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CAPABILITY INDICES FOR MULTIVARIATE PROCESSES

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ABSTRACT

To satisfactorily describe the capability of multivariate processes, a multivariate capability index is required. This paper describes three approaches to designing capability indices for multivariate normal processes. In particular, three bivariate process capability indices are proposed and some simple rules provided for interpreting the ranges of values they take. The development of one index involves the projection of a process ellipse, containing at least a specified proportion of products, on to its component axes. The second one is based on the Bonferroni inequality. The final index utilizes Sidak's multivariate normal probability inequality in its construction. Some comparisons of the three indices are provided. An approximate test is developed for the Sidak-type capability index. A possible method of forming robust multivariate capability indices based on multivariate Chebyshev-type inequalities is also considered.

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INTRODUCTION

Since the pioneering work of Kane (1986), there have been many articles published dealing with process capability indices. Some developments in process capability analysis are outlined by Rodriguez (1992) in a special issue of the Journal of Quality Technology, entirely devoted to the topic. In Marcucci et al. (1988), it was noted that 'an index for multidimensional situations.....is another outstanding problem.....'. Most of the relevant work to date has focussed on the developments of process capability indices for *single* product characteristics. In many manufacturing situations, the quality of a manufactured product is more often than not determined by reference to more than one product characteristic. Invariably manufacturing conditions are such that there is an inter-dependency in the development of these product characteristics. To discuss process capability under these circumstances then, requires a method that acknowledges this inter-dependency and constructs an index that incorporates knowledge of the covariance structure of the quality characteristics.

The most commonly used univariate capability indices are the C_p, C_{pk} and C_{pm} indices which are defined as :-

$$C_p = \frac{U-L}{6\sigma},$$

$$C_{pk} = Min\left\{\frac{U-\mu}{3\sigma}, \frac{\mu-L}{3\sigma}\right\}$$

$$\text{and} \quad C_{pm} = \frac{U-L}{6\sqrt{\sigma^2+(\mu-T)^2}}$$

where μ, σ, U, L and $T = \frac{U+L}{2}$ denote the process mean, standard deviation, upper and lower specification limits, and target respectively. The first is strictly concerned with *process potential* in that it makes no reference to the process mean, μ. However, they all essentially reflect process potential in that they implicitly assume a perfectly controlled process. For meaningful use of these indices to describe actual process behaviour consideration of their sampling distributions is necessary. Statistical issues of estimation and hypothesis testing and practical matters such as the use and interpretation of these indices have been extensively discussed in the literature (see for eg., Kushler et al.(1992), Franklin et al.(1992), Pearn et al.(1992), Barnett (1990) and Boyles (1991)). These indices are applicable for situations involving two-sided specifications but some adaptations for one-sided specifications can also be found in the literature.

After reviewing existing work on multivariate process capability indices, this paper explores further the possibility of assessing multivariate process performance by using a single composite measure and describes three approaches for doing so. In particular, three bivariate process capability indices are proposed and some simple rules provided for interpreting the values they take. The relative effectiveness of the proposed indices as a comprehensive summary of process performance, with respect to all of the measured characteristics, is also provided. An approximate test for one of the proposed indices is developed. Possible methods of developing robust capability indices are also considered. The paper focuses on the commonly encountered situations in which the measured characteristics of a process or a product have two-sided specifications forming a rectagular specification region. The extension of this work to situations involving unilateral or a mixture of unilateral and bilateral tolerances is a straightforward matter. The total discourse is given in the context of discrete item manufacturing.

A REVIEW OF MULTIVARIATE CAPABILITY INDICES

Chan et al.(1991) introduced a so-called multivariate version of the C_{pm} index which is defined as :

$$\underset{\sim}{C}_{pm} = \sqrt{\frac{np}{\sum_{i=1}^{n} (\mathbf{X}_{pi} - \mathbf{T}_p)' \mathbf{A}^{-1} (\mathbf{X}_{pi} - \mathbf{T}_p)}}$$

To do this, they made the assumption that the specification requirements for a p-variate process or product are prescribed in the form of an ellipsoidal region given by

$$(\mathbf{X}_p - \mathbf{T}_p)' \mathbf{A}^{-1} (\mathbf{X}_p - \mathbf{T}_p) \leq c^2$$

where $\mathbf{X}_p, \mathbf{T}_p, \mathbf{A}$ and c are respectively the p-characteristic random vector, some specified $p \times 1$ vector, a $p \times p$ positive definite matrix and a constant. As this definition involves the sample observations rather than being based on the process parameters (i.e the mean vector $\boldsymbol{\mu}_p$ and the covariance matrix Σ), Pearn et al.(1992) stated, quite correctly, that it should be taken as an estimator (denoted $\hat{\underset{\sim}{C}}_{pm}$) of the following revised index :

$$\underset{\sim}{C}_{pm} = \sqrt{\frac{p}{E\left[(\mathbf{X}_p - \mathbf{T}_p)' \mathbf{A}^{-1} (\mathbf{X}_p - \mathbf{T}_p)\right]}}$$

Much of the discussion of Chan et al.(1990) was devoted to the test of $\underset{\sim}{C}_{pm} = 1$ based on the univariate statistic,

$$D = \sum_{i=1}^{n} (\mathbf{X}_{pi} - \mathbf{T}_p)' \mathbf{A}^{-1} (\mathbf{X}_{pi} - \mathbf{T}_p),$$

which is distributed as a *Chi-square* variable with np degrees of freedom under the multinormal assumption, with $\mu_p = \mathbf{T}_p$ and $\Sigma = \mathbf{A}$. Such a test reduces to the *simultaneous* test of both $\mu_p = \mathbf{T}_p$ and $\Sigma = \mathbf{A}$ against all other possible alternatives including those cases where the process is incapable, barely capable and more-than-capable. A value greater or smaller than expected for the test statistic, D, is merely a consequence of the violation of this null hypothesis and does not make any definitive statement about the process capability. For instance, if the given specification boundary, c^2 is considerably smaller than $\chi^2_p(\delta)$, the upper 100δ th percentile of the *Chi-square* distribution with p degrees of freedom and the value of the test statistic is smaller than expected (which suggests that $\underset{\sim}{C}_{pm} > 1$), this does not ensure that the expected proportion of non-defective items is more than 1-δ, where δ represents the acceptable maximum proportion of defective items. On the other hand, if c^2 is larger than $\chi^2_p(\delta)$ and the value of the test statistic is greater than the predictable limit, this does not necessarily indicate that the process is incapable, although $\underset{\sim}{C}_{pm} < 1$ is suggested. Apart from the issue of interpretability and the unrealistic assumption of a specification ellipsoid, it is worth noting that this work is more concerned with 'process capability analysis' rather than with the design of a unitless capability measure. Other issues of importance were discussed by Pearn et al.(1992).

