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The Zero Crossing Problem

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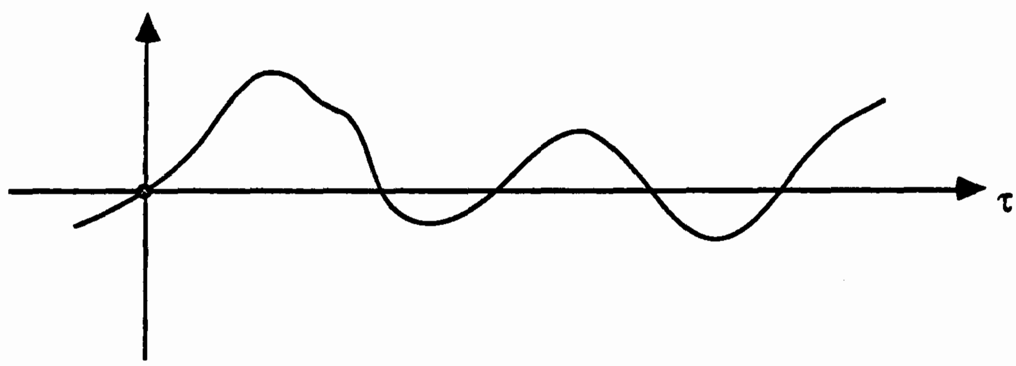
Abstract

We analyze a second order nonlinear differential equation that is useful in the zero crossing problem. We show that our calculated values are in agreement with the simulated values in the region of time τ, $0.1 < \tau < 0.9$.

Introduction

Consider a sample function of a random process as shown.



Researchers have been particularly interested in the statistics of the points where the sample function crosses the zero-axis, and in the probability density function of the intervals between these crossings.

Kac [1] in 1943 initiated some of the early investigations into this area, and more recently by Barnett [2]. Wong [3] has obtained some results based on approximate techniques.

The zero-crossing problem is a long standing one and it concerns the problem of determining the relevant statistics of the zero-crossings of random communication signals. This problem has relevance not only in communication theory, but also in the study of ocean waves, random vibrations, earthquakes, speech recognition, signal processing and frequency measurements.

Seumahu [4], through the application of the z-transformation has been able to derive a second order non-linear differential equation in the z-domain.

Gopalsamy and Lalli [5] have also investigated zero crossings in integrodifferential equations.

The Equation

Suppose that p_n is the probability that there are exactly n zero-crossings in a unit of time interval and that p is the generating function defined by

$$p = \sum_{n=0}^{\infty} p_n z^n$$

which is convergent for $z \in [-1, 1]$ and that p is twice differentiable term by term so that

$$p'' = \sum_{n=0}^{\infty} p_n n(n-1) z^{n-2} \qquad ; \qquad z \in [-1, 1]$$

then Seumahu [4], has derived the relation

$$p'' = \frac{c\mu^2 p\left(\beta^2 - p'^2\right)}{\mu^2\left(1 - p^2\right) - (1-c)\left(\beta^2 - p'^2\right)} \qquad \text{.......... (1)}$$

$$\text{with} \quad \left.\begin{aligned} p(1) &= \sum_{n=0}^{\infty} p_n = 1 \\ p'(1) &= \beta \\ p(-1) &= \sum_{n=0}^{\infty} (-1)^n p_n = r \end{aligned}\right\} \qquad \text{.......... (2)}$$

where c, μ, β are non-negative constants and r is a correlation coefficient such that $r \in [-1, 1]$.

The problem at hand is to solve this equation and see if it matches simulated results.

