VICTORIA UNIVERSITY OF TECHNOLOGY

DEPARTMENT OF COMPUTER AND MATHEMATICAL SCIENCES

Summing Series Arising
From Integro-Differential-Difference
Equations

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SUMMING SERIES ARISING FROM INTEGRO-DIFFERENTIAL-DIFFERENCE EQUATIONS

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ABSTRACT

By applying Laplace transform theory to solve first order homogeneous differential-difference equations it is conjectured that a resulting infinite sum of a series may be expressed in closed form. The technique used in obtaining a series in closed form is then applied to other examples in teletraffic theory, renewal processes, risk theory and neutron behaviour which may be represented by integral equations.

KEYWORDS : Differential-difference equations, series.

1. INTRODUCTION

Differential-difference equations occur in a variety of applications including : ship stabilization and automatic steering [19], the theory of electrical networks containing lossless transmission lines [7], the theory of biological systems [6], and in the study of distribution of primes [25].

The equation

$$f'(t)+\alpha f'(t-a)+\beta f(t)+\gamma f(t-a)+\delta f(t+a)=0$$

is termed a first order linear delay, or retarded, differential-difference equation for $\alpha=0, \delta=0$ and $a>0$. For $\alpha=0, \delta=0$ and $a<0$ it is termed an advanced equation.
In the case $\delta=0, a>0$ it is referred to as a neutral equation and when $\alpha=0, \beta=0, a>0$ an equation of mixed type.

Stability studies on general delay equations have been carried out in [5], and for neutral equations in [13]. Driver, Sasser and Slater [10] consider a first order linear delay equation and for a 'small' delay they show that it exhibits certain similarities associated with an equation without delay. Numerical studies have also been carried out in which chaos has been observed [14], and Seifert [22] hints strongly at a suspected chaotic interval function associated with discontinuous delays.

A great deal of the studies for the stability of differential-difference equations necessitate an investigation of its associated characteristic equation. Some of the early work in this area has been carried out by Pontryagin [21], Wright [28] and more recently by Cooke and van den Driessche [9] and Hao and Brauer [16].

The purpose of this paper is to show that, by using Laplace transform techniques together with a reliance of asymptotics, series representations for the solutions of delay equations may be expressed in closed form. The series, in its region of convergence, it is conjectured, applies for all values of the delay without necessarily relying on its association with the differential-difference equation.
Unlike some of the series that are listed as high procession fraud by Borwein and Borwein [3] the series in this paper will be shown to be exact by the use of Bürmann's theorem.
The analysis also relies on the exact location of the roots of the associated transcendental characteristic equation. This technique is then applied to particular examples.

2. METHOD

Consider the first order linear homogeneous differential-difference equation

$$\left.\begin{aligned} & f'(t)+bf(t)+cf(t-a)=0 \quad , \quad t \geq a \\ & f'(t)+bf(t)=0 \qquad\qquad\quad , \quad f(0)=1, \;\; 0 \leq t < a. \end{aligned}\right\} \qquad (1)$$

Taking the Laplace transform and using the initial condition, results in

$$\mathcal{L}[f(t)] = F(s) = \frac{1}{s+b+ce^{-as}} \qquad (2)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n c^n e^{-an(s+b)} e^{anb}}{(s+b)^{n+1}}.$$

The inverse Laplace transform

$$\mathcal{L}^{-1}[F(s)] = f(t) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n e^{-b(t-an)} (t-an)^n}{n!} H(t-an) \qquad (3)$$

where the Heaviside unit function

$$H(x) = \begin{cases} 1 \; , & \text{for } x \geq 0 \\ 0 \; , & \text{for } x < 0 \end{cases}$$

The solution to (1) by Laplace transform theory may be written as

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} [g(s)]^{-1} \, ds$$

for an appropriate choice of γ such that all the zeros of the characteristic equation

$$g(s) = s + b + ce^{-as} \qquad (4)$$

are contained to the left of the line in the Bromwich contour.

Now, using the residue theorem

$$f(t)=\sum \text{residues of } \left\{e^{st}\left[g(s)\right]^{-1}\right\}$$

which suggest the solution of $f(t)$ may be written in the form

$$f(t)=\sum_{r} Q_r\, e^{S_r t}$$

where the sum is over all the characteristic roots s_r of $g(s)=0$ and Q_r is the residue of $F(s)$ at $s=s_r$.

The poles of the expression (2) depend on the zeros of the characteristic equation (4), namely, the roots of $g(s)=0$.

The dominant root s_0 of $g(s)=0$ has the greatest real part and therefore asymptotically

$$f(t) \sim Q_0 e^{s_0 t} \quad , \qquad \text{and so from (3),}$$

$$f(t)=\sum_{n=0}^{\infty}(-1)^n\, c^n e^{-b(t-an)}\,\frac{(t-an)^n}{n!}\, H(t-an) \sim Q_0 e^{s_0 t}. \tag{5}$$

After some experimentation it is *conjectured* that:

$$\sum_{n=0}^{\infty}(-1)^n\, c^n e^{-b(t-an)}\,\frac{(t-an)^n}{n!} = Q_0 e^{s_0 t} \tag{6}$$

$\forall t \in R$ in the region where the series converges.

Bürmann's theorem will be used, a little later, to prove the identity (6).

By the use of the ratio test it can be shown that the series in (6) converges in the region

$$\left|ace^{1+ab}\right|<1. \tag{7}$$

In a similar fashion, the Laplace transform from (2) may be expressed as

$$F(s) = \frac{1}{s}\left[1+\frac{b+ce^{-as}}{s}\right]^{-1}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} (-1)^n b^{n-r} c^r \binom{n}{r} \frac{e^{-ars}}{s^{n+1}}$$

and the inverse Laplace transform may be written as

$$f(t) = \sum_{n=0}^{\infty} \sum_{r=0}^{n} (-1)^n b^{n-r} c^r \binom{n}{r} \frac{(t-ar)^n}{n!} H(t-ar) \sim Q_0 e^{s_0 t}.$$

As previous, it is conjectured that

$$\sum_{n=0}^{\infty} \sum_{r=0}^{n} (-1)^n b^{n-r} c^r \binom{n}{r} \frac{(t-ar)^n}{n!} = Q_0 e^{s_0 t} \tag{8}$$

whenever the double series converges.