As in Chan et al.(1990), Pearn et al.(1992) considered a ν-variate process with specification requirements formulated as an ellipsoidal region and proposed the following capability indices,

$$ {}_{v}C_{p}^{2} = \frac{c^{2}}{c_{v}^{2}} $$

and

$$ {}_{v}C_{pm}^{2} = \frac{{}_{v}C_{p}^{2}}{\left[1 + \frac{(\mu_{v} - T_{v})' A^{-1} (\mu_{v} - T_{v})}{v}\right]}, $$

as generalizations of the univariate C_p and C_{pm} indices. If $\mu_v = T_v$ and $\Sigma = A$, then c_v^2 in the above definitions is equated to $\chi_v^2(0.0027)$, otherwise, it is computed such that

$$ \Pr\left\{(X_v - T_v)' A^{-1} (X_v - T_v) \le c_v^2\right\} = 0.9973 . $$

In contrast to the $\underset{\sim}{C}_{pm}$ index, these indices correctly reflect process capability in the sense that their values decrease with declining process performance. However, as noted in their paper, the essential problem with these indices lies in the estimation and computation of them when $\mu_v \neq T_v$ and $\Sigma \neq A$ due to the complexity of the distribution of the quadratic form $(X_v - T_v)' A^{-1} (X_v - T_v)$. If these indices are to be of any practical use, therefore, computer programs for their estimation or computation must be available.

In view of the fact that it is unlikely to have specifications given as ellipsoids, Rodriguez (1992) suggested the direct estimation of the proportion of nonconforming items by integration of the multivariate normal density function over the specification rectangular region. Boyles (1994b) also considered this alternative of estimating process capability and discussed its statistical and practical merits over a competing procedure which is based on simple binomial estimates. The total discussion is in the context of repeated *lattice-structured* measurements.

Unlike others, Hubele et al.(1991) proposed a capability *vector* for a bivariate normal process which consists of three components. The first is the ratio of the area of the *specification rectangle* to that of the *projected process rectangle*, giving an analogue of the univariate C_p index. The second component, is defined as the significance level computed from a T^2-type statistic which measures the relative location of the process centre and the target. The last component is designed to capture situations where one or more of the *process limits* fall beyond the corresponding specification limits. Although some efforts were made to demonstrate the usefulness of this capability vector as a summary measure of the process performance, interpretation is sometimes difficult.

Other contributions come from Taam et al.(1993) who proposed a multivariate capability index defined as

$$MC_{pm} = \frac{\text{Volume of } R_1}{\text{Volume of } R_2},$$

where R_1 and R_2 represent respectively the *modified tolerance region* (modified according to the process distribution) and the scaled 99.73% process region (scaled by the mean squared error, $\Sigma_T = E[(\mathbf{X}_p - \mathbf{T}_p)(\mathbf{X}_p - \mathbf{T}_p)']$). If the process follows a multivariate normal distribution, then the modified tolerance region here is the largest ellipsoid inscribing the original specification region and the scaled process region, R_2, is an ellipsoidal region represented by $(\dot{\mathbf{X}}_p - \boldsymbol{\mu}_p)'\Sigma_T^{-1}(\mathbf{X}_p - \boldsymbol{\mu}_p) \le \chi^2_{p(0.0027)}$. Thus, under normality assumptions, this index becomes

$$MC_{pm} = \frac{\text{Vol.}(R_1)}{\text{Vol.}(R_3)} \times \frac{1}{\left[1+(\mu_p - \mathbf{T}_p)' \Sigma^{-1}(\mu_p - \mathbf{T}_p)\right]}$$

$$= \frac{MC_p}{D_{\mathbf{T}}}$$

where R_3 is the natural process ellipsoid containing 99.73% of items, $MC_p = \frac{\text{Vol.}(R_1)}{\text{Vol.}(R_3)}$ is an analogue of the univariate C_p (squared) index which measures the process potential and $D_{\mathbf{T}} = 1+(\mu_p - \mathbf{T}_p)'\Sigma^{-1}(\mu_p - \mathbf{T}_p)$ is a measure of process mean deviation from target. As stated by Taam et al.(1993), this is an analogue of the univariate C_{pm} (squared) index. Note also that this index is similar to ${}_vC^2_{pm}$, except in the manner in which the process potential and the deviation of mean from target are quantified. In terms of its ease of computation and general applicability, it is superior to the latter. Besides the fact that it can be used for different types of specification region (see the example on *geometric dimensioning and tolerancing* (GDT) in the same paper), this index can be extended to non-normal processes provided the specifications are two-sided. This, however, entails the determination of the proper process and modified tolerance region and the resulting computations are likely to be complex. In the same paper, Taam et al.(1993) considered the estimation of this capability index. However, they simply replace the unknown mean vector μ_p and the covariance matrix Σ in the expression for the proposed index with the usual unbiased estimates and use $\chi^2_{p(0.0027)}$ as the boundary of the process ellipsoid without taking into consideration issues such as unbiasedness, efficiency and uncertainty of the resulting capability index estimate. They also highlighted some similarities and differences between the proposed index (MC_{pm}), $\underset{\sim}{C}_{pm}$ and the bivariate capability vector proposed by Hubele et al.(1991). A major problem with this index is its likelihood of leading to misleading conclusions. For instance, if the measured characteristics are not independent and the index value is 1 (as a result of the process being on-target and the volume of the process ellipsoid being the same as that of the modified tolerance region), there is no assurance that the process under consideration is capable of meeting the specifications consistently or can be expected to produce 99.73% of conforming items. This is in conflict with the statement made by Taam et al.(1993) that, 'when the process is centered at the target and the capability index is 1, it indicates 99.73% of the process values lie inside the tolerance region.' The deficiency in this comment is illustrated in *Figure* 1 for a bivariate normal process.

Boyles (1994a) introduced the concept of *exploratory* capability analysis (ECA) which is aimed at capability improvement rather than assessment. This should be distinguished from the so-called *confirmatory* capability analysis (CCA) which involves formally assessing whether the process under consideration is capable of meeting the given specifications or not. ECA, essentially utilizes exploratory graphical data analysis techniques, such as boxplots, to reveal or to assist in identifying new opportunities for process improvement. Three real examples involving repeated measurements with lattice structure were used to illustrate the usefulness of the concept.

In another paper, Boyles (1994b) proposed an *expository* technique of analyzing multivariate data using repeated measurements with a lattice structure where the number of measurements for the same characteristic on each part or product, *p*, may possibly exceed the number of inspected parts or products, *n*. He developed a class of *Direct Covariance* (DC) models corresponding to a general class of lattices and obtained some positive definite estimates of the covariance matrix denoted by $\hat{\Sigma}_{DC}$ even when $n \lesssim p$. This property of positive definiteness for the estimated covariance matrix permits the computations of multivariate capability indices and estimated process yields which depend on Σ^{-1} when $n \leq p$ or when *n* is not much greater than *p*, in which case the usual sample covariance is illconditioned with respect to matrix inversion. He made some efforts to justify the use of the proposed model for process capability analysis. In particular, he demonstrated the superiority of employing $\hat{\Sigma}_{DC}$ to provide an estimate of the proportion of nonconforming units over the use of sample covariance and the 'empirical' approach of simple binomial estimates. To do this he used sets of data from Boyles (1994a) along with some simulation results.

