VICTORIA UNIVERSITY OF TECHNOLOGY

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Second-order Dynamic System

Venkatesan Gopalachary

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VICTORIA UNIVERSITY OF TECHNOLOGY
DEPARTMENT OF COMPUTER AND MATHEMATICAL SCIENCES
P O BOX 14428
MCMC
MELBOURNE, VICTORIA 8001
AUSTRALIA

TELEPHONE (03) 9688 4492
FACSIMILE (03) 9688 4050

Footscray Campus

A FEEDBACK CONTROL ALGORITHM FOR A SECOND-ORDER DYNAMIC SYSTEM

Venkatesan Gopalachary,

Victoria University of Technology, Melbourne, Australia.

Abstract

Statisticians and control engineers, often taking different perspectives, have long practised the 'art' of process control. In recent years, attention has been focussed on bringing the two control methodologies closer together. This report addresses some of the issues of automatic process control, with specific reference to continuous processes, such as stability of the feedback control path and the existence of dead time. A method to derive a control algorithm for making an adjustment that would yield the smallest possible mean square error (variance) of the output controlled variable is given for the critically damped second-order dynamic system.

1 INTRODUCTION

In manufacturing processes, uniform outputs are generally not attainable due to factors which cause unpredictable variation. The presence of non-random causes in a production system, (an assignable cause), may be detected as an unexpected variation in the system measurements. 'Disturbances' may afflict the system causing it to drift off target if no action is taken to eliminate them. A requirement for a process to be controlled satisfactorily is that its output should follow both some reference signal and remain unaffected by extraneous disturbances or parameter variations. Engineering control systems continually adjust processes on-line in an attempt to counteract the effects of disturbances. Statistical process control (SPC) aims to contain variation in output so that the level of quality is both predictable and satisfactory. It may also subsequently prompt action in an attempt to reduce variability. Automatic process control aims to maintain certain key process variables as near their desired values called *set points* for as much of the time as possible in order to satisfy production objectives. One approach is to *forecast* the deviation from target which would occur if no action were taken and then act so as to cancel out this deviation. In this report, an explanation is given to how a stochastic model can be used to describe the effects of disturbances and to develop a feedback control algorithm.

2 PROCESS CONTROL

Global production objectives are to achieve production targets at an acceptable cost and to manufacture products of a desired quality in a safe manner with the minimum possible harm to the environment. These goals are realised by monitoring and controlling production. *Control tools* are used:

(i) to detect changes in process performance from a stable state,

(ii) to identify assignable or special causes of variation indicated by these deviations and eliminate the same and/or

(iii) to adjust a process variable or variables so as to maintain a performance criterion in some desirable neighbourhood of a given target value (Box and Jenkins [1963]).

The first two control actions of process monitoring and control are achieved by statistical process control (SPC) techniques. The third process control action is possible by an appropriate *feedback* or *feedforward* control procedure which indicates when and by how much to adjust the process automatically, by either using the deviation from target of the output or by cancelling these deviations by using knowledge of the value of some fluctuating measured input variable, respectively. In some situations, a combination of feedforward and feedback controls is used.

Use of automatic control began in the late 1920's and the first technical publication appeared in 1932. Since then, there has been a steady growth in and use of automatic control systems. Various forms of feedback and feedforward control regulation schemes are used for process adjustment (control action) in automatic process control (APC). APC provides an instantaneous continuous response counteracting changes in the balance of a process and applying self-corrective action to bring the output close to the desired target often without human intervention (Keats and Hubele [1989]).

The term 'controlled process' is often used to mean a 'process state' that is (narrowly) interpreted as *stationary* having *iid variation about a target value*. An alternative has a 'state of control' as *a process state in which future behaviour can be predicted within probability limits determined by a common-cause system* (Box and Kramer [1992]).

If a state of statistical control is identified by a process generating independent and identically distributed (iid) random variables, control of such random processes by automatic means invariably leads to an undesirable increase in process variability. APC provides a consistent steady dynamic response in counteracting changes in the balance of a process and must be properly applied to obtain successful results.

In situations where the cost of making an adjustment to the process is considerable, APC can result in increased cost. In comparison with SPC, APC, referred to as 'engineering feedback control' is a short-term approach that attempts to minimise (output) variation by transferring the predictable component of the output variation to the input manipulated (control) variable [MacGregor] (Box and Kramer [1992]). The appropriate engineering control strategy depends upon (i) the *characteristics of the stochastic (statistical) component of the process modelled by a suitable time series and (ii) the costs associated with making dynamic adjustments.*

The general purpose of automatic control is to get satisfactory process operation by adjustment of a controller (control mechanism). By using a logical method for selecting controller adjustments and by suitable 'tuning', (which means, to have the freedom of choice to vary the parameters of control), there is the potential for high returns in the form of efficient process operation. By suitably modelling a process that is non-stationary but probabilistically predictable, it is possible to formulate a control mechanism leading to the 'adaptive control' situation (Box and Jenkins [1970, 1976]).

