{q-1}\right) \right|, \tag{5.99}$$

for any x*,* $y \neq 0$*,* xy*,* $|x|^p$*,* $|y|^q \in D(0,1)$ *and* $p > 1$ *with* $1/p + 1/q = 1$*.*

Corollary 109 *If $Li_n(x)$ is the polylogarithm function, then we have*

$$Li_n\left(y^{\nu}z^{1-\nu}\right)Li_n\left(y^{1-\nu}z^{\nu}\right) \leq Li_n(y)Li_n(z), \tag{5.100}$$

for any y, z, $y^{\nu}z^{1-\nu}$, $y^{1-\nu}z^{\nu} \in (0,1)$, $\nu \in [0,1]$ and $n \in \mathbb{Z}$. In particular, if $n=0$ in (5.100), then we get the inequality (4.72). Also, if $n=1$, then we get from (5.100) the inequality (4.73). Further, we obtain the following inequality by choosing $n=2$ in (5.100):

$$Li_2\left(x^{\nu}y^{1-\nu}\right)Li_2\left(x^{1-\nu}y^{\nu}\right) \leq Li_2(x)Li_2(y), \tag{5.101}$$

for y, z, $y^{\nu}z^{1-\nu}$, $y^{1-\nu}z^{\nu} \in (0,1)$ and $\nu \in [0,1]$.

Corollary 110 *If $Li_n(x)$ is the polylogarithm function, then we have*

$$\left(\frac{c}{a}\right)^{\frac{Li_{n-1}(a)}{Li_n(a)}} \leq \frac{Li_n(c)}{Li_n(a)} \leq \left(\frac{c}{a}\right)^{\frac{Li_{n-1}(c)}{Li_n(c)}}, \tag{5.102}$$

for any a, $c \in (0,1)$ and n is positive or negative integer. In particular, if $n=0$ in (5.102), then we have the simpler inequality

$$\left(\frac{c}{a}\right)^{\frac{a}{1-a}} \leq \frac{1-a}{1-c} \leq \left(\frac{c}{a}\right)^{\frac{c}{1-c}}, \tag{5.103}$$

for any a, $c \in (0,1)$. Also, if we take $n=1$ in (5.102), then we get

$$\left(\frac{c}{a}\right)^{-\frac{a}{(1-a)\ln(1-a)}} \leq \frac{\ln(1-c)}{\ln(1-a)} \leq \left(\frac{c}{a}\right)^{-\frac{c}{(1-c)\ln(1-c)}}, \tag{5.104}$$

for any a, $c \in (0,1)$. Further, we obtain the following inequality by choosing $n=2$ in (5.102):

$$\left(\frac{c}{a}\right)^{-\frac{\ln(1-a)}{Li_2(a)}} \leq \frac{Li_2(c)}{Li_2(a)} \leq \left(\frac{c}{a}\right)^{-\frac{\ln(1-c)}{Li_2(c)}}, \tag{5.105}$$

for any a, $c \in (0,1)$.

Further, we establish some inequalities involving the polylogarithms for different order of integers p, q and r. In [89], Cerone and Dragomir proved the following result via the de Bruijn inequality.

Theorem 111 (Cerone and Dragomir [89]) *$Li_n(z)$ is the polylogarithm function, $a \in (-1,1)$, $z \in D(0,1)$ and p, q, r integers such that the following series exist. Then*

$$|Li_{r+p+q}(az)|^2 \le \frac{1}{2} Li_{r+2p}\left(a^2\right)\left[Li_{r+2q}\left(|z|^2\right) + \left|Li_{r+2q}(z^2)\right|\right]. \tag{5.106}$$

We obtain the similar inequality to (5.106), by utilizing the Buzano's result (3.6) in inner product spaces (see [208], [217]). The result is given as follows:

Theorem 112 (Ibrahim and Dragomir [208]) *Let α, β, $x \in \mathbb{C}$ with $\alpha\overline{x}$, $\overline{\beta}x$, $|\alpha|^2$, $|\beta|^2$, $\alpha\overline{\beta}$, $|x|^2 \in D(0,1)$ and p, q, r integers such that the following series exist. Then*

$$\begin{aligned} &\left|Li_{r+p+q}(\alpha\overline{x})\, Li_{r+p+q}\left(\overline{\beta}x\right)\right| \\ &\le \frac{1}{2}\left(\left[Li_{r+2q}\left(|\alpha|^2\right) Li_{r+2q}\left(|\beta|^2\right)\right]^{1/2} + \left|Li_{r+2q}\left(\alpha\overline{\beta}\right)\right|\right) Li_{r+2p}\left(|x|^2\right). \end{aligned} \tag{5.107}$$

Proof. Utilising the Buzano inequality (3.6) for $p_k = 1/k^r$, $c_k = \alpha^k/k^q$, $b_k = \beta^k/k^q$ and $x_k = x^k/k^p$, we have

$$\begin{aligned} &\left|Li_{r+p+q}(\alpha\overline{x})\, Li_{r+p+q}\left(\overline{\beta}x\right)\right| \\ &= \left|\sum_{k=1}^{\infty}\frac{1}{k^r}\frac{\alpha^k}{k^q}\frac{(\overline{x})^k}{k^p}\sum_{k=1}^{\infty}\frac{1}{k^r}\frac{\left(\overline{\beta}\right)^k}{k^q}\frac{(x)^k}{k^p}\right| \\ &\le \frac{1}{2}\left(\left[\sum_{k=1}^{\infty}\frac{1}{k^r}\frac{\left(|\alpha|^2\right)^k}{k^{2q}}\sum_{k=1}^{\infty}\frac{1}{k^r}\frac{\left(|\beta|^2\right)^k}{k^{2q}}\right]^{1/2}\right. \\ &\left. + \left|\sum_{k=1}^{\infty}\frac{1}{k^r}\frac{\left(\alpha\overline{\beta}\right)^k}{k^{2q}}\right|\right)\sum_{k=1}^{\infty}\frac{1}{k^r}\frac{\left(|x|^2\right)^k}{k^{2p}} \\ &= \frac{1}{2}\left(\left[Li_{r+2q}\left(|\alpha|^2\right) Li_{r+2q}\left(|\beta|^2\right)\right]^{1/2}\right. \\ &\left. + \left|Li_{r+2q}\left(\alpha\overline{\beta}\right)\right|\right) Li_{r+2p}\left(|x|^2\right). \end{aligned} \tag{5.108}$$

Thus, the desired inequality (5.107) is proved. ■

Also, on utilising the refinement of the Schwarz inequality (3.54) in inner product spaces, we obtain the following result (see [209], [217]).

Theorem 113 (Ibrahim and Dragomir [209]) *Let x, y, $z \in \mathbb{C}$ with $|x|^2$, $|y|^2$, $|z|^2$, $x\overline{z}$, $z\overline{y}$, $x\overline{y} \in D(0,1)$ and p, q, r integers such that the following series exist. Then*

$$\begin{aligned}
&\left[Li_{r+2q}\left(|x|^2\right) Li_{r+2q}\left(|y|^2\right)\right]^{1/2} Li_{r+2p}\left(|z|^2\right) \\
&- \left|Li_{r+p+q}\left(x\overline{z}\right) Li_{r+p+q}\left(z\overline{y}\right)\right| \\
&\geq \left|Li_{r+2q}\left(x\overline{y}\right) Li_{r+2p}\left(|z|^2\right) - Li_{r+p+q}\left(x\overline{z}\right) Li_{r+p+q}\left(z\overline{y}\right)\right|. \qquad (5.109)
\end{aligned}$$

Proof. Utilising the discrete inequality (3.54) for $p_k = 1/k^r$, $x_k = x^k/k^q$, $y_k = y^k/k^q$, $z_k = z^k/k^p$, $k \in \{1, 2, ..., m\}$, we have

$$\begin{aligned}
&\left(\sum_{k=1}^{m} \frac{1}{k^r}\left|\frac{x^k}{k^q}\right|^2\right)^{1/2} \left(\sum_{k=1}^{m} \frac{1}{k^r}\left|\frac{y^k}{k^q}\right|^2\right)^{1/2} \sum_{k=1}^{m} \frac{1}{k^r}\left|\frac{z^k}{k^p}\right|^2 \\
&-\left|\sum_{k=1}^{n} \frac{1}{k^r}\frac{x^k}{k^q}\frac{(\overline{z})^k}{k^p} \sum_{k=1}^{n} \frac{1}{k^r}\frac{z^k}{k^p}\frac{(\overline{y})^k}{k^q}\right| \\
&\geq \left|\sum_{k=1}^{m} \frac{1}{k^r}\frac{x^k}{k^q}\frac{(\overline{y})^k}{k^q} \sum_{k=1}^{n} \frac{1}{k^r}\left|\frac{z^k}{k^p}\right|^2 - \sum_{k=1}^{m} \frac{1}{k^r}\frac{x^k}{k^q}\frac{(\overline{z})^k}{k^p} \sum_{k=1}^{m} \frac{1}{k^r}\frac{z^k}{k^p}\frac{(\overline{y})^k}{k^q}\right|. \qquad (5.110)
\end{aligned}$$

Hence

$$\begin{aligned}
&\left(\sum_{k=1}^{m} \frac{1}{k^{r+2q}}\left(|x|^2\right)^k\right)^{1/2} \left(\sum_{k=1}^{m} \frac{1}{k^{r+2q}}\left(|y|^2\right)^k\right)^{1/2} \sum_{k=1}^{m} \frac{1}{k^{r+2p}}\left(|z|^2\right)^k \\
&-\left|\sum_{k=1}^{m} \frac{1}{k^{r+p+q}}\left(x\overline{z}\right)^k \sum_{k=1}^{m} \frac{1}{k^{r+p+q}}\left(z\overline{y}\right)^k\right| \\
&\geq \sum_{k=1}^{m} \frac{\left(x\overline{y}\right)^k}{k^{r+2q}} \sum_{k=1}^{m} \frac{\left(|z|^2\right)^k}{k^{r+2p}} - \sum_{k=1}^{m} \frac{\left(x\overline{z}\right)^k}{k^{r+p+q}} \sum_{k=1}^{m} \frac{\left(z\overline{y}\right)^k}{k^{r+p+q}}, \qquad (5.111)
\end{aligned}$$

for $m \geq 0$. Taking the limit as $m \to \infty$ in the inequality (5.111), then we deduce the desired result (5.109). ∎

On making use of the results (5.107) and (5.109), we obtain some more simpler inequalities as follows:

(a) If $\alpha = z$, $\beta = \overline{z}$, then from (5.107) we can state that

$$\begin{aligned} &\left|Li_{r+p+q}\left(z\overline{x}\right) Li_{r+p+q}\left(zx\right)\right| \\ &\leq \frac{1}{2}\left[Li_{r+2q}\left(|z|^2\right) + \left|Li_{r+2q}\left(z^2\right)\right|\right] Li_{r+2p}\left(|x|^2\right). \end{aligned} \tag{5.112}$$

Further, if $x = a \in \mathbb{R}$, then from (5.112) we deduce the inequality (5.106).

(b) If $y = \overline{x}$, then from (5.109) we can state that

$$\begin{aligned} &Li_{r+2q}\left(|x|^2\right) Li_{r+2p}\left(|z|^2\right) - \left|Li_{r+p+q}\left(xz\right) Li_{r+p+q}\left(x\overline{z}\right)\right| \\ &\geq \left|Li_{r+2q}\left(x^2\right) L_{r+2p}\left(|z|^2\right) - Li_{r+p+q}\left(xz\right) Li_{r+p+q}\left(x\overline{z}\right)\right|, \end{aligned} \tag{5.113}$$

for $x, z \in \mathbb{C}$. Moreover, if $z = a \in \mathbb{R}$, then from (5.113) we deduce the inequality

$$\begin{aligned} &Li_{r+2q}\left(|x|^2\right) Li_{r+2p}\left(a^2\right) - \left|Li_{r+p+q}\left(ax\right)\right|^2 \\ &\geq \left|Li_{r+2q}\left(x^2\right) Li_{r+2p}\left(a^2\right) - Li^2_{r+p+q}\left(ax\right)\right|, \end{aligned} \tag{5.114}$$

for any $x \in \mathbb{C}$ and $a \in \mathbb{R}$.

In particular, we can state the following inequalities, which incorporate the Riemann zeta function $\zeta(z)$:

Corollary 114 *Let $z \in D(0,1)$ and p, q, r integer such that $r = 2p > 1$. Then*

$$\left|Li_{p+q+r}\left(z\right)\right|^2 \leq \frac{1}{2}\zeta\left(r+2p\right)\left[Li_{r+2q}\left(|z|^2\right) + \left|Li_{r+2q}\left(z^2\right)\right|\right]. \tag{5.115}$$

Proof. The proof follows by Theorem 111 for $a = 1$. ■

Corollary 115 *Let $\alpha, \beta \in D(0,1)$ and p, q, r integers such that $r + 2p > 1$. Then*

$$\begin{aligned} &\left|Li_{r+p+q}\left(-\alpha i\right) Li_{r+p+q}\left(\overline{\beta} i\right)\right| \\ &\leq \frac{1}{2}\zeta\left(r+2p\right)\left(\left[Li_{r+2q}\left(|\alpha|^2\right) Li_{r+2q}\left(|\beta|^2\right)\right]^{1/2} + \left|Li_{r+2q}\left(\alpha\overline{\beta}\right)\right|\right). \end{aligned} \tag{5.116}$$

Proof. The proof follows by Theorem 112 for $x = i$. ■

Corollary 116 *Let $\alpha,\ \beta \in D(0,1)$ and $p,\ q,\ r$ integers such that $r+2p>1$. Then*

$$\begin{aligned} &\zeta(r+2q)\, Li_{r+2q}\left(|x|^2\right) - |Li_{r+p+q}(x)|^2 \\ &\geq \left|\zeta(r+2q)\, Li_{r+2q}\left(x^2\right) - Li^2_{r+p+q}(x)\right|, \end{aligned} \tag{5.117}$$

for any $x \in \mathbb{C}$.

Proof. The proof follows by utilising the inequality (5.114) for $a=1$. ■

Next, on utilising the inequalities given by (5.115), (5.116) and (5.117), and taking into account that some particular values of $\zeta(n)$ are known such as $\zeta(2)$ and $\zeta(4)$ as mentioned in (5.22), we derive some particular inequalities involving the polylogarithm functions. The results are summarized as follows:

$$\begin{aligned}
&\text{(a)}\quad |Li_{q+1}(z)|^2 \leq \frac{\pi^2}{12}\left[Li_{2q}\left(|z|^2\right) + |Li_{2q}(z^2)|\right], \\
&\text{(b)}\quad |Li_{q+2}(z)|^2 \leq \frac{\pi^4}{180}\left[Li_{2q}\left(|z|^2\right) + |Li_{2q}(z^2)|\right], \\
&\text{(c)}\quad \left|Li_{q+1}(-\alpha i)\, Li_{q+1}\left(\overline{\beta} i\right)\right| \\
&\qquad \leq \frac{\pi^2}{12}\left(\left[Li_{2q}\left(|\alpha|^2\right) Li_{2q}\left(|\beta|^2\right)\right]^{1/2} + \left|Li_{2q}\left(\alpha\overline{\beta}\right)\right|\right), \\
&\text{(d)}\quad \left|Li_{q+2}(-\alpha i)\, Li_{q+2}\left(\overline{\beta} i\right)\right| \\
&\qquad \leq \frac{\pi^2}{180}\left(\left[Li_{2q}\left(|\alpha|^2\right) Li_{2q}\left(|\beta|^2\right)\right]^{1/2} + \left|Li_{2q}\left(\alpha\overline{\beta}\right)\right|\right), \\
&\text{(e)}\quad \left|Li_{q+3}(-\alpha i)\, Li_{q+3}\left(\overline{\beta} i\right)\right| \\
&\qquad \leq \frac{\pi^4}{180}\left(\left[Li_{2(q+1)}\left(|\alpha|^2\right) Li_{2(q+1)}\left(|\beta|^2\right)\right]^{1/2} + \left|Li_{2(q+1)}\left(\alpha\overline{\beta}\right)\right|\right), \\
&\text{(f)}\quad \left|\frac{\pi^2}{6} Li_{2q}(x^2) - \overset{2}{Li}_{q+1}(x)\right| \leq \frac{\pi^2}{6} Li_{2q}\left(|x|^2\right) - |Li_{q+1}(x)|^2, \\
&\text{(g)}\quad \left|\frac{\pi^4}{90} Li_{2q}(x^2) - \overset{2}{Li}_{q+2}(x)\right| \leq \frac{\pi^4}{90} Li_{2q}\left(|x|^2\right) - |Li_{q+2}(x)|, \\
&\text{(h)}\quad \left|\frac{\pi^4}{90} Li_{2(q+1)}(x^2) - \overset{2}{Li}_{q+3}(x)\right| \leq \frac{\pi^4}{90} Li_{2(q+1)}\left(|x|^2\right) - |Li_{q+3}(x)|^2,
\end{aligned} \tag{5.118}$$

for any α, β, x, $z \in D(0,1)$ and q is an integer.

Other studies related to the polylogarithm functions for further reading can be found in the literature, see for example ([1], [89], [105], [143], [295], [312], [391]) and the references cited therein.

5.3 Inequalities for Hypergeometric Functions

Since, the hypergeometric function (5.46) is a power series with nonnegative coefficients and convergent on the open disk $D(0,1)$, then all the results, which are established in Section 3.2 - Section 3.4 of Chapter 3, and Section 4.1.2, Section 4.1.3, Section 4.2.2 and Section 4.2.3 of Chapter 4 hold true. Therefore, for instance and thus, for the purpose of this section, we present some inequalities involving the hypergeometric functions by applying some of those results such as (3.118), (4.29), (4.70) and (4.156) for this function.

First, we obtain the following result by applying the inequality (3.118) for the hypergeometric functions (5.46), see ([214], [217]).

Corollary 117 *Let ${}_2F_1(a,b;c;z)$ be the hypergeometric function. Then*

$$\begin{aligned}
&{}_2F_1\left(a,b;c;|x|^2\right)\ {}_2F_1\left(a,b;c;|y|^2\right) - \left|{}_2F_1(a,b;c;xy)\right|^2 \\
&\geq \left|{}_2F_1(a,b;c;|x|\,x)\ {}_2F_1(a,b;c;|y|\,\overline{y})\right. \\
&\left.-{}_2F_1(a,b;c;|y|\,x)\ {}_2F_1(a,b;c;|x|\,\overline{y})\right|,
\end{aligned} \tag{5.119}$$

for any a, b, $c \in \mathbb{R}$, with $c \notin \{\ldots,-2,-1,0\}$ and x, $y \in \mathbb{C}$ such that $|x|$, $|y| < 1$.