CONSTRUCTING A MULTIVARIATE CAPABILITY INDEX

With the assumption that the process under focus follows a *multivariate normal* distribution, consider the following approaches to the design of a multivariate process capability index. Before proceeding, it should be pointed out that, although these approaches have been widely discussed in simultaneous interval estimation problems (see for eg., Johnson et al. (1988) and Nickerson (1994)), they are used here in a different context.

The first approach entails the construction of a conservative p-dimensional '*process rectangle*' from the projection of an exact ellipsoid (ellipse if bivariate) containing a specified proportion of items on to its component axes. The edges of the resulting process rectangle (the process limits) are then compared with their corresponding specification limits. The associated index is defined in such a way that it is 1 if the process rectangle is contained within the p-dimensional '*specification rectangle*' with at least one edge coinciding with its corresponding upper or lower specification limits, greater than 1 if the process rectangle is completely contained within the specification rectangle and less than 1 otherwise. A bivariate capability index developed using this approach is presented in the next section.

The second approach is based on the well known Bonferroni inequality. Unlike the first one, this approach actually requires only the weaker assumption of normality for each individual product characteristic. The capability index using this approach is defined in the same manner as above. The resulting process rectangular region having at least a specified proportion of conforming items is compared with the specification rectangle. The value of the proposed capability index reflects *conservatively* the process capability of meeting the specifications consistently. In fact, the assessment of process performance based on the Bonferroni inequality has been perceived by Boyles (1994b) but it is used in a different way and context. It should also be pointed out, despite his statement to the contrary, that the given inequality

$$\pi_l + \pi_u \geq \pi$$

where

$$1-\pi = \Pr\left(-D_l \leq X_j \leq D_u, 1 \leq j \leq p \mid \mu, \Sigma\right)$$
$$1-\pi_l = \Pr\left(X_j \geq -D_l, 1 \leq j \leq p \mid \mu, \Sigma\right)$$
$$1-\pi_u = \Pr\left(X_j \leq D_u, 1 \leq j \leq p \mid \mu, \Sigma\right)$$

is not generally true.

Another approach utilizes the multivariate normal probability inequality given by Sidak (1967). It will be seen later, that a capability index constructed based on this inequality and using arguments similar to the above, provides the best measure among all those proposed in this paper.

THREE BIVARIATE CAPABILITY INDICES

Suppose that the vector of the p product characteristics, $\mathbf{X}_p = (X_1, X_2, \ldots, X_p)'$ follows a multivariate normal distribution with mean vector $\boldsymbol{\mu}_p = (\mu_1, \mu_2, \ldots, \mu_p)'$ and covariance matrix Σ. Further, suppose that a manufactured product is considered *usable* if *all* its measured product characteristics are within their corresponding specification limits. Let δ denote the proportion of unusable items produced that can be tolerated. Our aim is to obtain the relationship between the component means, the elements of the covariance matrix, δ and the specification limits of all the measured characteristics by solving the following integral equation :

$$\int_{L_p}^{U_p} \cdots \int_{L_2}^{U_2} \int_{L_1}^{U_1} f(x_1, x_2, \ldots, x_p).dx_1 dx_2 \ldots dx_p = 1 - \alpha \,,$$

so that an index can be defined that reliably reflects the actual process capability. Directly attempting to solve this equation is generally inadvisable due to computational difficulties, so some approximations are presented.

(1) <u>Projection of Exact Ellipsoid Containing a Specified Proportion of Products</u>

It is known, for eg., Johnson et al.(1988) that, if $\mathbf{X}_p \sim N_p(\boldsymbol{\mu}_p, \Sigma)$, the quadratic form $(\mathbf{X}_p - \boldsymbol{\mu}_p)' \Sigma^{-1} (\mathbf{X}_p - \boldsymbol{\mu}_p) \sim \chi^2_p$. Thus, a region containing $100(1-\delta)\%$ of the products is the solid ellipsoid given by

$$(\mathbf{X}_p - \boldsymbol{\mu}_p)' \Sigma^{-1} (\mathbf{X}_p - \boldsymbol{\mu}_p) \leq \chi^2_p(\delta) \,.$$

As given by Nickerson (1994), the projection of the above ellipsoid on to each of its component axes is given by :

$$\left|x_j - \mu_j\right| \leq \sqrt{\chi_p^2(\delta)}\left[j\text{th } \textit{diagonal element of } \Sigma\right]^{\frac{1}{2}}$$

or

$$\mu_j - \sqrt{\chi_p^2(\delta)}\,\sigma_j \leq x_j \leq \mu_j + \sqrt{\chi_p^2(\delta)}\,\sigma_j \qquad \text{..........} \ (1)$$

$$j = 1,2\ldots,p$$

Note that rewriting (1) yields the well known 100(1-δ)% simultaneous confidence interval for $\boldsymbol{\mu}_p = (\mu_1, \mu_2, \ldots, \mu_p)'$ based on a sample of size $n = 1$ when $\sigma_1, \sigma_2, \ldots, \sigma_p$ are known (Johnson et al.(1988)). As a special case, consider developing a capability index for bivariate processes, though it can easily be extended to the more general case. Note that, for $p = 2$, $\chi_p^2(\delta) = -2\ln\delta$. Thus, we have from (1) that, the 'bivariate process limits' (i.e the limits beyond which at most 100δ % of items are expected to be produced) are,

$$x_j = \mu_j \pm \sigma_j\sqrt{-2\ln\delta} \qquad j = 1,2.$$

It follows, that for

$$\int_{L_2}^{U_2}\int_{L_1}^{U_1} f(x_1, x_2)\,dx_1 dx_2 \geq 1 - \delta,$$

the following conditions need to be simultaneously satisfied :-

$$U_1 \geq \mu_1 + \sigma_1\sqrt{-2\ln\delta}$$

$$L_1 \leq \mu_1 - \sigma_1\sqrt{-2\ln\delta}$$

$$U_2 \geq \mu_2 + \sigma_2\sqrt{-2\ln\delta}$$

$$L_2 \leq \mu_2 - \sigma_2\sqrt{-2\ln\delta}$$

or equivalently,

$$\frac{U_1 - L_1}{2\left\{\sigma_1\sqrt{-2\ln\delta} + \left|\mu_1 - \frac{U_1+L_1}{2}\right|\right\}} \geq 1$$

and

$$\frac{U_2 - L_2}{2\left\{\sigma_2\sqrt{-2\ln\delta} + \left|\mu_2 - \frac{U_2+L_2}{2}\right|\right\}} \geq 1 .$$