Analysis

One simple non-zero solution of (1) is obtained by inspection by noting that

$$(p')^2 = \beta^2.$$

Note $p(z) = 0$ is trivial hence, using (2)

$$p(z) = \beta(z-1) + 1, \text{ where } r = 1 - 2\beta \qquad \text{.........} \; (3)$$

Now, a first integral of (1) can be obtained as follows:

$$\left\{ \begin{aligned} p' &= q \\ q' &= \frac{c\mu^2 p\left(\beta^2 - q^2\right)}{\mu^2\left(1-p^2\right) - (1-c)\left(\beta^2 - q^2\right)} \end{aligned} \right\} \qquad \text{.........} \; (4)$$

Consider $w = p^2$ and $x = \beta^2 - q^2$

$$\frac{dw}{dx} = \frac{-p}{q'}$$

substitute for q', hence

$$\frac{dw}{dx} = \frac{-\mu^2(1-w) + (1-c)x}{c\mu^2 x}$$

now

$$\frac{d}{dx}\left(wx^{-1/c}\right) = \frac{(1-c)x^{-1/c}}{c\mu^2} - \frac{x^{-1-1/c}}{c}$$

so that

$$w = -\frac{x}{\mu^2} + 1 + Bx^{1/c} \quad ; \qquad \text{B is a constant}$$

replace w and x

$$\mu^2\left(1-p^2\right) = \left(\beta^2 - q^2\right) - A\left(\beta^2 - q^2\right)^{1/c} \qquad \text{............} \; (5)$$

A is a new constant

Substitute into (4) and we have

$$\left.\begin{aligned} p' &= q \\ q' &= \frac{\mu p}{1 - \frac{A}{c}\left(\beta^2 - q^2\right)^{\frac{1}{c} - 1}} \end{aligned}\right\} \qquad \text{..........} \quad (6)$$

or
$$\frac{dq}{dp} = \frac{\mu p}{q\left\{1 - \frac{A}{c}\left(\beta^2 - q^2\right)^{\frac{1}{c} - 1}\right\}}$$

provided that $\beta^2 - q^2 \neq 0$ which is always the case if $r \neq 1 - 2\beta$.

Also from the initial data, we can deduce that the constant A satisfies the inequality

$$\frac{A}{c}\left(\beta^2 - q^2\right) \leq 1 \quad ; \quad \forall z \in [-1, 1] \qquad \text{.........} \quad (7)$$

More over, it follows that

(i) p(z) is a Convex function of z whenever p(z) is positive and

(ii) p(z) is a Concave function of z whenever p(z) is negative.

This leads to the following theorem.

THEOREM

The constant of integration A, which appears in the first integral (5), of equation (1) has the property that

$$A \geq 0 \quad \text{if} \quad r \geq r_0$$

and
$$A \leq 0 \quad \text{if} \quad r \leq r_0.$$

PROOF

Consider $A \geq 0$, since the case $A \leq 0$ follows by reversing all the subsequent inequalities putting (5) into (1), we can write

$$p''(z) = \frac{\mu^2 p}{1 - \frac{A}{c}\left(\beta^2 - q^2\right)^{\frac{1}{c} - 1}}$$

the inequality at (7) now suggests that

$$p''(z) \geq \mu^2 p$$

multiplying by Cosh μz, we have

$$\frac{d}{dz}\left\{p' \operatorname{Cosh} \mu z - \mu p \operatorname{Sinh} \mu z\right\} \geq 0$$

integrating w.r.t. z and using $p'(1) = \beta$, we have

$$\beta \operatorname{Cosh} \mu - \mu \operatorname{Sinh} \mu \geq p' \operatorname{Cosh} \mu z - \mu p \operatorname{Sinh} \mu z$$

$$\left(\beta \operatorname{Cosh} \mu - \mu \operatorname{Sinh} \mu\right) \operatorname{Sech}^2 \mu z \geq \frac{d}{dz}\left\{p \operatorname{Sech} \mu z\right\}.$$

Integrating w.r.t. z and using the boundary condition $p(1) = 1$, yields

$$p(z) \geq \frac{\operatorname{Cosh} \mu z}{\operatorname{Cosh} \mu} - \left[\frac{\beta}{\mu} - \tanh \mu\right] \operatorname{Sinh} \mu (1 - z)$$

From the other boundary condition $p(-1) = r$, we have that

$$r = p(-1) \geq \operatorname{Cosh} 2\mu - \frac{\beta}{\mu} \operatorname{Sinh} 2\mu \equiv r_0$$

The proof is now complete.