As a natural consequence, the following result holds:

Corollary 118 *(i) If in (5.119), we choose $c = b$, then we have*

$$\begin{aligned}
&\frac{1}{\left[\left(1-|x|^2\right)\left(1-|y|^2\right)\right]^a} - \frac{1}{|1-xy|^{2a}} \\
&\geq \left|\frac{1}{\left[(1-|x|\,x)\,(1-|y|\,\overline{y})\right]^a} - \frac{1}{\left[(1-|y|\,x)\,(1-|x|\,\overline{y})\right]^a}\right|,
\end{aligned} \tag{5.120}$$

for any $a \in \mathbb{R}$, x, $y \in \mathbb{C}$ *such that* $|x|$, $|y| < 1$.

(ii) Also, if in (5.119), we choose $a = b = 1$, $c = 2$, *then we get*

$$\begin{aligned}
&\ln\left(\frac{1}{1-|x|^2}\right)\ln\left(\frac{1}{1-|y|^2}\right)-\left|\ln\left(\frac{1}{1-xy}\right)\right|^2\\
&\geq\left|\ln\left(\frac{1}{1-|x|\,x}\right)\ln\left(\frac{1}{1-|y|\,\overline{y}}\right)\right.\\
&\left.-\ln\left(\frac{1}{1-|y|\,x}\right)\ln\left(\frac{1}{1-|x|\,\overline{y}}\right)\right|,
\end{aligned}\tag{5.121}$$

for x, $y \in \mathbb{C}$ *with* $|x|$, $|y| < 1$.

Remark 119 *For* $a = 1$, *the inequality (5.120) reduces to (3.125).*

On applying the inequality (4.29) for the hypergeometric function (5.46), we also have (see [215]):

Corollary 120 *If* ${}_2F_1(a, b; c; z)$ *is a hypergeometric function, then for any* a, b, $c \in \mathbb{R}$ *we have*

$$\begin{aligned}
&{}_2F_1(a,b;c;|x|^p)\;{}_2F_1(a,b;c;|y|^q)\\
&\geq\left|{}_2F_1(a,b;c;xy)\;{}_2F_1\left(a,b;c;|x|^{p-1}|y|^{q-1}\right)\right|,
\end{aligned}\tag{5.122}$$

where x, $y \neq 0$ *with* xy, $|x|^p$, $|y|^q \in D(0,1)$ *and* $p > 1$, $1/p + 1/q = 1$. *In particular, if we choose* $a = 1$, $c = b$ *in (5.122), then we get the inequality (4.34). Also, if we choose* $a = b = 1$ *and* $c = 2$, *then the inequality (5.122) reduces to (4.38).*

Next, from the inequality (4.70) we obtain (see [216]):

Corollary 121 *If* ${}_2F_1(a, b; c; x)$ *is the hypergeometric function, then we have*

$$\begin{aligned}
&{}_2F_1\left(a,b;c;y^{\nu}z^{1-\nu}\right)\;{}_2F_1\left(a,b;c;y^{1-\nu}z^{\nu}\right)\\
&\leq\;{}_2F_1(a,b;c;y)\;{}_2F_1(a,b;c;z),
\end{aligned}\tag{5.123}$$

for y, z, $y^{\nu}z^{1-\nu}$, $y^{1-\nu}z^{\nu} \in (0,1)$ *and* $\nu \in [0,1]$. *In particular, if we choose* $a = b = c = 1$, *then the inequality (5.123) reduces to (4.72). In fact, the inequality (5.123) reduces to (4.72) for any* a, b, $c \in \mathbb{R}$ *such that* $c = b \notin \{0, -1, -2, \ldots\}$.

Finally, we get from (4.156) the following result (see [213]):

Corollary 122 *If* ${}_2F_1(a,b;c;x)$ *is the hypergeometric function, then for any* a, b, $c \in \mathbb{R}$ *and* x, $y \in (0,1)$ *we have*

$$\left(\frac{x}{y}\right)^{\frac{aby\ {}_2F_1(a+1,b+1;c+1;y)}{c\ {}_2F_1(a,b;c;y)}} \leq \frac{{}_2F_1(a,b;c;x)}{{}_2F_1(a,b;c;y)} \leq \left(\frac{x}{y}\right)^{\frac{abx\ {}_2F_1(a+1,b+1;c+1;x)}{c\ {}_2F_1(a,b;c;x)}}. \tag{5.124}$$

In particular, if we choose $b = c$ *in (5.124), then we obtain that*

$$\left(\frac{x}{y}\right)^{y/(1-y)} \leq \frac{1-y}{1-x} \leq \left(\frac{x}{y}\right)^{x/(1-x)}, \tag{5.125}$$

for any $a \in \mathbb{R}$, $x, y \in (0,1)$.

For other results devoted to the hypergeometric functions, see for example ([51], [86], [191], [260], [263], [366], [396]) and the references which are cited therein.

5.4 Inequalities for Bessel Functions

Similar to the polylogarithm and hypergeometric functions, the modified Bessel functions of the first kind (5.75) are also defined by the power series with nonnegative coefficients and convergent on the open disk $D(0,1)$, so that all the results in Section 3.2 - Section 3.4 of Chapter 3, and Section 4.1.2, Section 4.1.3, Section 4.2.2 and Section 4.2.3 of Chapter 4 hold true. Therefore, for instance from (3.118), (4.29), (4.70) and (4.156), we have the following inequalities involving the Bessel and modified Bessel functions of the first kind (see [213], [214], [215], [216]).

Corollary 123 *If* $J_\alpha(x)$ *and* $I_\alpha(x)$ *are the Bessel and modified Bessel functions of the first kind respectively, then we have*

$$\begin{aligned} &I_\alpha\left(|x|^2\right) I_\alpha\left(|y|^2\right) - |I_\alpha(xy)|^2 \\ &\geq |J_\alpha(|x|\,x)J_\alpha(|y|\,\overline{y}) - J_\alpha(|y|\,x)J_\alpha(|x|\,\overline{y})| \end{aligned} \tag{5.126}$$

for $\alpha \in \mathbb{R}$, x, $y \in \mathbb{C}$ with $|x|$, $|y| < 1$. In particular, if $y = \alpha = 0$, then we obtain from (5.126) that

$$I_0\left(|x|^2\right) - 1 \geq |J_0(|x|\,x) - 1)|\,, \tag{5.127}$$

where

$$J_0\left(z\right) = \sum_{k=0}^{\infty} \frac{(-1)^n}{(k!)^2}\left(\frac{z}{2}\right)^{2k} \quad \text{and} \quad I_0\left(z\right) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(\frac{z}{2}\right)^{2k} \tag{5.128}$$

for $|z| < 1$.

Corollary 124 *If $J_\alpha\left(z\right)$ and $I_\alpha\left(z\right)$ are the Bessel and modified Bessel functions of the first kind respectively, then for any α, x, $y \in \mathbb{C}$, we have*

$$I_\alpha\left(|x|^p\right) I_\alpha\left(|y|^q\right) \geq \left|J_\alpha\left(xy\right) J_\alpha\left(|x|^{p-1}\,|y|^{q-1}\right)\right|, \tag{5.129}$$

where x, $y \neq 0$, xy, $|x|^p$, $|y|^q \in D\left(0,1\right)$ and $p > 1$ with $1/p + 1/q = 1$. In particular, if $\alpha = 0$ in (5.129), then for p, $q > 1$ with $1/p + 1/q = 1$, we get

$$J_0\left(i\,|x|^p\right) J_0\left(i\,|y|^q\right) \geq \left|J_0\left(xy\right) J_0\left(|x|^{p-1}\,|y|^{q-1}\right)\right|. \tag{5.130}$$

Corollary 125 *If $I_\alpha\left(x\right)$ is the modified Bessel function of the first kind, then we have*

$$I_\alpha\left(y^\nu z^{1-\nu}\right) I_\alpha\left(y^{1-\nu} z^\nu\right) \leq I_\alpha(y) I_\alpha(z), \tag{5.131}$$

for y, z, $y^\nu z^{1-\nu}$, $y^{1-\nu} z^\nu \in (0,1)$ and $\nu \in [0,1]$. In particular, if $\alpha = 0$, then from (5.131) we get

$$I_0\left(y^\nu z^{1-\nu}\right) I_0\left(y^{1-\nu} z^\nu\right) \leq I_0(y) I_0(z), \tag{5.132}$$

for y, z, $y^\nu z^{1-\nu}$, $y^{1-\nu} z^\nu \in (0,1)$ and $\nu \in [0,1]$, where

$$I_0\left(x\right) = \sum_{k=0}^{\infty} \frac{x^{2k}}{4^k\left(k!\right)^2}. \tag{5.133}$$

If in (5.131) we choose $\alpha = \frac{1}{2}$, then we we obtain

$$\sinh\left(y^\nu z^{1-\nu}\right) \sinh\left(y^{1-\nu} z^\nu\right) \leq \sinh\left(y\right) \sinh\left(z\right), \tag{5.134}$$

for $y,\ z,\ y^{\nu}z^{1-\nu},\ y^{1-\nu}z^{\nu} \in (0,1)$ *and* $\nu \in [0,1]$*. If we take* $\alpha = 3/2$*, then we get from (5.131) that*

$$\begin{aligned}&\left[y^{\nu}z^{1-\nu}\cosh\left(y^{\nu}z^{1-\nu}\right)-\sinh\left(y^{\nu}z^{1-\nu}\right)\right]\\&\times\left[y^{1-\nu}z^{\nu}\cosh\left(y^{1-\nu}z^{\nu}\right)-\sinh\left(y^{1-\nu}z^{\nu}\right)\right]\\&\leq\left[y\cosh(y)-\sinh(y)\right]\left[z\cosh(z)-\sinh(z)\right],\end{aligned} \tag{5.135}$$

for $y,\ z,\ y^{\nu}z^{1-\nu},\ y^{1-\nu}z^{\nu} \in (0,1)$ *and* $\nu \in [0,1]$*.*

Remark 126 *For* $\alpha = -1/2$ *and* $-3/2$ *in (5.131), we get the dual results, namely*

$$\cosh\left(y^{\nu}z^{1-\nu}\right)\cosh\left(y^{1-\nu}z^{\nu}\right)\leq\cosh(y)\cosh(z) \tag{5.136}$$

and

$$\begin{aligned}&\left[y^{\nu}z^{1-\nu}\sinh\left(y^{\nu}z^{1-\nu}\right)-\cosh\left(y^{\nu}z^{1-\nu}\right)\right]\\&\times\left[y^{1-\nu}z^{\nu}\sinh\left(y^{1-\nu}z^{\nu}\right)-\cosh\left(y^{1-\nu}z^{\nu}\right)\right]\\&\leq\left[y\sinh(y)-\cosh(y)\right]\left[z\sinh(z)-\cosh(z)\right]\end{aligned} \tag{5.137}$$

respectively, for $y,\ z,\ y^{\nu}z^{1-\nu},\ y^{1-\nu}z^{\nu} \in (0,1)$ *and* $\nu \in [0,1]$*.*

Corollary 127 *If* $I_{\alpha}(x)$ *is the modified Bessel function of the first kind, then for any* $a,\ c,\ \alpha \in \mathbb{R}$ *we have*

$$\left(\frac{c}{a}\right)^{\frac{a\left[I_{\alpha-1}(a)+I_{\alpha+1}(a)\right]}{2I_{\alpha}(a)}}\leq\frac{I_{\alpha}(c)}{I_{\alpha}(a)}\leq\left(\frac{c}{a}\right)^{\frac{c\left[I_{\alpha-1}(c)+I_{\alpha+1}(c)\right]}{2I_{\alpha}(c)}}. \tag{5.138}$$

In particular, if $\alpha = 0$ *in (5.138), then for any* $a,\ c \in \mathbb{R}$*, we get*

$$\left(\frac{c}{a}\right)^{\frac{aI_1(a)}{I_0(a)}}\leq\frac{I_0(c)}{I_0(a)}\leq\left(\frac{c}{a}\right)^{\frac{cI_1(c)}{I_0(c)}}, \tag{5.139}$$

where

$$I_1(z)=\sum_{k=0}^{\infty}\frac{1}{k!\,(k+1)!}\left(\frac{z}{2}\right)^{2k+1}. \tag{5.140}$$

If in (5.138), we choose $\alpha = 1/2$, then we obtain

$$\left(\frac{c}{a}\right)^{a\coth(a)-\frac{1}{2}} \le \frac{\sinh(c)}{\sinh(a)} \le \left(\frac{c}{a}\right)^{c\coth(c)-\frac{1}{2}}, \tag{5.141}$$

for any a, $c \in \mathbb{R}$.

Remark 128 *For $\alpha = -1/2$ in (5.138), we get the dual result, namely*

$$\left(\frac{c}{a}\right)^{a\tanh(a)-\frac{1}{2}} \le \frac{\cosh(c)}{\cosh(a)} \le \left(\frac{c}{a}\right)^{c\tanh(c)-\frac{1}{2}}, \tag{5.142}$$

for any a, $c \in \mathbb{R}$.

Other inequalities related to the Bessel and Modified Bessel functions of the first kind for further reading can be found in the literature, see for example ([50], [227], [331], [447]) and the references which are cited therein.

Part II

SOME INEQUALITIES INVOLVING ANALYTIC AND UNIVALENT FUNCTIONS

Chapter 6

Elementary Theory of Univalent Functions

This chapter is of an introductory nature and gives the necessary background to Univalent Function Theory. Univalent functions are studied by many, not only in the area of Geometric Function Theory, but also in the classical studies of Complex Analysis as a whole.

Some fundamental concepts in the field of Univalent Function Theory, such as a simply connected domain, analyticity, conformality property, normalization conditions of analytic functions, and the Riemann Mapping Theorem, are given in Section 6.1. In Section 6.2, we introduce one of the most important subclasses of analytic functions, which is called the class $\mathcal{P}$ of all analytic functions with a positive real part. Some important and useful results related to this class are also given. The class S of analytic, univalent and normalized functions in a unit disk, and some of its well-known subclasses such as starlike, convex, close-to-convex, etc., are briefly discussed in Section 6.3 and Section 6.4. Some important properties of functions in these classes are given for completeness.

In Section 6.5, we state some of the classical results related to the class S. One of the famous differential operators, which is known as the Sălăgean differential operator, is mentioned at the end of this chapter. Several subclasses of S, which are characterized by this operator, are mentioned as well.

Some of the well-known results are mostly adopted from the standard texts of Ahlfors [6], Duren [144], Pommerenke [357], Graham and Kohr [170], Kühnau ([265], [266]) and other references, which are cited therein. Further reading and some additional results on this subject can also be found in the wide variety of classical and modern textbooks devoted to the Univalent Functions Theory and related topics, including ([5], [74], [169], [221], [262], [330], [386], [406]).

6.1 Basic Concepts

In a complex plane, an open set is said to be *connected* if every two points in the set can be joined by a continuous piecewise smooth curve (or polygonal path) that lies entirely in the set. An open connected set is a *domain*, while, a domain together with some, none or all of its boundary points, is often called a *region*. A domain E is said to be *simply connected* if it has the property that any *simple closed curve*[1] γ (or *Jordan curve*), which is parametrized by $z(t) = x(t) + iy(t)$ for $t \in [a, b]$, $z \in E$, lies completely in E. This means that the domain interior to γ lies wholly in E [5, p. 77-80]. In a simple language, a simply connected domain is one which is free of holes or cuts in its interior.

The open unit disk D, which is defined by (2.3), is one of the most important examples of a simply connected domain in $\mathbb{C}$. It plays an important role in the study of Univalent Function Theory, which is primarily concerned with analytic and univalent functions on this domain.

A set Λ_0 in a complex plane is called *starlike* (*star domain* or *star-shaped*) with respect to the fixed point $x_0 \in \Lambda_0$, for every $x \in \Lambda_0$, the line segment joining the point x_0 to x lies entirely in the set Λ_0. Geometrically, for any $x_0, x \in \Lambda_0$ with x_0 is a fixed point, then $tx + (1-t)x_0 \in \Lambda_0$, for every $t \in [0, 1]$. In the case if $x_0 = 0$, then we say that the set Λ_0 is a *starlike with respect to the origin*. The convex set can be described in a similar way. A set $\Lambda_1 \in \mathbb{C}$ is said to be *convex* if it is starlike with respect to each of its points; that is, the line segment joining any two points $x, y \in \Lambda_1$.

A complex-valued function $f : \Delta \to \mathbb{C}$ of a complex variable is *differentiable*

[1] A curve is said to be simple if it does not cross itself. It is called a simple closed curve if it closed and it does not cross itself except at the end points.

at a point $z_0 \in \Delta \subset \mathbb{C}$ if it has a derivative at z_0, i.e.,

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \tag{6.1}$$

A function $f(z)$ is said to be *analytic* at the point $z_0 \in \Delta$ if it is differentiable at every point in some neighborhood of $z_0 \in \Delta$. The function $f(z)$ is to be said analytic on the domain Δ provided that it is analytic at every point of $z \in \Delta$. We say that $f(z)$ is an *entire function* if is analytic on the whole complex plane.

One of the most important facts of complex analysis is that, if $f(z)$ is an analytic function on a certain domain in a complex plane, then it must have derivatives of all orders at z_0 in its domain, and $f(z)$ has a Taylor series expansion, which converges in an open disk centered at z_0 (see [144, p. 2], [166, p. 33]).

Theorem 129 ([166, p. 33]) *Suppose that the power series $\sum_{k=0}^{\infty} a_k z^k$ has radius of convergence R, $R > 0$. For $|z| < R$, let $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Then, the power series $\sum_{k=0}^{\infty} k a_k z^k$ has radius of convergence R and $f'(z) = \sum_{k=0}^{\infty} k a_k z^k$ for all $|z| < R$. Thus, a power series defines a function, which is analytic in its disk of convergence.*

This means that every analytic function is locally represented by its convergent power series in an open disk. Explicitly, we have [144, p. 2]

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{6.2}$$

which converges for all z sufficiently close to z_0. More precisely, if $f(z)$ is analytic in the disk $D(z_0, R)$, then the Taylor series expansion (6.2) converges to $f(z)$ for all $z \in D(z_0, R)$. Conversely, if the power series defined in (6.2) converges for every $z \in D(z_0, R)$, then the series (6.2) represents a function that is analytic on $D(z_0, R)$.