Accordingly, the bivariate process capability index, $C_{pk}^{(2)}$, is defined as :-

$$C_{pk}^{(2)} = Min\left\{\frac{C_{p1}}{\frac{1}{3}\sqrt{-2\ln\delta} + \frac{|\mu_1 - T_1|}{3\sigma_1}}, \frac{C_{p2}}{\frac{1}{3}\sqrt{-2\ln\delta} + \frac{|\mu_2 - T_2|}{3\sigma_2}}\right\},$$

where C_{pj} and T_j $(j = 1,2)$ are respectively the univariate process capability indices $(C_p)$and the target values for the two product characteristics. Note that if the process is on-target i.e $\mu_1 = T_1$ and $\mu_2 = T_2$,

$$C_{pk}^{(2)} = \frac{Min\{C_{p1}, C_{p2}\}}{\frac{1}{3}\sqrt{-2\ln\delta}},$$

which can be taken as a measure of the process potential. Although this capability index is conservative by nature and thus must be carefully interpreted, it does provide some insight into the practical capability of the process. A value of 1 or greater can safely be interpreted as the process producing at a satisfactory level provided there is no serious departure from normality. However, if it has a value smaller than 1, it does not necessarily indicate that the expected proportion of usable items produced is less than $1-\delta$, unless it is significantly different from 1. In this case, perhaps some simple guidelines or *ad hoc* rules would help to determine if the process capability is adequate. Note the interesting fact that, although the covariance structure of the product characteristics are considered in the development of the above, the proposed bivariate capability index does not involve the correlation coefficient ρ of the two characteristics. It is also noted that the proposed index has some similarity to the bivariate capability vector proposed by Hubele et al.(1991). It differs from the latter, however, in that it incorporates both the process potential and the deviation of the process mean from target into a unitless measure. Hubele et al's capability index consists of three components, one for measuring the process location, one for process dispersion (potential) and the other for indicating whether any of the process limits is beyond its corresponding specification limit(s). Whilst it may be argued that using separate indicators for each of the above factors to reflect the process status may make the interpretation clearer, this process capability vector involves more calculation and does not have any clear advantages over the proposed index.

(2) <u>Bonferroni-Type Process Rectangular Region</u>

According to the Bonferroni inequality, for a p-variate process for which the marginal distributions are normal, the p-dimensional centered process rectangle containing at least $100(1-\delta)\%$ of items is given by :

$$\mu_j - z_{\frac{\delta}{2p}}\sigma_j \le x_j \le \mu_j + z_{\frac{\delta}{2p}}\sigma_j, \qquad j=1,2,\ldots\ldots,p.$$

where $z_{\delta/2p}$ denotes the upper $100(\delta/2p)$th percentile of a standard normal distribution. By replacing p by 2 in the above, the bivariate process limits are obtained. Proceeding as previously, another bivariate capability index is obtained and defined as,

$$ {}^{B}C_{pk}^{(2)} = Min\left\{ \frac{C_{p1}}{\frac{1}{3}z_{\delta/4} + \frac{|\mu_1 - T_1|}{3\sigma_1}}, \frac{C_{p2}}{\frac{1}{3}z_{\delta/4} + \frac{|\mu_2 - T_2|}{3\sigma_2}} \right\} $$

which has a similar interpretation. If $p = 1$, $\delta = 0.0027$ and the process is on-target, this type of multivariate capability index reduces to the univariate C_p , C_{pk} and C_{pm} indices. It should be noted that this method of developing capability indices can be extended to non-normal processes by replacing $-z_{\delta/4}$ and $z_{\delta/4}$ by the appropriate quantiles of the process distribution.

(3) **Process Rectangular Region based on Sidak's Probability Inequality**

As given by Sidak (1967), the multivariate normal probability inequality is :-

$$ \Pr\left\{ \bigcap_{j=1}^{p} |Z_j| \le c_j \right\} \ge \prod_{j=1}^{p} \Pr\left\{ |Z_j| \le c_j \right\}, $$

where Z_j's are standard normal variables and c_j's denote some specified constants. For $c_j = c$ $(j = 1,2\ldots,p)$, this inequality becomes

$$\Pr\left\{\bigcap_{j=1}^{p} |Z_j| \le c\right\} \ge [2\Phi(c) - 1]^p,$$

where $\Phi(\bullet)$ denotes the cumulative distribution function of the standard normal variable. Setting the lower bound, $[2\Phi(c)-1]^p$ of the joint probability above equal to $1-\delta$ results in a p-dimensional process rectangle containing at least $100(1-\delta)\%$ of items given by :-

$$\mu_j - \sigma_j \Phi^{-1}\left(\tfrac{1}{2}\left[1+(1-\delta)^{1/p}\right]\right) \le x_j \le \mu_j + \sigma_j \Phi^{-1}\left(\tfrac{1}{2}\left[1+(1-\delta)^{1/p}\right]\right),$$
$$j = 1,2,\ldots,p,$$

where $\Phi^{-1}(\bullet)$ represents the inverse of the standard normal distribution function. A bivariate capability index is obtainable by replacing p with 2 and comparing the resulting bivariate process limits with the corresponding specification limits. This index is defined as :-

$$S_{C_{pk}}^{(2)} = Min\left\{\frac{C_{p1}}{\frac{1}{3}\Phi^{-1}\left(\frac{1}{2}\left[1+\sqrt{1-\delta}\right]\right)+\frac{|\mu_1-T_1|}{3\sigma_1}}, \frac{C_{p2}}{\frac{1}{3}\Phi^{-1}\left(\frac{1}{2}\left[1+\sqrt{1-\delta}\right]\right)+\frac{|\mu_2-T_2|}{3\sigma_2}}\right\}.$$

In the development of the above indices, it has been assumed that the tolerances are bilateral and that the target or nominal specification is the midpoint of the specification band, this, of course, is not always the case in practice. Under these circumstances, redefinition of the indices using similar arguments is straightforward and will not be discussed further.

As all of the above are of similar form, it is preferable to choose the one which is least conservative or that best reflects the actual process capability. In the following section, some comparisons between the three are provided in order to resolve this issue.