Lemma 1 *The poles of the expression (2) are all simple for the inequality (7).*

Proof: Assume on the contrary that there is a repeated root of

$$s+b+ce^{-as}=0 \tag{9}$$

then by differentiation it is required that $1-ace^{-as}=0$ in which case $s=\frac{1}{a}\ln(ac)$. Substituting in (9) results in, $\ln(ac)+ab+1=0$

and therefore $ace^{1+ab}=1$ which violates the inequality (7). Hence all roots of (9) are simple.

Now the residue Q_0 of the dominant simple root, $s_0=\xi$ is

$$\frac{1}{1+ab+a\xi}, \qquad \text{where} \qquad \xi+b+ce^{-a\xi}=0$$

and so the expressions (6) and (8) become

$$\sum_{n=0}^{\infty}(-1)^n c^n e^{-b(t-an)}\frac{(t-an)^n}{n!}$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{n}(-1)^n b^{n-r}c^r\binom{n}{r}\frac{(t-ar)^n}{n!}$$

$$=\frac{e^{\xi t}}{1+ab+a\xi} \tag{10}$$

whenever the single and double series converge in a mutual region.

Lemma 2

(i) *The single sum and the double sum in* (10) *are solutions to* (1) *in their region of convergence for* $t>a$.

(ii) *The closed form expression in* (10) *is a solution to* (1) for $t>a$.

(iii) *The single and double sum in* (10) *are equal in their mutual region of convergence, which is no larger than that region given by* (7).

Proof: (i) and (ii) can be shown to be a solutions of (1) by substitution.

(iii) To show $\sum_{n=0}^{\infty}\sum_{r=0}^{n}(-1)^n b^{n-r}c^r\binom{n}{r}\frac{(t-ar)^n}{n!}=\sum_{n=0}^{\infty}(-1)^n c^n e^{-b(t-an)}\frac{(t-an)^n}{n!}$,

expand the left hand side to give.

for $n=0$: $b^0c^0\binom{0}{0}\frac{(t-0)^0}{0!}$

$n=1$: $-b^1c^0\binom{1}{0}\frac{(t-0)^1}{1!}$ $-$ $b^0c^1\frac{(t-a)^1}{1!}$

$n=2$: $b^2c^0\binom{2}{0}\frac{(t-0)^2}{2!}+b^1c^1\binom{2}{1}\frac{(t-a)^2}{2!}+b^0c^2\binom{2}{2}\frac{(t-2a)^2}{2!}$

$n=3$: $-b^3c^0\binom{3}{0}\frac{(t-0)^3}{3!}-b^2c^1\binom{3}{1}\frac{(t-a)^3}{3!}-b^1c^2\binom{3}{2}\frac{(t-2a)^3}{3!}-b^0c^3\binom{3}{3}\frac{(t-3a)^3}{3!}$

$n=4$: . . . . . .

Summing each column results in

$$= \frac{c^0(t-0)^0}{0!}\left[\frac{1}{0!}-\frac{(-b)(t-0)}{1!}+\frac{(-b)^2(t-0)^2}{2!}-\frac{(-b)^3(t-0)^3}{3!}+\ldots\ldots\right]$$

$$-\frac{c(t-a)}{1!}\left[\frac{1}{0!}-\frac{(-b)(t-a)}{1!}+\frac{(-b)^2(t-a)^2}{2!}-\frac{(-b)^3(t-a)^3}{3!}+\ldots\ldots\right]$$

$$+\frac{c^2(t-2a)^2}{2!}\left[\frac{1}{0!}-\frac{(-b)(t-2a)}{1!}+\frac{(-b)^2(t-2a)^2}{2!}-\frac{(-b)^3(t-2a)^3}{3!}+\ldots\ldots\right]$$

$$-\frac{c^3(t-3a)^3}{3!}\left[\frac{1}{0!}-\frac{(-b)(t-3a)}{1!}+\frac{(-b)^2(t-3a)^2}{2!}-\frac{(-b)^3(t-3a)^3}{3!}+\ldots\ldots\right]$$

$$+\ \ldots\ldots$$

$$= \frac{c^0(t-0)^0}{0!}e^{-b(t-0)}-\frac{c(t-a)}{1!}e^{-b(t-a)}+\frac{c^2(t-2a)^2}{2!}e^{-b(t-2a)}-\frac{c^3(t-3a)^3}{3!}e^{-b(t-3a)}+\ldots\ldots$$

$$= \sum_{n=0}^{\infty}(-1)^n c^n e^{-b(t-an)}\frac{(t-an)^n}{n!}.$$

Bürmann's theorem [26] will now be used to prove the explicit form of relationship (6).

Bürmann's Theorem

Let ϕ be a simple function in a domain D, zero at a point β of D, and let

$$\theta(z)=\frac{z-\beta}{\phi(z)}\quad,\quad \theta(\beta)=\frac{1}{\phi'(\beta)}.$$

If $f(z)$ is Analytic in D then $\forall z\in D$

$$f(z)=f(\beta)+\sum_{r=1}^{n}\frac{\{\phi(z)\}^r}{r!}\frac{d^{r-1}}{dt^{r-1}}\left[f'(t)\{\theta(t)\}^r\right]_{t=\beta}+R_{n+1}$$

$$\textit{where}\quad R_{n+1}=\frac{1}{2\pi i}\int_\Gamma dv\int_C\left[\frac{\phi(v)}{\phi(t)}\right]^n\frac{f'(t)\phi'(v)}{\phi(t)-\phi(v)}\,dt.$$

The v - integral is taken along a contour Γ in D from β to z, and the t - integral along a closed contour C in D encircling Γ once positively.

Application of Bürmann's Theorem

The characteristic equation (4) may be shown to have a simple dominant zero at $s=0$ for $b+c=0$ and $(1+ab)>0$. Thus from (6)

$$\sum_{n=0}^{\infty}(-1)^n(-b)^n e^{-b(t-an)}\frac{(t-an)^n}{n!} = \frac{1}{1+ab}.$$

Let $t=-a\tau$, $ab=-\rho$, and hence from above

$$\sum_{n=0}^{\infty}\left(\rho e^{-\rho}\right)^n \frac{(\tau+n)^n}{n!} = \frac{e^{\rho\tau}}{1-\rho}. \qquad (11)$$

Equation (11) is shown to be true by applying Bürmann's theorem.