Besides the analyticity, the *conformality* of functions is another remarkable geometric property that must be possessed by the analytic complex-valued functions. A function f, which is analytic and univalent on D, also maps conformally the unit disk D onto the another domain in $\mathbb{C}$. In the other words, a complex-valued function is conformal if and only if it is analytic, and it has everywhere nonzero derivative on its domain. In particular, we have the following theorem [5, p. 259] (see also [44, p. 170]).

Theorem 130 ([5, p. 259]) *If the function $f(z)$ is analytic at $z_0 \in D$ and $f'(z_0) \neq 0$, then $f(z)$ is conformal at z_0. If $f(z)$ is analytic in D, then $f(z)$ is conformal at every point in D. The converse is also valid.*

The *conformal mapping,* which preserves the magnitude and the orientation of the angles, is also called the *conformal transformation*, *angle-preserving transformation* or *biholomorphic maps,* with its applications found in many areas of physics and engineering, electronics, medicines and other branches of applied mathematics (see for instance [153], [238], [245], [402]).

Riemann Mapping Theorem is one of the remarkable results in the Geometric Function Theory, which asserts that any simply connected domain in $\mathbb{C}$ can be mapped conformally onto the unit disk D (see [144, p. 11], [170, p. 5], [357, p. 10]). More precisely, we have (see also [6, p. 222])

Theorem 131 ([144, p. 11]) *Let U be a simply connected domain, which is a proper subset of a complex plane. Let ζ_0 be a given point in U. Then, there exists a unique analytic and one-to-one function $f : U \to D$, which maps U conformally onto the unit disk D, and has the properties $f(\zeta_0) = 0$, $f'(\zeta_0) > 0$.*

This theorem guarantees the existence of a conformal mapping, which maps the simply connected domain D onto any simply connected domain in $\mathbb{C}$. We note that since the inverse image of a conformal mapping is also conformal, then the Riemann Mapping Theorem implies that any two simply connected domains are conformally equivalent. That is, if $D \subsetneq \mathbb{C}$ and $U \nsubseteq \mathbb{C}$ are simply connected with $z \in D$ and $w \in U$, then there exists a unique conformal transformation $f : D \to U$ with $f(z) = w$ and $f'(z) > 0$.

An analytic function $w = f(z)$ that maps conformally a simply connected domain onto another domain in $\mathbb{C}$ is not unique. In view of the Riemman Mapping Theorem, the function $f(z)$ needs to be unique. Therefore, such functions need to satisfy the conditions for uniqueness, which are called *normalization.*

The standard normalization conditions of analytic functions in a complex plane are that, for $f : D_1 \to D_2$, where D_1 and D_2 are simply connected domains in $\mathbb{C}$, then the function $f(z)$ is called normalized such that [5, p. 261]

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1. \tag{6.3}$$

Other normalizations are possible, but the conditions in (6.3) are the most usual and the ones that we will be used throughout this study.

The next section discusses one of the important classes of analytic functions, which is called the class $\mathcal{P}$ consisting of all analytic functions with a positive real part.

6.2 Functions With a Positive Real Part

The analytic functions, which map the open unit disk D onto the right half-plane, are of particular interest in the study of Univalent Function Theory. In this section, we introduce the class $\mathcal{P}$ of analytic functions in the open unit disk D having a positive real part. Let us denote $\mathcal{H}(D)$, the class of all analytic functions in the unit disk D.

Definition 132 *Let $\mathcal{P}$ be the family of all analytic functions. Then,*

$$\mathcal{P} = \{p \in \mathcal{H}(D) : \operatorname{Re}[p(z)] > 0,\ p(0) = 1,\ z \in D\}. \tag{6.4}$$

The function in the class $\mathcal{P}$ is often called the *Carathédory function* [84]. It can be noted that all functions in the class $\mathcal{P}$ can be represented by the following power series:

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k, \tag{6.5}$$

for all $z \in D$, which is obviously that $p(0) = 1$. Due to Herglots [194], the family $\mathcal{P}$ is characterized by the set of functions $p(z)$, where they are represented by the *Herglotz formula* as follows:

$$p(z) = \int_0^{2\pi} \left(\frac{1 + e^{-it}z}{1 - e^{-it}z} \right) d\mu(t), \tag{6.6}$$

for $z \in D$, $\mu(t)$ is a real-valued nondecreasing function on $[0, 2\pi]$ and $\mu(2\pi) - \mu(0) = 1$. The most important example of a function in the class $\mathcal{P}$ is called the *Möbius transform*, which is of the form

$$m_0(z) = \frac{1+z}{1-z} = 1 + \sum_{n=1}^{\infty} 2z^n. \tag{6.7}$$

The function (6.7), which maps the unit disk D one-to-one onto the right half-plane, also provides one of the extremal functions of this class.

The following properties might be useful in the subsequent chapters (see [84], [144, p. 41], [240], [284], [357, p. 41]), with the first lemma due to Carathéodory [84]:

Lemma 133 ([84]) *If $p(z) \in \mathcal{P}$ and $p(z)$ is defined by the series (6.5), then $|p_n| \le 2$ for $n \in \{1, 2, 3, ...\}$. Equality is attained if and only if $p(z)$ is the Möbius function (6.7).*

Lemma 134 *Let $p(z) \in \mathcal{P}$ and be of the form (6.5). Then,*

$$\left| p_2 - \frac{p_1^2}{2} \right| \le 2 - \frac{|p_1|^2}{2}. \tag{6.8}$$

This result is sharp, with equality holding for the function

$$p(z) = \frac{1 + \frac{1}{2}(p_1 + \xi\overline{p_1})z + \xi z^2}{1 - \frac{1}{2}(p_1 - \xi\overline{p_1})z - \xi z^2}, \quad |\xi| = 1. \tag{6.9}$$

Lemma 135 (see [180]) *Let $p(z) \in \mathcal{P}$ and be of the form (6.5). Then,*

$$\begin{aligned} &\text{(i)} \quad |p_n| \le 2, \text{ for } n \ge 1, \\ &\text{(ii)} \quad |p_2 - \mu p_1^2| \le 2\max\{1, |1-2\mu|\}, \text{ for any } \mu \in \mathbb{C}. \end{aligned} \tag{6.10}$$

6.3 Analytic and Univalent Functions

A single-valued function $f : D \to \mathbb{C}$ is said to be *univalent*[2] in a domain $D \subset \mathbb{C}$ if it never takes the same value twice: that is, $f(z_1) \neq f(z_2)$ for all pairs of distinct points z_1 and z_2 in D. In other words, $f(z)$ is one-to-one (or injective) mapping of D onto another domain in a complex plane [144, p. 26]. A function $f(z)$ is called *locally univalent* at a point $z_0 \in D$, if it is univalent in some neighborhood of $z_0 \in D$. For an analytic function $f(z)$, the condition $f'(z_0) \neq 0$ is equivalent to the local univalence at z_0 [6, p. 131]. The locally univalent function is also known as conformal mapping because of its angle and sense preserving property (see [6, p. 10]).

As mentioned in the previous section, an analytic function $f(z)$ defined on the unit disk D has a Taylor series representation given by (6.2). Thus, we may assume without loss of generality that the series (6.2) is centered at the origin, and has been normalized by the conditions (6.3). Hence, each an analytic function has the power series of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{6.11}$$

for $z \in D$ and $a_k \in \mathbb{C}$, $k \in \{2, 3, \ldots\}$.

Let A denote the subclass of $\mathcal{H}(D)$ consisting of all analytic functions of the form (6.11), which are analytic and normalized by the conditions (6.3). We denote by S, the subclass of A consisting of all analytic, univalent and normalized functions in the unit disk D, and $f(z)$ has the form (6.11), i.e.,

$$S = \left\{ f \in A : f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ z \in D \right\}.$$

Remark 136 *Throughout this study, we shall be primarily concerned with the function $f(z)$ in the class A and S, and of the form (6.11), which are analytic and univalent in the unit disk D, and normalized by the conditions (6.3).*

Example 137 *The leading example of a function in the class S is called the Koebe function, which is given by*

[2]The terminologies such as '*simple*' or '*Schlicht*' are also used for univalent.

$$k(z) = \frac{z}{(1-z)^2} = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4} = \sum_{n=1}^{\infty} nz^n, \quad z \in D. \tag{6.12}$$

This function (6.12) maps the unit disk D, one-to-one and conformally onto the entire complex plane except for the negative real axis from $-1/4$ to ∞. The functions

$$e^{-i\beta} k_0 \left(e^{i\beta} z\right) = \frac{z}{(1-e^{i\beta}z)^2} = \sum_{n=1}^{\infty} ne^{i(n-1)\beta} z^n, \tag{6.13}$$

also belong to S and they are referred to as the rotations of Koebe function.

Example 138 *Other examples of functions belonging to the class S include*

(a) *The identity map, $f(z) = z$.*

(b) $f(z) = z/(1-z)$, *which maps the unit disk D conformally onto the half-plane,* $\mathrm{Re}\,(z) > -1/2$.

(c) $f(z) = \frac{1}{2}\log\left[(1+z)/(1-z)\right]$, *which maps the unit disk D onto the horizontal strip,* $-\pi/4 < \mathrm{Im}\,(z) < \pi/4$.

6.4 Subclasses of Analytic and Univalent Functions

In this section, we briefly discuss some of the well-known subclasses S, namely the starlike, convex and close-to-convex functions. These functions are defined by geometrical considerations, but they are very useful for analytic characterizations as well. The first subclass of S that is worthy in the study, is the class of starlike functions.

The class of starlike and convex functions: A function $f(z) \in S$ is said to be *starlike* in D with respect to origin if the image of D under f, i.e., $f(D)$ with $w = 0 \in f(D)$, is a starlike domain: that is, the line segment connecting the point $w = 0$ to any point of $f(D)$ lies entirely in $f(D)$. Similarly, a function

$f(z) \in S$ is said to be *convex* in D if the image $f(D)$ is a convex domain (see [357, p. 44]). We shall denote these classes of functions by S^* and C, which are starlike with respect to the origin and convex in the unit disk D, respectively. The class S^* was first studied by Alexander [15], later by Gronwall [174], Nevanlinna [332] and others (see [376], [319]).

The necessary and sufficient conditions for functions $f \in S$ to be starlike and convex have been given by Robertson [376] as follows (see also [144, p. 41], [319], [357, p. 42]):

Theorem 139 ([376]) *A function $f \in S$ of the form (6.11) is starlike in the unit disk D if and only if it satisfies*

$$z\frac{f'(z)}{f(z)} \in \mathcal{P}\ ,\ i.e.,\ \operatorname{Re}\left\{z\frac{f'(z)}{f(z)}\right\} > 0,\ z \in D. \tag{6.14}$$

Theorem 140 ([376]) *A function $f \in S$ of the form (6.11) is convex in the unit disk D if and only if it satisfies*

$$1 + z\frac{f''(z)}{f'(z)} \in \mathcal{P}\ ,\ i.e.,\ \operatorname{Re}\left\{1 + z\frac{f''(z)}{f'(z)}\right\} > 0,\ z \in D. \tag{6.15}$$

The *representation formula* for functions in the class of starlike, $f(z) \in S^*$ of the form (6.11) is given by

$$f(z) = z \exp\left[\int_0^{2\pi} \log\left(\frac{1}{1 - e^{-i\tau}z}\right) du(\tau)\right],\ z \in D, \tag{6.16}$$

for some increasing function $u(\tau)$ with $u(2\pi) - u(0) = 1$ [357, p. 43].

Alexander [15] observed that there exists a close connection between the convex and starlike functions: that is, a function $f \in S$ maps the unit disk D onto a convex domain if and only if $zf'(z)$ maps the unit disk D onto a starlike domain: i.e., $f(z) \in C$ if and only if $zf'(z) \in S^*$, $z \in D$. Thus, we have the inclusion $C \subset S^* \subset S$. Note that the Koebe function defined by (6.12) is starlike with respect to origin but not convex. The function $f(z)$ given by (1.21) is one of the examples of the convex functions, which maps the unit disk D onto a half plane, and it plays a central role in the class C.

In [376], Robertson also introduced the classes $S^{*}(\alpha)$ and $C(\alpha)$ of starlike functions of order α and convex functions of order α, for some α $(0 \leq \alpha < 0)$, which are defined by

$$S^{*}(\alpha) = \left\{ f \in S : \operatorname{Re}\left\{ z\frac{f'(z)}{f(z)} \right\} > \alpha,\ 0 \leq \alpha < 1,\ z \in D \right\}, \tag{6.17}$$

$$C(\alpha) = \left\{ f \in S : \operatorname{Re}\left\{ 1 + z\frac{f''(z)}{f(z)} \right\} > \alpha,\ 0 \leq \alpha < 1,\ z \in D \right\}. \tag{6.18}$$

In particular, we have $S^{*}(\alpha) \subset S^{*}(0) = S^{*} \subset S$ and $C(\alpha) \subset C(0) = C \subset S$. Strochhäcker [411] proved that every convex function is starlike of order $1/2$.

Further, an interesting unification of functions in the classes S^{*} and C was provided by Miller, Mocannu and Reade in [309]. They introduced the class $\mathcal{C}_{\lambda}$ of $\lambda-$convex functions (or $\lambda-$starlike functions) via the linear combinations of the representations of starlike and convex functions studied by Moccanu [319]. In [309], Miller, Mocannu and Reade showed that if $f(z)$ is an analytic function in the unit disk D with $f(z)f'(z)/z \neq 0$, and λ is a real number, then $f(z)$ is said to be in the class $\mathcal{C}_{\lambda}$ if and only if

$$\operatorname{Re}\left\{ (1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) \right\} > 0, \tag{6.19}$$

for $z \in D$. This more general class C_{λ} reduces to the standard starlike functions S^{*} and convex functions C when $\lambda = 0$ and 1, respectively.

Meanwhile, in [328], Nasr and Aouf introduced the class of starlike and convex functions of complex order b $(b \neq 0)$. Such classes of functions are denoted by S_b^{*} and C_b, and they are defined as follows:

$$S_b^{*} = \left\{ f \in S : \operatorname{Re}\left\{ 1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \right\} > 0 \ (z \in U) \right\}, \tag{6.20}$$

$$C_b = \left\{ f \in S : \operatorname{Re}\left\{ 1 + \frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right) \right\} > 0 \ (z \in U) \right\}, \tag{6.21}$$

where $f(z)/z \neq 0$.

The class of close-to-convex functions: Another important subclass of S is the class of close-to-convex functions, which was introduced by Kaplan [235] in 1952. The necessary and sufficient condition of functions in this class is stated in the following. A function $f(z) \in S$ and of the form (6.11) is said to be *close-to-convex* in D if and only if there exists a convex function $g(z)$ in D such that

$$\operatorname{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > 0, \ z \in D. \tag{6.22}$$

Since $h(z) = zg'(z) \in S^*$ with $g \in S$, then (6.22) is equivalent with

$$\operatorname{Re}\left\{\frac{zf'(z)}{h(z)}\right\} > 0, \ z \in D. \tag{6.23}$$

We shall denote by $\mathcal{K}$, the class of all functions, which are close-to-convex in D. It is clear that every convex function is close-to-convex, and every starlike function is close-to-convex as well. In [357, p. 51], Pommerenke showed that every close-to-convex function is univalent in D. Therefore, it can be seen that the proper inclusions $\mathcal{C} \subset S^* \subset \mathcal{K} \subset S$ hold.

The more general class $\mathcal{K}(\alpha, \beta)$ of close-to-convex functions of order α and type β are characterized as follows. A function $f(z) \in S$ is in the class $\mathcal{K}(\alpha, \beta)$ if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{h(z)}\right\} > \beta, \ z \in D, \tag{6.24}$$

for $h(z) \in S^*(\alpha)$, $0 \leq \alpha < 1$, $0 \leq \beta < 1$.

The class $\mathcal{K}$ and its generalizations have been studied by numerous researchers (see for instance [108], [109], [110], [164], [165], [206], [219], [292]) and recently by ([233], [261], [390], [430], [434], [446]).

The class of Bazilević functions: A more general class of analytic functions was introduced by Bazilevič [54] in 1955, which is called the class of *Bazilevič functions of type* α, and is denoted by $\mathcal{B}(\alpha)$, $\alpha > 0$. The functions in this class are characterized by

$$\operatorname{Re}\left(\frac{zf'(z)f(z)^{\alpha-1}}{g(z)^{\alpha}}\right) > 0 \ \ (z \in D)\,, \tag{6.25}$$

for $f(z) \in A$ and of the form (6.11), where $g(z) \in S^*$ in D, $\alpha > 0$. The class of Bazilevič functions has been studied by many authors with Thomas (see [418], [419]), Halim [182], Noor [336], Singh [400], Keogh and Miller [241] among others. We note that with specific choices of the associated parameter, the class $\mathcal{B}(\alpha)$ reduces to the well-known classes of starlike, convex and close-to-convex functions. Furthermore, if in (6.25) we choose the starlike function $g(z) = z$, then we have the family $\mathcal{B}_1(\alpha)$ of function satisfying [400]

$$\operatorname{Re}\left(\frac{zf'(z)f(z)^{\alpha-1}}{z^{\alpha}}\right) > 0, \ \ z \in D. \tag{6.26}$$

A more general class $\mathcal{B}(\alpha,\beta)$ of Bazilevič functions of order β and type α is defined by

$$\mathcal{B}(\alpha,\beta) = \left\{ f \in A : \operatorname{Re}\left(\frac{zf'(z)f(z)^{\alpha-1}}{g(z)^{\alpha}}\right) > \beta \ \ \ (z \in D) \right\}, \tag{6.27}$$

for some α $(\alpha > 0)$ and β $(0 \le \beta < 1)$, where $g(z) \in S^*$. In particular, if $\beta = 0$ and 1, then we have $\mathcal{B}(\alpha, 0) = \mathcal{B}(\alpha)$ and $\mathcal{B}(1,0) = \mathcal{B}(1) = \mathcal{K}$. In particular, we denote by $\mathcal{B}_1(\alpha,\beta)$ the subclass of $\mathcal{B}(\alpha,\beta)$ for which $g(z) = z \in S^*$, namely

$$\mathcal{B}_1(\alpha,\beta) = \left\{ f \in A : \operatorname{Re}\left(\frac{zf'(z)f(z)^{\alpha-1}}{z^{\alpha}}\right) > \beta, \ z \in D \right\}, \tag{6.28}$$

for some α $(\alpha > 0)$ and β $(0 \le \beta < 1)$. Also, we note that $\mathcal{B}(0,0) = \mathcal{B}_1(0,0) = S^*$, $\mathcal{B}(0,\beta) = \mathcal{B}_1(0,\beta) = S^*(\beta)$ and that $\mathcal{B}_1(1,\beta)$ is a subclass of A consisting of all functions for which $\operatorname{Re}\{f'(z)\} > \beta$, for $z \in D$.

