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Some Properties of the Raghavarao 13 Factor Foldover Design

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Some Properties of the Raghavarao 13 Factor Foldover Design

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Abstract

The projection properties of the Raghavarao 13 factor foldover design are examined in 3 and 4 space. Linear dependencies when k=2 are identified and where appropriate augmenting trials are listed to separate the confused models for both the error-free and the with error cases.

1 Introduction

If $X = ((x_{ij}))$ is an $n \times n$ matrix with $x_{ij} = x_{1m}$ where

$$m = \begin{cases} j - i + 1 & \text{if } j \geq i \\ n - (j - i + 1) & \text{otherwise} \end{cases}$$

then X is called a circulant matrix. Raghavarao (1959) presented a minimum run, non orthogonal, $2^{13}//13$ bias free weighing design. This resolution III design is of circulant form with first row given by

$$- \; - \; + \; - \; + \; + \; + \; + \; + \; - \; + \; + \; +$$

and has main effect trace efficiencies of 96.2%. These non-regular resolution III designs require few experimental runs but confound main effect and two-factor interaction estimates; for this reason designs of this class are useful in situations when the experimental budget is limited or in cases when effect sparsity is suspected and a screening design may be the best initial option.

By applying the foldover technique as defined by Box and Wilson (1951) this design can be moved from a bias free $2^{13}//13$ resolution III design, to a $2^{13}//26$ resolution IV design with bias. The Raghavarao 13 factor foldover design is presented in Table 1, and is in fact one circulant matrix augmented by another circulant matrix, being the negative of the first.

$$R = \begin{pmatrix} X \\ -X \end{pmatrix}, \quad \text{where } R \text{ is an } 2n \times n \text{ matrix.}$$

Whilst foldover designs requires doubling the number of runs, estimation of main effects now become unbiased by the two-factor interactions, it is also possible for a small number of two-factor interactions to be searched and estimated using methods as detailed below.

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2 Projection Properties

In order to examine every possible design in k space one would normally consider $\binom{n}{k}$ different possible combinations. Due to the circulant nature of this design any choice of k columns $(x_1, x_2, \ldots, x_k)$ has $(n-1)$ equivalent designs, found by $(x_{1+i}, x_{2+i}, \ldots, x_{k+i})$ for any $i = 1, \ldots, n$, with reduction modulo n being performed whenever neccesary. When considering the projection properties of the Raghavarao design in k space, therefore, only $\binom{n}{k}/n$ distinct choices of k columns need be considered, as all other permutations are derived from this base set.

1	2	3	4	5	6	7	8	9	10	11	12	13
−	−	+	−	+	+	+	+	+	−	+	+	+
+	−	−	+	−	+	+	+	+	+	−	+	+
+	+	−	−	+	−	+	+	+	+	+	−	+
+	+	+	−	−	+	−	+	+	+	+	+	−
−	+	+	+	−	−	+	−	+	+	+	+	+
+	−	+	+	+	−	−	+	−	+	+	+	+
+	+	−	+	+	+	−	−	+	−	+	+	+
+	+	+	−	+	+	+	−	−	+	−	+	+
+	+	+	+	−	+	+	+	−	−	+	−	+
+	+	+	+	+	−	+	+	+	−	−	+	−
−	+	+	+	+	+	−	+	+	+	−	−	+
+	−	+	+	+	+	+	−	+	+	+	−	−
−	+	−	+	+	+	+	+	−	+	+	+	−
+	+	−	+	−	−	−	−	−	+	−	−	−
−	+	+	−	+	−	−	−	−	−	+	−	−
−	−	+	+	−	+	−	−	−	−	−	+	−
−	−	−	+	+	−	+	−	−	−	−	−	+
+	−	−	−	+	+	−	+	−	−	−	−	−
−	+	−	−	−	+	+	−	+	−	−	−	−
−	−	+	−	−	−	+	+	−	+	−	−	−
−	−	−	+	−	−	−	+	+	−	+	−	−
−	−	−	−	+	−	−	−	+	+	−	+	−
−	−	−	−	−	+	−	−	−	+	+	−	+
+	−	−	−	−	−	+	−	−	−	+	+	−
−	+	−	−	−	−	−	+	−	−	−	+	+
+	−	+	−	−	−	−	−	+	−	−	−	+

Table 1: The Raghavarao 13 factor foldover design

2.1 In 3-Space

When examining the design in 3 space $\binom{13}{3}/13 = 22$ combinations must be considered. Direct examination of all 22 potentially different possibilities yields, for all cases, a 2^3 full factorial design replicated three times, and a 2^{3-2} design.