SOME COMPARISONS OF THE PROJECTED, BONFERRONI AND SIDAK-TYPE CAPABILITY INDICES

Following conventional practice, the relative merits of the proposed capability indices can be evaluated based on the following ratios :

$$I_{B:P} = \frac{\textit{Width of } 100(1-\delta)\% \textit{ projected Interval for } j\text{th } \textit{characteristic}}{\textit{Width of } 100(1-\delta)\% \textit{ Bonferroni Interval for } j\text{th } \textit{characteristic}}$$

$$= \frac{\sqrt{\chi_p^2(\delta)}}{z_{\delta/2p}}$$

and

$$I_{S:P} = \frac{\textit{Width of } 100(1-\delta)\% \textit{ projected Interval for } j\text{th } \textit{characteristic}}{\textit{Width of } 100(1-\delta)\% \textit{ Sidak Interval for } j\text{th } \textit{characteristic}}$$

$$= \frac{\sqrt{\chi_p^2(\delta)}}{\Phi^{-1}\left(\frac{1}{2}\left[1+(1-\delta)^{1/p}\right]\right)}.$$

Note that the expressions for $I_{B:P}$ and $I_{S:P}$ remain the same irrespective of the product characteristic being considered. One capability index is said to be less conservative than the other if its construction is based on a shorter interval for the same δ. Thus, according to the definitions above, if $I_{B:P}$ is greater than 1, the Bonferroni-type capability index is better (less conservative) than that which is based on projections. Similarly, a value of $I_{S:P}$ greater than 1 implies that the Sidak-type index is superior to the projected one. As for the relative effectiveness of the Bonferroni and Sidak-type indices, this is measured by the relative magnitude of their corresponding $I_{B:P}$ and $I_{S:P}$ values. The values of these indices are tabulated in *Table* 1 for some selected values of p and δ. It can be seen from this table that, in all the realistic cases considered, both the capability indices based on the Bonferroni and Sidak inequalities provide better measures than the Projection-type capability index. The table also shows that, as the number of measured characteristics, p increases, the better the Bonferroni or Sidak-type capability index becomes. Furthermore, as shown in the table, the Sidak-type capability index is marginally better than that based on the Bonferroni inequality. The following section is devoted to the development of a test concerning process capability based on the Sidak-type index.

TESTING THE CAPABILITY OF A BIVARIATE PROCESS

In practice, the assessment of process performance is often based on sample estimates of some capability indices which are subject to uncertainty. Unless the sample size is reasonably large, it is inappropriate to draw definite conclusions from these process capability estimates. Of course, the need for process stability before computation should also be emphasized, otherwise the interpretation of these indices is distorted, regardless of how large the sample is. If meaningful interpretation of the estimated capability is sought, it is important to take the sampling fluctuations of these estimates into consideration. A common approach is to employ confidence intervals. If point estimates are to be used, it is desirable that estimation is unbiased and that the minimum sample size required for an acceptable margin of estimation error is adhered to. Another approach is based on testing hypotheses. Either approach generally requires knowledge of the sampling distributions which are complicated. To circumvent this problem, we develop an *approximate* test for the Sidak-type index (${}^{S}C_{pk}^{(2)}$).

Consider the problem of testing the following hypotheses :

$$H_0 : {}^{S}C_{pk}^{(2)} \geq 1 \quad vs. \quad H_a : {}^{S}C_{pk}^{(2)} < 1$$

Under the null hypothesis, H_0, the process is capable and the worst scenario is when both the Sidak-type process and specification rectangles coincide, in which case ${}^{S}C_{pk}^{(2)} = 1$. On the other hand, the alternative hypothesis, H_a corresponds to situations where at least one edge of the process rectangle is beyond its corresponding specification limit. The test proposed here is designed to capture such a situation.

A reasonable choice of the test statistic for this problem is,

$${}^{S}\hat{C}_{pk}^{(2)} = Min\left\{ \frac{U_1 - L_1}{2cS_1 + 2|\overline{X}_1 - T_1|}, \frac{U_2 - L_2}{2cS_2 + 2|\overline{X}_2 - T_2|} \right\},$$

where

$$c = \Phi^{-1}\left(\tfrac{1}{2}\left[1 + \sqrt{1 - \delta}\right]\right)$$

and, $\overline{X}_j$ and S_j denote respectively the mean and standard deviation of the jth product characteristic based on a sample of size n. The decision rule is to reject H_0 in favor of H_a if

$$ {}^{S}\hat{C}_{pk}^{(2)} < k \,, $$

where k is some positive constant depending on the significance level of the test (α) and is determined from

$$ Max\ \Pr\left\{ {}^{S}\hat{C}_{pk}^{(2)} < k \mid H_0 \text{ is true} \right\} = \alpha \,. $$

The maximum value on the left-hand side of the above equation occurs when $\mu_j = T_j$ and $U_j - L_j = 2c\sigma_j$ for $j = 1,2$ (the worst situation under H_0). Thus, we have,

$$ \Pr\left\{ Min\left\{ \frac{c\sigma_1}{cS_1 + \left|\overline{X}_1 - \mu_1\right|}, \frac{c\sigma_2}{cS_2 + \left|\overline{X}_2 - \mu_2\right|} \right\} < k \right\} = \alpha $$

or

$$ \Pr\left\{ \bigcap_{j=1}^{2} \left[\frac{1}{\sqrt{n-1}} \sqrt{\frac{(n-1)S_j^2}{\sigma_j^2}} + \frac{1}{c\sqrt{n}} \sqrt{\left(\frac{\overline{X}_j - \mu_j}{\sigma_j/\sqrt{n}} \right)^2} < \frac{1}{k} \right] \right\} = 1 - \alpha \,. $$

.... (2)

According to the Bonferroni inequality,

$$ \Pr\left\{ \bigcap_{j=1}^{2} \left[\frac{1}{\sqrt{n-1}} \sqrt{\frac{(n-1)S_j^2}{\sigma_j^2}} + \frac{1}{c\sqrt{n}} \sqrt{\left(\frac{\overline{X}_j - \mu_j}{\sigma_j/\sqrt{n}} \right)^2} < \frac{1}{k} \right] \right\} \geq 1 - \sum_{j=1}^{2} \Pr\left\{ \frac{1}{\sqrt{n-1}} \sqrt{\frac{(n-1)S_j^2}{\sigma_j^2}} + \frac{1}{c\sqrt{n}} \sqrt{\left(\frac{\overline{X}_j - \mu_j}{\sigma_j/\sqrt{n}} \right)^2} \geq \frac{1}{k} \right\} $$

.... (3)

A *conservative* test of the hypotheses stipulated above may now be obtained by replacing the left-hand side of (2) with the right-hand side of (3) giving,

$$\sum_{j=1}^{2} \Pr\left\{ \frac{1}{\sqrt{n-1}} \sqrt{\frac{(n-1)S_j^2}{\sigma_j^2}} + \frac{1}{c\sqrt{n}} \sqrt{\left(\frac{\bar{X}_j - \mu_j}{\sigma_j / \sqrt{n}} \right)^2} \geq \frac{1}{k} \right\} = \alpha .$$

As

$$\frac{1}{\sqrt{n-1}} \sqrt{\frac{(n-1)S_j^2}{\sigma_j^2}} + \frac{1}{c\sqrt{n}} \sqrt{\left(\frac{\bar{X}_j - \mu_j}{\sigma_j / \sqrt{n}} \right)^2}, \qquad j = 1,2$$

are identically distributed, it follows that,

$$\Pr\left\{ \frac{1}{\sqrt{n-1}} \sqrt{\frac{(n-1)S_j^2}{\sigma_j^2}} + \frac{1}{c\sqrt{n}} \sqrt{\left(\frac{\bar{X}_j - \mu_j}{\sigma_j / \sqrt{n}} \right)^2} \geq \frac{1}{k} \right\} = \frac{\alpha}{2} .$$

