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On Digit Swaps and Shifts

P. Cerone

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VICTORIA UNIVERSITY OF TECHNOLOGY
(P O BOX 14428) MELBOURNE MAIL CENTRE
MELBOURNE, VICTORIA, 3000
AUSTRALIA

TELEPHONE (03) 688 4249 / 4492
FACSIMILE (03) 688 4050

Footscray Campus

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P. Cerone

Victoria University of Technology, Melbourne, Australia

ABSTRACT

A number of interesting results are obtained from performing on integers the operations of digit swaps and shifts followed by subtraction and addition of the results. Polynomial representation of numbers are shown to lead to product form expressions.

1. INTRODUCTION

A problem as outlined by Conlon (1985) was recently brought to my attention. The problem involved a two digit number and if the digits are swapped and the smaller number is subtracted from the larger, then we were required to find the original number given that the answer in 72. A variant of this problem is also discussed by Stacey and Southwell (1983).

A related problem to the above occurs if the digits of the answer are swapped and the result is added to the previous answer then we would get 99. If we started with a three digit number and the first and last digits are swapped, the smaller number is subtracted from the larger, the digits are swapped again and the results added then we would, according to Devi (1990), always get 1089. This result has belatedly been discovered by the author to have played a pivotal role in the work of Gardiner (1987) and his study of 'flips'.

The two problems outlined above are investigated in this paper. A number of novel results are obtained and considerable extensions and generalisations are investigated.

In particular cyclic digit shifts, both backward and forward, are investigated besides generalisations of the digit swap problems outlined above.

Operators are introduced to enhance the development, and a bevy of what are believed to be non-trivial and interesting results are obtained.

2. DIGIT SWAPS

If we take any two different two digit numbers and subtract the smaller from the larger then we would get two possible results depending on the four coefficients. If the smaller number is made up from the swapped digits of the larger then there will be no ambiguity in the result.

Let $\quad x = \lfloor a_1\, a_0 \rceil = 10a_1 + a_0$

and $\quad y = \lfloor b_1\, b_0 \rceil = 10b_1 + b_0.$

For definiteness let $x > y$ $(a_1 > b_1)$ then

$$x - y = \begin{cases} 10(a_1 - b_1) + a_0 - b_0 & , \quad a_0 > b_0 \text{ and } a_1 \geq b \\ 10[a_1 - (b_1 + 1)] + 10 + a_0 - b_0 & , \quad a_0 < b_0 \text{ and } a_1 > b_1 \end{cases}$$

$$= \begin{cases} \lfloor a_1 - b_1 \;\; a_0 - b_0 \rceil & , \quad a_0 > \mathrm{b}_0 \\ \lfloor a_1 - b_1 - 1 \;\; 10 - (b_0 - a_0) \rceil & , \quad a_0 < \mathrm{b}_0. \end{cases}$$

If y is made up from the swapped digits of x then $b_1 = a_0$ and $b_0 = a_1$ so that with $a_1 > a_0$

$$x - y = \lfloor a_1\, a_0 \rceil - \lfloor a_0\, a_1 \rceil = \lfloor (a_1 - a_0) - 1 \quad 10 - (a_1 - a_0) \rceil. \tag{1}$$

If we work with numbers to base b rather than base 10, then for two digit numbers x and y where y is made up of the swapped digits of x we get.

$$x - y = \lfloor a_1\, a_0 \rceil - \lfloor a_0\, a_1 \rceil = \lfloor (a_1 - a_0) - 1 \quad b - (a_1 - a_0) \rceil. \tag{2}$$

Thus (1) generalises to any base.

It is interesting to note that (2) can be written in the following form

$$x - y = (a_1 - a_0 - 1)b + (b - (a_1 - a_0))b^0$$

$$= (b-1)\alpha_1 \tag{3}$$

where $\quad \alpha_k = a_k - a_0 > 0. \quad$ (4)

Thus in the example outlined in the introduction we have that, from (3), $72 = 9 \text{ x } \alpha_1$ so that $\alpha_1 = 8$ and hence the original number is either 91 or 80. The same solution could have been obtained from using (2).

If we have a three digit number and a second number is formed from swapping the first and last digits, and the smaller number is subtracted from the larger $(a_2 > a_0)$, we obtain

$$\lfloor a_2\, a_1\, a_0 \rceil - \lfloor a_0\, a_1\, a_2 \rceil = \lfloor \alpha_2 - 1 \quad b - 1 \quad b - \alpha_2 \rceil. \tag{5}$$

The right hand side of (5) can be expressed in polynomial form

$$\lfloor \alpha_2 -1 \;\; b-1 \;\; b-\alpha_2 \rceil = (\alpha_2 -1)b^2 + (b-1)b + (b-\alpha_2)b^0,$$

which may be simplified to obtain a representation as a product of two numbers namely,

$$\lfloor \alpha_2 -1 \;\; b-1 \;\; b-\alpha_2 \rceil = (b^2 -1)\alpha_2. \tag{6}$$

The procedure can be extended to any k-degree polynomial to get for

$$\lfloor a_k \, a_{k-1} \ldots a_1 \, a_0 \rceil - \lfloor a_0 \, a_{k-1} \ldots a_1 \, a_k \rceil$$

$$= \lfloor \alpha_k -1 \;\; b-1 \ldots b-1 \;\; b-\alpha_k \rceil \tag{7}$$

where there are $k-1$ lots of $b-1$ digits since the number is of length $k+1$ and α_k is given by (4).