2.2 In 4-Space

There are $\binom{13}{4}/13 = 55$ potentially different choices of 4 columns which need to be considered when looking at the design in 4 space. Direct checking of each of the

55 cases yields 3 possible results:

(1) A 2^{4-1} (I= -1234)[1] , replicated three times and a 2^{4-3}

(2) A 2^4, a 2^{4-1} (I= 1234) and a 2^{4-3}

(3) A 2^4, a 2^{4-1} (I= -1234) and a 2^{4-3}

This immediately poses the problem, how to determine which of the three results is valid for a particular choice of 4 columns? Let $\alpha, \beta, \gamma, \delta$ be a unique choice of 4 columns in the foldover design and let X be the 78×26 matrix of interaction columns. If x_i is the column in X that corresponds to the $\alpha \times \beta$ interaction, and x_j is the column in X that corresponds to the $\gamma \times \delta$ interaction, the projection properties of the four columns corresponding to factors $\alpha, \beta, \gamma, \delta$ in the original foldover design is determined by examining the values of $x_i'x_j$ as follows:

$$x_i'x_j = \begin{cases} -22 & \text{Yields Result 1} \\ 10 & \text{Yields Result 2} \\ -6 & \text{Yields Result 3} \end{cases}$$

Determining the projection properties of the Raghavarao 13 factor foldover in 4 space is, therefore, as simple as looking up the relevant cell value for a given choice of four columns in the $X'X$ interaction matrix.

3 Linear Dependencies

Highly saturated resolution III designs are most often used as screening designs, a large number of factors are considered under the assumption that either main effects are the only active effects, or that few main effects are active and interactions are only considered between active main effects. This latter concept defines effect sparsity (Box and Meyer (1986)). Another concept, presented by Hamada and Wu (1992) is effect heredity, which is identical to effect sparsity except it also considers interactions between factors with one main effect active and the other main effect inert.

Srivastava (1975) developed the theory for search designs, and the concepts of effect sparsity and effect heredity fit neatly into this framework. He divided factorial effects into 3 categories:

1. Effects that can be assumed negligible.
2. Effects which require estimation.
3. All remaining effects, some which are negligible, some that will require estimation

Srivastava showed that, in the error-free case, when estimating all the effects in category 2 and k effects in category 3, the design is a strongly resolvable search design with resolution k iff every submatrix consisting of all the columns corresponding to category 2 and $2k$ of the columns corresponding to category 3 is of full rank.

3.1 When $k = 1$

The Raghavarao 13 factor foldover is strongly resolvable when $k = 1$ iff none of the interaction columns are identical to each other. This would correspond to a value of 26 in an off-diagonal element of the $X'X$ interaction matrix. As off-diagonal elements can only take values of 2,-6 and 10, the Raghavarao 13 factor foldover must be a strongly resolvable search design of resolution 1.

[1] For convenience Box, Hunter and Hunter (1978) notation will be used throughout this paper.

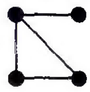
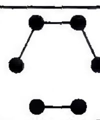
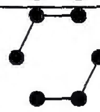
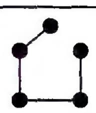
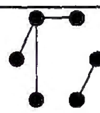
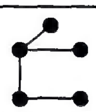
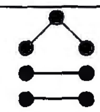
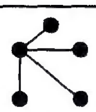
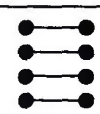
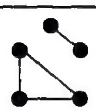
Case	Graph	Number of Isolated Vertices	Case	Graph	Number of Isolated Vertices
1		$(n-4)$	7		$(n-6)$
2		$(n-4)$	8		$(n-6)$
3		$(n-5)$	9		$(n-6)$
4		$(n-5)$	10		$(n-7)$
5		$(n-5)$	11		$(n-8)$
6		$(n-5)$			