Let $f(z)=\dfrac{e^{xz}}{1-z}$, $\theta(z)=\dfrac{z}{\phi(z)}=e^{z}$, $\phi(z)=ze^{-z}$, $f(\beta)_{\beta=0}=1$

and it may be shown that $R_{n+1}\to 0$ as $n\to\infty$. From $f(t)=\dfrac{e^{tx}}{1-t}$,

$$f'(t) = e^{tx}\left(\frac{x}{1-t}+\frac{1}{(1-t)^2}\right) = e^{tx}\left[\sum_{j=0}^{\infty}(x+1+j)t^j\right].$$

and so $\qquad f'(t)\{\theta(t)\}^r = e^{t(r+x)}\psi(t)$, where $\psi(t)=\sum_{j=0}^{\infty}(x+1+j)t^j$.

The coefficients in this expression are the same as those in a Taylor series expansion

$$\psi^{(j)}(0)=j!(x+1+j).$$

Now let $B_r(t) = \dfrac{d^{r-1}}{dt^{r-1}}\left[f'(t)\{\theta(t)\}^r\right] = \dfrac{d^{r-1}}{dt^{r-1}}\left[e^{t(r+x)}\psi(t)\right]$

$$= e^{t(r+x)}\left[(r+x)^{r-1}\binom{r-1}{0}\psi^{(0)}(t)+\binom{r-1}{1}(r+x)^{r-2}\psi'(t)+\binom{r-2}{2}(r+x)^{r-3}\psi''(t)+\ldots\ldots\right.$$

$$\left.\ldots\ldots+\binom{r-1}{r-2}(r+x)\psi^{(j-2)}(t)+\binom{r-1}{r-1}(r+x)^0\psi^{(j-1)}(t)\right].$$

$$\therefore\ B_r(0) = (r+x)^{r-1}\binom{r-1}{0}(x+1)+(r+x)^{r-2}\binom{r-1}{1}(x+2)+(r+x)^{r-3}\binom{r-1}{2}(x+3)+.....$$

$$.....+\binom{r-1}{r-2}(r+x)(r-2)!(x+r-1)+\binom{r-1}{r-1}(r-1)!(x+r).$$

Put $y = x+r$ giving

$$B_r(0) = y^{r-1}(y-r+1)+y^{r-2}(r-1)(y-r+2)+y^{r-3}(r-2)(r-1)(y-r+3)+...$$

$$...+(r-1)!\,y(y-1)+(r-1)!\,y$$

$$= y^r-(r-1)y^{r-1}+(r-1)y^{r-1}-(r-1)(r-2)y^{r-2}+(r-1)(r-2)y^{r-2}$$

$$-(r-1)(r-2)(r-3)y^{r-3}+..........+y^2(r-1)!-(r-1)!\,y+(r-1)!\,y$$

$$= y^r \quad = (x+r)^r.$$

Hence it follows that $$\frac{e^{xz}}{1-z} = 1+\sum_{r=1}^{\infty}\frac{\left(ze^{-z}\right)^r}{r!}\ (x+r)^r$$

Region of Convergence

The sum converges in the region $|\rho e^{1-\rho}|<1$, and so considering ρ as a complex variable, $\rho=x+iy$ then

$$\left[e^{2(1-x)}\left(x^2+y^2\right)\right]^{\frac{1}{2}} < 1.$$

The region is shown in figure 1.

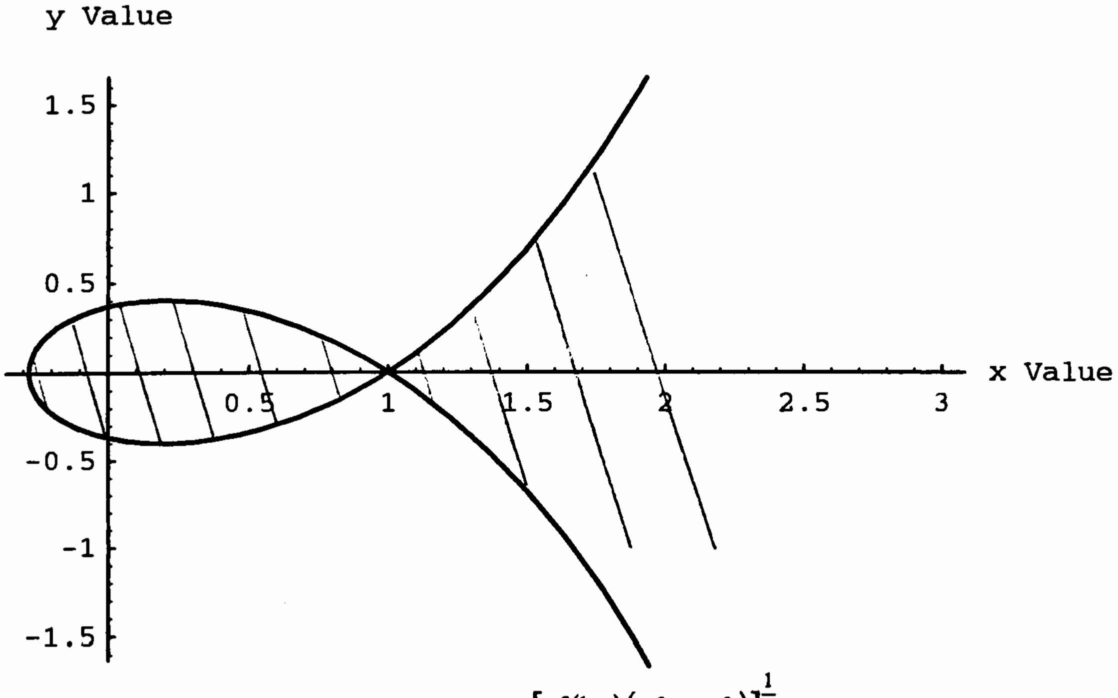

Figure 1: The region $\left[e^{2(1-x)}\left(x^2+y^2\right)\right]^{\frac{1}{2}} < 1$

On the boundary $\rho = 1$, from (11), the series

$$\sum_{n=0}^{\infty} e^{-(\tau+n)} \frac{(\tau+n)^n}{n!} \qquad \text{diverges.}$$

Consider the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$, then by the limit comparison test

$$\lim_{n\to\infty} \left(e^{-(\tau+n)} \frac{(\tau+n)^n}{n!} n \right) > 0$$

on utilizing Stirling's formula $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$ as $n \to \infty$.