The right hand side of (7) can be written in polynomial form to give, after simplification,

$$\lfloor \alpha_k -1 \;\; \underbrace{b-1 \,..\, b-1}_{k-1} \;\; b-\alpha_k \rceil = (b^k -1)\alpha_k. \tag{8}$$

Putting $k=2$ in (7) and (8) give (5) and (6).

The right hand side of (8) can be written in alternate forms on noting that $b-1$ is a factor giving:

$$\left.\begin{aligned} \lfloor \alpha_k -1 \;\; \underbrace{b-1 \ldots b-1}_{k-1} \;\; b-\alpha_k \rceil &= (b-1)(b^{k-1} + b^{k-2} + \ldots + b + 1)\,\alpha_k \\ &= (b-1)\lfloor \underbrace{1 \;\, 1 \ldots 1}_{k} \rceil \alpha_k. \end{aligned}\right\} \tag{9}$$

If we further notice that for k even both $b-1$ and $b+1$ are factors, then

$$\left.\begin{aligned} \lfloor \alpha_k -1 \;\; b-1 \ldots b-1 \;\; b-\alpha_k \rceil &= (b^2 -1)(b^{k-2} + b^{k-4} + \ldots + b^2 + 1)\alpha_k \\ &= (b^2 -1)\lfloor \underbrace{1\,0\,1 \ldots 0101}_{k-1} \rceil \alpha_k. \end{aligned}\right\} \tag{10}$$

The other problem outlined in the introduction was that the first and last digits of the result from the subtraction are swapped and the new number thus formed is added to the result of the subtraction. Thus from equation (7) we have

$$\lfloor \alpha_k -1 \;\; b-1 \ldots b-1 \;\; b-\alpha_k \rceil + \lfloor b-\alpha_k \;\; b-1 \ldots b-1 \;\; \alpha_k -1 \rceil$$

$$= \lfloor \; b-1 \;\; 2(b-1) \ldots . 2(b-1) \;\; b-1 \rceil$$

$$= \lfloor \; b \;\; b-1 \ldots \ldots b-1 \;\; b-2 \;\; b-1 \rceil$$

$$= \lfloor 1 \;\, 0 \;\; \underbrace{b-1 \ldots b-1 \;\, b-1}_{k-2} \;\; b-2 \;\; b-1 \rceil. \tag{11}$$

The right hand side of (11) can be written in polynomial form so that

$$\lfloor 1\,0\ \ b-1\ldots b-1\ b-2\ \ b-1 \rceil = b^{k+1}+b^{k}-b-1. \tag{12}$$

On noting that $b=\pm 1$ are roots of

$$q_1(b)=b^{k+1}+b^{k}-b-1 \tag{13}$$

then it becomes apparent through a division of $q_1(b)$ by b^2-1 that

$$b^{k+1}+b^{k}-b-1=\left(b^2-1\right)\left(b^{k-1}+b^{k-2}+\ldots+b+1\right).$$

Thus a product representation is obtained as

$$\left.\begin{aligned} \lfloor 1\,0\ \underbrace{b-1\ldots b-1}_{k-2}\ b-2\ \ b-1 \rceil &= \left(b^2-1\right)\left(b^{k-1}+b^{k-2}\ldots b+1\right) \\ &= \left(b^2-1\right)\lfloor \underbrace{1\ \ 1\ldots 1}_{k} \rceil. \end{aligned}\right\} \tag{14}$$

As an example, for any three digit number of base 10, from (7), (8) and (9) we get

$$\begin{aligned} \lfloor \alpha_2-1\ \ 9\ \ 10-\alpha_2 \rceil &= \left(10^2-1\right)\alpha_2 \\ &= 9\lfloor 1\ \ 1 \rceil \alpha_2 \end{aligned}$$

and from (11), (12) and (14)

$$\begin{aligned} \lfloor 1\,0\ \ 8\,9 \rceil &= 10^3+10^2-10-1 \\ &= 99\lfloor 1\ \ 1 \rceil. \end{aligned}$$

Gardiner (1987) devotes a substantial portion of the book to the number 1089 and its pivotal role in flips.