Other Subclasses: Some other well-known subclasses of analytic and univalent functions that have been studied repeatedly by many authors, are given in the following:

$$S_0 = \left\{ f(z) \in A : \operatorname{Re}\left\{\frac{f(z)}{z}\right\} > 0, \ z \in D \right\}, \tag{6.29}$$

$$B(\beta) = \left\{ f(z) \in A : \mathrm{Re}\left\{ \frac{f(z)}{z} \right\} > \beta,\ 0 \leq \beta < 1,\ z \in D \right\}, \qquad (6.30)$$

$$\delta(\beta) = \left\{ f(z) \in A : \mathrm{Re}\left\{ f'(z) \right\} > \beta,\ 0 \leq \beta < 1,\ z \in D \right\}. \qquad (6.31)$$

6.5 Some Classical Results

One of the most classical results in the Theory of Univalent Functions is *Bieberbach Conjecture*, proposed by Bieberbach [67] in 1961. This conjecture states the upper bounds for the coefficients of functions in the class S. Bieberbach was the first who established the bound for the second coefficient of functions in the class S: that is, $|a_2| \leq 2$ for $f(z) \in S$ and of the form (6.11). He conjectured that if $f(z) \in S$, then the coefficients a_n of $f(z)$ satisfied $|a_n| \leq n$, for all $n \geq 2$, with the equality holding if and only if $f(z)$ is the Koebe function (6.12) or one of its rotations (6.13). For many years, this conjecture has stood as a challenge to many mathematicians and has inspired the development of important new methods in complex analysis. However, the Bieberbach Conjecture was completely proved by de Brange [75] in 1985.

Since then, the estimation of the coefficients $|a_n|$, for $n \in \{2, 3, 4, \ldots\}$ have been investigated for various subclasses of S, in order to provide some basic properties of univalent functions (see [18], [58], [83], [115], [236], [294], [357, p. 53], [371], [379]). For instance, the following results showed that the Bieberbach conjecture also holds for functions in the subclass of S.

Theorem 141 ([357, p. 46]) *If $f \in S^*$ of the form (6.11), then $|a_n| \leq n$. Equality occurs if and only if $f(z)$ is the rotation of Koebe function.*

Theorem 142 ([371]) *If $f \in K$ of the form (6.11), then $a_n \leq n$.*

The necessary and sufficient condition for functions in the starlike and convex classes that has been proved by Alexander [15], is given as follows (see also [372]):

Theorem 143 ([15]) *Let $f(z)$ be of the form (6.11). If $\sum_{n=2}^{\infty} n\,|a_n| \leq 1$, then $f(z)$ is univalent in $|z| < 1$ and maps that region onto a region that is starlike* *with respect to the origin. If* $\sum_{n=2}^{\infty} n^2 |a_n| \leq 1$, *then* $f(z)$ *is univalent in* $|z| < 1$ *and maps that region onto a convex region.*

Silverman [398] provided the following properties of the starlikeness and convexity of functions in the class $S^*(\alpha)$ and $C(\alpha)$.

Theorem 144 ([398]) *Let* $f(z)$ *be of the form (6.11). If* $\sum_{n=2}^{\infty} (n-\alpha)|a_n| \leq 1-\alpha$, $0 \leq \alpha < 1$, *then* $f \in S^*(\alpha)$. *If* $\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq 1-\alpha$, $0 \leq \alpha < 1$, *then* $f \in C(\alpha)$.

Another closely related problem in the Theory of Univalent Functions is the determination of the sharp upper bounds for the nonlinear functional $|a_3 - ua_2^2|$, for both real and complex values of the parameter u. It is popularly known as the *Fekete-Szegő problem*, and also arise in the investigation of univalency of the analytic functions. The Fekete-Szegő problem has its origin in a conjecture of Littlewood and Parley [289] in 1932 that the bound on the coefficients $|a_3 - a_2^2|$ of an odd univalent function is 1. However, this result was disproved by Fekete and Szegő [152] (see also [144, p. 104]). Since them, the Fekete-Szegő problem has continued to receive attention of large number of researchers in Geometric Function Theory.

In 1933, Fekete and Szegő [152] first obtained the sharp upper bounds of the functional $|a_3 - a_2^2|$ for $f \in S$, for each fixed u in the interval $0 \leq \lambda < 1$. The result is stated as follows:

Theorem 145 ([152]) *If* $f(z)$ *in* S *and of the form (6.11), then*

$$|a_3 - ua_2^2| \leq \begin{cases} 3-4u, & \text{for } u \leq 0; \\ 1 + 2\exp\left(\dfrac{-2u}{1-u}\right), & \text{for } 0 \leq u < 1; \\ 4u-3, & \text{for } u \geq 1. \end{cases} \tag{6.32}$$

This bound is sharp when u is real.

Pfluger (see [353], [354]) has considered the Fekete-Szegő problems for $f \in S$ when u is complex numbers. In the cases of functions in the classes S^*, C and $\mathcal{K}$ of starlike, convex and close-to-convex functions, the above inequality (6.32) has been improved by several researchers. For instance, Keogh and Merker [240] proved the Fekete-Szegő problem for the close-to-convex functions as follows:

Theorem 146 ([240]) *For $f(z) \in \mathcal{K}$ and of the form (6.11), we have*

$$\left|a_3 - ua_2^2\right| \leq \begin{cases} 3-4u, & \text{if } u \leq \frac{1}{3}, \\ \dfrac{1}{3}+\dfrac{4}{9u}, & \text{if } \frac{1}{3} < u \leq \frac{2}{3}, \\ 1, & \text{if } \frac{2}{3} < u \leq 1, \\ 4u-3, & \text{if } u \geq 1, \end{cases} \tag{6.33}$$

where for each u real, there is a function in $\mathcal{K}$ such that the equality holds.

They also proved the Fekete-Szegö problem for functions in the class of starlike of order α as follows:

Theorem 147 ([240]) *Let $f \in S^*(\alpha), 0 \leq \alpha < 1$ and of the form (6.11), then for u real*

$$\left|a_3 - ua_2^2\right| \leq (1-\alpha)\max\{1, |3-2\alpha-4u(1-\alpha)|\}. \tag{6.34}$$

The study of the Fekete-Szegő functional has continued to receive attention from many researchers for more general classes of functions with varied success. For instance, Koepf (see [254], [255]) established the Fekete-Szegő inequalities for the class $\mathcal{K}(\alpha)$, while London [292] extended the results of Koepf to the strongly class of close-to-convex functions of order α. Meanwhile, Darus and Thomas (see [108], [109], [110]) solved the Fekete-Szegő problems for the strongly class of close-to-convex functions and the class M^α of α-logarithmic convex functions. A similar result for the close-to-convex functions of order α and type β has been proved by Ibrahim and Darus [206]. For other results related to the Fekete-Szegő problem, see ([63], [155], [234], [314], [347], [370], [392]) and the references which are cited therein.

The remainder of this chapter discusses the Sălăgean differential operator and its related subclasses of functions defined involving this operator.

6.6 Sălăgean Differential Operator and Related Subclasses

For functions $f(z)$ belonging to the class A of analytic functions in the unit disk D, Sălăgean [383] was introduced the differential operator $D^n f$, $n \in \mathbb{N}_0$, which is popularly known as *Sălăgean differential operator*. In this section, we discuss this well-known differential operator and its generalization, and provide some subclasses of analytic functions that are characterized involving this operator.

Definition 148 ([383]) *For a function $f(z) \in A$, $n \in \mathbb{N}_0$, the Sălăgean differential operator $D^n : A \to A$, is defined by*

$$D^n f(z) = D\left[D^{n-1} f(z)\right] = z\left[D^{n-1} f(z)\right]', \tag{6.35}$$

where

$$D^0 f(z) = f(z) \quad \text{and} \quad D^1 f(z) = zf'(z). \tag{6.36}$$

The operator D^n has been employed by various authors to defined several subclasses of analytic and univalent functions (see for instance [41], [69], [180], [232], [339], [383], [343]).

The following operator was introduced by Al-Oboudi [17], which generalized the Sălăgean differential operator.

Definition 149 ([17]) *For a function $f(z) \in A$, $n \in \mathbb{N}_0$ and $\lambda \geq 1$, the generalized Sălăgean operator $D_\lambda^n : A \to A$, is defined by*

$$\begin{aligned} D_\lambda^n f(z) &= D_\lambda\left[D_\lambda^{n-1} f(z)\right] \\ &= (1-\lambda) D_\lambda^{n-1} f(z) + \lambda z\left[D_\lambda^{n-1} f(z)\right]', \end{aligned} \tag{6.37}$$

where

$$D_\lambda^0 f(z) = f(z), \tag{6.38}$$

$$D_\lambda^1 f(z) = D_\lambda f(z) = (1-\lambda) f(z) + \lambda z f'(z). \tag{6.39}$$

We note that $D_1^n f(z) = D^n f(z)$. Some properties of these operators have been investigated by several authors (see for instance [207], [228], [243]).

We observe that for $f(z) \in A$ and of the form (6.11), and applying the operator (6.35) for (6.11), we have

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \tag{6.40}$$

for $z \in D$, $n \in \mathbb{N}_0$. Also, suppose that $\alpha > 0$, then form (6.11) we can write

$$f(z)^\alpha = \left(z + \sum_{k=2}^{\infty} a_k z^k \right)^\alpha. \tag{6.41}$$

Using *binomial expansion* for the function (6.41), we get

$$f(z)^\alpha = z^\alpha + \sum_{k=2}^{\infty} A_k(\alpha) z^{\alpha+k-1}, \tag{6.42}$$

where the coefficients $A_k(\alpha)$, $k \in \{2, 3, \ldots\}$ with $A_k(1) = a_k$, depend on the coefficients a_k of $f(z)$ and the parameter $\alpha > 0$. Now, if we apply the operator (6.35) for the function (6.42), then we obtain

$$D^n f(z)^\alpha = \alpha^n z^\alpha + \sum_{k=2}^{\infty} (\alpha + k - 1)^n A_k(\alpha) z^{\alpha+k-1}, \tag{6.43}$$

for $z \in D$, $\alpha > 0$, $n \in \mathbb{N}_0$, where $D^0 f(z)^\alpha = f(z)^\alpha$ and $D^1 f(z)^\alpha = z\left[f(z)^\alpha\right]'$.

Similarly, for $f(z) \in A$ and of the form (6.11), we observe that from (6.37) - (6.39)

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\lambda]^n a_k z^k, \tag{6.44}$$

for $z \in D$, $\lambda \geq 1$, $n \in \mathbb{N}_0$. Also, on utilizing the differential operator (6.37) for the function (6.42), we get

$$\begin{aligned} D_\lambda^n f(z)^\alpha &= [1 + (\alpha - 1)\lambda]^n z^\alpha \\ &\quad + \sum_{k=2}^{\infty} [1 + (\alpha + k - 2)\lambda]^n A_k(\alpha) z^{\alpha+k-1}, \end{aligned} \tag{6.45}$$

for $\alpha > 0$, $\lambda \geq 1$, $n \in \mathbb{N}_0$, $z \in D$.

Many subclasses of analytic functions, which are characterized involving the Sălăgean differential operator (6.35), have been explored by numerous researchers. Sălăgean [383] used the operator introduced the class S_n, $n \in \mathbb{N}_0$, which generalized the convexity and starlikeness of analytic and univalent functions in the unit disk. He defined that a function $f(z) \in A$ belongs to the class S_n, $n \in \mathbb{N}_0$ if and only if it satisfies the condition

$$\operatorname{Re}\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) > 0, \tag{6.46}$$

for $z \in D$. We note that $S_0 = S^*$ and $S_1 = C$, that is the standard class of starlike and convex functions, respectively. The class S_n of order α, i.e., $S_n(\alpha)$, $0 \leq \alpha < 1$, has been studied by Owa et al. ([349], [350]), Lin and Owa [285] and others.

In 2005, Oros [348] defined the class $M_n(\alpha)$ consisting of all analytic functions in D and satisfying the inequality

$$\operatorname{Re}\left[D^n f(z)\right]' > \alpha, \tag{6.47}$$

for some α $(0 \leq \alpha < 1)$, $z \in D$, $n \in \mathbb{N}_0$. This class has been further studied by Tăut, Oros and Şendruţiu [415]. Halim in [180] introduced and studied the a generalization of analytic functions satisfying (6.26) as follows:

$$\operatorname{Re}\left(\frac{D^n f(z)^\alpha}{z^\alpha}\right) > 0, \tag{6.48}$$

for $\alpha > 0$, $z \in D$. He denoted this class of functions by $\mathcal{B}_n(\alpha)$. Obviously, this class reduces to the class of the Bazilević functions (6.26) with logarithmic growth studied in ([338], [400], [418], [419]), when $n = 1$. The class $\mathcal{B}_0(\alpha)$ was initiated by Yamaguchi [435].

Opoola [343] further generalized functions defined by geometric condition (6.48) by choosing a real number β for $0 \leq \beta < 1$ that will be discussed in details in the following chapter.

Chapter 7

Properties for Certain Subclasses of Analytic Functions

Chapter 7 is devoted to some results concerning analytic, univalent and normalized functions on a unit disk. The class $T_n^\alpha(\beta)$, $\alpha > 0$, $0 \leq \beta < 1$, $n \in \mathbb{N}_0$, of analytic and univalent functions, which was introduced by Opoola [343] in 1994, is one of the interested subclasses of functions throughout this chapter. The properties of functions in the class $T_n^\alpha(\beta)$ have been explored by numerous researchers in the last few decades. In this chapter, we present further properties of functions in this class.

Section 7.1 discusses some known results concerning the class $T_n^\alpha(\beta)$ that have been proved by Opoola [343], Opoola et al. [344], Babalola and Opoola [42] and others. In Section 7.2, we provide some other properties of functions in the this class. The properties of functions in the classes $\widetilde{T}_n^\alpha(\beta)$ and $T_n^\alpha(\beta, \lambda)$, $\lambda \geq 1$ are also investigated. The coefficient bounds for functions in the strongly class of Opoola's functions, $\widetilde{T}_n^\alpha(\beta)$, $\alpha > 0$, $0 < \beta \leq 1$, $n \in \mathbb{N}_0$, have been presented in Section 7.3. Finally, we end this chapter by providing some results on Fekete-Szegő problem for a certain subclass of analytic and univalent functions.

All the results presented in this chapter, are mainly taken from the author's research papers in collaboration with Darus, Dragomir and Joseph (see [210], [211], [212]).

7.1 Introduction

In [343], Opoola introduced the subclass $T_n^\alpha(\beta)$ of analytic functions, which are defined involving the Sălăgean differential operator (6.35) as follows (see also [39], [40], [42]):

Definition 150 ([343]) *A function $f(z) \in T_n^\alpha(\beta)$ if and only if it satisfies the condition*

$$\operatorname{Re}\left\{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\right\} > \beta, \tag{7.1}$$

for $\alpha > 0$ is real, $0 \le \beta < 1$, $z \in D$, where D^n, $n \in \mathbb{N}_0$ is the Sălăgean differential operator and the powers in (7.1) meaning principal determinations only.

The geometric condition (7.1) slightly modifies the one given originally in [343] (see also [41], [345]) with the number α^n is considered. The class $T_n^\alpha(\beta)$ was studied by Opoola [343] as well as by Babalola [39], Babalola and Opoola (see [40], [41]), Halim [181], Oladipo et al. [341] and others (see [344], [345], [346]). Some interesting properties of functions in this class were established in the literature. For instance, Opoola [343] proved that the class T_n^α is a subclass of univalent functions, and the members of the family $T_n^\alpha(\beta)$ satisfied the inclusion relations, $T_{n+1}^\alpha(\beta) \subset T_n^\alpha(\beta)$, $\alpha > 0$, $n \in \mathbb{N}_0$. In [344], Opoola et al. showed that if $f(z)$ belongs to the class $T_n^\alpha(\beta)$, then the coefficient bounds are given by

$$|a_2| \le \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+1)^n} \quad \text{for } \alpha > 0, \tag{7.2}$$

$$|a_3| \le \begin{cases} 2(1-\beta)\alpha^{n-1}\left[\dfrac{1}{(\alpha+2)^n} - \dfrac{(\alpha-1)^{n-1}(1-\beta)}{(\alpha+1)^{2n}}\right], & \text{for } 0<\alpha<1, \\ \dfrac{2(1-\beta)\alpha^{n-1}}{(\alpha+2)^n}, & \text{for } \alpha \ge 1. \end{cases} \tag{7.3}$$

The coefficient bounds for $|a_4|$ and $|a_5|$ are investigated as well. Earlier results, Halim [180] provided the coefficient bounds for $|a_n|$, $n = 2, 3, 4$ for the class

$T_n^\alpha(0) = B_n(\alpha)$. Recently, Babalola and Opoola [42] obtained certain coefficient inequalities and solved the Fekete-Szegö problem for functions in the class $T_n^\alpha(\beta)$ as well.

One of the aims of this chapter is to establish some other properties concerning functions in the class $T_n^\alpha(\beta)$. Instead of this class, in this chapter, we also studied the subclasses of analytic functions, which are defined as follows.

Definition 151 *Let $\widetilde{T}_n^\alpha(\beta)$ denote the subclass of A consisting of analytic functions, which satisfy the condition*

$$\left|\arg\left\{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\right\}\right| \leq \frac{\beta\pi}{2}, \tag{7.4}$$

for some $\alpha > 0$, $0 < \beta \leq 1$, $z \in D$, where D^n, $n \in \mathbb{N}_0$ is the Sălăgean differential operator, and all the index in (7.4) meant principal determination only.

Using the Al-Oboudi differential operator (6.37), we give the definition of a more general class of Opoola's functions.