A parametric solution of (1) from the first integral (5) can be obtained as follows:

write $$p^2 = 1 - \frac{\left(\beta^2 - q^2\right)}{\mu^2} + \frac{A}{\mu^2}\left(\beta^2 - q^2\right)^{1/c}.$$

Define

$$p^2 \equiv H(q) = 1 - \frac{\left(\beta^2 - q^2\right)}{\mu^2} + \frac{A}{\mu^2}\left(\beta^2 - q^2\right)^{1/c} \qquad \text{.......... (8)}$$

hence

$$p = \frac{H'(q)\; q'}{2q}$$

and

$$1 = \frac{H'(q)}{2q\sqrt{H(q)}} \cdot \frac{dq}{dz} \qquad \text{since } p = \sqrt{H(q)}\ .$$

Given that $p'(1) = \beta$ integrate, with respect to the variable z.

$$z = 1 + \int_{\beta}^{q} \frac{H'(q)}{2q\sqrt{H(q)}}\, dq$$

$$z = 1 + \int_{x=\beta}^{q} \frac{1}{x}\frac{d}{dx}\left[\sqrt{H(x)}\right] dx$$

$$z = 1 + \frac{\sqrt{H(q)}}{q} - \frac{\sqrt{H(\beta)}}{\beta} + \int_{x=\beta}^{q} \frac{\sqrt{H(x)}}{x^2}\, dx$$

From (8), $H(\beta) = 1$, so that

$$z = 1 - \frac{1}{\beta} + \frac{\sqrt{H(q)}}{q} + \int_{x=\beta}^{q} \frac{\sqrt{H(x)}}{x^2}\, dx.$$

For the special case of $A \equiv 0$, the solution of (1) follows from (5)

$$q = p' = \frac{dp}{dz} = \sqrt{\beta^2 - \mu^2 + \mu^2 p^2}$$

$$z = \int \frac{dp}{\sqrt{\beta^2 - \mu^2 + \mu^2 p^2}}$$

<u>Case (i)</u> If $\beta = \mu > 0$

$$p(z) = \exp\left[\beta(z-1)\right] \qquad \text{for } x \in [-1, 1]$$

provided that $r = \exp(-2\beta)$.

<u>Case (ii)</u> If $\beta > \mu > 0$

$$p(z) = \frac{\left(\beta^2 - \mu^2\right)^{1/2}}{\mu} \operatorname{Sinh} \mu\,(z - z_0)$$

where
$$z_0 = 1 - \frac{1}{2\mu} \log\left(\frac{\beta + \mu}{\beta - \mu}\right)$$

provided that
$$r = r_0 = \operatorname{Cosh} 2\mu - \frac{\beta}{\mu} \operatorname{Sinh} 2\mu.$$

[Note that the singular solution (3) corresponds to letting $\mu \to 0$]

<u>Case (iii)</u> If $0 < \beta < \mu$

$$p(z) = \frac{\left(\mu^2 - \beta^2\right)^{1/2}}{\mu} \operatorname{Cosh} \mu(z - z_0)$$

where
$$z_0 = 1 - \frac{1}{2\mu} \log\left(\frac{\mu + \beta}{\mu - \beta}\right)$$

provided that $r = r_0$.

Numerical Results

For $0 < c < 1$, the right hand side of (6) is continuous in a neighbourbood of $p = 1, q = \beta$, hence has a solution. Moreover, for $0 < c < \frac{1}{2}$, it is differentiable and hence there is a unique solution satisfying the given initial conditions. Thus the solution of the initial value problem (1) is not unique. There is an infinite family of solutions, one for each choice of the constant A.

Thus we have two independent parameters c and A (regarding μ and β as fixed) to be determined from the one boundary condition $p = r$ when $z = -1$.

For $0.1 < t < 0.9$, the following computer method has been used to estimate the constant A (but unfortunately does not apply for larger values of τ).