Up until now we have only looked at swaps of the first and last digit. What would happen if we also allowed a swap of the 2nd and 2nd last digit, and so on? It turns out that there would be different possibilities depending on the relationship between the digits. If instead of starting with the general number

$$\lfloor a_k\, a_{k-1}\ldots a_1\, a_0 \rceil$$

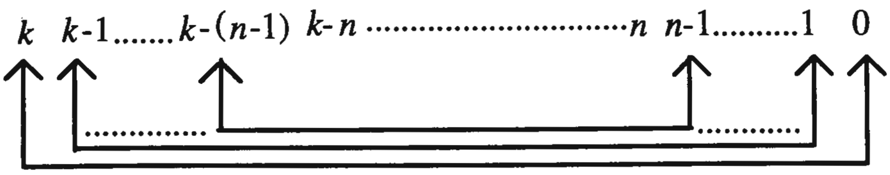
we let $a_i=i$ giving

$$x=\lfloor k\ k-1\ldots 1\,0 \rceil\ ,\ \ k+1\leq b\ , \tag{15}$$

then we may allow up to n swaps, where

$$n\leq\frac{k+1}{2}. \tag{16}$$

We clarify here that an n-swap is as represented in the diagram below



(17)

Figure 1 : An n-swap of (15).

If we subtract the n-swap number, (17) from x, as given by (15), we obtain

$$\lfloor \gamma_0\ \gamma_1\ \gamma_2 \ldots \gamma_{n-2}\ \gamma_{n-1}-1\ \underbrace{b-1 \ldots b-1}_{k-(2n-1)}\ b-\gamma_{n-1}-1 \ldots b-\gamma_1-1\ b-\gamma_0 \rceil \tag{18}$$

where

$$\gamma_j = k-2j. \tag{19}$$

Performing an n-swap on (18) and then adding gives:

$$\lfloor 1\ 0\ \underbrace{b-1 \ldots b-1}_{k-(n+1)}\ b-2\ b-1\ \underbrace{0 \ldots 0}_{n-1} \rceil. \tag{20}$$

Note that if we put $n=1$, and thus allowing for only 1 swap, into (18) and (10) we obtain (7) $\left(\text{with } \alpha_k = k\right)$ and (11) respectively.

Equation (18) may be expressed in polynomial form to give

$$\lfloor \gamma_0\, \gamma_1 \ldots \gamma_{n-2}\, \gamma_{n-1}-1\ b-1 \ldots b-1\ b-\gamma_{n-1}-1 \ldots b-\gamma_1-1\ b-\gamma_0 \rceil$$

$$= \sum_{j=0}^{n-1} \gamma_j \left(b^{k-j}-b^j\right) = p_n(b) \quad \text{(say)} \tag{21}$$

where γ_j is as given by (19).

If we put $n=1$ in (21) then we get (8), with $\alpha_k = k$.

From (21) we note that

$p_n(1)=0$ and $p_n(-1)=0$ only when k is even.

Using equations (8) and (9) we have

$$p_n(b) = (p-1)\sum_{j=0}^{n-1} \gamma_j \left(b^{k-j-1}+b^{k-j-2}+\ldots+b^j\right). \tag{22}$$

Equation (20) may be expressed in polynomial form to give

$$\lfloor 1\ 0\ \underbrace{b-1 \ldots b-1}_{k-(n+1)}\ b-2\ \ b-1\ \underbrace{0 \ldots 0}_{n-1} \rceil = b^{k+1}+b^k-b^n-b^{n-1} = q_n(b) \text{ (say)} \tag{23}$$

From (20) we note that $q(\pm 1)=0$ so that $q_n(b)$ may be factorised to give a product form

$$q_n(b) = \left(b^2-1\right)\left(b^{k-1}+b^{k-2}+\ldots+b^n+b^{n-1}\right)$$

$$= \left(b^2-1\right)\lfloor \underbrace{1\ 1 \ldots 1}_{k-(n-1)} \rceil b^{n-1}. \tag{24}$$

Putting $n=1$ in (23) and (24) gives (12), (13) and (14).

For example, with $b=10, k=9$ we have, from (15),

$$x=\lfloor 9\ 8\ 7\ldots 1\ 0 \rceil.$$

Performing an n-swap, subtract, another n-swap and add would give from (23) and (24)

$$\lfloor 1\ 0\ \underbrace{9\ldots 9}_{9-(n+1)}\ 8\ 9\ \underbrace{0\ldots 0}_{n-1}\rceil = 99 \times \lfloor \underbrace{1\,1\ldots 1}_{9-(n-1)} \rceil \times 9^{n-1}.$$

3. DIGIT SHIFTS

In Section 2 we investigated some problems regarding digit swaps. In this section both backward and forward shifts are examined. Consider the monotonically decreasing number x as given by equation (15) and subtract from this the number with the first n_B digits shifted backward, to the end. Then,

$$\begin{aligned}
&\lfloor k \quad k-1 \ldots\ldots n_B+1 \quad n_B \quad n_B-1 \ldots\ldots 1 \quad 0 \rceil - \\
&\lfloor k-n_B \ \ k-(n_B+1) \ldots\ldots 1 \quad 0 \quad k \quad \ldots\ldots \ k-(n_B-1) \rceil \\
&= \lfloor \underbrace{n_B \ldots n_B}_{k-n_B} \ \ n_B-1 \ \ \underbrace{b-(k-(n_B-2)) \ldots b-(k-(n_B-2))}_{n_B-1} \ \ b-(k-(n_B-1)) \rceil. \qquad (25)
\end{aligned}$$