The divergence of the above series can also be ascertained from the closed form representation of a modified right hand side in (11).

The Double Pole

The characteristic equation (4) may be shown to have a dominant double zero at $s = 0$ for $b + c = 0$ and $1 + ab = 0$. From the general theory of linear functional differential equations [15] it follows that there exists constants α and β such that

$$\lim_{t\to\infty} [f(t) - \alpha t] = \beta.$$

From residue theory, the constants α and β can be shown to be $\frac{2}{a}$ and $\frac{2}{3}$ respectively, in which case

$$\lim_{t\to\infty} \left[f(t) - \frac{2t}{a} \right] = \frac{2}{3}.$$

The Degenerate Case

From (10) and (2) it can be seen that

$$\lim_{a\to 0} \left[\sum_{n=0}^{\infty} (-1)^n c^n e^{-b(t-an)} \frac{(t-an)^n}{n!} \right] = e^{-(b+c)t}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} (-1)^n b^{n-r} c^r \binom{n}{r} \frac{t^n}{n!}.$$

This result can be ascertained directly from the differential-difference equation (1).

3. APPLICATIONS

A number of examples are investigated in which the method of the previous section is applicable.

(A) Bruwier Series

Bellman and Cooke [1] refer to

$$f(t) = \sum_{n=0}^{\infty} \frac{v^n}{n!} (t+n\omega)^n$$

as the *Bruwier* series, which is a solution to the advanced equation

$$f'(t) - vf(t+\omega) = 0 \quad , \quad f(0) = 1. \tag{12}$$

Comparing (12) with (1) it can be seen that $b=0$, $c=-v$, $a=-\omega$ and from the series at (6)

$$\sum_{n=0}^{\infty} \frac{v^n}{n!} (t+\omega n)^n = \frac{e^{\xi t}}{1-\omega\xi}$$

where ξ is the dominant real root of $\xi - ve^{\omega\xi} = 0$ and when $|v\omega e| < 1$, the region of convergence of the series.

(B) Teletraffic example

Erlang [11] considers the delay in answering of telephone calls. The problem is to determine the function $f(t)$, representing the probability of the waiting time not exceeding time t. Hence for an $M/M/1$ regimen Erlang shows

$$f(t) = \int_{y=0}^{\infty} f(t+y-a)e^{-y}dy.$$

The probability that, at the moment a call arrives, the time having elapsed since the preceeding call confined between y and $y+\delta y$, is $e^{-y}dy$. The probability that the waiting time of the preceeding call has been less than $(t+y-a)$ is $f(t+y-a)$, where a is the connection time of a call. Differentiating the integral equation with respect to t and partially integrating the result gives the differential-difference equation

$$\left.\begin{array}{ll} f'(t)-f(t)+f(t-a)=0 \quad , & t\geq a \\ f'(t)-f(t)=0 \quad , f(0)=1 \; , & 0\leq t<a. \end{array}\right\} \qquad (13)$$

The system (13) is compared with (1) where $b=-1$ and $c=1$.
Hence a solution of (13) is, from section 2

$$f(t) = \sum_{n=0}^{\infty}(-1)^n e^{t-an}\frac{(t-an)^n}{n!} = \frac{e^{\xi t}}{1-a+a\xi}$$

in the region of convergence $\left|ae^{1-a}\right| < 1$ and ξ is the dominant real root of $\xi-1+e^{-a\xi}=0$.

It can be shown that the characteristic equation of (13),

$$s-1+e^{-as}=0$$

has the following real root distribution:

(i) One root at $s=0$ for $a\leq 0$,
(ii) One negative root plus $s=0$ for $0<a<1$,
(iii) A double (repeated) root at $s=0$ for $a=1$,
(iv) One positive root plus $s=0$ for $a>1$.

In view of the convergence criteria for the single sum $\left|ae^{1-a}\right|<1$, the following results apply for all real values of t .

$$\sum_{n=0}^{\infty}(-1)^n e^{t-an}\frac{(t-an)^n}{n!} = \begin{cases} \dfrac{e^{\xi t}}{1-a+a\xi} & \text{for} \quad a>1 \\ \dfrac{1}{1-a} & \text{for} \quad a<1 \end{cases} \qquad (14)$$

which on putting $t=-a\tau$, the sum can be written as

$$\sum_{n=0}^{\infty}\left(ae^{-a}\right)^n\frac{(\tau+n)^n}{n!} = \begin{cases} \dfrac{e^{a\tau(1-\xi)}}{1-a+a\xi} & \text{for} \quad a>1 \\ \dfrac{e^{a\tau}}{1-a} & \text{for} \quad a<1 \end{cases}$$

where ξ is the positive root of $\xi-1+e^{-a\xi}=0$.

Erlang [11] considered only the case $0 < a < 1$.
In the case when $a = 1$ there is a double pole which results in, from a previous statement, $\lim_{t\to\infty} [f(t) - 2t] = \frac{2}{3}$.
This fact has also been noted, in a different context, by Feller [12].
Bloom [2] proposes the problem of evaluating

$$\lim_{t\to\infty} \{f(t) - 2t\}$$

given that , for t a positive integer

$$f(t) = \sum_{0 \le n \le t} \frac{(-1)^n}{n!} e^{t-n} (t-n)^n .$$

The W.M.C. problems group [27] and Holzsager [17] both solve this problem, and in particular Holzsager considers $f(t)$, $\forall t > 0$. Now, $f(t)$ satisfies the differential-difference equation

$$f'(t) = f(t) - f(t-1) \quad , \quad t \ge 1$$

using the theory of linear functional differential equations, Holzsager shows that

$$\lim_{t\to\infty} \{f(t) - 2t\} = \frac{2}{3}.$$

This work relates only to the asymptotic of the finite sum whereas in this paper it is shown that the infinite sum is equal to the asymptotic expression for all t .

(C) Neutron behaviour example

In the slowing down of Neutrons Teichmann [23] introduces Laplace transform techniques to analyze the 'renewal' equation. This example involves the Placzek function

$$F(s) = \frac{1 - e^{-(s+1)u_0}}{(s+1)(1-\alpha) - 1 + e^{-u_0(s+1)}} \qquad (15)$$

before inversion, where α is a constant depending on the mass of the moderating nuclei and $u_0 = -\ln \alpha$ is the maximum lethargy change in a single collision.

