VICTORIA UNIVERSITY OF TECHNOLOGY

DEPARTMENT OF COMPUTER AND MATHEMATICAL SCIENCES

Visualisations of Bairstow's Basins

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25 MATH 4

February 1993

TECHNICAL REPORT

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VISUALISATIONS OF BAIRSTOW'S BASINS

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Summary. Computer experiments with dynamical systems generated by Bairstow's method for finding quadratic factors of a polynomial have been carried out. Basins of attraction for the known factors of some low degree polynomials have been obtained. The method is applied to polynomials with real coefficients where only real quadratic factors are considered giving a variety of pictures, some exhibiting a fractal-like property of self-similar regions at different scales. Others have chaotic characteristics. Convergence of the algorithm is not dependent on the coefficients of the quadratic factors being real and some results allowing for complex coefficients are presented.

1. Introduction

Iteration, a technique associated with many numerical methods, has found a new application in recent years - the basis for the creation of pictures (or images). These pictures may be used to convey information about the mathematical problem underlying the iterative process or merely looked upon as a form of art for which no mathematical knowledge is required.

Since the generation of a crude black/white picture by B. Mandelbrot, of an image now named after him, the development of this area of research has continued aided by the improved quality and availability of modern computers and their graphics displays. A description of the discovery of the Mandelbrot set may be found in Peitgen and Richter (1986). Numerous pictures, both black/white and color, generated by iterative processes may be seen in that reference and also in Barnsley (1988), Becker and Dörfler(1989), Devaney (1989), and Peitgen and Saupe (1988). Even undergraduate texts such as Berkey and Blanchard (1992) and Ellis and Gulick (1990) contain iteratively generated pictures, with accompanying explanations of their source.

2. Preliminaries

Dynamical systems of the form

$$r_{n+1} = f(r_n,s_n) \quad , \quad s_{n+1} = g(r_n,s_n) \tag{1}$$

$$(n = 0,1,2,3, \dots)$$

will be used to generate sequences of points $\{(r_n,s_n)\}$ starting with an initial point (r_0,s_0). If the sequence converges to (r^*,s^*) then this limit is a fixed point of the dynamical system since it satisfies $f(r^*,s^*) = r^*$ and $g(r^*,s^*) = s^*$. When there is more than one fixed point the choice of initial values (r_0,s_0) will determine which fixed point, if any, is obtained as the limit of the sequence. Of course, it is also possible for some choices of (r_0,s_0) that the sequence does not converge or that either of the functions $f(r,s)$ and $g(r,s)$ become undefined at one of the sequence points.

The basin of attraction for a fixed point is defined as the set of initial points such that the sequence $\{(r_n,s_n)\}$ generated by (1) converges to that fixed point.

For a given dynamical system the basins of attraction are visualised by performing the calculations on a Cartesian grid of (r_0,s_0) values corresponding to pixel locations on a computer graphics monitor and setting the color of each pixel according to which fixed point is obtained. Examples 1-42 are reproductions of images generated by this method.

Bairstow's iterative algorithm, commonly referred to as Bairstow's method, is used on polynomials of low degree with known quadratic factors to generate dynamical systems of the above type whose fixed points are in one-one correspondence with coefficients of the quadratic factors. The algorithm is a special application of Newton's method for solving non-linear equations.

3. Newton's method

Newton's method for finding a solution of the non-linear equations

$$u(r,s) = 0 \quad , \quad v(r,s) = 0 \tag{2}$$

starting with an initial approximation (r_0,s_0) is to successively calculate the sequence $\{(r_n,s_n)\}$ from the iterative scheme

$$r_{n+1} = r_n + \Delta r_n \quad , \quad s_{n+1} = s_n + \Delta s_n \qquad (n = 0,1,2, \dots)$$

where Δr_n, Δs_n are obtained by replacing u(r,s) and v(r,s) in (2) with their Taylor expansions at (r_n,s_n) and solving the resulting linear approximations

$$u_r(r_n,s_n)\Delta r_n + u_s(r_n,s_n)\Delta s_n + u(r_n,s_n) = 0$$
$$v_r(r_n,s_n)\Delta r_n + v_s(r_n,s_n)\Delta s_n + v(r_n,s_n) = 0. \qquad (3)$$

4. Synthetic division

In Bairstow's method for finding quadratic factors of a given polynomial p(x) the source of the functions u and v are the two components of the linear remainder term when the polynomial is divided by $x^2 - rx - s$:

$$p(x) = (x^2 - rx - s)q(x) + u(x - r) + v. \qquad (4)$$

This quadratic will be a factor of p(x) when both u(r,s) = 0 and v(r,s) = 0, non-linear equations which may be solved by Newton's method described above. Evaluation of the functions and their partial derivatives for numerical values of r and s, as required by (3), for a general p(x) does not need explicit formulas for u and v as the Bairstow iterative algorithm uses a synthetic division technique similar to Horner's synthetic division evaluation of a polynomial and its derivative at a specified x-value.