If on the other hand the last n_F digits of x are shifted forward and then the resulting number is subtracted from x we have

$$\begin{aligned}
&\lfloor \ k \ \ k-1 \ldots k-(n_F-2)\, k-(n_F-1) \ \ k-n_F \ \ k-(n_F+1) \ldots 1 \quad 0 \rceil - \\
&\lfloor \ n_F-1 \ n_F-2 \ldots \quad 1 \quad 0 \quad k \quad k-1 \ldots\ldots n_F+1 \quad n_F \rceil \\
&= \lfloor \underbrace{k-(n_F-1) \ldots k-(n_F-1)}_{n_F-1} \ \ k-n_F \ \ \underbrace{b-(n_F+1) \ldots b-(n_F+1)}_{k-n_F} \ \ b-n_F \rceil. \qquad (26)
\end{aligned}$$

Putting $n_B = k+1-n_F$ into (25) gives (26), and the forward shift result equals the backward shift result when the shift $n=\dfrac{k+1}{2}$, an integer. This can only occur when k is odd. For example,

for $b=10$ and $k=7$ we have

$$x=\lfloor 7\ 6\ 5\ldots 1\ 0 \rceil$$

so that with a shift of $n=3$

backward, from (25), $\quad \underbrace{3\ldots 3}_{4} \quad 2 \quad \underbrace{4\ 4}_{2} \quad 5$

and forward, from (26), $\quad \underbrace{5\ 5}_{2} \quad 4 \quad \underbrace{6\ldots 6}_{4} \quad 7.$

It is interesting to note that (25) can be written in product form to give

$$\lfloor n_B \ldots n_B \; n_B - 1 \; b-(k-(n_B-2)) \ldots b-(k-(n_B-2)) \; b-(k-(n_B-1)) \rceil$$

$$= n_B(b^k + \ldots + b^n) - (k-(n_B-1))(b^{n-1} + b^{n-2} + \ldots + 1)$$

$$= n_B(b^k + b^{k-1} + \ldots + 1) - (k+1)(b^{n-1} + b^{n-2} + \ldots + 1)$$

$$= n_B \lfloor \underbrace{1 \; 1 \ldots 1}_{k+1} \rceil - (k+1) \lfloor \underbrace{1 \; 1 \ldots 1}_{n} \rceil \tag{27}$$

Equation (26) can also be written in product form to give

$$\lfloor k-(n_F-1) \ldots k-(n_F-1) \; k-n_F \; b-(n_F+1) \ldots b-(n_F+1) \; b-n_F \rceil$$

$$= [k-(n_F-1)](b^k + b^{k-1} \ldots b^{k-(n_F-2)}) - n_F(b^{k-(n_F-1)} + \ldots + b + 1)$$

$$= (k-(n_F-1))(b^k + b^{k-1} + \ldots + b + 1) - (k+1)(b^{k-(n_F-1)} + \ldots + b + 1)$$

$$= (k-(n_F-1)) \lfloor \underbrace{1 \; 1 \ldots 1}_{k+1} \rceil - (k+1) \lfloor \underbrace{1 \; 1 \ldots 1}_{k-(n_F-1)} \rceil. \tag{28}$$

Note that in equations (27) and (28) if $k+1=b$ then $(k+1)\lfloor 1 \; 1 \ldots 1 \rceil$ should be written as $\lfloor 1 \; 1 \ldots 1 \; 0 \rceil$.

For example,

for $b=10$ and $k=7$ with a shift of $n_B = n_F = 3$ from (27) and (28) we have

$$\lfloor 3 \; 3 \; 3 \; 3 \; 2 \; 4 \; 4 \; 5 \rceil = 3\lfloor 1 \; 1 \ldots 1 \rceil - 8\lfloor 1 \; 1 \; 1 \; 1 \rceil$$

and

$$\lfloor 5 \; 5 \; 4 \; 6 \; 6 \; 6 \; 6 \; 7 \rceil = 5\lfloor 1 \; 1 \ldots 1 \rceil - 8\lfloor 1 \; 1 \; 1 \; 1 \; 1 \; 1 \rceil.$$

If the base, $b=7$ and $k=6$, with a shift of $n_B = n_F = 3$ (27) and (28) give

$$\begin{aligned} \lfloor 3 \; 3 \; 3 \; 2 \; 2 \; 2 \; 3 \rceil &= 3\lfloor 1 \; 1 \; 1 \; 1 \; 1 \; 1 \; 1 \rceil - 7\lfloor 1 \; 1 \; 1 \; 1 \rceil \\ &= 3\lfloor 1 \; 1 \; 1 \; 1 \; 1 \; 1 \; 1 \rceil - \lfloor 1 \; 1 \; 1 \; 0 \rceil \end{aligned}$$

and

$$\begin{aligned} \lfloor 4 \; 4 \; 3 \; 3 \; 3 \; 3 \; 4 \rceil &= 4\lfloor 1 \; 1 \; 1 \; 1 \; 1 \; 1 \; 1 \rceil - 7\lfloor 1 \; 1 \; 1 \; 1 \; 1 \rceil \\ &= 4\lfloor 1 \; 1 \; 1 \; 1 \; 1 \; 1 \; 1 \rceil - \lfloor 1 \; 1 \; 1 \; 1 \; 0 \rceil. \end{aligned}$$