Using the techniques of the previous section Keane's [18] result is confirmed as

$$f(t)=\sum_{n=0}^{\infty}\frac{e^{\alpha t/(1-\alpha)}}{1-\alpha}\frac{(-1)^n}{n!}\left[\frac{(t-u_0n)^n}{(1-\alpha)^n}+\frac{n(t-u_0n)^{n-1}}{(1-\alpha)^{n-1}}\right]e^{-\frac{u_0n}{1-\alpha}}H(t-u_0n)$$

where t is lethargy and $H(t-u_0n)$ is the normal Heaviside function.
The contribution from the residue of (15), from the simple dominant pole at $s=0$, when $(1-\alpha+\alpha\ln\alpha)\neq 0$ is

$$\frac{1}{A}=\frac{1}{1+\frac{\alpha}{1-\alpha}\ln\alpha}.$$

Using the previous results, it will now be shown that

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left[\left(\frac{t-u_0n}{1-\alpha}\right)^n+n\left(\frac{t-u_0n}{1-\alpha}\right)^{n-1}\right]e^{-\frac{u_0n}{1-\alpha}}=\frac{(1-\alpha)e^{-\frac{\alpha}{1-\alpha}t}}{A}. \tag{16}$$

From (6), for $b=0$ and $c=1$ results in

$$g(p)=\sum_{n=0}^{\infty}(-1)^n\frac{(p-an)^n}{n!}=\frac{e^{\eta p}}{1+a\eta}\quad\text{where}\quad \eta+e^{-a\eta}=0. \tag{17}$$

Rewriting the left hand side of (16) gives

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left[t\frac{e^{-\frac{u_0}{1-\alpha}}}{1-\alpha}-\frac{u_0e^{-\frac{u_0}{1-\alpha}}}{1-\alpha}n\right]^n-\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left[\frac{te^{-\frac{u_0}{1-\alpha}}}{1-\alpha}-\frac{u_0(n+1)e^{-\frac{u_0}{1-\alpha}}}{1-\alpha}\right]^n e^{-\frac{u_0}{1-\alpha}}$$

$$=g(t)-g(t-u_0)e^{-\frac{u_0}{1-\alpha}}, \tag{18}$$

and it is required to show that (18) is identically equal to the right hand side of (16).

Let $p=\dfrac{te^{-\frac{u_0}{1-\alpha}}}{1-\alpha}$, $a=\dfrac{u_0}{1-\alpha}e^{-\frac{u_0}{1-\alpha}}$ and $B=\dfrac{u_0}{1-\alpha}$ then $a=Be^{-B}$,

so that from (17)

$$g(t)=\frac{e^{\frac{\eta t}{1-\alpha}e^{-B}}}{1+a\eta}.$$

From $\eta + e^{-\eta B e^{-B}} = 0$ put $a\eta = -E$ then $E = ae^{E}$ and hence $Ee^{-E} = Be^{-B}$, which is satisfied by the relationship $E = \alpha B$.

Now from (18) $$\frac{e^{-\frac{\alpha}{1-\alpha}t}}{1+\frac{\alpha}{1-\alpha}\ln\alpha} - \frac{e^{-\frac{\alpha}{1-\alpha}(t-u_0)}e^{-\frac{u_0}{1-\alpha}}}{1+\frac{\alpha}{1-\alpha}\ln\alpha}$$

$$= \frac{(1-\alpha)e^{-\frac{\alpha}{1-\alpha}t}}{1+\frac{\alpha}{1-\alpha}\ln\alpha} \qquad \text{as required.}$$

Equations (18) and (16) hold, in the region of convergence $\left|\frac{u_0}{1-\alpha}e^{\frac{(1-u_0-\alpha)}{1-\alpha}}\right| < 1$.

From (15) a double pole occurs at $s = 0$ when $1-\alpha+\alpha\ln\alpha = 0$, therefore

$$\lim_{t\to\infty}\left[f(t) - \frac{2\alpha}{1-\alpha}t\right] = \frac{2}{3}\left(\frac{\alpha(2\alpha+1)}{1-\alpha}\right).$$

(D) A Renewal example

In determining the availability of a renewed component Pagẽs and Gondran [20] consider the case of a constant failure rate.

Given that $A(t)$ is the availability of a Markovian component, λ is the constant failure rate, and $g(t)$ is a density function, then the integro-differential equation satisfied by $A(t)$ is

$$\frac{d}{dt}A(t) = -\lambda A(t) + (1-A_0)g(t) + \lambda\int_0^t g(u)A(t-u)du \quad , \quad A(0) = A_0.$$

Taking the Laplace Transform results in

$$\overline{A}(s) = \mathscr{L}\{A(t)\} = \frac{A_0 + (1-A_0)g(s)}{s+\lambda-\lambda g(s)}$$

Considering the case of constant repair time, that is Mean Time To Repair, M.T.T.R., is a , then $g(t) = \delta(t-a)$, where $\delta(t)$ is the Impulse function.

Hence,

$$\bar{A}(s) = \frac{A_0 + (1-A_0)e^{-as}}{s+\lambda-\lambda e^{-as}} \tag{19}$$

$$= \sum_{n=0}^{\infty} \frac{\lambda^n}{(s+\lambda)^{n+1}} \left\{A_0 e^{-asn} + (1-A_0)e^{-as(n+1)}\right\}$$

and by inversion

$$A(t) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \left\{A_0 e^{-\lambda(t-an)}(t-an)^n H(t-an) + (1-A_0)e^{-\lambda(t-a(n+1))}(t-a(n+1))^n H(t-a(n+1))\right\}$$

where $H(x)$ is the Heaviside function.

From (19) the residue at the dominant root $s=0$, of the characteristic equation $s+\lambda-\lambda e^{-as}=0$ for $a>0$ and $1+a\lambda \neq 0$, is $\dfrac{1}{1+a\lambda}$,

hence, by utilizing the previous section, the result becomes

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \left\{A_0 e^{-\lambda(t-an)}(t-an)^n + (1-A_0)e^{-\lambda(t-a(n+1))}(t-a(n+1))^n\right\} = \frac{1}{1+a\lambda}$$

in its region of convergence $\left|a\lambda e^{1+a\lambda}\right| < 1$ and $\forall t \in R$.