If $\quad V = \dfrac{(n-1)S_j^2}{\sigma_j^2} \quad$ and $\quad W = \left(\dfrac{\bar{X}_j - \mu_j}{\sigma_j / \sqrt{n}} \right)^2,$

the problem reduces to finding the $\left(1 - \frac{\alpha}{2}\right)$th quantile of

$$\frac{1}{\sqrt{n-1}} \sqrt{V} + \frac{1}{c\sqrt{n}} \sqrt{W},$$

which is a linear combination of the square root of two *independent Chi-Square* variables with n-1 and 1 degrees of freedom respectively. A closed-form representation of the probability density of this linear combination is not available. However, it is possible to obtain the approximate values of the required quantiles and thus the critical values, k, using *Cornish-Fisher expansions*. Johnson et al. (1970) outlined the method of obtaining these expansions and provided a formula which expresses the standardized quantiles of any distribution in terms of its standardized cumulants and the corresponding standard normal quantiles. However, it is found that there are some inconsistencies in the results obtained by using the expression provided by these authors. As an alternative, numerical solutions are obtained from the following integral equation :-

$$\int_0^\infty F_{n-1}\left[(n-1)\left(1/k-\sqrt{w}/c\sqrt{n}\right)^2\right]f_1(w)dw = 1-\frac{\alpha}{2},$$

where $f_v(\bullet)$ and $F_v(\bullet)$ respectively denote the probability density and the cumulative distribution function of a *Chi-Square* variable with v degrees of freedom. The approximate critical value, k, is obtained in this way using *Mathematica* software and given to 4 significant digits in *Table* 2 for various combinations of tolerable proportion of unusable items (δ), sample size (n) and significance level (α).

ROBUSTNESS TO DEPARTURES FROM NORMALITY ---- SOME CONSIDERATIONS

Various attempts have been made to extend the definitions of the standard univariate capability indices to situations where the process distribution is non-normal and corresponding estimation procedures have been proposed. These are intended to correctly reflect the proportion of items out of specification irrespective of the form of the process distribution. No attempts have appeared in the literature, however, to develop multivariate capability indices which are insensitive to departures from multivariate normality. Some robust univariate capability indices and procedures for assessing process performance currently available are briefly reviewed and an approach outlined for designing robust multivariate capability indices.

Chan et al. (1988) suggested the use of a *tolerance interval* approach similar to that of Guenther (1985) to estimate, with a certain level of confidence, the interval within which at least a specified proportion of items is contained. This estimated interval is then used in place of the normal-theory based interval (some multiple of σ) in the expressions for C_p, C_{pk} and C_{pm}. The $100(1-\alpha)\%$ confidence β-content tolerance interval is designed to capture at least $100\beta\%$ of the process distribution, $100(1-\alpha)\%$ of the time by using appropriate order statistics. However, it was found by Chan et al. (1988) that the natural choice of β, 0.9973 and α, 0.05 results in the requirement of taking sample sizes, n of 1000 or larger. To circumvent this problem, they proposed the use of a tolerance interval with smaller β, specifically, that with $\beta = 0.9546$ and $\beta = 0.6826$ in place of 4σ and 2σ respectively in the expressions for C_p, C_{pk} and C_{pm}, and provided the corresponding 95% confidence estimators for sample sizes less than 300. Although this modification greatly reduces the minimum sample size required, Pearn et al. (1992) pointed out that 'it depends on the (somewhat doubtful) assumption that the ratios of distribution-free tolerance interval lengths for different β are always approximately the same as that for normal tolerance intervals'. Furthermore, the proposed extensions retain the process mean, μ in the original definitions of C_{pk} and C_{pm} rather than replacing it by the median. This complicates the interpretation of the resulting indices since the median may differ considerably from the process mean for heavily skewed distributions.

Another approach to analysing process capability for non-normal processes (especially unimodal and fairly smooth distributions) is based on *systems* or *families* of distributions. Having redefined the standard C_p and C_{pk} indices as

$$C_p = \frac{\mathrm{U}-\mathrm{L}}{P_{0.99865} - P_{0.00135}}$$

and

$$C_{pk} = Min\left\{\frac{\mathrm{U}-\mathrm{M}}{P_{0.99865}-\mathrm{M}}, \frac{\mathrm{M}-\mathrm{L}}{\mathrm{M}-P_{0.00135}}\right\}$$
$$= Min\left\{\frac{\mathrm{U}-P_{0.5}}{P_{0.99865}-P_{0.5}}, \frac{P_{0.5}-\mathrm{L}}{P_{0.5}-P_{0.00135}}\right\},$$

where P_δ denotes the 100δth percentile of the distribution, Clement (1989) proposed fitting a Pearson-type curve to the observed data using the method of moments and the percentiles required for computation of these indices are then obtained from the fitted distribution. The required standardized percentiles were tabulated for various combinations of the coefficients of skewness and kurtosis. Some potential difficulties with this approach were given by Rodriguez (1992). In view of the complexity and difficulty of interpreting the equations for fitted Pearson and Johnson-type curves, Rodriguez (1992) suggested the fitting of a particular parametric family of distributions such as the Gamma, Lognormal or Weibull distribution to the process data. For checking the adequacy of the distributional model, he recommended the use of statistical methods based on the empirical distribution function (EDF) including the Kolmogorov-Smirnov test, the Cramer Von Mises test and the Anderson-Darling test. As for the graphical checking of distributional adequacy, he stated that this can be accomplished by means of Quantile-Quantile plots or probability plots. In the same paper, he also briefly described the use of Kernel Density estimates for process capability analysis, especially for non-normal distributions.

Pearn et al. (1992) suggested a possible approach to obtain a robust capability index by defining an index

$$C_\theta = \frac{U - L}{\theta\sigma},$$

where θ is chosen such that

$$P_\theta = \Pr[\mu - \theta\sigma < X < \mu + \theta\sigma],$$

is as insensitive as possible to the form of the distribution of X. He showed that, for $P_\theta = 0.99$ the choice of $\theta = 5.15$ is quite adequate for a wide range of distributions.