Let

$$p(x) = a_m x^m + a_{m-1}x^{m-1} + \dots + a_1 x + a_0$$

and

$$q(x) = b_m x^{m-2} + b_{m-1}x^{m-3} + \dots + b_3 x + b_2$$

then substituting into (4) and equating coefficients of powers of x gives

$$b_k = a_k + rb_{k+1} + sb_{k+2} \qquad (5)$$

$$(k = m, m - 1, \dots ,1, 0)$$

where $u = b_1$, $v = b_0$ and we define $b_{m+1} = b_{m+2} = 0$.

Further, dividing xq(x) by $x^2 - rx - s$ gives

$$xq(x) = (x^2 - rx -s)t(x) + c_2(x - r) + c_1 \qquad (6)$$

where

$$t(x) = c_m x^{m-3} + c_{m-1}x^{m-4} + \dots + c_4 x + c_3$$

Equating powers of x in (6) gives

$$c_k = b_k + rc_{k+1} + sc_{k+2} \tag{7}$$

$$(k = m, m - 1, \ldots , 1)$$

where we define $c_{m+1} = c_{m+2} = 0$.

It is proved that $v_r = c_1$, $v_s = c_2 = u_r$ and $u_s = c_3$ in Mathews (1987) from which the following summary of Bairstow's method is taken.

5. Bairstow's iterative algorithm

The steps in Bairstow's method for a given p(x) are :

(i) Start with an initial quadratic $x^2 - rx - s$.

(ii) Calculate the coefficients b_k and c_k from (5) and (7).

(iii) Solve the equations

$$c_1\Delta r + c_2\Delta s = - b_0 \quad , \quad c_2\Delta r + c_3\Delta s = - b_1$$

unless

$$D = c_1c_3 - c_2^2 = 0$$

when the algorithm fails.

(iv) Form new values r' and s' from

$$r' = r + \Delta r \quad , \quad s' = s + \Delta s$$

which become the coefficients of the new quadratic $x^2 - r'x - s'$.

(v) Repeat steps (i) - (iv) with r' and s' replacing r and s respectively until the iteration process converges, or some specified maximum number of iterations have been performed.

If the algorithm converges then it does so to a quadratic factor of the given polynomial, but if the polynomial has more than one distinct quadratic factor then the factor obtained will depend on the choice of the starting values of r and s in (i).