3 STOCHASTIC MODELS AND STOCHASTIC DISTURBANCES

In process control, it is common to come across disturbances (noise) that are drifting or nonstationary in nature. The importance of considering disturbances of this nature was known to control system engineers from the early stages of the development of *deterministic control theory*. Deterministic control theory was developed to provide tools to analyse and synthesise a large variety of feedback control systems. Results from various branches of applied mathematics and control problems were used in developing this theory. The early development focussed on *stability theory* and the *theory* of *analytic functions*. Due to complexity and the stringent performance criteria required of controlled processes, the theory of *optimal control* of *deterministic processes* was developed using the tools of the calculus of variations. In controlling deterministic processes, no significant distinction was made between a feedback control system and a feedforward control system and no dynamics (inertia) was assumed in the feedback. There were some drawbacks in using deterministic control theory such as not using realistic models for disturbances and when a disturbance was introduced, it was postulated as a function which is known a priori. Many of the classical methods were capable of dealing with disturbances in only an heuristic manner (Hall

[1956]). The effects of disturbances were required to be predicted by suitable models and there existed the need to model the disturbances in a proper and fitting manner.

Since analytic functions are limited in their capacity to accurately model, the potential for the use of 'statistical models' became apparent. Barnard [1959] and Bather [1963] linked the control problem and SPC charts. Barnard [1959] suggested that for a wandering industrial process, by using control charts and its signals, improvements in process adjustments can be made by means of a model that closely described the disturbance and an estimate of the current process mean connected with the control problem. Other statisticians, Box [1970], Jenkins [1970] and Astrom [1970] endeavoured to provide an answer to the problem of how to characterise and model the disturbance.

A 'deterministic model' makes possible exact calculation of the value of some time-dependent quantity at any instant of time. In many process control problems, unknown factors make this unrealistic. However, it is often possible to derive a realistic model that can be used to calculate the probability of a future value lying between two specified limits. Such a model is called a 'probabilistic' or a 'stochastic model'. Box and Jenkins [1970,1976] adopted this approach and made a major contribution to stochastic control.

Since a disturbance causes a process to drift off target, it is necessary to compensate this by taking proper control action. A process in which the mean is varying in nature with respect to time can be described as a non-stationary disturbance. A stationary disturbance represents the situation where there is no drift in the mean and the process is in a perfect state of control.

Disturbances entering at various points of a process are often persistent in nature, however, in many instances, it may not be economically possible or physically feasible to eliminate them. Such disturbances can be envisaged as the result of a sequence of independent random shocks which can be represented by a first order differential equation, such a system being referred to as a 'first-order dynamic system'.

Control systems engineers described the system model behaviour in which the response of a system to a given input is certain and well defined (deterministic). They used (linear) differential equations to represent the dynamic (feedback) control systems in continuous time and used Laplace transforms to obtain simplified solutions (Deshpande and Ash [1981]. The linearity assumption supplies an approximation for many practical situations. In a similar manner, in dealing with discrete processes, linear difference equations were employed to represent the processes in which the sampling intervals are short enough so that the dynamic or inertial properties of the process cannot be ignored. A first-order process may be represented by the first-order difference equation when sampled at discrete intervals or by the first-order transfer function or 'filter', (the term used in engineering terminology, by control system engineers), (cf. MacGregor [1987]).

Box and Jenkins [1965] set up realistic and flexible stochastic models for the disturbances which force the system, unless controlled, away from their optimal operating conditions. They used process knowledge and took care of the inertia or the dynamics of the system which makes the control actions, needed to combat these disturbances, more complex in nature. In doing so, they found methods for estimating the unknown parameters in the models from process input-output data. They also used the models, after fitting the parameters, to design control schemes. Box, Jenkins and MacGregor [1974], describe how stochastic and (dynamic) transfer function models may be brought together to design feedforward and feedback control schemes. They showed how the parameters in the stochastic and dynamic models may be simultaneously estimated from measurements made in the operating system.

4 FEEDFORWARD AND FEEDBACK CONTROL MODELS

A feedforward control model is proposed when the major disturbances to a production system can be measured. Feedback control may be applied, when the primary sources of disturbance are either not known or cannot be measured. Making use of the available knowledge of the production process and the serially occurring data (which are very likely correlated), it is often possible to build stochastic models to realistically represent and model the disturbances. Box and Jenkins [1970] expressed the process inputs and outputs in terms of time series and described the disturbances by time series models in order to manipulate the system for control purposes.

Feedforward or open-loop control is used to eliminate the effect of some fluctuating measured input by making an adjustment from direct calculation of its effect on the output. For a specified target value of the output, the feedforward control model gives an estimate of the required change to be made in the compensating variable to minimise the *mean square error (sum of the squared deviations between an output value and the target value)*. When the time series model predicts an out-of-control signal for shifts in the mean of the quality deviations from target, changes are made to the compensating variable to offset the effects of the predicted situation (Keats and Hubele [1989]).

Feedback or closed-loop control uses past output deviations from target to determine a process adjustment. This approach makes use of the error (difference between the output and the target values) as the means of identifying changes to the input. Using time series analysis, the effect of the disturbance in the absence of a control action is estimated and a dynamic model is developed linking the input and the process output.

5 ARIMA MODELS

The class of stochastic time series nonstationary models, called Autoregressive Integrated Moving Average (ARIMA) Models developed by Box and Jenkins [1970,1976] are used to describe the stochastic disturbances to the system and provide a means of modelling disturbances and process dynamics (inertia). These time series models characterise and forecast the drifting behaviour of process disturbances when no control action is taken and describe the dynamic relationship between the controlled variables (outputs) and the manipulated variables (control inputs). From these models, a feedback control algorithm is derived which minimises the variance of the output controlled variable at every sample point which exactly compensates for the forecasted disturbance. Models of this kind are used in inventory control problems, in econometrics and to characterise certain disturbances that regularly occur in industrial processes.