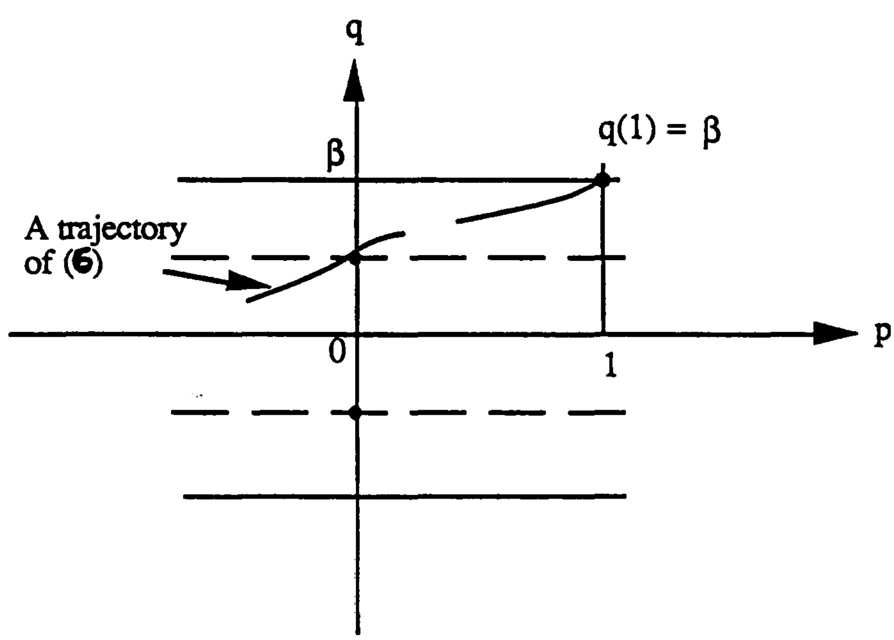


Fig. 1: Singularities of the system (6)

A trajectory of the system (6) can only cross the singularities (dotted line - Fig. 1) at a point on the q-axis. This gives the formula for A in terms of c.

$$A = c\left(\frac{1-c}{\mu}\right)^{\frac{1}{c}-1}$$

For $0.1 < \tau < 0.9$ this gives calculated results of p(0), provided by Seumahu [4] (See table 1).

Table 1:

τ	p(0) Calculated	p(0) Simulated	c	A
.1	.88478	.88478		
.2	.77115	.77116	.5047	54.83
.3	.66096	.66097	.5005	16.84
.5	.4614	.46147	.4963	3.304
.59	.3865	.38669	.4936	1.8926
.7	.312	.31248	.4887	1.0597
.8	.260	.26123	.4811	.6796
.9	.22	.22195	.4686	.46156

For larger values of τ we lose the singularities, hence there is no sensible way to determine A in terms of c. We now have two parameters but only one boundary condition to determine them.

If $\tau = 1.2$

$$\left.\begin{array}{l} c = .4 \\ A = .145338 \end{array}\right\} \text{gives} \left\{\begin{array}{l} p(-1) = .08063 \\ p(0) = .11996 \end{array}\right.$$

$$\left.\begin{array}{l} c = .41 \\ A = .151324 \end{array}\right\} \text{gives} \left\{\begin{array}{l} p(-1) = .08162 \\ p(0) = .12622 \end{array}\right.$$

$$\left.\begin{array}{l} c = .42 \\ A = .157272 \end{array}\right\} \text{gives} \left\{\begin{array}{l} p(-1) = .08026 \\ p(0) = .132186 \end{array}\right.$$

$$\left.\begin{array}{l} c = .43 \\ A = .163124 \end{array}\right\} \text{gives} \left\{\begin{array}{l} p(-1) = .080319 \\ p(0) = .137866 \end{array}\right.$$

$$\left.\begin{array}{l} c = .44 \\ A = .169025 \end{array}\right\} \text{gives} \left\{\begin{array}{l} p(-1) = .080752 \\ p(0) = .137866 \end{array}\right.$$

$$\left.\begin{array}{l} c = .45 \\ A = .1748 \end{array}\right\} \text{gives} \left\{\begin{array}{l} p(-1) = .079568 \\ p(0) = .148310 \end{array}\right.$$

For $\tau = 1.2$, the simulated value of $p(0)$ is in fact .14019.

Hence the problem in hand is 'how do we determine the parameter A in terms of c ?'

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