Table 2: All graphs with n vertices and 4 edges

3.2 When $k = 2$

Let X be a matrix of 4 interaction columns and consider any two columns of X say x_i and x_j; $i, j = 1, 2, 3, 4$. The vector product $x_i'x_j$ can take on one of five different values depending on which interaction columns x_i and x_j represent. If the interaction corresponding to x_i has no letters in common with the interaction x_j, $x_i'x_j$ can take one of 10,-6 or -22, likewise if the interaction corresponding to x_i has one letter in common with interaction x_j, $x_i'x_j = 2$. Finally if $x_i = x_j$, $x_i'x_j = 26$. In summary :-

$$x_i'x_j = \begin{cases} 26 & \text{if i=j} \\ 2 & \text{if one letter in common} \\ \begin{pmatrix} 2 \pm 8 \\ -22 \end{pmatrix} & \text{if no letters in common} \end{cases}$$

To determine if the Raghavarao 13 factor foldover design is a strongly resolvable search design of resolution 2, every possible $X'X$ generated from the design must be of full rank, and each $X'X$ takes the following form:

$$X'X = \begin{pmatrix} 26 & x_{12} & x_{13} & x_{14} \\ x_{12} & 26 & x_{23} & x_{24} \\ x_{13} & x_{23} & 26 & x_{34} \\ x_{14} & x_{24} & x_{34} & 26 \end{pmatrix}$$

where $x_{ij} = 2, -6, 10, -22; i \neq j$.

Since x_{ij} can only take 4 possible values, there are $4^6 = 4096$ possible $X'X$ matrices to consider. Direct examination of these 4096 possibilities yields 3 essentially different rank difficient matrices. The upper diagonals of these matrices are

General Case	$\alpha\times\beta+\gamma\times\delta$	$\alpha\times\gamma+\beta\times\delta$	$\alpha\times\delta+\beta\times\gamma$
1	$1\times 2+4\times 10$	$1\times 4+2\times 10$	$1\times 10+2\times 4$
2	$1\times 3+9\times 13$	$1\times 9+3\times 13$	$1\times 13+3\times 9$
3	$1\times 5+6\times 8$	$1\times 6+5\times 8$	$1\times 8+5\times 6$
4	$1\times 7+11\times 12$	$1\times 11+7\times 12$	$1\times 12+7\times 11$
5	$2\times 3+5\times 11$	$2\times 5+3\times 11$	$2\times 11+3\times 5$
6	$2\times 6+7\times 9$	$2\times 7+6\times 9$	$2\times 9+6\times 7$
7	$2\times 8+12\times 13$	$2\times 12+8\times 13$	$2\times 13+8\times 12$
8	$3\times 4+6\times 12$	$3\times 6+4\times 12$	$3\times 12+4\times 6$
9	$3\times 7+8\times 10$	$3\times 8+7\times 10$	$3\times 10+7\times 8$
10	$4\times 5+7\times 13$	$4\times 7+5\times 13$	$4\times 13+5\times 7$
11	$4\times 8+9\times 11$	$4\times 9+8\times 11$	$4\times 11+8\times 9$
12	$5\times 9+10\times 12$	$5\times 10+9\times 12$	$5\times 12+9\times 10$
13	$6\times 10+11\times 13$	$6\times 11+10\times 13$	$6\times 13+10\times 11$

Table 3: Linear Dependencies in the Raghavarao 13 factor foldover design

as follows:

$$\begin{pmatrix} 26 & 2 & 2 & -22 \\ & 26 & -22 & 2 \\ & & 26 & 2 \\ & & & 26 \end{pmatrix} \begin{pmatrix} 26 & -6 & 10 & -22 \\ & 26 & -22 & 10 \\ & & 26 & -6 \\ & & & 26 \end{pmatrix} \begin{pmatrix} 26 & -6 & 10 & 10 \\ & 26 & 10 & 10 \\ & & 26 & -6 \\ & & & 26 \end{pmatrix}$$

Diamond (1993) showed that every X can be represented by one of eleven graphs, given in Table 2, involving n vertices and 4 edges. Each vertex represents a factor whilst each edge represents a two-factor interaction. Note that for the Raghavarao 13 factor foldover design if two edges are co-incident at one of the vertices, the corresponding vector product $x_i'x_j$ must be 2. The first dependent matrix above involves two-factor interactions between 4 factors, and can therefore be illustrated as Graph 2 in Table 1, whilst the second and third dependent matrices involve two-factor interactions between 8 factors and correspond to Graph 11 in Table 1. For the Raghavarao 13 factor foldover design, therefore, only two possible linearly dependent cases need to be examined, when $k = 2$.