6 CONTROLLERS

6.1 Proportional Integral Derivative (PID) Controllers

In feedback control systems, the process adjustments (control actions) are performed either manually or by automatic means through the use of 'controllers'. Often a digital computer connected directly to the process accomplishes the execution of the control action by observing the system so that the available data appears in discrete-time.

Slow changes are encountered in many chemical processes. Under such circumstances, it may be adequate to monitor and take whatever control action is necessary at convenient time intervals. For many automatic controllers, as soon as the measurements are made, the control actions are initiated immediately. By means of the discrete data, the adjustments are made to bring the process into a state of control. With the process data available, it is possible to control the mean square error about the target by **proportional-integral feedback control** schemes (Box and Kramer [1992]).

The proportional plus integral (PI) controller makes a compensation (correction) (which lags behind the trend, if any, in the disturbance) proportional to a (linear) combination of terms involving the deviation and the integral of all the previous errors. A PI controller is a 'standard linear controller'. A special case of such a controller is, regulation based on the control-modified **EWMA** statistic.

The **proportional integral derivative (PID) controller** is a modified form of the PI controller in which an additional term involving the first derivative with respect to the time of the error is included. This type of automatic control action makes a correction which is proportional to a (linear) combination of *(i) the first derivative of the current deviation ('the difference between value of the output controlled variable and position of the final controller set point'), (ii) the deviation itself and (iii) the integral of the deviations over all past history* (Box and Kramer [1990]). PID controllers, also known as *three-term* controllers, are automatic continuous time controllers. These controllers (i) are not capable of providing tight control over processes in which the effect of an adjustment is delayed until the following sample due to time taken to deliver material from the point of adjustment to the sample point (called, the 'dead time'), (ii) tend to perform poorly unless 'detuned' in the face of dead times in order to take necessary action at each sampling instant (page 428, Harris, MacGregor & Wright [1982]), and (iii) are not suited for direct-digital (discrete) control. PID controllers are also not capable of producing control actions that might be called for by a minimum variance feedback controller (page 437, Box and Jenkins [1970,1976]). Controllers employing stochastic characteristics to regulate production processes are called time series controllers. With time series controllers, it is possible to provide tight control of processes with dead time and to provide minimum variance at the same time.

6.2 Time Series Controllers - Characteristics and features

Time series controllers are used in the chemical and process industries for regulating quality variables measured at discrete time intervals. Their 'stochastic feedback control algorithms'[1] are used to calculate a series of adjustments which compensate the disturbances. Recourse to ARIMA models

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1 The terms 'stochastic feedback control algorithm' and 'statistical time series control algorithm' are synonymous and used alternatively in this report at appropriate places.

is made to forecast their drifting behaviour and the stochastic feedback control algorithm or equation derived from these ARIMA models is computerised. Thus 'time series control algorithms' by calculating a series of adjustments compensate for the disturbances by making an adjustment at every sample point.

7 MINIMUM VARIANCE CONTROL

A feedback control strategy to minimise the mean square of the output deviation (error) from the target is by minimum mean square error or variance control. Minimum variance control is the best possible control for processes described by linear functions with disturbances which can be added together and treated as a single disturbance for purposes of mathematical analysis and convenience. Its implementation may demand aggressive control much in excess of what is (normally) required and so may not be practically desirable. However, minimum variance control provides a convenient bound on achievable performance against which the performance of other controllers may be compared. Such a basis is important in the context of deciding corrective control actions. Harris [1989] described a technique to ascertain the best theoretically achievable feedback control performance as measured by the output mean square error.

Time series controllers are capable of giving a minimum control error variance even when there are dynamics (inertia) and delay in the process control system. It may be possible to restrict sampling a process until an acceptable minimum of control error variance can be achieved by making use of the time series controller's minimum variance control property and to minimise monitoring and adjustment costs.

8 DEVELOPMENT OF TIME SERIES MODELS

8.1 Feedback Control Difference Equation

The 'stochastic difference equation' for the feedback control model is derived with the help of a block diagram shown in figure 1.