4. OPERATORS FOR DIGIT MANIPULATION

It is instructive to introduce operators that cater for the various digit manipulations covered thus far. We let *I*, *B*, *F* and *S* represent the identity, backward shift, forward shift and swap operators respectively. For x, as given by equation (15), whose digits are monotonically decreasing we have

$$I(x)=x=\lfloor k\ k-1\ldots 1\ 0\rceil,\ k+1\le b \tag{29}$$

$$B(x)=\lfloor k-1\ k-2\ldots 1\ 0\ k\rceil \tag{30}$$

$$F(x)=\lfloor 0\ k\ k-1\ldots\ldots 2\ 1\rceil \tag{31}$$

and $$S(x)=\lfloor 0\ k-1\ \ k-2\ldots 1\ k\rceil. \tag{32}$$

We now discuss the backward shift operator, *B* in more detail and both relate it to the problem treated in section 3 and provide some new results. It is possibly best to begin by listing some properties of the backward shift operator, *B*:

(*B*1) $B^{(n)}(x)=\lfloor k-n\ k-(n+1)\ldots 2\ 1\ 0\ k\ k-1\ldots k-(n-2)\ \ k-(n-1)\rceil$

(*B*2) $B^{(m)}(x)>B^{(n)}(x), 0\le m<n<k\ ,\ \ B^{(0)}(x)=I(x)$

(*B*3) $B^{(k+1)}(x)\ \ =\ \ I(x)=x$

(*B*4) $(I-B)(x)\ \ =\ \ \lfloor\underbrace{1\,1\ldots 1}_{k-1}\,0\ b-k\rceil$

(*B*5) $\left(I-B^{(n)}\right)(x)\ \ =\lfloor\underbrace{n\ldots n}_{k-n}\ \ n-1\ \ \underbrace{b-k+(n-2)\ldots b-k+(n-2)}_{n-1}\ b-k+(n-1)\rceil$

(*B*6) $B^{(m)}\left(I-B^{(n)}\right)(x)=\lfloor\underbrace{n\ldots n}_{k-n-m}\ \ n-1\ \ \underbrace{b-k+(n-2)\ldots b-k+(n-2)}_{n-1}$

$$b-k+(n-1)\ \ \underbrace{n\ldots n}_{m}\rceil$$

(*B*7) $\left(B^{(m)}-B^{(n)}\right)(x)=\lfloor\underbrace{(n-m)\ldots(n-m)}_{k-n}\,n-m-1\ \ \underbrace{b-k+(n-m-2)\ldots b-k+(n-m-2)}_{n-m-1}$

$$b-k+(n-m-1)\,\underbrace{n-m\ldots n-m}_{m}\rceil,\ n>m$$

$$=B^{(m)}\left(I-B^{(n-m)}\right)(x)=B^{(m)}(x)-B^{(n)}(x).$$

(*B*8) $\left(I\pm B^{(m)}\right)\left(I-B^{(n)}\right)(x)=\left(I-B^{(n)}\right)(x)\pm B^{(n)}\left(I-B^{(n)}\right)(x),\quad k+1>m+n$

(*B*9) $\displaystyle\sum_{m=0}^{N}B^{(n)}(I-B)(x)=\begin{cases}\left(I-B^{(N+1)}\right)(x) & ,\quad N<k\\ 0 & ,\quad N=k\end{cases}$

$$(B10)\quad \sum_{n=0}^{N}(-1)^n\, B^{(n)}(x) = \begin{cases} \displaystyle\sum_{n=0}^{\frac{N-1}{2}} B^{(2n)}(I-B)(x) & , \quad N \text{ odd} \\ \displaystyle\sum_{n=0}^{\frac{N-1}{2}} B^{(2n)}(I-B)(x)+B^{(N)}(x) & , \quad N \text{ even} \end{cases}$$

$$(B11)\quad \left(I+B^{(n)}\right)\left(I-B^{(n)}\right)(x)=\left(I-B^{2n}\right)(x), \qquad k+1<2n,$$

Property $(B1)$ is simply a cyclic backward shift of the first n digits and we see that $B^{(n)}(x)=B\,B^{(n-1)}(x)$. Property $(B2)$ states that B is a strictly monotonically decreasing operator and $(B3)$ demonstrates the cyclic nature of the shifts.

Properties $(B4)$ and $(B5)$ are equivalent to the $n_B=1$ and $n_B=n$ problem treated in Section 3 as represented by equation (25). Properties $(B6)$ and $(B7)$ can be shown to be true from the use of properties $(B1)$ - $(B5)$. Property $(B8)$ may be proved using $(B5)$ and $(B6)$. For the + version we need to use the extra fact that $k<2n+1$ to show that each of the digits on the right hand side of $(B8)$ is less than the base, b. Properties $(B9)$ and $(B10)$ are straight forward and $(B11)$ is a special case of $(B8)$ with $m=n$.