For non-normal multivariate processes, it seems reasonable to use capability indices constructed based on multivariate Chebyshev-type inequalities (see Johnson et al. (1976)) to reflect the process performance as no normality assumption is required. The most basic type of these inequalities is obtained by combining the Bonferroni and Chebyshev inequalities as follows :-

For our purpose here, the Bonferroni inequality is given by

$$\Pr\left\{\bigcap_{j=1}^{p}\left|\frac{X_j-\mu_j}{\sigma_j}\right| \le k\right\} \ge 1-\sum_{j=1}^{p}\Pr\left\{\left|\frac{X_j-\mu_j}{\sigma_j}\right| \ge k\right\} \qquad \text{.............. (4)}$$

Upon applying the Chebyshev inequality to each term in the summation on the right-hand side of (4), the following is obtained :-

$$\Pr\left\{\bigcap_{j=1}^{p}\left|\frac{X_j-\mu_j}{\sigma_j}\right|\le k\right\}\ge 1-\sum_{j=1}^{p}\frac{1}{k^2}$$

$$\ge 1-\frac{p}{k^2} \qquad \text{................ (5)}$$

Note that, for the same k, the lower bound for (5) is smaller than that for (4). However, this does not imply that the capability index constructed based on inequality (5) is less conservative than that which is based on (4). For the same lower bound, $1-\delta$, the process rectangle based on the multivariate Chebyshev-type inequality (5) is always larger (as a result of larger k) than that of (4) irrespective of the underlying distribution. Note, however, that the Bonferroni-type capability index proposed in this paper is obtained by imposing a normality condition on the marginal distributions of the process and thus it can be either too liberal or too stringent as a performance measure for non-normal processes. For instance, a value greater than 1 for this index does not guarantee that the expected proportion of non-defective items is more than $1-\delta$ if the process distribution is heavy-tailed (such as a multivariate-*t* distribution) unless it is significantly different from 1.

There are some improvements to the above multivariate Chebyshev-type inequality, however, the expressions involved are complicated, causing the construction of multivariate capability indices based on them to be difficult except for situations where there are relatively few variables. It is also found that these capability indices are only marginally better than that based on inequality (5). Thus, it is reasonable to use (5) whenever the use of distribution-free capability indices is warranted.

CONCLUSIONS AND FURTHER REMARKS

In this paper, three bivariate capability indices have been proposed based on the relative area and position of the conservative process rectangle containing at least a specified proportion of items, and the specification rectangle. The development of the first involves the projection of a process ellipse containing a specified percentage of products on to its component axes whereas the other two are based on the Bonferroni and Sidak inequalities respectively. Some calculations that fairly compare the three reveal that the latter two are superior to the former and that the Sidak-type capability index is marginally better than that based on the Bonferroni inequality. A reasonable test for the Sidak-type index has also been proposed and critical values provided for some chosen levels of significance, sample sizes and acceptable percentages of nonconforming items. The computation of these indices is easier than other proposed indices and capability analysis methods. However, as with other multivariate capability indices, it has not yet been possible to obtain the unbiased estimators and appropriate confidence intervals for the proposed indices except to note that for large sample sizes, it seems appropriate to replace the parameters involved with the usual sample estimates.

Further research in this area should perhaps focus on developing *exact* multivariate capability indices which accurately reflect the process status (i.e the expected proportion of usable items produced) and the expected costs incurred. There may also be situations where not all the measured characteristics are equally important in determining the product quality and it seems, therefore, reasonable to develop some index which takes this factor into consideration. A final concern about multivariate capability indices is their robustness to departures from normality. A conservative type of distribution-free capability index may be obtained by use of the multivariate Chebyshev-type probability inequalities as demonstrated. Although this is no better (more conservative) than the Bonferroni-type capability index, the process rectangle containing at least a specified proportion of items used for defining the index can be constructed easily for any type of process distribution. If the underlying distribution for each quality characteristic is known to belong to some well-known system or family of distributions and hence appropriate quantiles may be obtained, it is advisable to consider the use of the capability index constructed based on the Bonferroni inequality (4) although in some cases, this might not be practical.

REFERENCES

BARNETT, N. S. (1990). 'Process Control and Product Quality - The C_p and C_{pk} Revisited'. *International journal of Quality and Reliability Management*, Vol.7, No.4, pp.34-43.

BOYLES, R. A. (1991). 'The Taguchi Capability Index'. *Journal of Quality Technology*, Vol.23, No.1, pp.17-26.

BOYLES, R. A. (1994a). 'Exploratory Capability Analysis'. Submitted to *Journal of Quality Technology*.

BOYLES, R. A. (1994b). 'Covariance Models for Repeated Measurements with Lattice Structure'. Submitted to *Technometrics*.

CHAN, L. K.; CHENG, S. W. and SPIRING, F. A. (1988). 'A Graphical Technique for Process Capability'. *ASQC Quality Congress Transactions* - Dallas, pp.268-275.

CHAN, L. K.; CHENG, S. W. and SPIRING, F. A. (1991). 'A Multivariate Measure of Process Capability'. *Journal of Modeling and Simulation*, Vol.11, pp.1-6.

CLEMENTS, J. A. (1989). 'Process Capability Calculations for Non-Normal Distributions'. *Quality Progress*, September, pp.95-100.

FRANKLIN, L. A. and WASSERMAN, G. S. (1992). 'Bootstrap Lower Confidence Limits for Capability Indices'. *Journal of Quality Technology*, Vol.24, No.4, pp.196-210.

GUENTHER, W. H. (1985). 'Two-Sided Distribution-Free Tolerance Intervals and Accompanying Sample Size Problems'. *Journal of Quality Technology*, Vol.17, No.1, pp.40-43.

HUBELE, N. F.; SHAHRIARI, H. and CHENG, C-S. (1991). 'A Bivariate Process Capability Vector'. in *Statistical Process Control in Manufacturing*, Eds. J.B. Keats and D.C. Montgomerry, Marcel Dekker, New York, NY & ASQC Press, Milwaukee, Wisconsin.

JOHNSON, N. L. and KOTZ, S. (1970). *Distributions in Statistics : Continuous Univariate Distributions* 1, Vol.2, John Wiley & Sons, New York, NY.

JOHNSON, N. L. and KOTZ, S. (1976). *Distributions in Statistics : Continuous Multivariate Distributions*, Vol.4, John Wiley & Sons, New York, NY.

JOHNSON, R. A. and WICHERN, D. W. (1988). *Applied Multivariate Statistical Analysis*, 2nd ed., Prentice-Hall, Englewood Cliffs, NJ.

KANE, V. E. (1986). 'Process Capability Indices'. *Journal of Quality Technology*, Vol.18, No.1, pp.41-52, Corrigenda p.265.

KUSHLER, R. H. and HURLEY, P. (1992). 'Confidence Bounds for Capability Indices'. *Journal of Quality Technology*, Vol.24, No.4, pp.188-195.

MARCUCCI, M. O. and BEAZLEY, C. C. (1988). 'Capability Indices : Process Performance Measures'. *ASQC Quality Congress Transactions*, pp.516-523.

NICKERSON, D. M. (1994). 'Construction of a Conservative Region From Projections of an Exact Confidence Region in Multiple Linear Regression'. *The American Statistician*, Vol.48, No.2, pp.120-124.