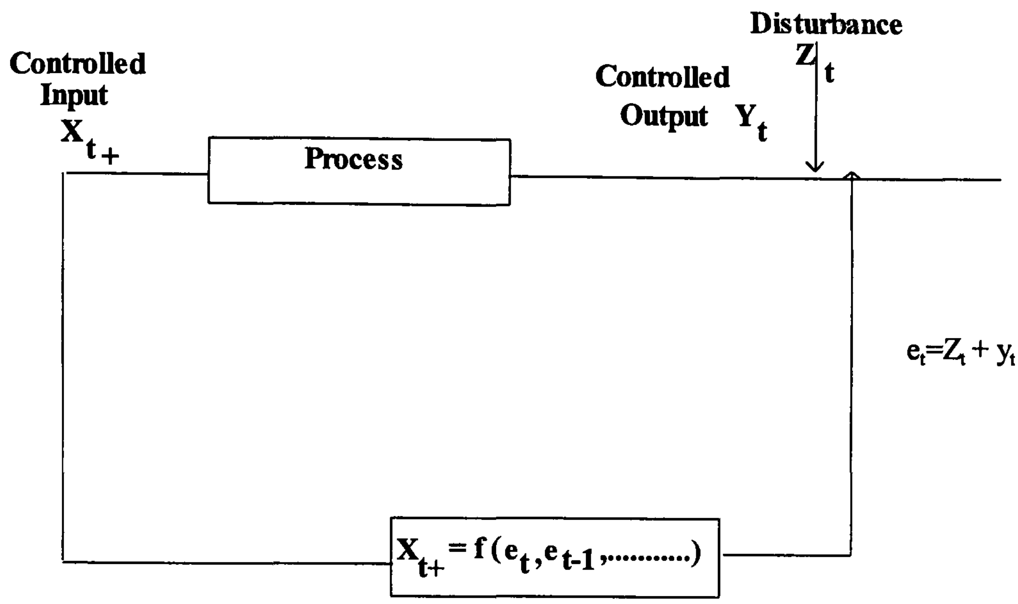

Figure 1 Block Diagram for the Feedback Control Model

In the feedback control scheme shown in figure 1, the process is regulated by manipulating the input variable X_t which in turn affects the controlled output Y_t. X_{t+} is the setting of the controlled input variable (the plus sign on the subscript of X_{t+} implies that the adjustment is made during the interval between t and t+1). A definite deterministic relationship exists between the process input X_t and its output Y_t which does not exhibit stochastic characteristics. Z_t, the non-stationary disturbance is the output of the (linear) system, when subjected to a sequence of uncorrelated random shocks $\{a_t\}$ where $a_t \sim N(0, \sigma_a^2)$.

8.1.1 Symbols used in the Feedback Control model block diagram of figure 1.

a_t Random shocks NID $(0, \sigma_a^2)$,

Z_t Disturbance[2],

e_t Forecast Error,

X_t Input Manipulative Variable (Linear function of e_t and of integral over time of past errors),

Y_t Output or Controlled Variable,

$e_t = Z_t + Y_t$

Box and Jenkins [1970,1976] described some dynamic models of the order (r,s) by

$$\delta_r(B)Y_t = \omega_s(B)B^bX_t.$$

(Table 10.1, page 350, Box and Jenkins [1970,1976]), b being the number of whole periods of dead time (delay), where

$\delta_r(B)$ and $\delta_s(B)$ are polynomials in B and $BX_t = X_{t-1}$, $B^bX_t = X_{t-b}$; B is the backward shift operator.

A first-order *single input single output* (SISO) dynamic system is parsimoniously, represented by the general (linear) difference equation

$$(1+\xi\nabla)Y_t = gX_{t-b}, \qquad 0 \le \delta < 1 \qquad (1)$$

where $\xi = \delta/1\text{-}\delta$ and ∇ is the backward difference operator, $\nabla = 1\text{-}B$.

The terms g(ain) and δ are explained subsequently.

This *discrete dynamic* model of the order $(r,s) \cong (1,0)$ has the form

$$(1-\delta B)Y_t = \omega_0 B^bX_t \qquad 0 \le \delta < 1$$

With s = 0, the *impulse response* tails off exponentially (geometrically) from the initial starting value

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[2] Note that we will be using 'z' to denote the *stochastic variable* and 'Z' to represent the stochastic disturbance. The same logic holds good for a_t which denotes the *variable* and $\{a_t\}$ represents the sequence of *random variables*.

$$\omega_0 = g/(1 + \xi) = g/(1 + (\delta/(1 - \delta)) = g(1 - \delta)$$

(page 352 Box and Jenkins [1970,1976]).

where g, the (steady-state) gain denotes the ratio of change in the steady-state process output to the change in the input which caused it (Deshpande and Ash [1981], (Shinskey [1988]). δ represents the (dynamics) inertial capacity of a process to recover back to its equilibrium conditions after an adjustment is made to the process and due to which the adjustments do not have an immediate effect on the process. It is connected to the sampling interval and the time constant by means of the relation $\delta = e^{-t/T}$ where t is the sampling interval of the discrete process and T, the process time constant. Time constant is the time required for the process output to complete 63.2% of its final steady-state value after a change is made in the input.

So, we write the *recursive control equation* for the first-order dynamic model with b units of delay (dead time) in the form

$$(1-\delta B)Y_t = \omega_0 X_{t-b} = g(1-\delta)X_{t-b} = g(1-\delta)B^b X_t \quad 0 \leq \delta < 1 \quad (2)$$

This is the feedback control first-order difference equation for the dynamic model for which the output change asymptotically approaches 'g' for a unit change in the input, where 'g' is called also the 'system' or 'pure' gain (Box and Kramer [1992]).

8.2 Justification For Second-Order Dynamic Models

For feedback control (closed-loop) stability, the parameter δ must satisfy the condition that $0 < \delta < 1$ for the discrete dynamic model of the process and the gain should be less than or equal to 1.0 (Shinskey [1988]).

The first-order dynamic model characterised by the linear difference equation (2) can be written as

$$Y_t = \delta \times Y_{t-1} + g(1 - \delta)X_{t-b} \quad 0 \leq \delta < 1 \quad (3)$$

The MMSE (minimum mean square error) or minimum variance control schemes based on the first-order dynamic model and the ARIMA (0,1,1) disturbance model produce the minimum mean square error (MMSE) at the output requiring excessive control action in the following situations in which (i) the values of δ are not fairly small; and in (ii) as δ becomes larger and in particular, as it approaches unity (Box and Kramer [1992]). As δ becomes larger, the minimum variance control exhibits large 'alternating' character in the required adjustments (control actions) to give minimum output variance (Box and Kramer [1992]). It is believed and understood that the properties of the noise reflects the system inertia as well (Box and Jenkins [1970, 1976]).