To consider the case consisting of four interactions with no letters in common there are

$$\binom{13}{8} \times \frac{\binom{8}{2}\times\binom{6}{2}\times\binom{4}{2}\times\binom{2}{2}}{4!} = 135,135$$

cases. Direct checking shows that these 135,135 possible cases generated from the Raghavarao 13 factor foldover are all linearly independent.

To examine the case in which dependent models exist in four factors, the projection properties in 4-space can be utilised. Section 2.2 identified 3 distinct results for any choice of 4 columns, identified by $x_i'x_j = \{-22, 10, -6\}$; where x_i and x_j represent unique two-factor interactions in the design. Any choice of four factors in the design yielding $x_i'x_j = 10$ or -6, forms a full 2^4 factorial design, and thus provides unbiased estimation of all interactions between the four factors. Only choices of four columns yielding $x_i'x_j = -22$, therefore, require examination for linearly dependent models.

For any two-factor interaction $\alpha\times\beta$, corresponding to a column x_i in the $X'X$ interaction matrix, -22 will appear in only one of it's 78 cells. If x_j is the row in

$X'X$ corresponding to this cell, and x_j represents the two-factor interaction $\gamma \times \delta$, each linear dependency is of following form:

$$\alpha \times \beta + \gamma \times \delta = \alpha \times \gamma + \beta \times \delta = \alpha \times \delta + \beta \times \gamma \tag{1}$$

and by using the circulant properties of the design any one dependency of this form has $(k-1)$ equivalent designs found by:

$$\begin{aligned} &\{(\alpha + k) \times (\beta + k)\} + \{(\gamma + k) \times (\delta + k)\} = \{(\alpha + k) \times (\gamma + k)\} + \\ &\quad \{(\beta + k) \times (\delta + k)\} = \{(\alpha + k) \times (\delta + k)\} + \{(\beta + k) \times (\gamma + k)\} \end{aligned} \tag{2}$$

where $k = 1 \ldots 13$ and reduction modulo 13 is performed as neccessary.

Each dependency in the design is of the form given in equation 1. In total 13 linear dependencies exist and these form a closed set in which each of the 78 two-factor interactions of the design appear in one dependent model only. These 13 linear dependencies are listed in Table 3 for completeness, however by using equation 2 if any one linear dependency is known any other dependency can be derived.

Eg: If $(1\times 2) + (4\times 10) = (1\times 4) + (2\times 10) = (1\times 10) + (2\times 4)$ is a known linear dependency in the design, and it is desired to find the dependency which contains the 1×7 interaction. The interaction 1×7 can be expressed as $(4 + 10) \times (10 + 10)$ and substituted into equation 2 as follows:

$$\begin{aligned} &\{(1 + 10) \times (2 + 10)\} + \{(4 + 10) \times (10 + 10)\} = \{(1 + 10) \times (4 + 10)\} + \\ &\{(2 + 10) \times (10 + 10)\} = \{(1 + 10) \times (10 + 10)\} + \{(2 + 10) \times (10 + 10)\} \end{aligned}$$

Evaluating this equation and taking modulo 13 as required the linear dependency $(1 \times 7) + (11 \times 12) = (1 \times 11) + (7 \times 12) = (1 \times 12) + (7 \times 1)$ is identified, which is the only linear dependency in which the 1×7 interaction appears.

From the above results it is apparent that the Raghavarao 13 factor foldover is not a strongly resolvable, resolution 2 search design, and to estimate two-factor interactions in some cases requires the addition of augmenting trials.

4 Augmenting Design

If $(\alpha \times \beta + \gamma \times \delta)$ is the true model but is completely confused with the models $(\alpha \times \gamma + \beta \times \delta)$ and $(\alpha \times \delta + \beta \times \gamma)$ as described in section 3.2, then the addition of augmenting trials is required to identify the true model.