The value of the availability limit sum is independent of the initial value A_0 and the closed form solution is independent of the value of t .

It can be seen that

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda(t-an)}(t-an)^n = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda(t-a(n+1))}(t-a(n+1))^n = \frac{1}{1+a\lambda}$$

by putting $t-a=T$ in the second sum. Utilizing (8) and putting $t=-a\tau$ results in

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (\lambda a e^{a\lambda})^n (\tau+n)^n = \frac{e^{-\lambda a\tau}}{1+a\lambda} = e^{-a\lambda\tau} \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(-1)^r}{n!} \binom{n}{r} (a\lambda)^n (\tau+r)^n$$

whenever the double sum converges.

From (19) a double pole occurs at $s=0$ when $1+a\lambda=0$, and in this case

$$\lim_{t\to\infty}\left\{A(t)+\frac{2}{a}t\right\}=\frac{2}{3}(3A_0-2).$$

(E) **Ruin Problems in compound Poisson processes**

The integro-differential equation

$$R'(t)=(\frac{\alpha}{c_1})\left\{R(t)-\int_0^t R(t-x)dF(x)\right\} \tag{20}$$

is derived by Tijms [24] and Feller [12] and has applications to collective risk theory, storage problems and scheduling of patients. Here, α is the Poisson parameter and c_1 a positive rate.

Taking the Laplace transform of (20), it follows that

$$\overline{R}(s)=\mathscr{L}\{R(t)\} = \frac{R(0)}{1-\frac{\alpha}{c_1 s}(1-\overline{F}(s))}\cdot\frac{1}{s}.$$

Given that F is a distribution concentrated at the point a , μ is the expectation of F and $R(0)=1-k\mu$, where $k=\dfrac{\alpha}{c_1}$ results in

$$\overline{R}(s) = \frac{1-k\mu}{s-k+ke^{-as}}. \tag{21}$$

Comparing (21) with (2), $b=-k$, $c=k$ results in

$$R(t) = (1-k\mu)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}k^n e^{k(t-an)}(t-an)^n H(t-an).$$

The characteristic equation $s-k+ke^{-as}=0$ has a simple real dominant root at $s=0$, for $1-ak\neq 0$ and therefore

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}k^n e^{k(t-an)}(t-an)^n = \frac{1}{1-ak} \quad , \quad \forall t\in R$$

and in the region of convergence $\left|ak\,e^{1-ak}\right|<1.$

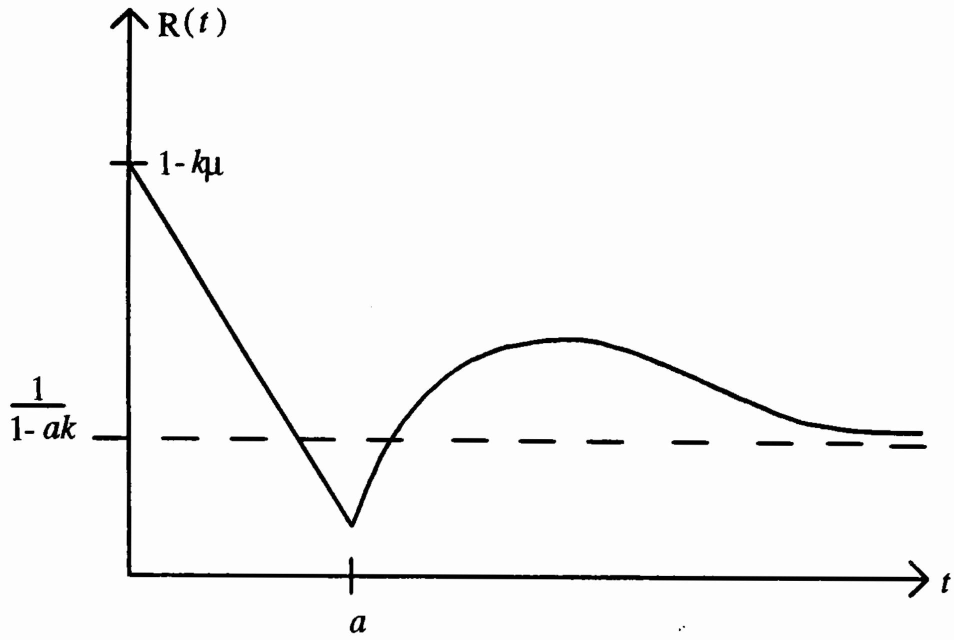

Figure 2: Sketch of the Ruin function $R(t)$.
The Ruin function $R(t)$ is continuous $\forall t \geq 0$, differentiable $\forall t \geq 0 \setminus \{t = a\}$ and approaches its limiting value $\dfrac{1}{1-ak}$.

From (21) a double pole occurs at $s = 0$ when $1 - ak = 0$, therefore

$$\lim_{t \to \infty} \left\{ R(t) + \frac{2}{a}(1 - k\mu)t \right\} = \frac{2}{3}(1 - k\mu).$$

4. ZEROS OF THE TRANSCENDENTAL EQUATION

Equation (4) is the transcendental equation associated with the differential-difference equation (1). The zeros of this equation are well documented and since many research papers have been interested in the stability of the solution of the differential-difference equation, conditions are given for the existence of complex conjugate roots with negative real part.
From Bellman and Cooke [1] a necessary and sufficient condition for (4) to have roots with negative real part is

(i) $ab > 1$.

(ii) $-ab < ac < \sqrt{\xi^2 + (ab)^2}$ where ξ is the root of $\xi + ab \tan \xi = 0$; $0 < \xi < \pi$ or $\xi = \dfrac{\pi}{2}$ if $ab = 0$.

Lemma 3

Equation (4) has at most 2 real zeros.

Proof: From (4), let $z = as$, $\alpha = ab$, $\beta = ac$, $a > 0$, $g(as) = G(z)$ and so

$$G(z) = z + \alpha + \beta e^{-z} = 0 \tag{22}$$

Let $Y(z) = \dfrac{1}{\beta}(z+\alpha)e^{z} = -1$ then at the turning point $z^* = -(1+\alpha)$,

$Y(z^*) = -\dfrac{1}{\beta e^{1+\alpha}}$. Hence, since $\left|\beta e^{1+\alpha}\right| < 1$, if $Y(z^*) < -1$ there exists at most 2 real roots as can be seen from figure 3.