For higher values of δ, the general recourse is to go in for constrained variance control schemes. In such control schemes, reduced control action may be achieved at a cost of small increases in the mean squared error at the output by placing a constraint on the input manipulated variable. Kramer [1990] developed a constrained variance control scheme in which he showed the effect on both adjustment variance and the specified output variance in order to evaluate the trade-offs between the two variances.

The processes found in practice are complex because of their dynamic characteristics which change with time. Approximating such processes by first-order dynamic models does not always seem to be satisfactory. It can be shown from the simulation study results of the time series controller for a first-order (plus dead time) dynamic model and ARIMA (0,1,1) model that for drifting processes, for values of δ from 0 to 1, the required adjustments are of alternating character and sometimes with huge increases in control error standard deviation and its adjustment variance. It is likely that some processes may have more than one dynamic element and the exact mathematical model relating the output and the input could be greater than the first-order. Many complicated dynamic systems can be fairly closely approximated to a greater extent by second-order systems with delay (dead time). It is conjectured that the dynamic system is better described by a second-order system represented by a dynamic model of the order (2,1), ('a discretely coincident' continuous system, page 358, Box and Jenkins [1970,1976]). Detailed analysis and identification of the dynamic models and their suitability can be found in Box and Jenkins [1970,1976].

It may therefore, be appropriate to use higher order dynamic models. Many more complex processes can be closely approximated by second-order systems with dead time (delay) than by the first-order dynamic model (Page 345, Box and Jenkins [1970,1976]). This view is shared also by MacGregor [1988].

The recent methodologies, suggested by MacGregor [1988], Box and Kramer [1992] to superimpose statistical process control charts to monitor the performance of closed-loop controlled systems give rise to stability problems of the feedback control loop. Under these circumstances, it will then be appropriate and justify our action to consider a second-order dynamic model

$$(1-\delta_1 B-\delta_2 B^2)Y_t=(\omega_0-\omega_1 B)\,B^b\,X_t. \tag{4}$$

For stability reasons, only the 'critically damped' behaviour of the second-order dynamic system is considered, (for which the time constants T_1 and T_2 are real and equal) and not the behaviour of the system when it is said to be 'overdamped' or 'underdamped' for which their respective time constants T_1 and T_2 can be either real or complex.

The second-order dynamic system can then be thought of as equivalent to two discrete first-order systems arranged in series.

The second-order model will be

(i) underdamped, when the roots are complex, that is, when

$\delta_1^2+4\,\delta_2<0$;

(ii) overdamped, when the roots are real (and not equal), that is, when

$\delta_1^2+4\,\delta_2>0$; and

(iii) critically damped, when the roots are real and equal, that is when

$\delta_1^2 + 4\,\delta_2 = 0.$

Stability is achieved when the point (δ_1, δ_2) lies in a triangular region defined by the conditions $\delta_2 - \delta_1 = 1$, $\delta_1 + \delta_2 = 1$ and $\delta_2 < 1$. This is shown in figure 2.

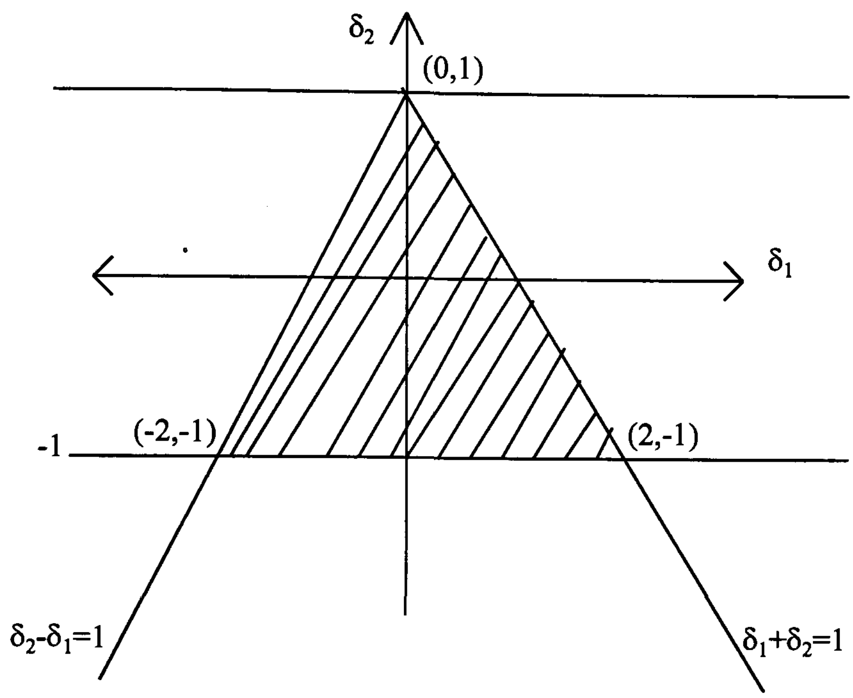

Figure 2 Triangular region defined by the inequality conditions for achieving stability.

We approximate equation (4) by

$$(1 - \delta_1 B - \delta_2 B^2) Y_t = \omega\, B^{b+1}\, X_t \qquad (5)$$

where $(\omega_0 - \omega_1) = \omega$, for mathematical convenience in dealing with a single term, being, the magnitude of the process response to a unit step change in the first period following the dead time carrying over into additional sample periods. This is possible since we are considering only whole periods of deadtime (delay) and not fractional periods.

Moreover, equation (5) reduces to that of Baxley's [1991] first-order dynamic model, namely,

$$Y_t = \delta Y_{t-1} + \omega\, B^{b+1}\, X_t$$

that describes the first-order system with dead time (delay) when $\delta_2 = 0$ and $\delta_1 = \delta$.