Definition 152 *Let $T_n^\alpha(\beta, \lambda)$ denote the subclass of A consisting of analytic functions, which satisfy the inequality*

$$\operatorname{Re}\left\{\frac{D_\lambda^n f(z)^\alpha}{\alpha^n z^\alpha}\right\} > \beta, \tag{7.5}$$

for some $\alpha > 0$, $0 \leq \beta < 1$, $z \in D$ and D_λ^n, $\lambda \geq 1$, $n \in \mathbb{N}_0$ is the Al-Oboudi differential operator.

The class $T_n^\alpha(\beta, \lambda)$ serves as a new generalization of many known subclasses of analytic functions in the direction by means of various choices of the parameters involved. For instance, several subclasses of functions are given in the following remark.

Remark 153 *(a) We note that $\widetilde{T}_n^\alpha(1) := T_n^\alpha(0)$.*

(ii) For $\lambda = 1$ in (7.5), we have $T_n^\alpha(\beta, 1) := T_n^\alpha(\beta)$, which is the class of functions studied by Opoola [343] and others (see [41], [341], [345]).

(iii) For $\lambda = 1$, $\beta = 0$ in (7.5), we have $T_n^\alpha(0, 1) := T_n^\alpha(0) = B_n(\alpha)$, which is the class of functions studied by Halim in [180].

(iv) For $\lambda = 1$, $\beta = 0$, $n = 1$ *in (7.5), we have* $T_1^\alpha(0,1) := T_1^\alpha(0) = B_1(\alpha)$, *which is the class of functions studied by Bazilević [54] and Singh [400].*

(v) For $\alpha = 1$, $\lambda = 1$, $\beta = 0$, $n = 0$ *in (7.5), we have* $T_0^\alpha(0,1) := T_0^\alpha(0) = S_0$, *which is the class of functions studied by Yamaguchi in [435].*

In order to prove the main results, we shall need the following well-known lemma due to Hayami et al. [188].

Lemma 154 *A function* $p(z) \in \mathcal{P}$ *satisfies the condition*

$$\operatorname{Re}[p(z)] > 0, \;\; z \in D \tag{7.6}$$

if and only if

$$p(z) \neq \frac{\psi - 1}{\psi + 1}, \;\; z \in D, \;\; \psi \in \mathbb{C}, \;\; |\psi| = 1. \tag{7.7}$$

The following result is well-known as *Schwarz Lemma* (see [6, p. 135], [144, p. 3]).

Lemma 155 *Let* $f(z)$ *be an analytic function in the unit disk* D*, with* $f(0) = 0$ *and* $|f(z)| < 1$*. Then,* $|f'(0)| \leq 1$ *and* $|f(z)| \leq |z|$ *in* D*. Strict inequality holds in both estimates unless* $f(z)$ *is a rotation of the disk:* $f(z) = e^{i\theta}z$*. If* $|f(z)| = |z|$ *for some* $z \neq 0$*, then* $f(z) = cz$*, with a constant* c *of absolute value* 1.

7.2 The Properties

In this section, we establish some other properties of functions in the class $T_n^\alpha(\beta)$. The properties of functions in the class $\widetilde{T}_n^\alpha(\beta)$ and $T_n^\alpha(\beta,\lambda)$ are also presented. First, we prove a sufficient condition for functions in the class $T_n^\alpha(\beta)$.

Theorem 156 (Ibrahim, Darus, Dragomir and Joseph [212]) *A function* $f(z) \in A$ *is in the class* $T_n^\alpha(\beta)$ *if and only if*

$$1 + \sum_{k=2}^{\infty} Q_k z^{k-1} \neq 0, \tag{7.8}$$

where

$$Q_k = \frac{(\psi+1)}{2(1-\beta)} \left(\frac{\alpha+k-1}{\alpha} \right)^n A_k(\alpha), \tag{7.9}$$

for some $\alpha > 0$, $0 \leq \beta < 1$, $\psi \in \mathbb{C}$ *with* $|\psi| = 1$.

Proof. From Definition 150, it suggests that there exists a function $p(z) \in \mathcal{P}$ such that

$$\frac{D^n f(z)^\alpha}{z^\alpha} = \alpha^n [\beta + (1-\beta) p(z)], \tag{7.10}$$

for $z \in D$, $\alpha > 0$, $0 < \beta \leq 1$, $n \in \mathbb{N}_0$. Upon setting

$$p(z) = \frac{\dfrac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - \beta}{1-\beta}, \tag{7.11}$$

then from Lemma 154, we have

$$\frac{\dfrac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - \beta}{1-\beta} \neq \frac{\psi-1}{\psi+1}, \tag{7.12}$$

for $z \in D$, $\psi \in \mathbb{C}$ with $|\psi| = 1$, which is equivalent with

$$(\psi+1) D^n f(z)^\alpha - [(\psi+1)\beta + (1-\beta)(\psi-1)] \alpha^n z^\alpha \neq 0. \tag{7.13}$$

Substituting (6.43) into (7.13), we get

$$\begin{aligned} &(\psi+1) \left[\alpha^n z^\alpha + \sum_{k=2}^{\infty} (\alpha+k-1)^n A_k(\alpha) z^{\alpha+k-1} \right] \\ &- [(\psi+1)\beta + (1-\beta)(\psi-1)] \alpha^n z^\alpha \neq 0, \end{aligned} \tag{7.14}$$

which gives us that

$$2(1-\beta) \alpha^n z^\alpha + \sum_{k=2}^{\infty} (\psi+1)(\alpha+k-1)^n A_k(\alpha) z^{\alpha+k-1} \neq 0. \tag{7.15}$$

Now, dividing both side of (7.15) by $2(1-\beta)\alpha^n z^\alpha \neq 0$, we obtain

$$1+\sum_{k=2}^{\infty}\frac{(\psi+1)}{2(1-\beta)}\left(\frac{\alpha+k-1}{\alpha}\right)^{n}A_{k}(\alpha)\,z^{k-1}\neq 0, \tag{7.16}$$

for any $\psi\in\mathbb{C}$ such that $|\psi|=1$, $\alpha>0$, $0\leq\beta<1$, $z\in D$, $n\in\mathbb{N}_0$. ■

Corollary 157 *A function $f(z)\in A$ is in the class $\mathcal{B}_n(\alpha):=T_n^{\alpha}(0)$ if and only if*

$$1+\sum_{k=2}^{\infty}Q_k z^{k-1}\neq 0,$$

where

$$Q_k=\frac{(\psi+1)}{2}\left(\frac{\alpha+k-1}{\alpha}\right)^{n}A_k(\alpha), \tag{7.17}$$

for any $\psi\in\mathbb{C}$ with $|\psi|=1$, $\alpha>0$, $z\in D$, $n\in\mathbb{N}_0$.

Remark 158 *The Corollary 157 has been proved by Singh et al. [401].*

The following property of functions in the class $T_n^{\alpha}(\beta)$ is also established.

Theorem 159 (Ibrahim, Darus, Dragomir and Joseph [212]) *Let $f(z)$ belongs to the class $T_n^{\alpha}(\beta)$. Then, there exists an analytic function $\phi(z)$ with $|\phi(z)|\leq 1$, $z\in D$ such that*

$$\frac{D^n f(z)^{\alpha}}{\alpha^n z^{\alpha}}=2\beta-1+\frac{2(1-\beta)}{1-z\phi(z)}, \tag{7.18}$$

for $z\in D$, $\alpha>0$, $0\leq\beta<1$, $n\in\mathbb{N}_0$.

Proof. Let us define the functions $A(z)$ and $B(z)$ as follows:

$$A(z)=\frac{D^n f(z)^{\alpha}}{\alpha^n z^{\alpha}}-\beta \tag{7.19}$$

and

$$B(z)=z\left(\frac{A(z)-(1-\beta)}{A(z)+(1-\beta)}\right), \tag{7.20}$$

for $\alpha > 0$, $0 \leq \beta < 1$, $z \in D$, $n \in \mathbb{N}_0$. Substituting the equation (7.19) into (7.20), we get

$$B(z) = z \left(\frac{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - 1}{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - (2\beta - 1)} \right), \tag{7.21}$$

for any $z \in D$. $B(z)$ is an analytic function for $z \in D$. Also, since $f(0) - 0$ and $f'(0) = 1$, we have that $B(0) = 0$ and $|B(z)| < 1$, for $z \in D$. Hence, by Schwarz's Lemma (Lemma 155), $|B(z)| < |z|$ for $z \in D$, which gives that

$$\left| \frac{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - 1}{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - (2\beta - 1)} \right| < |z|, \; z \in D, \tag{7.22}$$

or equivalently,

$$\left(\frac{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - 1}{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - (2\beta - 1)} \right) = z\phi(z), \tag{7.23}$$

where $\phi(z)$ is analytic, and $|\phi(z)| \leq 1$ for $z \in D$. Therefore

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - 1 = z\phi(z) \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - (2\beta - 1)\, z\phi(z). \tag{7.24}$$

Solving for $D^n f(z)^\alpha / \alpha^n z^\alpha$, we get

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = \frac{1 - (2\beta - 1)\, z\phi(z)}{1 - z\phi(z)} = (2\beta - 1) + \frac{2\,(1 - \beta)}{1 - z\phi(z)}. \tag{7.25}$$

Thus, we obtain the desired result (7.18). ■

The following result is called the integral representation theorem for functions in the class $T_n^\alpha(\beta)$, which provides further property of functions in this class.

Theorem 160 (Ibrahim, Darus, Dragomir and Joseph [212]) *Let $f(z)$ belongs to the class $T_n^\alpha(\beta)$. Then, there exists an analytic function $\phi(z)$, with $|\phi(z)| \leq 1$, $z \in D$ such that*

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = \frac{1}{\alpha^n z^\alpha} \exp \int_0^z \left(\frac{\alpha}{t} - C(t)\right) dt, \tag{7.26}$$

for $\alpha > 0$, $0 \leq \beta < 1$, $n \in \mathbb{N}_0$, *where*

$$C(t) = \frac{2[\beta - 1][z\phi'(t) + \phi(t)]}{1 - 2\beta t\phi(t) + (2\beta - 1)t^2\phi^2(t)}. \tag{7.27}$$

Proof. Let $f(z) \in T_n^\alpha(\beta)$. Then, from Theorem 159, we have

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = (2\beta - 1) + \frac{2(1-\beta)}{1 - z\phi(z)} = \frac{1 - (2\beta - 1) z\phi(z)}{1 - z\phi(z)}, \tag{7.28}$$

Taking the logarithmic differentiation, we get from (7.28)

$$\frac{[D^n f(z)^\alpha]'}{D^n f(z)^\alpha} = \frac{\alpha}{z} - \frac{2[\beta - 1][z\phi'(z) + \phi(z)]}{1 - 2\beta z\phi(z) + (2\beta - 1)z^2\phi^2(z)}. \tag{7.29}$$

Now, integrating both sides of (7.29) along the line segment from 0 to z, we obtain

$$\ln[D^n f(z)^\alpha] = \int_0^z \left(\frac{\alpha}{t} - \frac{2[\beta - 1][t\phi'(t) + \phi(t)]}{1 - 2\beta t\phi(t) + (2\beta - 1)t^2\phi^2(t)}\right) dt. \tag{7.30}$$

This gives us that

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = \frac{1}{\alpha^n z^\alpha} \exp \int_0^z \left(\frac{\alpha}{t} - C(t)\right) dt, \tag{7.31}$$

for $z \in D$, $\alpha > 0$, $0 \leq \beta < 1$, $n \in \mathbb{N}_0$, where

$$C(t) = \frac{2[\beta - 1][t\phi'(t) + \phi(t)]}{1 - 2\beta t\phi(t) + (2\beta - 1)t^2\phi^2(t)}, \tag{7.32}$$

for an analytic function $\phi(z)$ with $|\phi(z)| \leq 1$, $z \in D$. Thus, the proof of Theorem 160 is completed. ■

Theorem 161 (Ibrahim, Darus and Dragomir [211]) *A function $f(z) \in A$ is in the class $\widetilde{T}_n^{\alpha}(\beta)$ if and only if*

$$1+\sum_{k=2}^{\infty} Q_k z^{k-1} \neq 0, \tag{7.33}$$

where

$$Q_k = \frac{(\psi+1)^{\beta}}{\left[(\psi+1)^{\beta}-(\psi-1)^{\beta}\right]}\left(\frac{\alpha+k-1}{\alpha}\right)^{n} A_k(\alpha), \tag{7.34}$$

for some α $(\alpha>0)$, $0<\beta\leq 1$, $\psi \in C$, $|\psi|=1$.

Proof. From (7.4), it suggests that there exists a function $p(z) \in \mathcal{P}$ such that

$$\frac{D^n f(z)^{\alpha}}{z^{\alpha}} = \alpha^n \left[p(z)\right]^{\beta}, \tag{7.35}$$

for $z \in D$, $\alpha > 1$, $0<\beta\leq 1$. Upon setting

$$p(z) = \left(\frac{D^n f(z)^{\alpha}}{\alpha^n z^{\alpha}}\right)^{1/\beta}, \tag{7.36}$$

then from Lemma 154, we have

$$\frac{D^n f(z)^{\alpha}}{\alpha^n z^{\alpha}} \neq \frac{(\psi-1)^{\beta}}{(\psi+1)^{\beta}}, \tag{7.37}$$

for $z \in D$, $\psi \in \mathbb{C}$ with $|\psi|=1$. It is equivalent with

$$(\psi+1)^{\beta} D^n f(z)^{\alpha} - (\psi-1)^{\beta}\alpha^n z^{\alpha} \neq 0. \tag{7.38}$$

Substituting (6.43) into (7.38) yields

$$(\psi+1)^{\beta}\left[\alpha^n z^{\alpha} + \sum_{k=2}^{\infty}(\alpha+k-1)^n A_k(\alpha) z^{\alpha+k-1}\right] - (\psi-1)^{\beta}\alpha^n z^{\alpha} \neq 0, \tag{7.39}$$

which give us that

$$\begin{aligned}&\left[(\psi+1)^{\beta}-(\psi-1)^{\beta}\right]\alpha^{n}z^{\alpha}\\&+(\psi+1)^{\beta}\textstyle\sum_{k=2}^{\infty}(\alpha+k-1)^{n}A_{k}(\alpha)\,z^{\alpha+k-1}\neq 0.\end{aligned} \tag{7.40}$$

Now, dividing both sides of (7.40) by $\left[(\psi+1)^{\beta}-(\psi-1)^{\beta}\right]\alpha^{n}z^{\alpha}\neq 0$, we obtain

$$1+\sum_{k=2}^{\infty}\frac{(\psi+1)^{\beta}}{\left[(\psi+1)^{\beta}-(\psi-1)^{\beta}\right]}\left(\frac{\alpha+k-1}{\alpha}\right)^{n}A_{k}(\alpha)\,z^{k-1}\neq 0, \tag{7.41}$$

for all $z\in D$, $\psi\in\mathbb{C}$ such that $|\psi|=1$. Thus, the proof of Theorem 161 is completed. ■

The similar results to (7.9), (7.18) and (7.26) for more general class $T_{n}^{\alpha}(\beta,\lambda)$ also have been obtained.

Theorem 162 (Ibrahim, Darus, Dragomir and Joseph [212]) *A function $f(z)\in A$ is in the class $T_{n}^{\alpha}(\beta,\lambda)$ if and only if*

$$1+\sum_{k=2}^{\infty}Q_{k}z^{k-1}\neq 0, \tag{7.42}$$

where

$$Q_{k}=\frac{(\psi+1)\left[1+(\alpha+k-2)\lambda\right]^{n}}{(\psi+1)\left[1+(\alpha-1)\lambda\right]^{n}-(2\beta+\psi-1)\alpha^{n}}A_{k}(\alpha)\,, \tag{7.43}$$

for some $\alpha>0$, $\lambda\geq 1$, $0<\beta\leq 1$, $\psi\in\mathbb{C}$ with $|\psi|=1$.

Proof. From Definition 152, it suggests that there exists a function $p(z)\in\mathcal{P}$ such that

$$\frac{D_{\lambda}^{n}f(z)^{\alpha}}{\alpha^{n}z^{\alpha}}=\left[\beta+(1-\beta)\,p(z)\right], \tag{7.44}$$

for $z\in D$, $\alpha>0$, $\lambda\geq 1$, $0\leq\beta<1$, $n\in\mathbb{N}_{0}$. Upon setting

$$p(z)=\frac{\frac{D_{\lambda}^{n}f(z)^{\alpha}}{\alpha^{n}z^{\alpha}}-\beta}{1-\beta}, \tag{7.45}$$

Hence from Lemma 154, we have

$$\frac{\frac{D_\lambda^n f(z)^\alpha}{\alpha^n z^\alpha} - \beta}{1-\beta} \neq \frac{\psi - 1}{\psi + 1}, \tag{7.46}$$

for $z \in D$, $\psi \in \mathbb{C}$ with $|\psi| = 1$, which is equivalent with

$$(\psi + 1)\, D_\lambda^n f(z)^\alpha - [(\psi + 1)\, \beta + (\psi - 1)\,(1-\beta)]\, \alpha^n z^\alpha \neq 0. \tag{7.47}$$

Substituting (6.45) into (7.47), we get

$$\begin{aligned} &(\psi + 1) \left[[1 + (\alpha - 1)\,\lambda]^n\, z^\alpha + \sum_{k=2}^{\infty} [1 + (\alpha + k - 2)\,\lambda]^n\, A_k\,(\alpha)\, z^{\alpha+k-1} \right] \\ &- (2\beta + \psi - 1)\, \alpha^n z^\alpha \neq 0. \end{aligned} \tag{7.48}$$

Hence, it gives us that

$$\begin{aligned} &[(\psi + 1)\,[1 + (\alpha - 1)\,\lambda]^n - (2\beta + \psi - 1)\,\alpha^n]\, z^\alpha \\ &+ \sum_{k=2}^{\infty} (\psi + 1)\,[1 + (\alpha + k - 2)\,\lambda]^n\, A_k\,(\alpha)\, z^{\alpha+k-1} \neq 0. \end{aligned} \tag{7.49}$$

Now, dividing both side of (7.49) by

$$[(\psi + 1)\,[1 + (\alpha - 1)\,\lambda]^n - (2\beta + \psi - 1)\,\alpha^n]\, z^\alpha \neq 0,$$

we obtain

$$1 + \sum_{k=2}^{\infty} \frac{(\psi + 1)\,[1 + (\alpha + k - 2)\,\lambda]^n}{(\psi + 1)\,[1 + (\alpha - 1)\,\lambda]^n - (2\beta + \psi - 1)\,\alpha^n} A_k\,(\alpha)\, z^{k-1} \neq 0, \tag{7.50}$$

for all $\psi \in \mathbb{C}$ such that $|\psi| = 1$, $\alpha > 0$, $\lambda \geq 1$, $0 \leq \beta < 1$, $z \in D$, $n \in \mathbb{N}_0$. ■

Remark 163 *In particular, if $\lambda = 1$ in Theorem 162, then it reduce to the result as shown in Theorem 156. Also, we have Corollary 157 for $\lambda = 1$ and $\beta = 0$.*

The following corollaries also hold for some particular values of the parameters involved in Theorem 162.