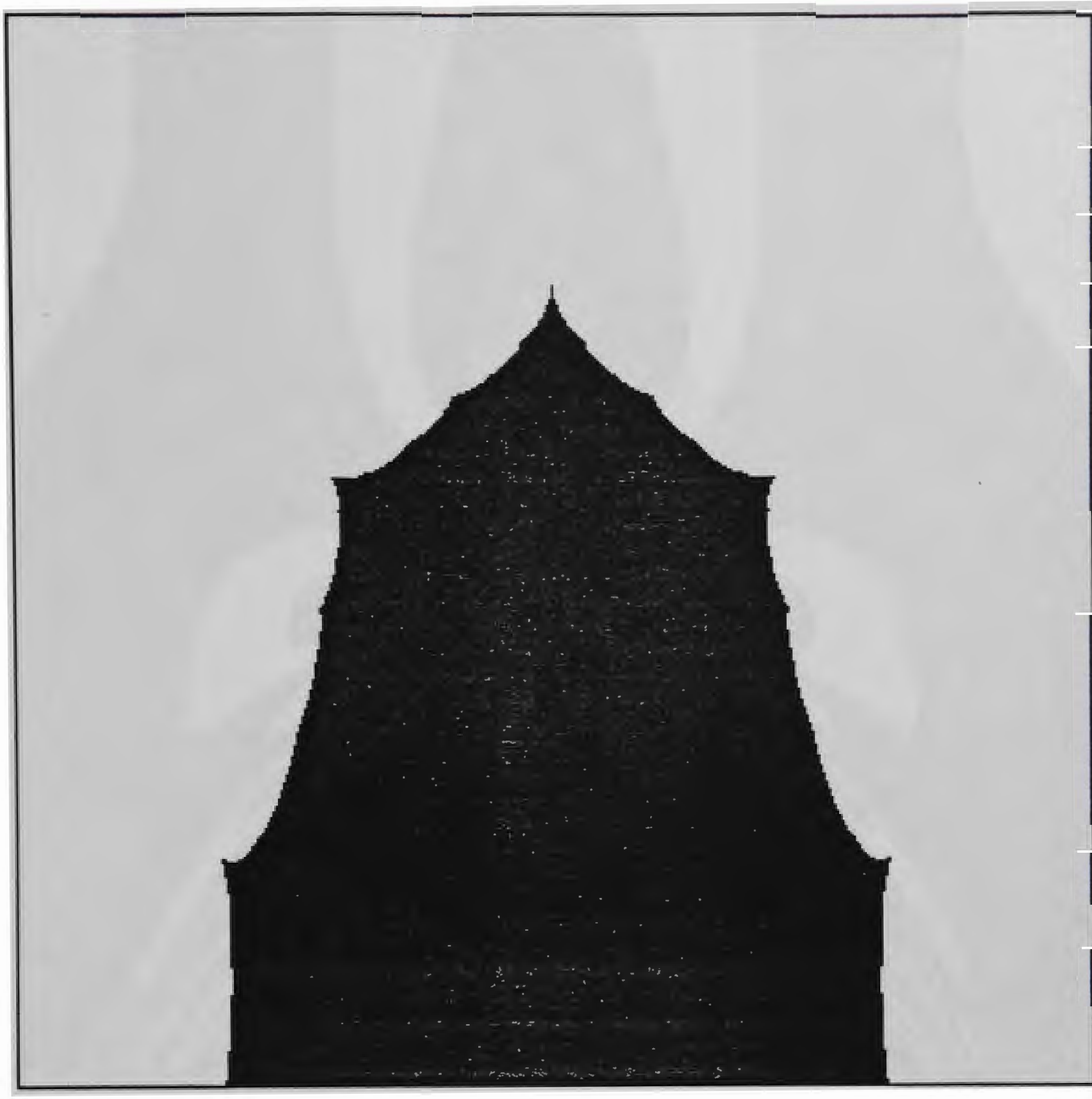
6. Examples

The above Bairstow algorithm is a dynamical system of the form (1), with the subscripts on r and s omitted for reasons of convenience only, except it is not necessary to obtain explicit formulas for f(r,s) and g(r,s). The quadratic $x^2 - rx - s$ corresponds to the point (r,s) in an r-s plane and if $x^2 - r^*x - s^*$ is a factor of the polynomial then (r^*,s^*) is the corresponding fixed point of the dynamical system, and vice-versa. It follows that the basins of attraction of the fixed points may interchangeably be referred to as basins of attractions for the quadratic factors.

In the following examples, calculations were performed to double precision and the images reproduced from a PC monitor in VGA 640 X 480 mode. Each pixel location corresponds to a single (r,s) point and the screen represents a rectangular grid of points.

The window refers to the domain of initial (r,s) values and it should be emphasised that the images are discrete approximations to the true situation, since calculations are only started at points in the window corresponding to pixel locations, but the whole pixel is colored once the algorithm is terminated. For this reason basins that actually are continuous narrow bands may appear as a collection of un-connected points and a family of narrow bands as a dust of points. Similarly, some basins appear to have rough edges, but closer inspection would show that this is due to narrow bands of the same color being in close proximity to the main basin boundary.

Examples 1-13 have exactly two fixed points and the images are shown in black/white. The basin of the fixed point with the asterisk (*) is colored black. Examples 1-35 only consider quadratic factors with real coefficients, the domain of (r,s) values being 2-dimensional. There is no real reason why this restriction on the coefficients cannot be removed, except that visualisation of the basins of attraction becomes difficult since they are 4-dimensional in their most general form. Imposing two constraints reduces the problem back to 2-dimensions and allows the presentation of examples 36-41.



Example 1.