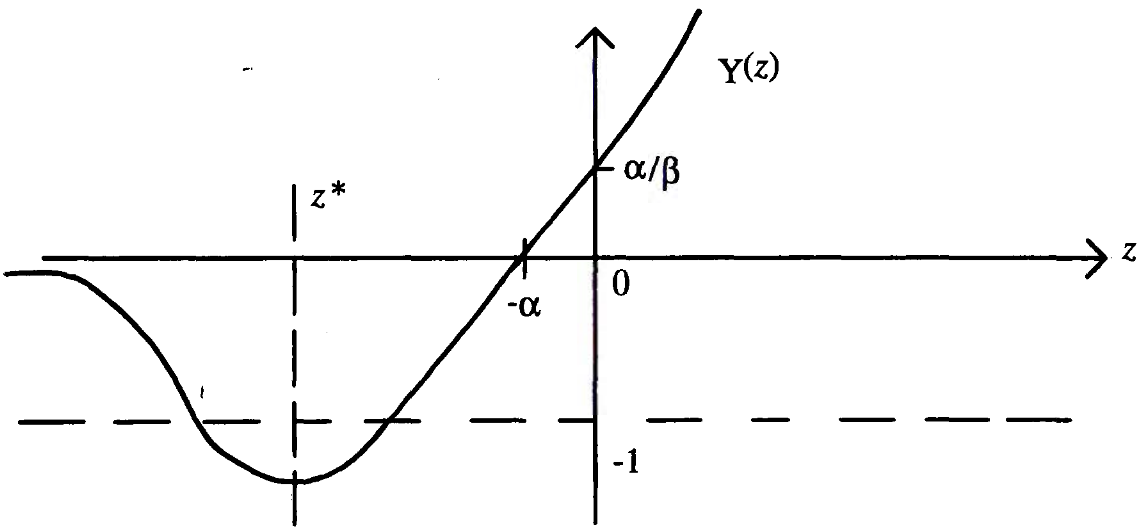

Figure 3 : The real roots of $G(z)$

Lemma 4

Equation (22) has a finite number of complex roots with positive real part.

Proof: Let $z = x + iy$, then from (22)

$$\left.\begin{aligned} x+\alpha+\beta e^{-x}\text{Cos}y &= 0 \\ y-\beta e^{-x}\text{Sin}y &= 0 \end{aligned}\right\} \tag{23}$$

The zeros of $G(z)$ depend continuously on β , and for $\beta > 0$ all zeros will be in the half plane $Re\ (z) \le \beta$. If $G(z) = G'(z) = 0$ there will be a double root at $z+1+\alpha = 0$ and therefore zeros cannot bifurcate or merge, as β varies, in the half plane $x > -1$.

Utilizing similar arguments to that of Cooke and Grossman [8] it can be seen that if $z = z(\beta)$ is an isolated simple zero with $Re\ (z) \geq 0$, then it moves to the right of the half plane for increasing β, since

$$\frac{dz}{d\beta} = -\frac{dG/d\beta}{dG/dz} = \frac{z+\alpha}{\beta(1+z+\alpha)}$$

and
$$Re\left(\frac{dz}{d\beta}\right) = \frac{(x+\alpha)(x+\alpha+1)+y^2}{(x+1+\alpha)^2+y^2} > 0.$$

Suppose a pure imaginary root exists, then $z = iy$ and a manipulation of (23) results in

$$y^2 = \beta^2 - \alpha^2$$

$$\alpha + \beta\, Cos\sqrt{\beta^2-\alpha^2} = 0.$$

For β increasing from α to ∞ these exists an increasing sequence

$$0 < \beta_1(\alpha) < \beta_2(\alpha) < \ldots\ldots, \quad \lim_{k\to\infty} \beta_k(\alpha) = \infty$$

$$\text{with } \operatorname{Sin}\sqrt{\beta^2-\alpha^2} > 0$$

such that for $\beta \in (\beta_k(\alpha)\,,\ \beta_{k+1}(\alpha))$ equation (22) has precisely k complex roots with positive real part. Also, whenever $\beta = \beta_k(\alpha)$ there exists a pair of complex conjugate imaginary roots $\pm i y_k$ such that

$$(4k+1)\frac{\pi}{2} < y_k < (2k+1)\pi \ , \quad k = 0, 1, 2, 3, \ldots .$$

It appears from (23) that a zero must remain in the region where $Sin y > 0$ and $Cos\, y < 0$.
In the specific case where $\alpha = 0$ then

$$\beta = y_k = (4k+1)\frac{\pi}{2} \ ; \quad k = 0, 1, 2, 3, \ldots\ldots$$

5. NUMERICAL EXAMPLES

The roots of the characteristic equation $s+b+ce^{-as}=0$ can be located using *Mathematica.*
Let $s=x+iy$ then

$$\begin{aligned} R(x,y)=0 &= x+b+ce^{-ax}\,Cos\,ay \\ I(x,y)=0 &= y-ce^{-ax}\,Sin\,ay. \end{aligned} \tag{24}$$

Note that in (24) if for any x, y is a solution then so is $-y$. Hence the non-real zeros occur in complex conjugate pairs.

Putting $t=-a\tau$, then (10) can be restated as

$$\sum_{n=0}^{\infty}\left(ace^{ab}\right)^n\frac{(\tau+n)^n}{n!} = \frac{e^{-a\tau(b+\xi_1)}}{1+ab+a\xi_1} = e^{-ab\tau}\sum_{n=0}^{\infty}\frac{(ab)^n}{n!}\sum_{r=0}^{n}\left(\frac{c}{b}\right)^r\binom{n}{r}(\tau+r)^n \tag{25}$$

where ξ_1 is the dominant root of the characteristic equation.

(a,b,c)	ξ_1	τ	Single Sum at (25)
.5, .2, .6	-1.421013107061	2	8.705206947716
	"	-2	.757217409662
8, -1, 6	.997954008152	2	1.050471634335
	"	-2	.983898536807
-4, 3, 1	-3.000006144061	2	.999926274283
	"	-2	1.000024576849
.1, 2, -2	0	2	.558600038363
	"	-2	1.243187248034
.15, -4, 4	0	2	8.300292306841
	"	-2	.752985529781
.99, -1, 1	0	2	724.274298516101
	"	-2	13.806292373109

Table 1: Examples of sums.