Corollary 164 *A function $f(z) \in A$ is in the class $B_1(\alpha)$ if and only if*

$$1 + \sum_{k=2}^{\infty} Q_k z^{k-1} \neq 0,$$

where

$$Q_k = \frac{(\psi + 1)[\alpha + k - 1]}{2\alpha} A_k(\alpha), \tag{7.51}$$

for any $\psi \in \mathbb{C}$ with $|\psi| = 1$, $\alpha > 0$, $z \in D$.

Corollary 165 *A function $f(z) \in A$ is in the class S_0 if and only if*

$$1 + \sum_{k=2}^{\infty} Q_k z^{k-1} \neq 0,$$

where

$$Q_k = \frac{(\psi + 1)}{2\alpha} A_k(\alpha), \tag{7.52}$$

for any $\psi \in \mathbb{C}$ with $|\psi| = 1$, $\alpha > 0$, $z \in D$, $n \in \mathbb{N}_0$.

Theorem 166 (Ibrahim, Darus, Dragomir and Joseph [212]) *If $f(z)$ belong to the class $T_n^{\alpha}(\beta, \lambda)$, then there exists an analytic function $\phi(z)$ with $|\phi(z)| \leq 1$, $z \in D$ such that*

$$\frac{D_{\lambda}^{n} f(z)^{\alpha}}{\lambda \alpha^{n} z^{\alpha}} = (2\beta - 1) + \frac{2(1 - \beta)}{1 - z\phi(z)}, \tag{7.53}$$

for $z \in D$, $\alpha > 0$, $\lambda \geq 1$, $0 \leq \beta < 1$, $n \in \mathbb{N}_0$.

Proof. Define the functions $F(z)$ and $G(z)$ as follows:

$$F(z) = \frac{D_{\lambda}^{n} f(z)^{\alpha}}{\lambda \alpha^{n} z^{\alpha}} - \beta \tag{7.54}$$

and

$$G(z) = z \left(\frac{F(z) - (1 - \beta)}{F(z) + (1 - \beta)} \right). \tag{7.55}$$

Substituting the equation (7.54) into (7.55), we get

$$G(z) = z\left(\frac{\dfrac{D_\lambda^n f(z)^\alpha}{\lambda\alpha^n z^\alpha} - 1}{\dfrac{D_\lambda^n f(z)^\alpha}{\lambda\alpha^n z^\alpha} - (2\beta - 1)}\right), \; z \in D. \tag{7.56}$$

$G(z)$ is an analytic function for $z \in D$. Also, since $f(0) - 0$ and $f'(0) = 1$, we have that $G(0) = 0$ and $|G(z)| < 1$ for $z \in D$. Hence, by Schwarz's Lemma (Lemma 155), $|G(z)| < |z|$, $z \in D$, gives us that

$$\left|\frac{\dfrac{D_\lambda^n f(z)^\alpha}{\lambda\alpha^n z^\alpha} - 1}{\dfrac{D_\lambda^n f(z)^\alpha}{\lambda\alpha^n z^\alpha} - (2\beta - 1)}\right| < |z|, \quad z \in D, \tag{7.57}$$

or equivalently,

$$\left(\frac{\dfrac{D_\lambda^n f(z)^\alpha}{\lambda\alpha^n z^\alpha} - 1}{\dfrac{D_\lambda^n f(z)^\alpha}{\lambda\alpha^n z^\alpha} - (2\beta - 1)}\right) = z\phi(z), \tag{7.58}$$

where $\phi(z)$ is analytic, and $|\phi(z)| \leq 1$ for $z \in D$. Therefore

$$\begin{aligned}\frac{D_\lambda^n f(z)^\alpha}{\lambda\alpha^n z^\alpha} &= \frac{[1 - (2\beta - 1)\, z\phi(z)]}{1 - z\phi(z)} \\ &= 2\beta - 1 + \frac{2\,(1-\beta)}{1 - z\phi(z)},\end{aligned} \tag{7.59}$$

which completes the proof of Theorem 166. ■

Theorem 167 (Ibrahim, Darus, Dragomir and Joseph [212]) *If $f(z)$ belongs to the class $T_n^\alpha(\beta, \lambda)$, then there exists an analytic function $\phi(z)$ with $|\phi(z)| \leq 1$, $z \in D$ such that*

$$\frac{D_\lambda^n f(z)^\alpha}{\alpha^n z^\alpha} = \frac{1}{\alpha^n z^\alpha} \exp \int_0^z \left(\frac{\alpha\lambda}{t} - E(t)\right) dt, \tag{7.60}$$

for $\alpha > 0$, $\lambda \geq 1$, $0 \leq \beta < 1$, $n \in \mathbb{N}_0$, where

$$E(t) = \frac{2\left[t\phi'(t) + \phi(t)\right]\left[\beta - 1\right]}{1 - 2\beta t\phi(t) + (2\beta - 1)\, t^2\phi^2(t)}. \tag{7.61}$$

Proof. Let $f(z) \in T_n^\alpha(\beta, \lambda)$. Then, from Theorem 166, we have

$$\frac{D_\lambda^n f(z)^\alpha}{\lambda\alpha^n z^\alpha} = \frac{1 - (2\beta - 1)\, z\phi(z)}{1 - z\phi(z)}. \tag{7.62}$$

Taking the logarithmic differentiation, we get from (7.62)

$$\frac{\left[D^n f(z)^\alpha\right]'}{D^n f(z)^\alpha} = \frac{\alpha\lambda}{z} - \frac{\left[z\phi'(z) + \phi(z)\right]\left[2\beta - 2\right]}{1 - 2\beta z\phi(z) + (2\beta - 1)\, z^2\phi^2(z)}. \tag{7.63}$$

Integrating the both sides of (7.29) along the line segment from 0 to z, we obtain

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = \frac{1}{\alpha^n z^\alpha} \exp \int_0^z \left(\frac{\alpha}{t} - E(t)\right) dt, \tag{7.64}$$

where

$$E(t) = \frac{2\left[t\phi'(t) + \phi(t)\right]\left[\beta - 1\right]}{1 - 2\beta t\phi(t) + (2\beta - 1)\, t^2\phi^2(t)}. \tag{7.65}$$

Thus, the proof of Theorem 167 is completed. ■

7.3 Coefficient Bounds

In this section, we establish certain coefficient bounds for functions in the Class $\widetilde{T}_n^\alpha(\beta)$.

Theorem 168 (Ibrahim, Darus and Dragomir [211]) *Let $f(z)$ given by (6.11) belongs to the class $\widetilde{T}_n^\alpha(\beta)$, $\alpha > 0$, $0 < \beta \leq 1$, $n \in \{0, 1, 2, ...\}$. Then, the following inequalities hold*

$$|a_2| \leq \frac{2\beta\alpha^{n-1}}{(\alpha+1)^n}; \quad \textit{for } \alpha > 0, \tag{7.66}$$

$$|a_3| \leq \begin{cases} \dfrac{2\beta\alpha^{n-1}}{(\alpha+2)^n} \\ \times\left(1+\dfrac{(\alpha+2)^n}{(\alpha+1)^{2n}}(1-\alpha)\,\alpha^{n-1}\beta\right); & \textit{for } 0<\alpha<1, \\ \dfrac{2\beta\alpha^{n-1}}{(\alpha+2)^n}; & \textit{for } \alpha \geq 1, \end{cases} \tag{7.67}$$

$$|a_4| \leq \begin{cases} F_1+F_2+F_3; & \textit{for } 0<\alpha<1, \\ F_1+F_2+F_4+F_5; & \textit{for } 1\leq\alpha<2, \\ F_1+F_2+F_4; & \textit{for } \alpha\geq 2, \end{cases} \tag{7.68}$$

where

$$F_1 = \frac{2\beta\alpha^{n-1}}{3(\alpha+3)^n}\left[3+2(\beta-1)(\beta-5)\right], \tag{7.69}$$

$$F_2 = \frac{4\beta^2(1-\alpha)^2\alpha^{3n-3}}{(1+\alpha)^{3n}}, \tag{7.70}$$

$$F_3 = \frac{4\beta^2(1-\alpha)\,\alpha^{2n-2}}{(\alpha+1)^n(\alpha+2)^n}, \tag{7.71}$$

$$F_4 = \frac{4\beta^2(\beta-1)(1-\alpha)\,\alpha^{2n-2}}{(\alpha+1)^n(\alpha+2)^n}, \tag{7.72}$$

$$F_5 = \frac{4\beta^3(1-\alpha)(\alpha-2)\,\alpha^{3n-3}}{3(\alpha+1)^{3n}}. \tag{7.73}$$

Proof. From Definition 151, for $f \in \widetilde{T}_n^{\alpha}(\beta)$, it suggests that there exists an analytic function $p(z) \in \mathcal{P}$ such that

$$\frac{D^n f(z)^\alpha}{z^\alpha} = \alpha^n\left[p(z)\right]^\beta, \tag{7.74}$$

for $\alpha > 0$, $0 < \beta \leq 1$, $z \in D$, $n \in \mathbb{N}_0$. Making use the Sălăgean differential operator (6.35), i.e., $D^n f(z)^\alpha$ as $z\left[D^{n-1}f(z)^\alpha\right]'$ and $p(z)$ given by the series (6.5), we get

$$\left[D^{n-1} f(z)^{\alpha}\right]' = \alpha^n z^{\alpha-1}\left[1+\sum_{i=1}^{\infty} c_i z^i\right]^{\beta}$$
$$= \alpha^n\left[z^{\alpha-1}+\mu_1 z^{\alpha}+\mu_2 z^{\alpha+1}+\mu_3 z^{\alpha+2}+\mu_4 z^{\alpha+3}+...\right], \quad (7.75)$$

where

$$\mu_1 = \beta_1 p_1, \quad (7.76)$$
$$\mu_2 = \beta_1 p_2+\beta_2 p_1^2, \quad (7.77)$$
$$\mu_3 = \beta_1 p_3+2\beta_2 p_1 p_2+\beta_3 p_1^3, \quad (7.78)$$
$$\mu_4 = \beta_1 p_4+2\beta_2 p_1 p_3+\beta_2 p_2^2+3\beta_3 p_1^2 p_2+\beta_4 p_1^4, \quad (7.79)$$

and

$$\beta_j = \binom{\beta}{j} = \frac{\beta!}{j!\,(\beta-j)!},\ j\in\{1,2,3,...\}. \quad (7.80)$$

Integrating both sides of (7.75) along the line segment from 0 to z, we obtain

$$\frac{D^{n-1} f(z)^{\alpha}}{z^{\alpha}} = \alpha^{n-1}\left[1+\frac{\mu_1\alpha}{(\alpha+1)}z+\frac{\mu_2\alpha}{(\alpha+2)}z^2\right.$$
$$\left.+\frac{\mu_3\alpha}{(\alpha+3)}z^3+\frac{\mu_4\alpha}{(\alpha+4)}z^4+...\right], \quad (7.81)$$

where $f(0)=0$.

Noting that $D^{n-1} f(z)^{\alpha} = z\left[D^{n-2} f(z)^{\alpha}\right]'$ and so on, and repeating the process as above, we are able to produce the following relation, which holds in general for any $k\in\{0,1,2,...\}$, that is,

$$\frac{D^{n-k} f(z)^{\alpha}}{z^{\alpha}} = \alpha^{n-k}\left[1+\frac{\mu_1\alpha^k}{(\alpha+1)^k}z+\frac{\mu_2\alpha^k}{(\alpha+2)^k}z^2\right.$$
$$\left.+\frac{\mu_3\alpha^k}{(\alpha+3)^k}z^3+\frac{\mu_4\alpha^k}{(\alpha+4)^k}z^4+...\right].$$

In particular, for $n=k$, we have the following equation:

$$\left(\frac{f(z)}{z}\right)^{\alpha}=1+\frac{\mu_1\alpha^n}{(\alpha+1)^n}z+\frac{\mu_2\alpha^n}{(\alpha+2)^n}z^2+\frac{\mu_3\alpha^n}{(\alpha+3)^n}z^3$$
$$+\frac{\mu_4\alpha^n}{(\alpha+4)^n}z^4+\dots\,. \tag{7.82}$$

Comparing the coefficients of z, z^2 and z^3 in (7.82), we get

$$\alpha_1 a_2=\frac{\mu_1\alpha^n}{(\alpha+1)^n}, \tag{7.83}$$

$$\alpha_1 a_3=\frac{\mu_2\alpha^n}{(\alpha+2)^n}-\alpha_2 a_2^2, \tag{7.84}$$

$$\alpha_1 a_4=\frac{\mu_3\alpha^n}{(\alpha+3)^n}-2\alpha_2 a_2 a_3-\alpha_3 a_2^3, \tag{7.85}$$

where

$$\alpha_j \quad = \quad \binom{\alpha}{j}=\frac{\alpha!}{j!\,(\alpha-j)!},\ j\in\{1,2,3,...\}\,. \tag{7.86}$$

Since $|p_1|\leq 2$ from Lemma 133, thus it follows from (7.83) that

$$|a_2|\leq\frac{2\beta\alpha^{n-1}}{(1+\alpha)^n}, \tag{7.87}$$

for $\alpha>0$, $0<\beta\leq 1$.

Next, we eliminate a_2 in (7.84) to get

$$a_3=\frac{\beta\alpha^{n-1}}{(\alpha+2)^n}p_2+\frac{\beta(\beta-1)\,\alpha^{n-1}}{2\,(\alpha+2)^n}p_1^2+\frac{(1-\alpha)\,\beta^2\alpha^{2n-2}}{(\alpha+1)^{2n}}p_1^2. \tag{7.88}$$

Again, from Lemma 133 with $|p_2|\leq 2$ and $0<\beta\leq 1$, we have

$$|a_3|\leq\frac{2\beta\alpha^{n-1}}{(\alpha+2)^n}\left(1+(\beta-1)+\frac{2\,|1-\alpha|\,(\alpha+2)^n\,\beta\alpha^{n-1}}{(\alpha+1)^{2n}}\right). \tag{7.89}$$

Considering the cases of α in the intervals $0<\alpha<1$ and $\alpha\geq 1$, then the both desired inequalities in (7.67) are easily obtained.

Now, we prove the inequalities in (7.68). Substituting the equalities (7.83) and (7.84) into (7.85), gives us that

$$
\begin{aligned}
a_4 &= \frac{\beta\alpha^{n-1}}{(\alpha+3)^n}p_3 + \frac{\beta(\beta-1)\alpha^{n-1}}{(\alpha+3)^n}p_1p_2 \\
&+ \frac{\beta(\beta-1)(\beta-2)\alpha^{n-1}}{6(\alpha+3)^n}p_1^3 \\
&+ \frac{\beta^2(1-\alpha)\alpha^{2n-2}}{(\alpha+1)^n(\alpha+2)^n}p_1p_2 + \frac{\beta^2(\beta-1)(1-\alpha)\alpha^{2n-2}}{2(\alpha+1)^n(\alpha+2)^n}p_1^3 \\
&+ \frac{\beta^3(1-\alpha)^2\alpha^{3n-3}}{2(\alpha+1)^{3n}}p_1^3 + \frac{\beta^3(1-\alpha)(\alpha-2)\alpha^{3n-3}}{6(\alpha+1)^{3n}}p_1^3. \qquad (7.90)
\end{aligned}
$$

Hence, for $\alpha > 0$, $0 < \beta \leq 1$, we obtain

$$
\begin{aligned}
|a_4| \leq\ & \frac{2\beta\alpha^{n-1}}{3(\alpha+3)^n}\left[3+2(\beta-1)(\beta-5)\right] \\
&+\frac{4\beta^3(1-\alpha)^2\alpha^{3n-3}}{(1+\alpha)^{3n}} + \frac{4\beta^2|(1-\alpha)|\alpha^{2n-2}}{(\alpha+1)^n(\alpha+2)^n} \\
&-\frac{4\beta^2|(1-\alpha)|(\beta-1)\alpha^{2n-2}}{(\alpha+1)^n(\alpha+2)^n} \\
&+\frac{4\beta^3|(1-\alpha)(\alpha-2)|\alpha^{3n-3}}{3(\alpha+1)^{3n}}, \qquad (7.91)
\end{aligned}
$$

by applying the well-known inequality $|p_k| \leq 2$, $k \in \{1, 2, ...\}$ from Lemma 133. Thus, the coefficient bounds $|a_4|$ depends on the parameter α in the intervals $(0, 1)$, $[1, 2)$ and $[2, \infty)$. Let us denote

$$
\begin{aligned}
F_1 &= \frac{2\beta\alpha^{n-1}}{3(\alpha+3)^n}\left[3+2(\beta-1)(\beta-5)\right], \\
F_2 &= \frac{4\beta^2(1-\alpha)^2\alpha^{3n-3}}{(1+\alpha)^{3n}}, \\
F_3 &= \frac{4\beta^2(1-\alpha)\alpha^{2n-2}}{(\alpha+1)^n(\alpha+2)^n}, \\
F_4 &= \frac{4\beta^2(\beta-1)(1-\alpha)\alpha^{2n-2}}{(\alpha+1)^n(\alpha+2)^n},
\end{aligned}
$$

$$F_5 = \frac{4\beta^3 (1-\alpha)(\alpha-2)\alpha^{3n-3}}{3(\alpha+1)^{3n}}.$$

Then, from (7.91) we get the desired inequalities (7.68), and the proof of Theorem 168 is completed. ■

Remark 169 *If we choose $\beta = 1$ in Theorem 168, we deduce the result proved by Halim in [181].*

7.4 Fekete-Szegő Inequalities

7.4.1 Introduction and Preliminary results

In this section, we consider functions $f : D \to G \subset \overline{\mathbb{C}}$ that are univalent in D, analytic at zero and have the expansion given by (6.11). The class of those functions, which map D into the domain whose complement with respect to G is convex, is denoted by C_0 and we call it the class of *concave univalent functions*. Since ∂G is a curve, the function $f(z)$ has continuous extension to $\overline{D}$ onto G, which is infinite at exactly one point on $T = \partial D$. We can choose the map such that $f(1) = \infty$. The class $C_0(1)$ consists of all concave univalent functions of the form (6.11) and normalized such that $f(1) = \infty$.