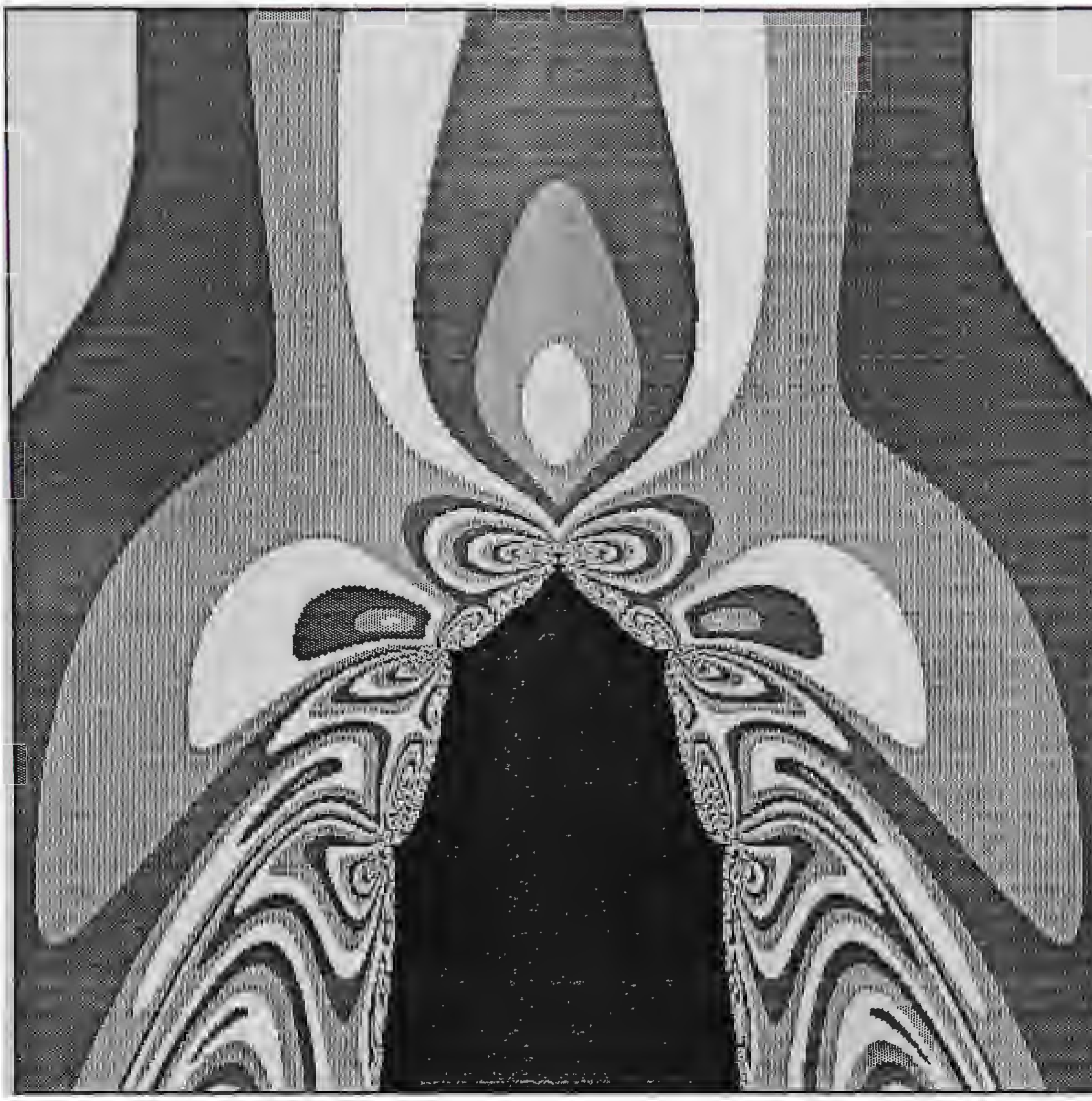
Polynomial : $x^4 - 1$

Factors/ Fixed points :	$x^2 - 1$	/	(0, 1)
	$x^2 + 1$	/	(0,-1)*

Window : $-2 < r < +2$, $-3 < s < +1$

Comments :

Points in the black region belong to the basin of attraction for the fixed point (0,-1). Alternatively, if the starting quadratic in Bairstow's algorithm has its coefficients taken from the same black region then it converges to the factor $x^2 + 1$. Under magnification the boundary appears to be a smooth curve except at cusps.