For example, let $b=10$ and $k=9$ so that

$$x = \lfloor 9\ 8\ 7 \ldots 2\ 1\ 0 \rceil$$

and $B^{(2)}(x)=\lfloor 7\ 6\ 5 \ldots\ 0\ 9\ 8 \rceil$

$B^{(3)}(x)=\lfloor 6\ 5\ 4 \ldots 0\ 9\ 8\ 7 \rceil$

$B^{(4)}(x)=\lfloor 5\ 4 \ldots 0\ 9\ 8\ 7\ 6 \rceil$

then from $(B5)$ $\quad \left(I-B^{(2)}\right)(x) = \lfloor 2\ 2\ 2\ 2\ 2\ 2\ 2\ 1\ 1\ 2 \rceil$

$\left(I-B^{(3)}\right)(x) = \lfloor 3\ 3\ 3\ 3\ 3\ 3\ 2\ 2\ 2\ 3 \rceil$

$\left(I-B^{(4)}\right)(x) = \lfloor 4\ 4\ 4\ 4\ 4\ 3\ 3\ 3\ 3\ 4 \rceil$

and from $(B6)$ $\quad B^{(2)}\left(I-B^{(2)}\right)(x)= \lfloor 2\ 2\ 2\ 2\ 2\ 1\ 1\ 2\ 2\ 2 \rceil$

$B^{(2)}\left(I-B^{(3)}\right)(x)= \lfloor 3\ 3\ 3\ 3\ 2\ 2\ 2\ 3\ 3\ 3 \rceil.$

Hence $\quad \left(I-B^{(2)}\right)\left(I-B^{(3)}\right)(x) = \left(I-B^{(3)}\right)(x)-B^{(2)}\left(I-B^{(3)}\right)(x)$

$=\lfloor 0\ 0\ 0\ 0\ 1\ 0\ 9\ 8\ 9\ 0 \rceil,$

and $\quad \left(I+B^{(2)}\right)\left(I-B^{(2)}\right)(x) = \left(I-B^{(2)}\right)(x)+B^{(2)}\left(I-B^{(2)}\right)(x)$

$=\lfloor 4\ 4\ 4\ 4\ 4\ 3\ 3\ 3\ 3\ 4 \rceil$

$=\left(I-B^{(4)}\right)(x).$

We now turn to discussing the forward shift operator, F as represented by equation (31) and using (29). The operator, F has the following properties:

(F1) $F^{(n)}(x)=\lfloor n-1\; n-2 \ldots 1\; 0\; k\;\; k-1 \ldots n+1\; n\rceil$

(F2) $F^{(m)}(x)<F^{(n)}(x),\; 0<m<n$

(F3) $F^{(k+1)}(x)=I(x)=1$

(F4) $(I-F)(x)=\lfloor k-1\;\; b-2 \ldots b-2\;\; b-1\rceil$

(F5) $\left(I-F^{(n)}\right)(x)=\lfloor k-\underbrace{(n-1)\ldots k-}_{n-1}(n-1)\;\; k-n\;\; b-\underbrace{(n+1)\ldots b-}_{k-n}(n+1)\;\; b-n\rceil$

($\tilde{F}6$) $$F^{(m)}\left(I-F^{(n)}\right)(x)=\lfloor b-\underbrace{(n+1)\ldots b-}_{m-1}(n+1)\; b-n\; k-\underbrace{(n-1)\ldots k-(n-1)}_{n-1}$$
$$k-n\;\; b-\underbrace{(n+1)\ldots b-}_{k-n-m}(n+1)\rceil$$
$$\neq F^{(n)}(x)-F^{(m+n)}(x)$$

(F7) $$\left(F^{(m)}-F^{(n)}\right)(x)\;=\lfloor\underbrace{m-n\ldots m-n}_{n-1}\;\; m-n-1\;\; \underbrace{b-k+(m-n)-2\ldots b-k+(m-n)-2}_{m-n-1}$$
$$b-(k+1)+m-n\;\underbrace{m-n\ldots m-n}_{k+1-m}\rceil$$
$$=F^{(m)}(x)-F^{(n)}(x)\qquad,\qquad m>n$$

We notice that property ($F1$) is a cyclic forward shift of the last n digits of equation (15) (or(29)) and if $n=k+1$, as demonstrated by ($F3$), the result is x.

The extensive results of the backward shift operator ($B1$) - ($B11$) are not obtained here since $F^{(n)}(x)$ is a monotonically increasing operator for $m>0$ as shown in ($F2$). However, $I(x)=F^{(0)}(x)=x>F(x)$ but $F(x)<F^{(2)}(x)$. Thus, the monotonicity of the operator $F^{(m)}(x)$ changes as m changes from zero to not zero.

There are problems with property $\left(\tilde{F}6\right)$. The left hand side represents a forward shift of m digits from $(F5)$ which is not equal to the right hand side since it does not exist as it is negative.