The steady-state gain, 'g', of such a second-order discrete dynamic model is given by

$$g = (\omega_0 - \omega_1)/(1-\delta_1-\delta_2).$$

(equation 10.2.5, page 346, Box and Jenkins [1970]).

To evaluate the values for ω_0 and ω_1 equation (4) can be written as

$$(1-\delta_1 B-\delta_2 B^2)Y_t = (\omega_0-\omega_1 B)B^b X_t$$

$$= (1-S_1 B)(1-S_2 B)Y_t$$

$$= (\omega_0 - \omega_1 B)\, B^b\, X_t$$

where

$$\omega_0 = [PG/(T_1-T_2)]\{(T_1(1-S_1)-T_2(1-S_2)\}$$

$$\omega_1 = [PG/(T_1-T_2)]\{(S_1+S_2)(T_1-T_2)+T_2 S_2(1+S_1) - T_1 S_1(1+S_2)\},$$

(Palmor and Shinnar [1979]),

$$S_1 = e^{-1/T_1},$$

$$S_2 = e^{-1/T_2},$$

$$\delta_1 = S_1 + S_2 = e^{-1/T_1} + e^{-1/T_2},$$

$$\delta 2 = - S_1 \times S_2 = - e^{-(1/T_1 - 1/T_2)} \text{ and}$$

PG represents the process gain, realised by the total effect in output caused by a unit change in the input variable after the completion of the dynamic response (Baxley [1991]).

Now,

$$\omega = (\omega_0 - \omega_1) = [[PG/(T_1 - T_2)]\{(T_1(1 - S_1) -T_2(1 - S_2)\}] -$$

$$[[PG/(T_1 - T_2)]\{(S_1 + S_2)(T_1 - T_2) +T_2 S_2(1 + S_1) -T_1 S_1(1+S2)\}]$$

which on simplification, gives for a critically damped system,

$$\omega = PG[1 - S_1 - S_2 + S_1 S_2]$$

$$= PG[1-(S_1+S_2)+S_1 S_2]$$

$$= PG[1-(e^{-1/T1} + e^{-1/T2}) +(e^{-1/T1} \times e^{-1/T2})]$$

$$= PG[1 - \delta_1 - \delta_2]$$

Therefore, the steady-state or system gain

$$g = (\omega_0 - \omega_1)/(1 - \delta_1 - \delta_2) = PG[1- \delta_1- \delta_2]/[1- \delta_1- \delta_2] = PG, \text{ the process gain.}$$

Baxley [1991] used PG =1/1-δ and made PG = 1.0 by setting $\delta = 0$, meaning that there are no carry-over effects (inertia) and seems to have tackled the problem of feedback control stability in a convincing manner in his simulation studies for drifting processes. Kramer [1992], derived expressions for the disturbance and the output effect of control actions as functions of random shocks, *independent* of the *control scheme.* Moreover, Kramer [1992] considered approaches for reducing adjustment variability. Since, interest here is in reducing product variability at the output, it may be worthwhile to consider the critically damped behaviour of the second-order dynamic system for which the time constants are real and equal thus ensuring closed-loop stability. Furthermore, it is shown that the steady state gain of such critically damped second-order systems is PG, the process gain itself.

An additional term in the parameter δ, (δ_2) of the second-order dynamic model makes it possible to account for more of the process dynamics for both small and large values of δ and to better represent the dynamic nature of the process. The additional term Y_{t-2} defines the input-output relationship in a better manner than the first-order dynamic model.

For stability of the second-order dynamic model, the parameters δ_1 and δ_2 must satisfy the following inequality conditions given by

$$\delta_1 + \delta_2 < 1$$

$$\delta_2 - \delta_1 < 1$$

$$-1 < \delta_2 < 1$$

For the second-order dynamic system, when the roots of the characteristic equation $(1-\delta_1 B-\delta_2 B^2)=0$, are real, that is, when $\delta_1^2 + 4\,\delta_2 \geq 0$, the solution of this equation will be the sum of two exponentials.

The roots of the characteristic equation determine the stability of the second-order dynamic system. When these roots are real and positive, the step response, which is the sum of two exponential terms, approaches its asymptote g, the steady-state gain, without crossing it. When the roots are complex, as can be seen from figure 3, (Reproduced from figure 10.5, page 344, Box and Jenkins [1970]), the step response overshoots the value g . From figure 3, we see also that the system has no overshoot when the characteristic equation has real positive roots. This explains the focus on the critically damped second-order discrete dynamic model which ensures closed-loop (feedback control) stability.

The values of δ_1 and δ_2 are given by

$$\delta_1 = e^{-1/T_1} + e^{-1/T_2}$$

$$\delta_2 = e^{-1/T_1} \times e^{-1/T_2}$$

It is known that for the critically damped second-order dynamic system, $T_1 = T_2 = T$.