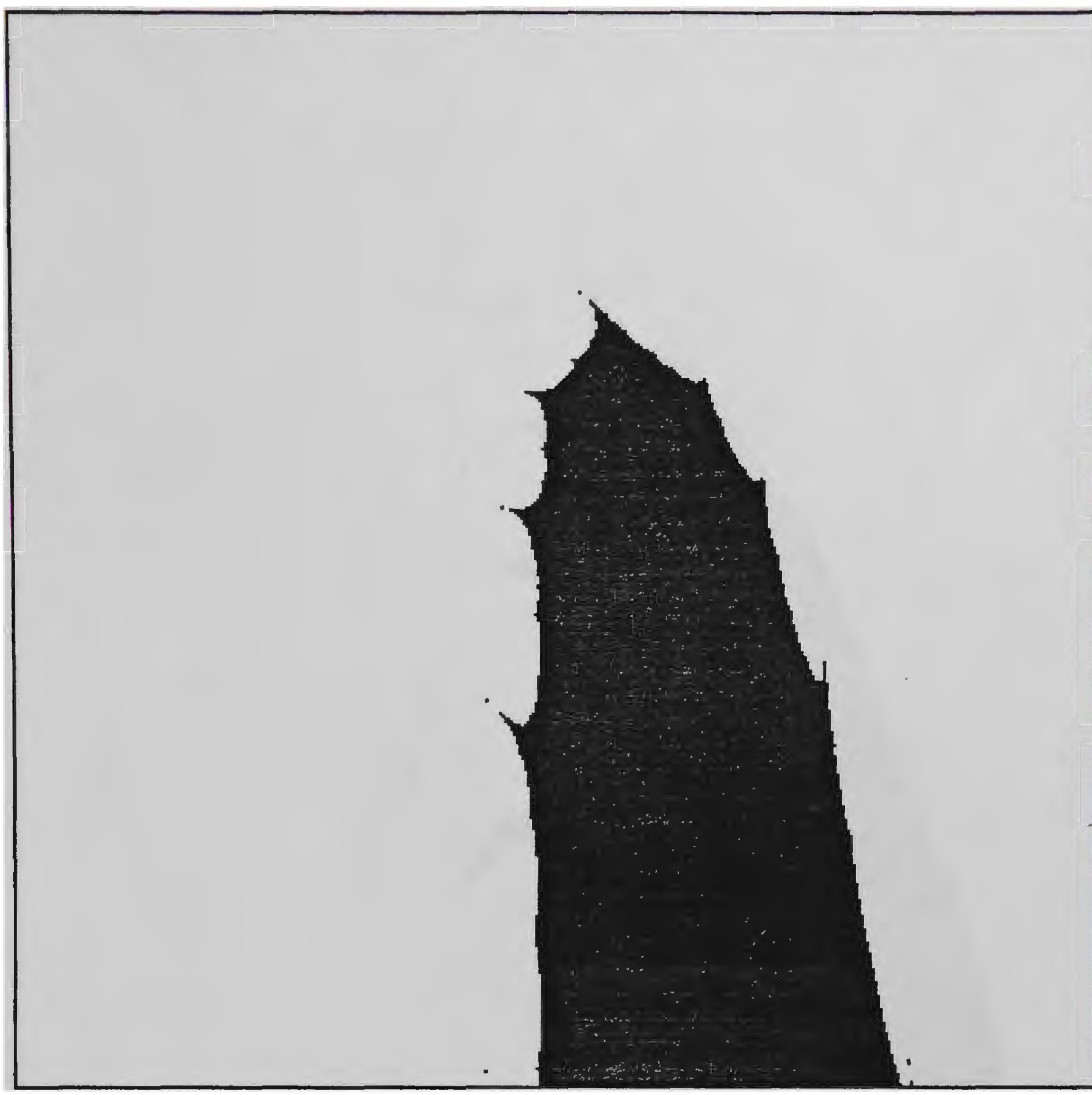


Example 2.

Polynomial :	$x^4 - 1$		
Factors/ Fixed points :	$x^2 - 1$ $x^2 + 1$	/ /	(0, 1) (0,-1)*
Window :	$-4 < r < +4$	,	$-4 < s < +4$

Comments :

A different window to example 1. The shaded parts in the previously all white basin of the fixed point (0,1) illustrate how the number of iterations required for convergence to a specified accuracy, varies with the initial point. Progression from white to light grey to dark grey to white etc., indicates extra iterations



Example 3.