In table 1, the single sum converges to the closed form term at (25) to within a truncation error of $\epsilon = 10^{-12}$.

Using the technique developed by Braden [4] the single positive sum needs at least 245 terms so that its sum is within a truncation error of $\epsilon = 10^{-12}$ for $(a,b,c) = (.15,-4,4)$ and over one million terms for $(a,b,c) = (.99,-1,1)$ and $\tau = 2$.

The double sum in (10), when it converges, generally requires many more terms in it's series than does the single sum to converge to some prescribed truncation error. In Table 1 the double series converges only for the cases (a, b, c) = (.5, .2, .6) and (.1, 2, -2).

Using *Mathematica* a three dimensional plot of the surfaces $R(x,y) = z$ and $I(x,y) = z$ can be obtained as demonstrated in figures 4 and 5 respectively.

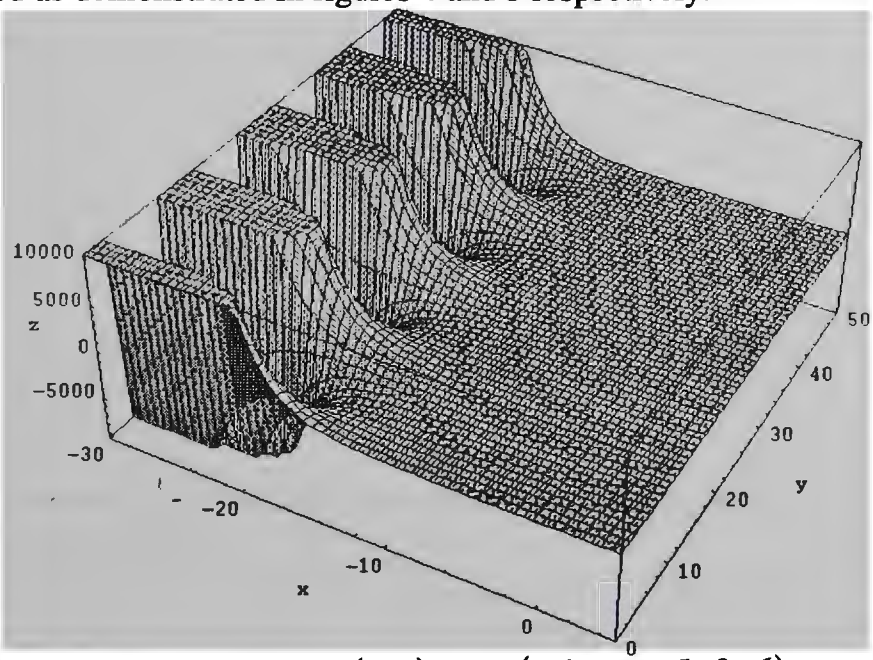

Figure 4: The Surface $z = R(x,y)$ for $(a,b,c) = (.5,.2,.6)$.

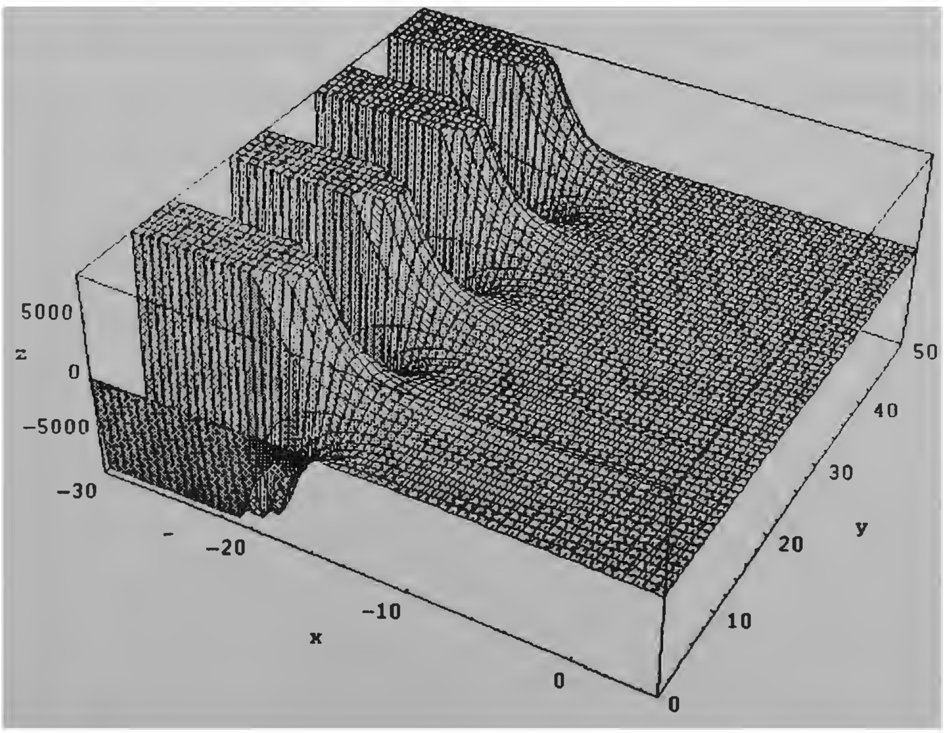

Figure 5: The Surface $z = I(x,y)$ for $(a,b,c) = (.5,.2,.6)$.

CONCLUSION

A technique has been demonstrated whereby series may be represented in closed form. An association was made between a differential-difference equation and its characteristic equation, however a starting point may simply be taken as a Laplace transform equation of the type

$$F(s) = \frac{1}{P_n(s) + Q_n(s)e^{-as}}$$

for $P_n(s)$ and $Q_n(s)$ polynomials in s.

In a follow up paper more general systems of the type

$$\sum_{k=0}^{R} \binom{R}{k} (-1)^k f^{(k)}(t) - f(t-a) = g(t)$$

and

$$\sum_{k=0}^{R} \binom{R}{k} (-1)^k f^{(R-k)}(t-ka) = h(t)$$

will be considered, together with equations that have more than one delay and equations of neutral and mixed types.

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