Another case of interest is when $\overline{\mathbb{C}} \backslash G$ is a bounded convex set. Then, the conformal map $f(z)$ of D onto G has a pole in D. The class $C_0(p)$ consists of all such functions of the form (6.11) and $f(p) = \infty$, where $0 < p < 1$. For the details discussion about this class of functions, we refer to the works of ([35], [36], [103], [291], [431]) and the references which are cited therein. In a recent work by Avkhadiev and Wirths [37] (see also [36], [64]), they consider a concave domain G with given angle $\pi\alpha$ at ∞, where $\alpha \in [1, 2]$, and if $\alpha = 1$, then G is a halfplane.

To be precise, we say that a function $f : D \to \overline{\mathbb{C}}$ belongs to the family $C_0(\alpha)$, $\alpha > 1$ if $f(z)$ satisfies the following conditions: (a) $f(z)$ is analytic in D with the standard normalization conditions (6.3). In addition, it satisfies $f(1) = \infty$; (b) $f(z)$ maps conformally onto a set whose complement with respect to $\overline{\mathbb{C}}$ is convex;

(c) The opening angle of $f(D)$ at ∞ is less than or equal to $\pi\alpha$, $\alpha \in (1,2]$. We observe that $C_0(2)$ contains the classes $C_0(\alpha)$, $\alpha \in (1,2]$.

In [36], Avkhadiev and Wirths showed that an analytic function in the class $C_0(\alpha)$, $\alpha \in (1,2]$ if and only if it satisfies the following condition

$$\operatorname{Re}\left(\frac{2}{\alpha-1}\left[\frac{(\alpha+1)}{2}\frac{1+z}{1-z}-1-z\frac{f''(z)}{f'(z)}\right]\right)>0. \tag{7.92}$$

The class $C_0(\alpha)$ was recently studied by Bhowmik, Ponnusamy and Wirths (see [64], [65]). In [65], they have used the above characterization (7.92) and proved the following theorem.

Theorem 170 ([65]) *Let $\alpha \in (1,2]$. A function $f(z) \in C_0(\alpha)$ if and only if there exists a starlike function $\phi \in S^*$ such that $f(z) = \Lambda_\phi(z)$ where*

$$\Lambda_\phi(z) = \int_0^z \frac{1}{(1-t)^{\alpha+1}}\left(\frac{t}{\phi(t)}\right)^{(\alpha-1)/2} dt, \tag{7.93}$$

and S^* denote the family of starlike functions $g \in S^*$ satisfying (6.14).

Theorem 171 ([65]) *Let $\alpha \in (1,2]$. A function $f(z) \in C_0(\alpha)$ if and only if there exists a function $s \in \Pi_{\alpha-1}$ such that*

$$f(z) = \int_0^z \frac{s(t)}{(1-t)^{\alpha+1}}dt. \tag{7.94}$$

In this section, we introduce the class $C_0(\alpha,n)$, $\alpha \in (1,2]$, $n \in \mathbb{N}_0$ consisting of all the concave univalent functions, which are defined as follows. A function $f(z) \in C_0(\alpha,n)$ if and only if there exists a function $\psi \in S_n$ such that $f(z) = \Lambda_\psi(z)$ where

$$\Lambda_\psi(z) = \int_0^z \frac{1}{(1-t)^{\alpha+1}}\left(\frac{t}{\psi(t)}\right)^{(\alpha-1)/2} dt, \tag{7.95}$$

and S_n denote the family of functions $g \in S_n$, which satisfy the condition (6.46). We note that $C_0(\alpha,0) = C_0(\alpha)$.

The purpose of this section is to determine the sharp upper bounds of the Fekete-Szegő functional $|a_3 - \mu a_2^2|$, for functions in the class $C_0(\alpha,n)$. In particular, we extend the result of the Fekete-Szegő problem established by Bhowmik et al. in [64], for a real and complex parameter μ. In order to prove our main results, we need to recall the following results.

Corollary 172 (See [207]) *Let $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in S_n$. Then*

$$|b_k| \leq k^{1-n}, \quad \textit{for } k = 2, 3, \ldots \tag{7.96}$$

and

$$\left|b_3 - \mu b_2^2\right| \leq \max\{1, |3 - 4\mu|\}. \tag{7.97}$$

7.4.2 Main Results and Their Proofs

The Fekete-Szegő inequalities $|a_3 - \lambda a_2^2|$ for a real and complex parameter λ are incorporated in the following theorems. First for a real λ, we have

Theorem 173 (Ibrahim, Darus and Dragomir [210]) *Let $f \in C_0(\alpha, n)$, $\alpha \in (1, 2]$, $n \in \{0, 1, 2, \ldots\}$, have the expansion given by (6.11). If λ is a real number, then we have*

$$12\left|a_3 - \lambda a_2^2\right|$$
$$\leq \begin{cases} \begin{array}{l}(3 + 2^{2n})(2 - 3\lambda)\alpha^2 \\ +3(1 - 2^{2n})(1 - 2\alpha)\lambda \\ +6(1 - 3^n)\alpha + 2(3^{n+1} - 2^{2n}),\end{array} & \textit{if } \lambda \leq \lambda_0; \\ 4[(2 - 3\lambda)\alpha^2 + 1], & \textit{if } \lambda_0 \leq \lambda \leq \dfrac{2}{3\alpha}(\alpha - 1); \\ \dfrac{4[(10 - 9\lambda)\alpha + (2 - 3\lambda)]}{3(2 - \lambda) - (2 - 3\lambda)\alpha}, & \textit{if } \dfrac{2}{3\alpha}(\alpha - 1) \leq \lambda \leq \dfrac{2}{3}; \\ 12(1 - \lambda)\alpha\sqrt{\frac{12(1-\lambda)}{(4-3\lambda)^2 - (3\lambda-2)^2\alpha^2}}, & \textit{if } \dfrac{2}{3} \leq \lambda \leq \lambda_2; \\ 4[(3\lambda - 2)\alpha^2 - 1], & \textit{if } \lambda_2 \leq \lambda \leq \dfrac{2(\alpha + 2)}{3(\alpha + 1)}; \\ \begin{array}{l}(3 + 2^{2n})(3\lambda - 2)\alpha^2 \\ +3(2^{2n} - 1)(1 - 2\alpha)\lambda \\ +6(3^n - 1)\alpha + 2(2^{2n} - 3^{n+1}),\end{array} & \textit{if } \lambda \geq \dfrac{2(\alpha + 2)}{3(\alpha + 1)}; \end{cases} \tag{7.98}$$

where

$$\lambda_0 = \frac{2^{2n-2}(\alpha+1) - 3^n}{3\left(2^{2n-3}\right)(\alpha-1)} \quad \textit{and} \quad \lambda_2 = \frac{2}{3} + \frac{1}{6\alpha^2}\left(\sqrt{8\alpha^2+1} - 1\right). \tag{7.99}$$

The inequalities are sharp.

We have the following result for the complex parameter λ.

Theorem 174 (Ibrahim, Darus and Dragomir [210]) *Let $f \in C_0(\alpha, n)$ have the expansion given by (6.11), $\alpha \in (1, 2]$, $n \in \{0, 1, 2, ...\}$. If λ is a complex number, then we have*

$$\left|a_3 - \lambda a_2^2\right| \leq \max\left\{1,\ \frac{1}{12}(\alpha+1)\,\nu(\alpha,\lambda)\ \right\}, \tag{7.100}$$

where

$$\begin{aligned}\nu(\alpha,\lambda) &= |(2-3\lambda)(\alpha+1)+2| + 2(\alpha-1)|3\lambda-2| \\ &\quad + \left(\frac{\alpha-1}{\alpha+1}\right)|6+[2-3(\alpha-1)\lambda]|.\end{aligned} \tag{7.101}$$

Proof. We recall that for $f \in C_0(\alpha, n)$ if and only if there exist a function $\psi(z) = z + \sum_{n=2}^{\infty} \psi_n z^n \in S_n$, $n \in \{0, 1, 2, ...\}$ such that

$$f'(z) = \frac{1}{(1-z)^{\alpha+1}}\left(\frac{z}{\psi(z)}\right)^{(\alpha-1)/2}, \tag{7.102}$$

where $f(z)$ has the form given by (6.11). Comparing the coefficients of z and z^2 on the both sides of the series expansion (7.102), we obtain

$$a_2 = \frac{(\alpha+1)}{2} - 2^{n-2}(\alpha-1)\psi_2 \tag{7.103}$$

and

$$\begin{aligned}a_3 &= \frac{1}{6}(\alpha+1)(\alpha+2) - \frac{2^{n-1}}{3}\left(\alpha^2-1\right)\psi_2 \\ &\quad - \frac{3^{n-1}}{2}(\alpha-1)\psi_3 + \frac{2^{2n-3}}{3}(\alpha-1)\psi_2^2,\end{aligned} \tag{7.104}$$

respectively. A computation yields that

$$\begin{aligned} a_3 - \lambda a_2^2 &= \frac{(\alpha+1)^2}{4}\left[\frac{2(\alpha+2)}{3(\alpha+1)} - \lambda\right] \\ &+2^{n-2}\left(\alpha^2-1\right)\left(\lambda - \frac{2}{3}\right)\psi_2 - \frac{3^{n-1}}{2}(\alpha-1) \\ &\times\left[\psi_3 - \left(\frac{2^{2n-2}(\alpha+1) - 3\lambda\left(2^{2n-3}\right)(\alpha-1)}{3^n}\right)\psi_2^2\right]. \quad (7.105) \end{aligned}$$

Hence, for $\alpha > 1$, we get

$$\begin{aligned} \left|a_3 - \lambda a_2^2\right| &\leq \frac{(\alpha+1)^2}{4}\left|\frac{2(\alpha+2)}{3(\alpha+1)} - \lambda\right| \\ &+2^{n-2}\left(\alpha^2-1\right)\left|\lambda - \frac{2}{3}\right||\psi_2| - \frac{3^{n-1}}{2}(\alpha-1) \\ &\times\left|\psi_3 - \left(\frac{2^{2n-2}(\alpha+1) - 3\lambda\left(2^{2n-3}\right)(\alpha-1)}{3^n}\right)\psi_2^2\right|. \quad (7.106) \end{aligned}$$

Now, we need to investigate the maximum values of the functional $\left|a_3 - \lambda a_2^2\right|$ by considering several cases of λ real.

Case 1: First, consider the case for all $\lambda \leq \dfrac{2^{2n-2}(\alpha+1) - 3^n}{3\left(2^{2n-3}\right)(\alpha-1)}$. We observe that the assumption on λ is seen to be equivalent to

$$2^{2n-2}(\alpha+1) - 3\lambda\left(2^{2n-3}\right)(\alpha-1) \geq 1, \quad (7.107)$$

and the first term in the right hand-side of (7.105) is nonnegative. Hence, from (7.97) we have that

$$\begin{aligned} &\left|\psi_3 - \left(\frac{2^{2n-2}(\alpha+1) - 3\lambda\left(2^{2n-3}\right)(\alpha-1)}{3^n}\right)\psi_2^2\right| \\ &\leq \frac{2^{2n}(\alpha+1) - 3\lambda\left(2^{2n-1}\right)(\alpha-1)}{3^n} - 3, \quad (7.108) \end{aligned}$$

and also, noticing that from (7.96), for $\psi(z) \in S_n$, $|\psi_k| \leq k^{1-n}$ for $k \in \{2, 3, ...\}$. Hence we have from (7.106) that

$$\begin{aligned}
\left|a_3 - \lambda a_2^2\right| &\leq \frac{(\alpha+1)^2}{4}\left[\frac{2(\alpha+2)}{3(\alpha+1)} - \lambda\right] + 2^{n-2}\left(\alpha^2-1\right)\left(\frac{2}{3}-\lambda\right)|\psi_2| \\
&+ \frac{3^{n-1}}{2}(\alpha-1)\left|\psi_3 - \left(\frac{2^{2n-2}(\alpha+1) - 3\lambda\left(2^{2n-3}\right)(\alpha-1)}{3^n}\right)\psi_2^2\right| \\
&= \frac{(\alpha+1)(\alpha+2)}{6} - \frac{\lambda}{4}(\alpha+1)^2 + \frac{\left(\alpha^2-1\right)}{2}\left(\frac{2}{3}-\lambda\right) \\
&+ \frac{3^{n-1}}{2}(\alpha-1)\left(\frac{2^{2n}(\alpha+1) - 3\lambda\left(2^{2n-1}\right)(\alpha-1)}{3^n} - 3\right). \qquad (7.109)
\end{aligned}$$

It can be simplified to

$$\begin{aligned}
\left|a_3 - \lambda a_2^2\right| \leq \frac{1}{12}&\left[\left(3+2^{2n}\right)(2-3\lambda)\alpha^2 + 3\left(1-2^{2n}\right)(1-2\alpha)\lambda\right. \\
&\left.+6\left(1-3^n\right)\alpha + 2\left(3^{n+1}-2^{2n}\right)\right], \qquad (7.110)
\end{aligned}$$

for any $\lambda \in \left(\infty, \dfrac{2^{2n-2}(\alpha+1) - 3^n}{3\left(2^{2n-3}\right)(\alpha-1)}\right)$.

Case 2: Let $\lambda \geq \dfrac{2(\alpha+2)}{3(\alpha+1)}$. For this case, the first term of right hand-side of (7.105) is nonnegative. The condition on λ in particular, gives $\lambda \geq 2/3$, and therefore, our assumption on λ implies that

$$\frac{2^{2n-2}(\alpha+1) - 3\lambda\left(2^{2n-3}\right)(\alpha-1)}{3^n} \leq \frac{2^{2n}}{3^n}\left(\frac{1}{2}\right). \qquad (7.111)$$

Again, it follows from (7.97) that

$$\begin{aligned}
&\left|\psi_3 - \left(\frac{2^{2n-2}(\alpha+1) - 3\lambda\left(2^{2n-3}\right)(\alpha-1)}{3^n}\right)\psi_2^2\right| \\
&\leq 3 - \frac{2^{2n}(\alpha+1) - 3\lambda\left(2^{2n-1}\right)(\alpha-1)}{3^n}. \qquad (7.112)
\end{aligned}$$

In view of these observation and the use of the inequality that $|\psi_2| \leq 2^{1-k}$, the inequality (7.106) gives

$$\left|a_3 - \lambda a_2^2\right| \le \frac{(\alpha+1)^2}{4}\left[\lambda - \frac{2(\alpha+2)}{3(\alpha+1)}\right] + 2^{n-2}\left(\alpha^2-1\right)\left(\lambda - \frac{2}{3}\right)\left(2^{1-n}\right)$$
$$+\frac{3^{n-1}}{2}(\alpha-1)\left(3 - \frac{2^{2n}(\alpha+1) - 3\lambda\left(2^{2n-1}\right)(\alpha-1)}{3^n}\right). \qquad (7.113)$$

Thus, simplifying the right hand-side expression in (7.113), we obtain

$$\left|a_3 - \lambda a_2^2\right| \le \frac{1}{12}\left[\left(3+2^{2n}\right)(3\lambda-2)\alpha^2 + 3\left(2^{2n}-1\right)(1-2\alpha)\lambda\right.$$
$$\left.+6\left(3^n-1\right)\alpha - 2\left(3^{n+1}-2^{2n}\right)\right], \qquad (7.114)$$

for any $\lambda \in [2(\alpha+2)/3(\alpha+1), \infty)$.