We digress temporarily to note, from equations (29) - (31), that B and F are inverse operators of each other satisfying

$$BF(x)=FB(x)=I(x). \tag{33}$$

That is,

$$\left.\begin{aligned} F^{-1}(x)&=B(x)\\ B^{-1}(x)&=F(x). \end{aligned}\right\} \tag{34}$$

We are now in a position to state a corrected form of property ($F6$) namely,

$$(F6)\qquad F^{(m)}\left(I-F^{-(n)}\right)(x)=\left(F^{(m)}-F^{(m-n)}\right)(x)\quad ,m>n$$
$$=F^{(m)}\left(I-B^{(n)}\right)(x)$$
$$=\lfloor \underbrace{n...n}_{n-m-1}\ \ n-1\ \underbrace{b-k+n-2...b-k+n-2}_{n-1}\quad b-k+n-1\ \underbrace{n...n}_{k+1-n}\rceil$$

where we have used (33), (34) and

$$F^{-(n)}(x)=\left(F^{-1}\right)^{(n)}(x)=B^{(n)}(x). \tag{35}$$

The expression for ($F6$) may be obtained from either an m forward shift of ($B5$) or from using ($F7$).

It is interesting to note the similarity between properties ($B7$) and ($F7$). If we temporarily use subscripts B and F to distinguish the parameters m and n in these two equations then we may observe that if we put

$$m_F=k+1-m_B$$

and $\quad n_F=k+1-n_B$

in ($F7$) we obtain ($B7$), upon using ($F3$) and (35).

Corresponding results to ($B8$) - ($B11$) do not seem to hold for the forward shift operator resulting in either negative numbers or digits bigger than the base.

We now turn our attention to the swap operator, S which unlike both B and F, is not cyclic. The swap operator S has the following properties when operating on x as given by equation (15).

$$(S1)\qquad S^{(n)}(x)=\lfloor 0\ 1\ 2\ ...n-1\ \ k-n...n\ \ k-(n-1)...k-1\ k\rceil$$

$$(S1)\qquad S^{(m)}(x)>S^{(n)}(x)\qquad ,\qquad 0\le m<n\le\frac{k+1}{2}$$

$$(S3)\qquad (I-S)(x)=\lfloor k-1\ \ b-1...b-1\ \ b-k\rceil$$

$$(S4)\qquad \left(I-S^{(n)}\right)(x)=\lfloor\underbrace{\gamma_0\,\gamma_1\cdots\gamma_{n-2}}_{n-1}\ \gamma_{n-1}-1\ \ \underbrace{b-1...b-1}_{k-(2n-1)}\ \underbrace{b-\gamma_{n-1}-1...b-\gamma_1-1}_{n-1}\ \ b-\gamma_0\rceil$$

$$(S5)\qquad \left(S^{(m)}-S^{(n)}\right)(x)=\lfloor\underbrace{0...0}_{m-1}\ \underbrace{\gamma_m\ \gamma_{m+1}\cdots\gamma_{n-2}}_{n-m-1}\ \gamma_{n-1}-1\ \ \underbrace{b-1...b-1}_{k-(2n-1)}$$
$$\underbrace{b-\gamma_{n-2}-1....b-\gamma_{m+1}-1}_{n-m-1}\ b-\gamma_m\ \ \underbrace{0...0}_{m-1}\rceil\ ,\ n>m.$$

Property ($S1$) is as represented in Figure 1 and ($S2$) shows S to be a decreasing operator when applied to a monotonically decreasing number x, given in (15). ($S3$) is equivalent to equation (8) with $\alpha_k = k$ and ($S4$) is equation (18) with γ_j given by (19).

Expression ($S5$) may be obtained from either ($S1$) or ($S4$). For the swap operator there is no expression corresponding to ($B6$).There are no zeros in ($S4$) and an application of the swap operator will not produce anything like ($S5$) with appropriately chosen m and n. Equivalent expressions to ($B8$) - ($B11$) also do not hold. The expression $\left(I+S^{(n)}\right)\left(I-S^{(n)}\right)(x)$ does give equation (20), and it is not equal to $\left(I-S^{(2n)}\right)(x)$.

Further expressions may be obtained by combining the three operators. For example,

$$(B-F)(x)=\lfloor k-2\ b-3 \ldots\ldots b-3\ \ b-2\ \ k-1\rceil$$

$$(B-S)(x)=\lfloor k-2\ b-2 \ldots\ldots b-2\ \ b-1\ \ \ 0\ \rceil$$

and $$(F-S)(x)=\lfloor 0\ \ 1 \ldots\ldots\ldots\ldots 1\ \ 0\ \ b-(k-1)\rceil$$

Care must be taken however to ensure that the terms are well defined.

5. EXTENSIONS

We will now investigate a number of extensions of previous results.

(A) Reverse Order

In earlier sections we investigated the manipulation of a $k+1$ monotonically decreasing number with unique digits, x, as given by equation (15). We now examine operations on the corresponding monotonically increasing number

$$y=\lfloor 0\ \ 1\ \ 2 \ldots\ldots k-1\ \ k\rceil .$$

If we note that

$$y=K-x \tag{36}$$

where $$K=\lfloor \underbrace{k \ldots\ldots k}_{k+1} \rceil \tag{37}$$

and x is as given by (15),
then

$$\left.\begin{aligned}(B-I)(y)&=(B-I)(K-x)=(I-B)(x)\\(F-I)(y)&=(I-F)(x)\\ \text{and}\quad (S-I)(y)&=(I-S)(x).\end{aligned}\right\} \tag{38}$$

Thus previous results may be utilised with the appropriate modifications as indicated by (36) - (38).