Let A be a 16×13 matrix with columns $\alpha, \beta, \gamma, \delta$ forming a full factorial design and all other columns held constant, and let $a^{[\alpha\beta]}$ correspond to the interaction column $\alpha \times \beta$ generated from A. Now let $C = (c^{[1]}, c^{[2]}, c^{[3]})$ be a 16×3 matrix with columns defined as follows:

$$\begin{aligned} c^{[1]} &= a^{[\alpha\beta]} + a^{[\gamma\delta]} - (a^{[\alpha\gamma]} + a^{[\beta\delta]}) \\ c^{[2]} &= a^{[\alpha\beta]} + a^{[\gamma\delta]} - (a^{[\alpha\delta]} + a^{[\beta\gamma]}) \\ c^{[3]} &= a^{[\alpha\delta]} + a^{[\beta\gamma]} - (a^{[\alpha\gamma]} + a^{[\beta\delta]}) \end{aligned}$$

Any $n \times 3$ submatrix of C forms an augmenting set of trials that will seperate the models iff each column of the submatrix satisfies the following:-

1. Each column is not equal to the null vector.

2. Each column is not proportional to the unit vector (if a block term is required).

3. No one column is proportional to any other column.

Eg: In order to seperate the $(1 \times 2) + (4 \times 10) = (1 \times 4) + (2 \times 10) = (1 \times 10) + (2 \times 4)$ linear dependency the following A matrix is generated:

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]	[,9]	[,10]	[,11]	[,12]	[,13]
[1,]	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
[2,]	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
[3,]	-1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
[4,]	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
[5,]	-1	-1	-1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1
[6,]	1	-1	-1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1
[7,]	-1	1	-1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1
[8,]	1	1	-1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1
[9,]	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	-1	-1	-1
[10,]	1	-1	-1	-1	-1	-1	-1	-1	-1	1	-1	-1	-1
[11,]	-1	1	-1	-1	-1	-1	-1	-1	-1	1	-1	-1	-1
[12,]	1	1	-1	-1	-1	-1	-1	-1	-1	1	-1	-1	-1
[13,]	-1	-1	-1	1	-1	-1	-1	-1	-1	1	-1	-1	-1
[14,]	1	-1	-1	1	-1	-1	-1	-1	-1	1	-1	-1	-1
[15,]	-1	1	-1	1	-1	-1	-1	-1	-1	1	-1	-1	-1
[16,]	1	1	-1	1	-1	-1	-1	-1	-1	1	-1	-1	-1

and the corresponding C matrix is as follows:

	[,1]	[,2]	[,3]
[1,]	0	0	0
[2,]	0	0	0
[3,]	0	0	0
[4,]	4	4	0
[5,]	0	0	0
[6,]	-4	0	-4
[7,]	0	-4	4
[8,]	0	0	0
[9,]	0	0	0
[10,]	0	-4	4
[11,]	-4	0	-4
[12,]	0	0	0
[13,]	4	4	0
[14,]	0	0	0
[15,]	0	0	0
[16,]	0	0	0

A submatrix of C that satisfies the three criteria required to seperate the dependency is:

	[,1]	[,2]	[,3]
[4,]	4	4	0
[7,]	0	-4	4

which corresponds to the augmenting trials $(4, 10)$ and $(1, 10)$.[2]

This result can be generalised as follows: If $\alpha \times \beta + \gamma \times \delta = \alpha \times \gamma + \beta \times \delta = \alpha \times \delta + \beta \times \gamma$ is a linear dependency present in the Raghavarao 13 factor foldover, addition of the augmenting trials $(\gamma\delta\ ,\ \alpha\gamma)$ is sufficient to both seperate the dependency, and to estimate a block effect.

5 Conclusion

In this paper it has been shown that the Raghavarao 13 factor foldover design gives a 2^3 and a 2^{3-2} in every set of three factors. When examining the design in 4 factors

[2] The notation$(4, 10)$represents an experimental run with factors 4 and 10 set at their high level and all other factors set at their low level respectively.

three possible results can be obtained, these three results are listed in section 2.2. The design has been shown to be a strongly resolvable search design when $k = 1$, but not when $k = 2$ as in some cases a number of models fit the data equally well. Each linearly dependent case has been listed and a general result given to generate the augmenting trials required to seperate the dependent models.

These results show that when a large number of factors need to be considered with a minimum amount of experimental observations the Raghavarao 13 factor foldover provides an effective option to the experimenter.

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