Polynomial : $x^5 - 1$

Factors/ Fixed points :

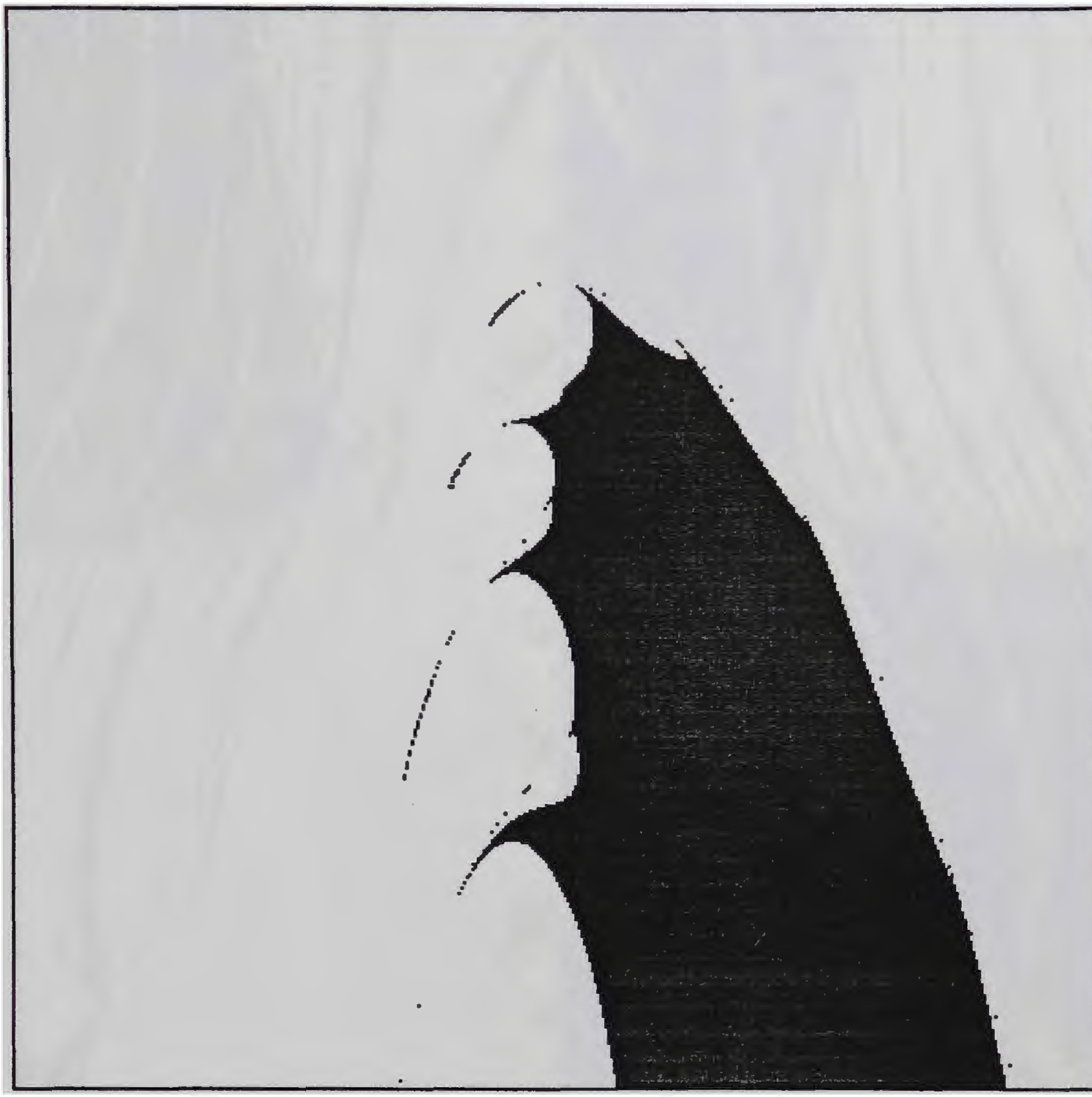
$x^2 + 0.5(1 - \sqrt{5})x + 1$ / (0.616,-1)*

$x^2 + 0.5(1 + \sqrt{5})x + 1$ / (-1.616,-1)

Window : $-4 < r < +4$, $-6 < s < +2$

Comments :

In common with examples 4-7 the two fixed points lie on the line $s = -1$ and the basin for the fixed point with positive r is shown in black. Unlike examples 5-7 the fixed points are not symmetrically placed about the vertical axis.



Example 4.