(B) Truncated Monotonically Decreasing Number

Let $x_k = \lfloor k\ k-1 \ldots\ldots 1\ \ 0 \rceil\ ,\ |x_k| = k+1 \le b,$ (39)

and consider

$$u = \lfloor k\ \ k-1 \ldots . r \rceil \tag{40}$$

$$= k\,b^{k-r} + (k-1)b^{k-r-1} + \ldots\ldots + (r+1)b + r\,b^0. \tag{41}$$

We notice that the magnitude of u is,

$$|u| = k - r + 1 \tag{42}$$

and from (41)

$$u = (x_k - x_{r-1})b^{-r} = \lfloor k\ \ k-1 \ldots\ldots . r\ \ \underbrace{0 \ldots . 0}_{r-1} \rceil\, b^{-r}. \tag{43}$$

Now, applying a backward shift to (40) gives

$$B(u) = \lfloor k-1\ \ \ k-2 \ldots r+1\ \ r\ \ k \rceil \tag{44}$$

and so $(I-B)(u) = \lfloor \underbrace{1\ 1 \ldots . 1}_{|u|-2}\ \ 0\ \ \ b-(k-r) \rceil,$ (45)

where $|u|$ is given in (42).

Previous results may be used if we note that

$$x_{k-r} = u - R \tag{46}$$

where $R = \lfloor r\ \ r \ldots\ldots r \rceil\ ,\ |R| = |u| = k - r + 1\ ,$ (47)

and $(I-B)(x_{k-r}) = (I-B)(u-R) = (I-B)(u).$ (48)

Hence (45) may be obtained by replacing k by $k-r$ in ($B4$). Similar developments may be made with the other operators and results.

Using (46) and (47) developments may be made with the operators F and S on u similar to those obtained in Sections 3 and 4.

(C) Truncated Monotonically Increasing Number

This involves both generalisations (A) and (B).

Consider

$$\mathrm{v} = \lfloor r \ \ r+1 \ldots k-1 \ \ k \rceil \tag{49}$$

then $$\mathrm{v}-R=\lfloor 0 \ \ 1 \ldots k-r \rceil = \gamma_{k-r} = K - x_{k-r} \tag{50}$$

where R, K, x_k are given by (47), (37), (39) respectively,

and $$\gamma_k = \lfloor 0 \ \ 1 \ \ldots \ k-r-1 \ \ k-r \rceil. \tag{51}$$

Now, from (50),

$$(B-I)(\mathrm{v}-R)=(B-I)(\mathrm{v})=(B-I)\,y_{k-r}=(I-B)\,x_{k-r},$$

and so we may again obtain results for v similar to those developed earlier on using the equations (49) - (51).

(D) Decreasing Digit Numbers

Consider the number

$$z = \lfloor a_k \ \ a_{k-1} \ \ldots \ a_1 \ \ a_0 \rceil \tag{52}$$

where $\Delta_i^{(1)} = a_i - a_{i-1} \geq 1 \quad , \quad i = 1, 2, \ldots k \quad ,$

and the number z is decreasing in a non-uniform fashion.

Now,

$$(I-B)(z)=\lfloor \Delta_k^{(1)} \ldots \Delta_2^{(1)} \ \Delta_1^{(1)} - 1 \ \ b - (a_k - a_0) \rceil , \tag{53}$$

which is equivalent to $(B4)$ when $\Delta_i^{(1)} \equiv 1$ and $a_i = k$.

Further,

$$(I+B)(I-B)(z)=\lfloor \Delta_k^{(2)} \ldots \ \Delta_3^{(2)} \ \Delta_2^{(2)} - 1 \quad b-(a_k - a_1) - 1 \quad b - (a_{k-1} - a_0) \rceil$$

where $\Delta_k^{(2)} = \Delta_k^{(1)} + \Delta_{k-1}^{(1)} = a_k - a_{k-2}$.

For example,

$$\text{Let } z_1 = \lfloor 7\ 5\ \ 4\ 3\ 1 \rceil$$
$$\Delta^{(1)} = \lfloor 2\ \ 1\ \ 1\ \ 2 \rceil$$
$$\Delta^{(2)} = \lfloor 3\ \ 2\ \ 3 \rceil$$

then $(I-B)(z_1) = \lfloor 2\ \ 1\ \ 1\ \ 2-1\ \ 4 \rceil$

and so $(I+B)(I-B)(z_1) = \lfloor 3\ \ 2\ \ 2\ \ 5\ \ 6 \rceil$.

The above development could equally as well be performed for non-uniform increasing digit numbers. Similar, permissible results may be obtained for the other operators.

6. CONCLUSION

Some interesting and novel results have been obtained through the operations of digit swaps and shifts together with subtraction and addition of the results.

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