So, $\delta_1 = 2e^{-1/T}$ and

$\delta_2 = -e^{-2/T}$

As shown in figures 2 and 3, the values of δ_1 and δ_2 should satisfy

$$-2 < \delta_1 < 2$$

$$-1 < \delta_2 < 1.$$

9 EXPRESSION FOR THE CONTROL ADJUSTMENT IN THE INPUT VARIABLE OF A TIME SERIES CONTROLLER

An expression is derived for the *feedback control adjustment* required in the input manipulated variable of a time series controller for a dynamic process with dead time (delay). This expression is different from equation (2) which explains the feedback control model. Figure 4 shows the feedback control scheme to compensate a disturbance Z_t by means of a time series controller. Baxley [1991] considered the dead time equal to one period when deriving the feedback control equation. In this Section, the feedback control (adjustment) algorithm is derived considering b periods of deadtime. It is shown that it conforms to the minimum variance (mean square error) control equation derived by Kramer [1990] for a system in which adjustments to the input variable are made after the process is observed and so their effects are first seen at the next observation (b= 0).

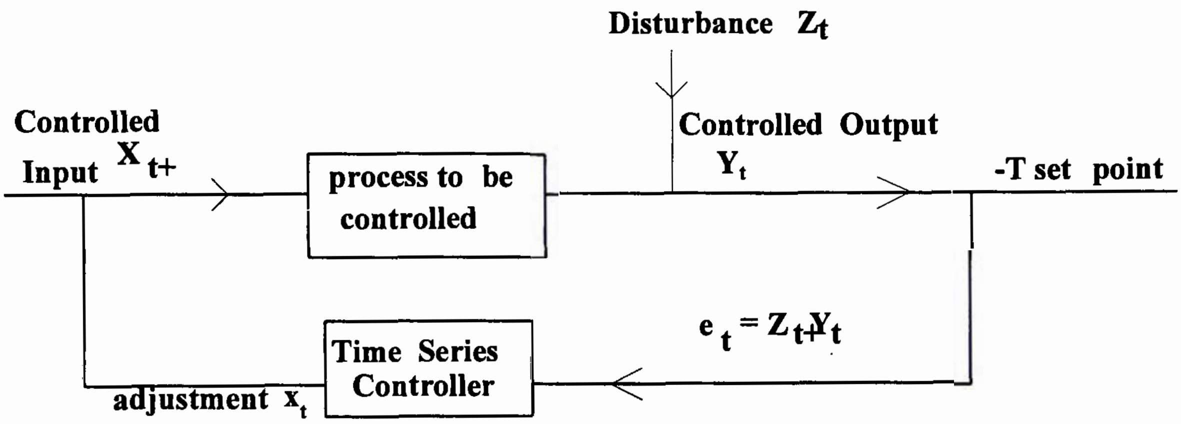

Figure 4 Feedback Control scheme to compensate disturbance Z_t in a Time Series Controller in the existence of Dynamics and Deadtime

From (5),

$$(1-\delta_1 B-\delta_2 B^2)Y_t=\omega B^{b+1} X_t.$$

Changes are made in the controlled input X at times t,t-1,t-2,---, immediately after observing the disturbances z_t, z_{t-1}, z_{t-2},---.

Because of this, a pulsed input results and the level of X in the interval t to t+1 is denoted by X_{t+}.

For this pulsed input, assume that the dynamic model which connects the input manipulated variable X_{t+} and the controlled output Y_t is

$$Y_t=L_{1-1}(B)L_2(B)B^{b+1}X_{t+}, \tag{6}$$

where,

$L_1(B)$ is a polynomial in B of degree r,

$L_2(B)$ is a polynomial in B of degree s and

b is the number of complete intervals of pure delay before an adjustment in the input X_{t+} begins to affect the output Y_t.

The nonstationary disturbance is represented by the ARIMA (0,1,1) model

$$\nabla Z_t = (1-\Theta B)a_t.$$

Z_t measures the effect at the output of an unobserved disturbance, that is, an uncompensated nonstationary disturbance that reaches the output before it is possible for the compensating control action to become effective, this causes the process to wander off target. It is defined as the deviation from the target that would occur if no control action was taken. The effect of the disturbance would be cancelled if it was possible to set

$$X_{t+} = -L_1(B)L_{2-1}(B)Z_{t+b+1}$$

This control action is not realisable since (b+1) is positive; but, the minimum mean square error of the deviation of the output from its target value can be obtained by replacing Z_{t+b+1} by its forecast estimate $\hat{Z}_t(b+1)$ made at time t.

That is, by taking the minimum variance control action

$$X_{t+} = -L_1(B)L_{2-1}(B)\hat{Z}_t(b+1) \qquad (7)$$

The change or adjustment to be made in the input manipulated variable is then

$$x_t = -L_1(B)L_{2-1}(B)\{\hat{Z}_t(b+1)-\hat{Z}_{t-1}(b+1)\} \qquad (8)$$

The error at the output or deviation from the target at time (t+b+1) is the forecast error $e_t(b+1)$ at lead time b+1 for the Z_t disturbance.

That is,

$$e_t(b+1) = Z_{t+b+1} - \hat{Z}_t(b+1)$$

made (b+1) steps ahead at time t.

The error observed at time t is

$$\varepsilon_t = e_{t-b-1}(b+1)$$
$$= Z_t - \hat{Z}_{t-b-1}(b+1)$$

$\hat{Z}_t(b+1) - \hat{Z}_{t-1}(b+1)$ can be deduced from the observed error sequence $\varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2}, ---$.

e_t (b+1) and $\hat{Z}_t(b+1)$ are linear functions of the $\{a_t\}$'s.