PEARN, W. L.; KOTZ, S. and JOHNSON, N. L. (1992). 'Distributional and Inferential Properties of Process Capability Indices'. *Journal of Quality Technology*, Vol.24, No.4, pp.216-231.

RODRIGUEZ, R. N. (1992). 'Recent Developments in Process Capability Analysis'. *Journal of Quality Technology*, Vol.24, No.4, pp.176-187.

SIDAK, Z. (1967). 'Rectangular Confidence Regions For The Means of Multivariate Normal Distributions'. *Journal of the American Statistical Association*, Vol.62, pp.626-633.

TAAM, W.; SUBBAIAH, P. and LIDDY, J. W. (1993). 'A Note on Multivariate Capability Indices'. *Journal of Applied Statistics*, Vol.20, No.3, pp.339-351.

Table 1.

P	δ	$I_{B:P} = \frac{\sqrt{\chi_p^2(\delta)}}{z_{\delta/2p}}$	$I_{S:P} = \frac{\sqrt{\chi_p^2(\delta)}}{\Phi^{-1}\left(\frac{1}{2}\left[1+(1-\delta)^{1/p}\right]\right)}$
2	0.0025	1.0726	1.0727
	0.005	1.0767	1.0768
	0.01	1.0811	1.0815
	0.02	1.0859	1.0867
	0.05	1.0921	1.0945
3	0.0025	1.1325	1.1326
	0.005	1.1397	1.1398
	0.01	1.1475	1.1479
	0.02	1.1561	1.1570
	0.05	1.1677	1.1708
5	0.0025	1.2319	1.2320
	0.005	1.2438	1.2440
	0.01	1.2569	1.2574
	0.02	1.2713	1.2724
	0.05	1.2917	1.2953
10	0.0025	1.4218	1.4219
	0.005	1.4419	1.4421
	0.01	1.4641	1.4646
	0.02	1.4886	1.4899
	0.05	1.5243	1.5283

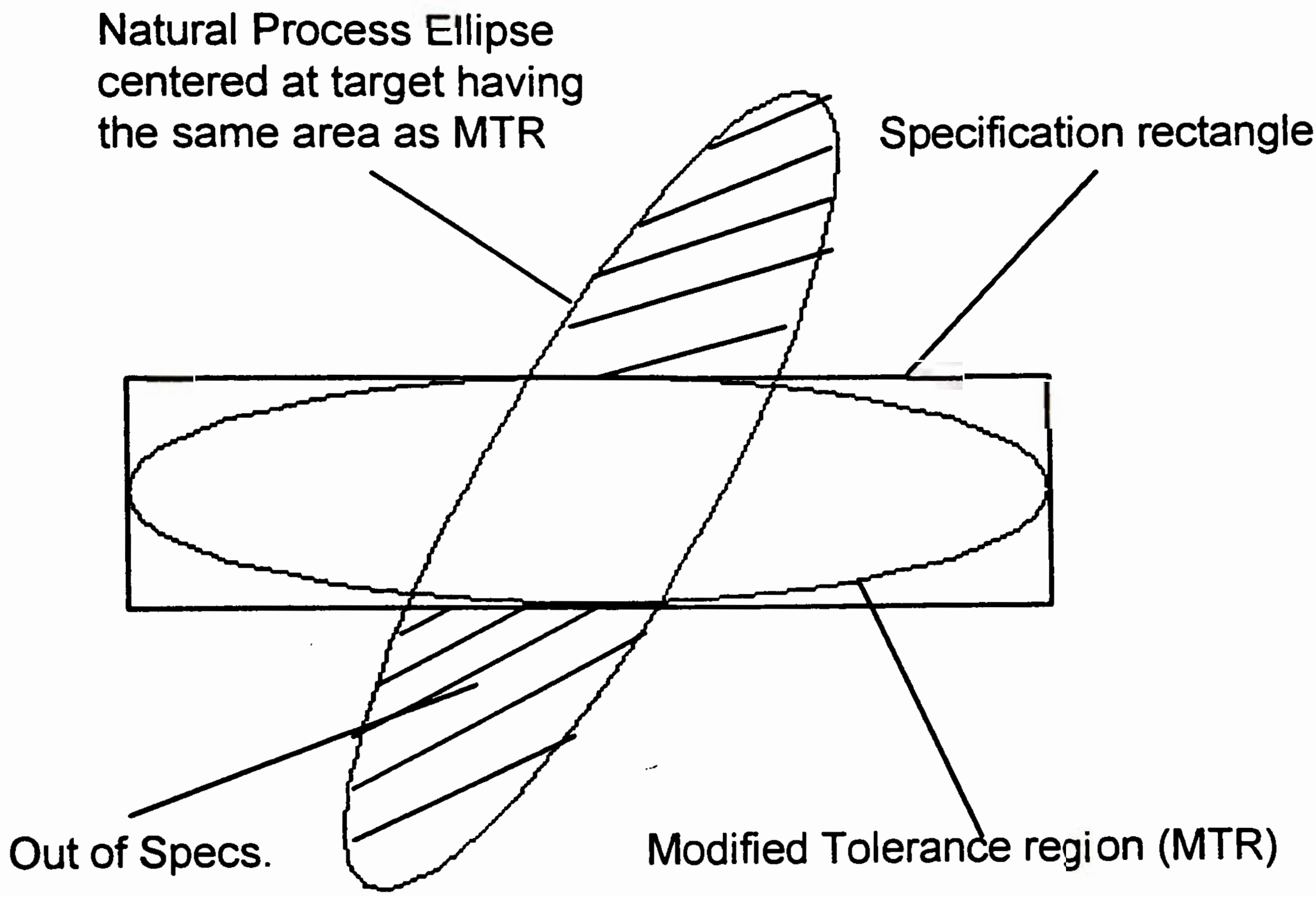
Table **2. Critical Values for Testing** $^{S}C_{pk}^{(2)}$

	α			
n	0.01	0.025	0.05	0.1
10	* 0.5763 † (0.5636)	0.6093 (0.5960)	0.6397 (0.6258)	0.6770 (0.6624)
15	0.6284 (0.6158)	0.6590 (0.6461)	0.6869 (0.6737)	0.7206 (0.7070)
20	0.6630 (0.6507)	0.6918 (0.6794)	0.7178 (0.7052)	0.7490 (0.7362)
25	0.6884 (0.6765)	0.7158 (0.7038)	0.7403 (0.7283)	0.7695 (0.7574)
50	0.7594 (0.7489)	0.7820 (0.7717)	0.8020 (0.7918)	0.8254 (0.8154)
100	0.8178 (0.8091)	0.8359 (0.8275)	0.8516 (0.8434)	0.8698 (0.8619)

* unbracketed values correspond to $\delta = 0.01$.

† bracketed values correspond to $\delta = 0.05$.

Figure 1.


Natural Process Ellipse
centered at target having
the same area as MTR
Specification rectangle
Out of Specs.
Modified Tolerance region (MTR)