Case 3: Consider λ, where $\lambda \in \left(\frac{2^{2n-2}(\alpha+1)-3^n}{3\left(2^{2n-3}\right)(\alpha-1)}, \frac{2(\alpha+2)}{3(\alpha+1)}\right)$. Now we deal with the case by using the formulas (7.102) and (7.105) together with the representation formula for $\psi(z) \in S_n$. Let us define the function $w(z)$ by

$$\frac{D^{n+1}\psi(z)}{D^n\psi(z)} = \frac{1+zw(z)}{1-zw(z)}; \ (w(z) \ne 1), \qquad (7.115)$$

where $w : D \to \overline{D}$ is a function analytic in D with the Taylor series

$$w(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad (7.116)$$

and D^n, $n \in \mathbb{N}_0$ is the Sălăgean differential operator defined by (6.35). Comparing the coefficients of z and z^2 in the series expansion (7.115), we get

$$\psi_2 = 2^{1-n} c_0 \qquad (7.117)$$

and

$$\psi_3 = \frac{1}{3^n}\left(c_1 + 3c_0^2\right). \qquad (7.118)$$

Inserting these resulting formulas (7.117) and (7.118), into (7.105) yields

$$
\begin{aligned}
a_3-\lambda a_2^2 \leq{} & \frac{(\alpha+1)^2}{4}\left[\frac{2(\alpha+2)}{3(\alpha+1)}-\lambda\right] \\
& +2^{n-2}\left(\alpha^2-1\right)\left(\lambda-\frac{2}{3}\right)\left(2^{1-n}c_0\right) \\
& +\frac{3^{n-1}}{2}(\alpha-1)\left[\frac{1}{3^n}\left(c_1+3c_0^2\right)\right. \\
& \left.-\left(\frac{2^{2n-2}(\alpha+1)-3\lambda\left(2^{2n-3}\right)(\alpha-1)}{3^n}\right)\left(2^{2-2n}\right)c_0^2\right] \\
& =A+Bc_0+Cc_0^2+Dc_1, \qquad (7.119)
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \frac{1}{6}(\alpha+2)(\alpha+1)-\frac{\lambda}{4}(\alpha+1)^2, && (7.120)\\
B &= \frac{1}{6}\left(\alpha^2-1\right)(3\lambda-2), && (7.121)\\
C &= -\frac{1}{12}(\alpha-1)\left[4-2\alpha+3\lambda(\alpha-1)\right], && (7.122)\\
D &= -\frac{1}{6}(\alpha-1). && (7.123)
\end{aligned}
$$

Hence, by using the well-known inequalities that $|c_0|\leq 1$ and $|c_1|\leq 1-|c_0|^2$, we obtain from (7.119) that

$$
\left|a_3-\lambda a_2^2\right| \leq \left|A+Bc_0+Cc_0^2\right|+\frac{1}{6}(\alpha-1)\left(1-|c_0|^2\right). \qquad (7.124)
$$

To find the maximum value of $|a_3-\lambda a_2^2|$ in (7.124), we let $c_0=re^{i\theta}$, and consider the following quadratic expression:

$$
\begin{aligned}
f(r,\theta) &= \left|A+Bc_0+Cc_0^2\right|^2 \\
&= \left(A-Cr^2\right)^2+B^2r^2+2Br\left(A+Cr^2\right)\cos\theta+4ACr^2\cos^2\theta, \qquad (7.125)
\end{aligned}
$$

where $\cos\theta\in[-1,1]$, $r\in(0,1]$. For getting the upper bounds of $|a_3-\lambda a_2^2|$, we have to find the maximum value of $f(r,\theta)$ for r in the interval $(0,1]$. So, let $x=\cos\theta$, then from (7.125) we have

$$h(x) = \left(A - Cr^2\right)^2 + B^2r^2 + 2Br\left(A + Cr^2\right)x + 4ACr^2x^2. \tag{7.126}$$

In order to determine the maximum value of $h(x)$ for $x \in [-1, 1]$, we need to consider several subintervals of λ, where

$$\lambda \in \left(\frac{2^{2n-2}(\alpha+1) - 3^n}{3\left(2^{2n-3}\right)(\alpha-1)}, \frac{2(\alpha+2)}{3(\alpha+1)}\right). \tag{7.127}$$

Case 3A: First, consider $\lambda \in \left(\frac{2^{2n-2}(\alpha+1) - 3^n}{3\left(2^{2n-3}\right)(\alpha-1)}, \frac{2(\alpha-2)}{3(\alpha-1)}\right)$. We observe that for λ in this interval, we have $A > 0$, $B < 0$, $C > 0$ and $A + Cr^2 > 0$ for $r \in (0, 1]$, and (7.126) attains its maximum value at $x = -1$. Therefore, it gives that

$$\left|a_3 - \lambda a_2^2\right| \le g(r) = A - Br + Cr^2 + \frac{1}{3}(\alpha-1)\left(1 - r^2\right). \tag{7.128}$$

By a simple calculation, we show that the maximum value of (7.128) attains at the boundary of r, i.e., $r = 1$. Thus, we have

$$g(r) \le g(1) = A - B + C = \frac{1}{3}\left[(2 - 3\lambda)\alpha^2 + 1\right]. \tag{7.129}$$

Case 3B: Let $\lambda = 2(\alpha-2)/3(\alpha-1)$. In this case, we have $C = 0$, therefore from (7.126), $h(x)$ becomes a linear function as follows,

$$h(x) = A^2 + B^2r^2 + 2BrAx. \tag{7.130}$$

It is easy to show that the maximum value of (7.130) occurs at $x = -1$ and $r = 1$. Again we get the maximum value of $|a_3 - \lambda a_2^2|$ as given in the previous case.

Case 3C: Let $\lambda \in \left(\frac{2(\alpha-2)}{3(\alpha-1)}, \frac{2(\alpha-1)}{3\alpha}\right)$. In this interval, the quadratic function (7.126) has the maximum value at

$$x\left(r\right)=-\frac{B}{4}\left(\frac{1}{Cr}+\frac{r}{A}\right), \tag{7.131}$$

where $x\left(r\right)$ is a monotonic increasing function in $r\in\left(0,1\right]$, and $x\left(1\right)<-1$. Hence we get the upper bound of $|a_3-\lambda a_2^2|$ as given in Cases 3A and 3B. As conclusion, Cases 3A, 3B and 3C give

$$\left|a_3-\lambda a_2^2\right|\leq\frac{1}{3}\left[\left(2-3\lambda\right)\alpha^2+1\right], \tag{7.132}$$

for all $\lambda\in\left(\dfrac{2^{2n-2}\left(\alpha+1\right)-3^n}{3\left(2^{2n-3}\right)\left(\alpha-1\right)},\dfrac{2\left(\alpha-1\right)}{3\alpha}\right)$.

Case 3D: Let $\lambda\in\left[\dfrac{2\left(\alpha-1\right)}{3\alpha},\dfrac{2}{3}\right)$. From the Case 3C, the inequality $x\left(1\right)<-1$ gives us that

$$\frac{2\left(3\lambda+4\alpha^2-12\alpha^2\lambda+9\alpha^2\lambda^2-4\right)}{\left[\left(3\lambda-4\right)+\alpha\left(3\lambda-2\right)\right]\left[\alpha\left(3\lambda-2\right)-\left(3\lambda-4\right)\right]}<0, \tag{7.133}$$

hence it shows that

$$p(\lambda)=9\alpha^2\lambda^2+\left(3-12\alpha^2\right)\lambda+4\left(\alpha^2-1\right)<0, \tag{7.134}$$

where $\lambda<2/3$. Factorizing $p(\lambda)$ in (7.134), we have

$$\lambda_1=\frac{2}{3}-\frac{1}{6\alpha^2}\left(1+\sqrt{8\alpha^2+1}\right) \tag{7.135}$$

and

$$\lambda_2=\frac{2}{3}-\frac{1}{6\alpha^2}\left(1-\sqrt{8\alpha^2+1}\right). \tag{7.136}$$

It is clear that $\lambda_1<\lambda_2$. Therefore, for $\lambda\in\left[2\left(\alpha-1\right)/3\alpha,\lambda_1\right)$, the functions (7.126) and (7.128) have their maximum values at

$$x=-1\text{ and }r_m=\frac{-3B}{-6C+\alpha-1}\in\left(0,1\right], \tag{7.137}$$

respectively. Hence the upper bound of the Fekete-Szegő functional is given by

$$\begin{aligned}\left|a_3-\lambda a_2^2\right| \leq g(r_m) &= A - Br_m + Cr_m^2 + \frac{1}{3}(\alpha-1)\left(1-r_m^2\right)\\ &= \frac{4\left[(10-9\lambda)\alpha+(2-3\lambda)\right]}{3(2-\lambda)-(2-3\lambda)\alpha}. \qquad (7.138)\end{aligned}$$

Next, we consider $\lambda \in [\lambda_1, 2/3)$. In this interval, the quadratic equation (7.126) attains its maximum value at

$$x(r) = \frac{-B\left(A+Cr^2\right)}{4ACr}, \qquad (7.139)$$

with

$$h(x(r)) = -\frac{1}{4AC}\left(B^2-4AC\right)(A-Cr)^2. \qquad (7.140)$$

Hence, the Fekete-Szegő functional satisfies the following inequality

$$\begin{aligned}\left|a_3-\lambda a_2^2\right| &\leq \sqrt{h(x(r))} + \frac{(\alpha-1)}{6}\left(1-r^2\right)\\ &= (A-Cr)\sqrt{1-\frac{B^2}{4AC}} + \frac{(\alpha-1)}{6}\left(1-r^2\right) = k(r). \qquad (7.141)\end{aligned}$$

The maximum value of $g(r)$, where

$$g(r) = A - Br + Cr^2 + \frac{(\alpha-1)}{6}\left(1-r^2\right) \qquad (7.142)$$

and the function (7.141) occur at

$$r_m = \frac{-B}{-2C+\frac{(\alpha-1)}{3}} \text{ and } r_0 = \frac{B}{2C+\sqrt{1-\frac{B^2}{4AC}}}, \qquad (7.143)$$

respectively. It is easy to show that (7.141) is a monotonic decreasing function for $r \geq r_0$. Hence, the maximum value of $\left|a_3-\lambda a_2^2\right|$ is also given by (7.138).

For $\lambda = 2/3$, we get $B = 0$ and $C = (1-\alpha)/6$. Thus, the maximum value

$$\left|a_3 - \lambda a_2^2\right| = \frac{\alpha}{3}, \tag{7.144}$$

occurs at $x = \cos\theta = 0$ and $r \in (0,1]$. Therefore, from (7.138) and (7.144) we conclude that

$$\left|a_3 - \lambda a_2^2\right| \leq \frac{4\left[(10-9\lambda)\,\alpha + (2-3\lambda)\right]}{3\,(2-\lambda) - (2-3\lambda)\,\alpha}, \tag{7.145}$$

for all $\lambda \in [2\,(\alpha-1)\,/3\alpha,\ 2/3]$.

Case 3E: Let $\lambda \in (2/3, \lambda_2]$, where λ_2 is given by (7.136). In this interval, we have $B > 0$. So that the function (7.126) attains its maximum value at $x = 1$. Then, we have

$$l\,(r) = h\,(1) = A + Br + Cr^2 + \frac{(\alpha-1)}{6}\left(1-r^2\right). \tag{7.146}$$

Again, by a simple calculation shows that the maximum value of $l\,(r)$ occurs at

$$r_n = \frac{B}{-2C + \frac{(\alpha-1)}{3}}. \tag{7.147}$$

Hence the maximum of the function (7.141) is attained at

$$r_1 = \frac{B}{-2C\left(1+\sqrt{1-\frac{B^2}{4AC}}\right)} \in (0,1]\,. \tag{7.148}$$

It is easily to prove that $r_1 < r_n \leq 1$. Since $k(r)$ is monotonic increasing function, then

$$k\,(r) \leq k\,(1) = (A-C)\sqrt{1-\frac{B^2}{4AC}}, \tag{7.149}$$

which gives that

$$\left|a_3 - \lambda a_2^2\right| \leq k(1) = (1-\lambda)\,\alpha\sqrt{\frac{12\,(1-\lambda)}{(4-3\lambda)^2 - (3\lambda-2)^2\,\alpha^2}}, \tag{7.150}$$

for $\lambda \in (2/3, \lambda_2]$.

Case 3F: Finally, we consider the case for $\lambda \in (\lambda_2, 2(\alpha+2)/3(\alpha+1))$. For λ in this interval, we see that $A < 0$, $B > 0$, $C < 0$, $A + Cr^2 < 0$ and the maximum value of function (7.124) is attained for $x = -1$, i.e.,

$$\eta(x) = -A + Br - Cr^2 + \frac{(\alpha-1)}{6}\left(1 - r^2\right). \tag{7.151}$$

We get $\eta(r) \leq \eta(1)$ for all λ in these interval, and hence

$$\left|a_3 - \lambda a_2^2\right| \leq -A + B - C = \frac{1}{3}\left[(3\lambda - 2)\alpha^2 - 1\right]. \tag{7.152}$$

Thus, the proof of Theorem 173 is completed.

Further, substituting (7.117) and (7.118) into (7.105), we get

$$\begin{aligned} 12\left(a_3 - \lambda a_2^2\right) = {} & (\alpha+1)\left[(2-3\lambda)(\alpha+1)+2\right] \\ & + 2\left(\alpha^2-1\right)(3\lambda-2)c_0 + 2(1-\alpha)c_1 \\ & + (\alpha-1)\left(6 + \left[2 - 3(\alpha-1)\lambda\right]\right)c_0^2. \end{aligned} \tag{7.153}$$

Hence for λ complex numbers, we have

$$\begin{aligned} 12\left|a_3 - \lambda a_2^2\right| \leq {} & (\alpha+1)\left|(2-3\lambda)(\alpha+1)+2\right| \\ & + 2(1-\alpha)|c_1| + 2\left(\alpha^2-1\right)|3\lambda-2||c_0| \\ & + (\alpha-1)\left|6 + \left[2 - 3(\alpha-1)\lambda\right]\right||c_0|^2. \end{aligned} \tag{7.154}$$

Using the well-known inequality that $|c_0| \leq 1$ and $|c_1| \leq 1 - |c_0|^2$, then from (7.154) we get

$$12\left|a_3 - \lambda a_2^2\right| \leq \frac{1}{12}(\alpha+1)\,\nu(\alpha,\lambda), \tag{7.155}$$

for $\operatorname{Re}\{\nu(\alpha,\lambda)\} > 0$, where

$$\begin{aligned} \nu(\alpha,\lambda) = {} & \left|(2-3\lambda)(\alpha+1)+2\right| + 2(1-\alpha)|3\lambda-2| \\ & + \frac{(\alpha-1)}{\alpha+1}\left|6 + \left[2 - 3(\alpha-1)\lambda\right]\right|. \end{aligned} \tag{7.156}$$

Thus, the proof of Theorem 174 is completed. ■

Remark 175 *Taking $n = 0$ and λ is a real numbers in Theorem 173, we deduce a result of Bhowmik et al. [64]..*

Other problems related to the Fekete-Szegő functional for further reading can be found in (see for instance [107], [148], [151], [206], [234], [244], [321], [347], [393], [394]).

Chapter 8

Summary and Conclusion

This dissertation is composed of eight chapters in which the research has been carried out. This work and the main achievements throughout this study are summarized in this chapter.

8.1 Summary

The power series $f(z)$ in a complex variable with real or complex coefficients is defined by

$$f(z) = \sum_{k=1}^{\infty} a_k z^k, \tag{8.1}$$

which is convergent on the open disk $D(0,R) = \{z \in \mathbb{C} : |z| < R\}$, $R > 0$, where R is called the radius of convergence of the series (8.1). If an analytic function, which is defined by the convergent power series (8.1), is univalent on the open unit disk $D(0,1)$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$, then we have the power series of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{8.2}$$

for any $z \in D(0,1)$.

The classical Cauchy-Bunyakovsky-Schwarz (CBS) inequality for two sequences of complex numbers states that

$$\left|\sum_{k=1}^{n} a_k b_k\right|^2 \leq \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 , \tag{8.3}$$

for $a_k, b_k \in \mathbb{C}$, $k \in \{1, 2, \ldots, n\}$, with equality holds in (8.3) if and only if there is a complex number $c \in \mathbb{C}$ such that $a_k = c\overline{b}_k$ for any $k \in \{1, 2, \ldots, n\}$. A direct generalization of the (CBS)-inequality (8.3) provided by Hölder [201] in 1889, is

$$\left|\sum_{k=1}^{n} a_k b_k\right| \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |b_k|^q\right)^{1/p} , \tag{8.4}$$

where p and q satisfy the equation $1/p + 1/q = 1$ with $p > 1$. The equality occurs in (8.4) if and only if the sequences $\{|a_k|^p\}$ and $\{|b_k|^q\}$, $k \in \{1, 2, \ldots, n\}$ are proportional. This inequality is well-known as the Hölder inequality.

In the real case, the (CBS) and Hölder inequalities can be derived by utilising the Jensen inequality for certain underlying convex functions (see [422], [301, p. 457], [381, p. 63-64]). The Jensen inequality, which connects with the notion of convexity of functions, asserts that

$$f\left(\frac{1}{P_n} \sum_{j=1}^{n} p_j x_j\right) \leq \frac{1}{P_n} \sum_{j=1}^{n} p_j f\left(x_j\right) \tag{8.5}$$

provided that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on I, $x_j \in I$ such that $x_j \geq 0$, $j \in \{1, 2, \ldots, n\}$, $p_j \geq 0$ with $P_n := \sum_{j=1}^{n} p_j > 0$, $n \geq 2$. The equality holds in (8.5) if and only if $x_j = x_k$ for all $j, k \in \{1, 2, \ldots, n\}$.

The (CBS)-inequality has been generalized to integrals and inner product spaces. It can be generalized for functions defined by the power series as well. These analogous inequalities are highlighted in the following:

$$|f(zw)|^2 \leq f\left(|z|^2\right) f\left(|w|^2\right), \tag{8.6}$$

for $zw, |z|^2, |w|^2 \in D(0, R)$, where f is a function defined by the power series

(8.1) with nonnegative coefficients, and

$$|f(zw)|^2 \leq f_A\left(|z|^2\right) f_A\left(|w|^2\right), \tag{8.7}$$

for $zw, |z|^2, |w|^2 \in D(0,R)$, where f is a function defined by the power series (8.1) with real coefficients. Similarly, the analogous versions of the Hölder inequality for functions defined by the power series (8.1) can be stated as follows:

$$f(zw) = f^{1/p}(|z|^p) f^{1/q}(|w|^q) \tag{8.8}$$

and

$$f(zw) = f_A^{1/p}(|z|^p) f_A^{1/q}(|w|^q), \tag{8.9}$$

for $zw, |z|^2, |w|^2 \in D(0,R)$.

In the literature, many results concerning generalizations, extensions, refinements, etc., of the classical (CBS), Hölder and Jensen type inequalities, have been established by a number of remarkable researchers over the years. However, most of the established results are discrete and involve finite sums. In this study, we derive some inequalities related to the (CBS), Hölder and Jensen type for the power series. In particular, some refinements, improvements, etc., of the inequalities (8.6) - (8.9) have been developed by utilising some tools that have been available in the literature.

8.2 Main Achievements

The main contributions of this dissertation are in Inequalities Theory and Univalent Function Theory: that is, to develop some inequalities involving the power series functions (8.1) and to investigate some properties of functions (8.2) in certain subclasses of analytic and univalent functions.

In the Theory of Inequalities, this dissertation contributes, firstly, in the improvements, as well as the refinements of the (CBS)-type inequality for functions defined by the power series with real or nonnegative coefficients. Applications for some fundamental functions such as exponential, logarithm, trigonometric and hyperbolic functions are highlighted. Some of the refinements, which are contained in Chapter 3, generalize the results by Cerone and Dragomir [89]. The results in the first part of Chapter 4 also contribute to the improvements of the (CBS)-type inequality for the power series. Particularly, the results that are related to classical Young's inequality provide an improvement of the Hölder's type inequality for the power series with real coefficient. Whereas the results in the second part of Chapter 4 contribute to the Jensen type inequalities through their improvements and refinements, as well as their reverses of the Jensen type inequalities for the real power series with positive coefficients. Lastly, the results in Chapter 5 give some particular inequalities involving special functions such as polylogarithm, hypergeometric, Bessel and Modified Bessel functions of the first kind, that have been derived from some of the results in Chapter 3 and Chapter 4.

This dissertation has also contributed in the study of Univalent Function Theory. The results in Chapter 7 give some new properties of functions which are analytic and univalent in the unit disk. The coefficients inequalities and Fekete-Szegő theorem, which provide the univalence properties of certain subclasses of analytic and univalent functions, have been investigated.

In addition, the results established in this dissertation will contribute to the development of the new problems of these areas and the related topics, and will explore further applications in various fields of pure and applied mathematics.

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