So,

$$Z_{t+b+1} = L_4(B)a_{t+b+1} + L_3(B)a_t, \text{ where}$$

$L_3(B)$ and $L_4(B)$ are operators in B which can be deduced from the relations

$$e_{t-b-1}(b+1) = L_4(B)a_t \text{ and}$$
$$\hat{Z}_t(b+1) = L_3(B)a_t.$$

From these,

$$\hat{Z}_t(b+1) = L_3(B)/ L_4(B)e_{t-b-1}(b+1) = \{L_3(B)/ L_4(B)\}\varepsilon_t$$

and

$$\hat{Z}_t(b+1) = \{1-\Theta/ 1-B\}a_t = L3(B)a_t, \text{ giving } L_3(B) = 1-\Theta/1-B.$$

Similarly, $L_4(B)$ is found by expressing the forecast errors as a linear function of future shocks (Box and Jenkins [page 128, 1970,1976]).

Then,

$$L_1(B) = (1-\delta_1 B - \delta_2 B^2),$$
$$L_2(B) = PG(1-\delta_1 - \delta_2)$$
$$L_3(B) = (1-\Theta)/(1-B) \text{ and}$$
$$L_4(B) = 1 + (1-\Theta)B.$$

So, for a time series controller, when the disturbance is described by the ARIMA (0,1,1) model and there are <u>definite</u> carry over effects, the adjustment (x_t) in the input manipulated variable required to make the control and forecast error variances equal, is given by

$$X_{t+} = -\{L_1(B)L_3(B)/(L_2(B)L_4(B))\}\varepsilon_t.$$

(Box and Jenkins [1970,1976])

The control action in terms of the adjustment $x_t = x_{t+} - x_{t-1+}$ to be made at time t is

$$x_t = \frac{-L_1(B)L_3(B)(1-B)}{L_2(B)L_4(B)}\varepsilon_t$$

(equation 12.2.8 page 435 Box and Jenkins [1970,1976]).

This 'feedback control equation defines the adjustment to be made to the process at time t which would produce the feedback control action yielding the smallest possible mean square error since it exactly compensates the predicted deviation from target' (page 213, Box and Jenkins [1968b]).

The above equation, on substituting the expressions for $L_1(B)$, $L_2(B)$, $L_3(B)$ and $L_4(B)$, results in

$$x_t = -\frac{(1-\delta_1 B-\delta_2 B^2)(1-\Theta)}{PG(1-\delta_1-\delta_2)(1+(1-\Theta)B)}\varepsilon_t \qquad (9)$$

where Θ is the *moving average* (operator) parameter.

The control (forecast) errors which turn out to be the one-step ahead forecast errors are measured in practice.

It is known that the forecast error ε_t at the output at time t is the forecast error at lead time b+1 for the Z_t disturbance.

So,

$$\varepsilon_t = e_{t-b-1}(b+1) = \psi_0 a_t + \psi_1 a_{t-1} \qquad (10)$$

For the ARIMA (0, 1, 1) model, the weights are $\psi_0 = 1$ and

$\psi_1 = 1 - \Theta$, so

$$\varepsilon_t = a_t + (1 - \Theta)\, a_{t-1}$$

$$= (1 + (1 - \Theta)B)\, a_t$$

and further,

$$x_t = -\frac{(1 - \delta_1 B - \delta_2 B^2)(1 - \Theta)}{PG(1 - \delta_1 - \delta_2)(1 + (1 - \Theta)B)}(1 + (1 - \Theta)B)\, a_t$$

Since $(1 - \Theta) \times 100$ per cent of the control error will affect the future process behaviour as per the disturbance model, for a dead time b,

$$e_t = a_t + (1 - \Theta)a_{t-b}$$

$$= a_t[1 + (1 - \Theta)B^b]$$

and so

$$a_t = e_t / [1 + (1 - \Theta)B^b] \tag{11}$$

Therefore, the control adjustment equation for b periods of deadtime is

$$x_t = -\frac{(1 - \delta_1 B - \delta_2 B^2)(1 - \Theta)}{PG(1 - \delta_1 - \delta_2)} \times \frac{e_t}{(1 + (1 - \Theta)B^b)}$$

That is,

$$x_t + (1 - \Theta)x_{t-b} = -\frac{(1 - \delta_1 B - \delta_2 B^2)(1 - \Theta)}{PG(1 - \delta_1 - \delta_2)} e_t$$

giving

$$x_t = -\frac{(e_t - \delta_1 e_{t-1} - \delta_2 e_{t-2})(1 - \Theta)}{PG(1 - \delta_1 - \delta_2)} - (1 - \Theta)x_{t-b} \tag{12}$$

The control adjustment action given by (12) minimises the variance of the output controlled variable.

Equation (12) is in conformance with the feedback control action adjustment equation of Kramer [1990] when the output variance is made equal to the variance (σ_{a2}) of the random shocks, the a_t's, for achieving minimum variance or mean square control when $b = 0$. The control adjustment action is made up of the current deviation (e_t) and the past adjustment action x_{t-b} (Kramer [1990]). It is observed also that this is similar to the feedback control action adjustment equation for one period of deadtime derived by Baxley [1991] on taking a value 1 for b, the deadtime and when there are no carry-over effects for a 'standard' time series controller. On comparison with the equation of Baxley [1991], it is found that the first term in equation 12 gives the integral action and the second term, the deadtime compensator, developed by Smith [1959] (Baxley [1991]).

Some simulation results of equation 12 obtained when b = 1, (Table 1) match closely with that of Baxley's [1991] values for the time series controller.

10 CONCLUSION

This report, having discussed briefly the need for stochastic models has provided a brief discussion of minimum variance control and time series controllers. A general feedback control equation has been derived and a statistical control algorithm developed for the critically damped second-order dynamic system.

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