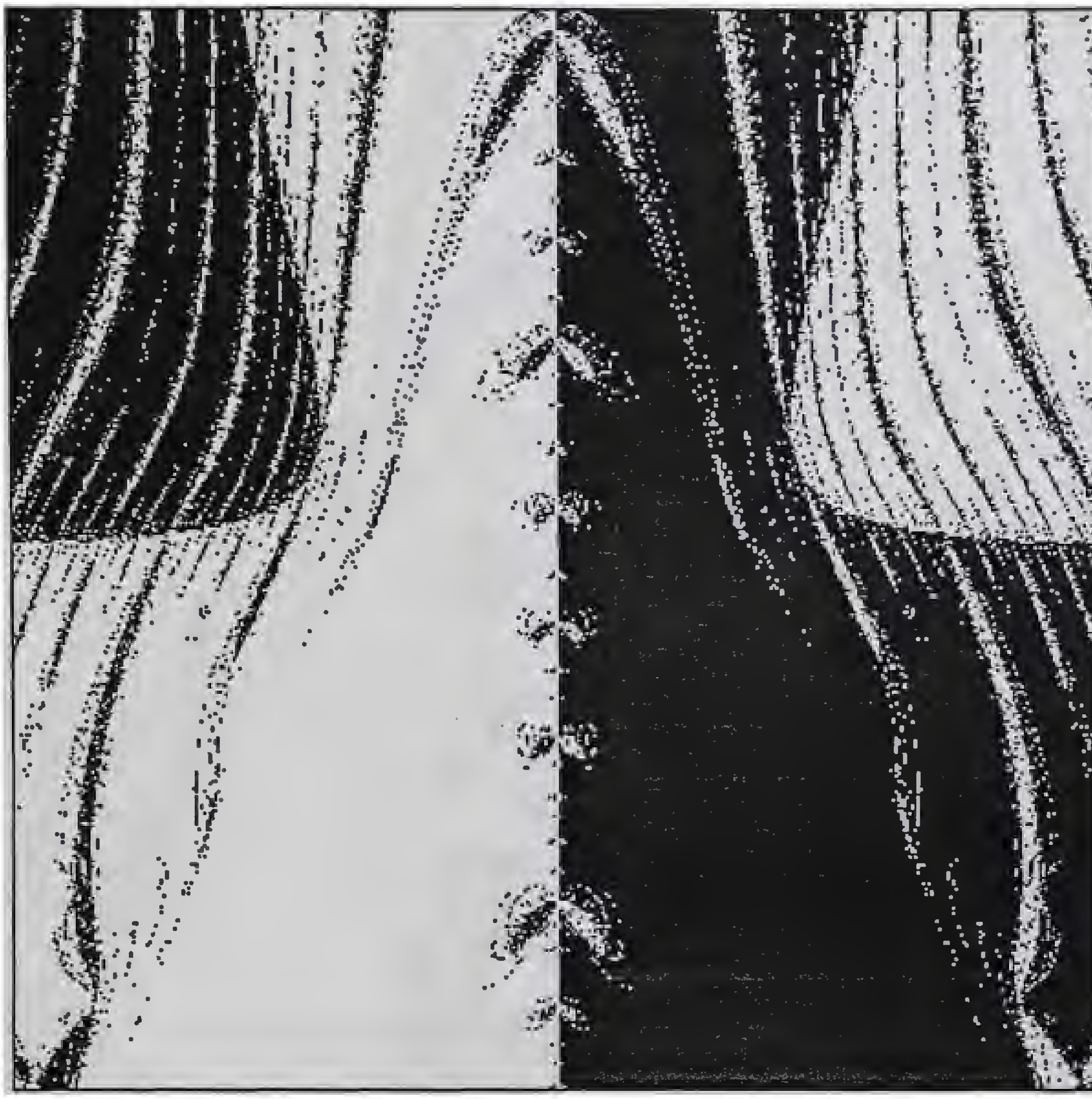
Polynomial : $x^4 + x^3 + x^2 + x + 1$

Factors/	$x^2 + 0.5(1 - \sqrt{5})x + 1$ /	(0.616,-1)*
Fixed points :	$x^2 + 0.5(1 + \sqrt{5})x + 1$ /	(-1.616,-1)

Window : $-4 < r < +4$, $-6 < s < +2$

Comments :

Removing the linear factor from the previous example does not alter the fixed points and similar basins are obtained, but the number of iterations required for convergence is generally less.



Example 5.

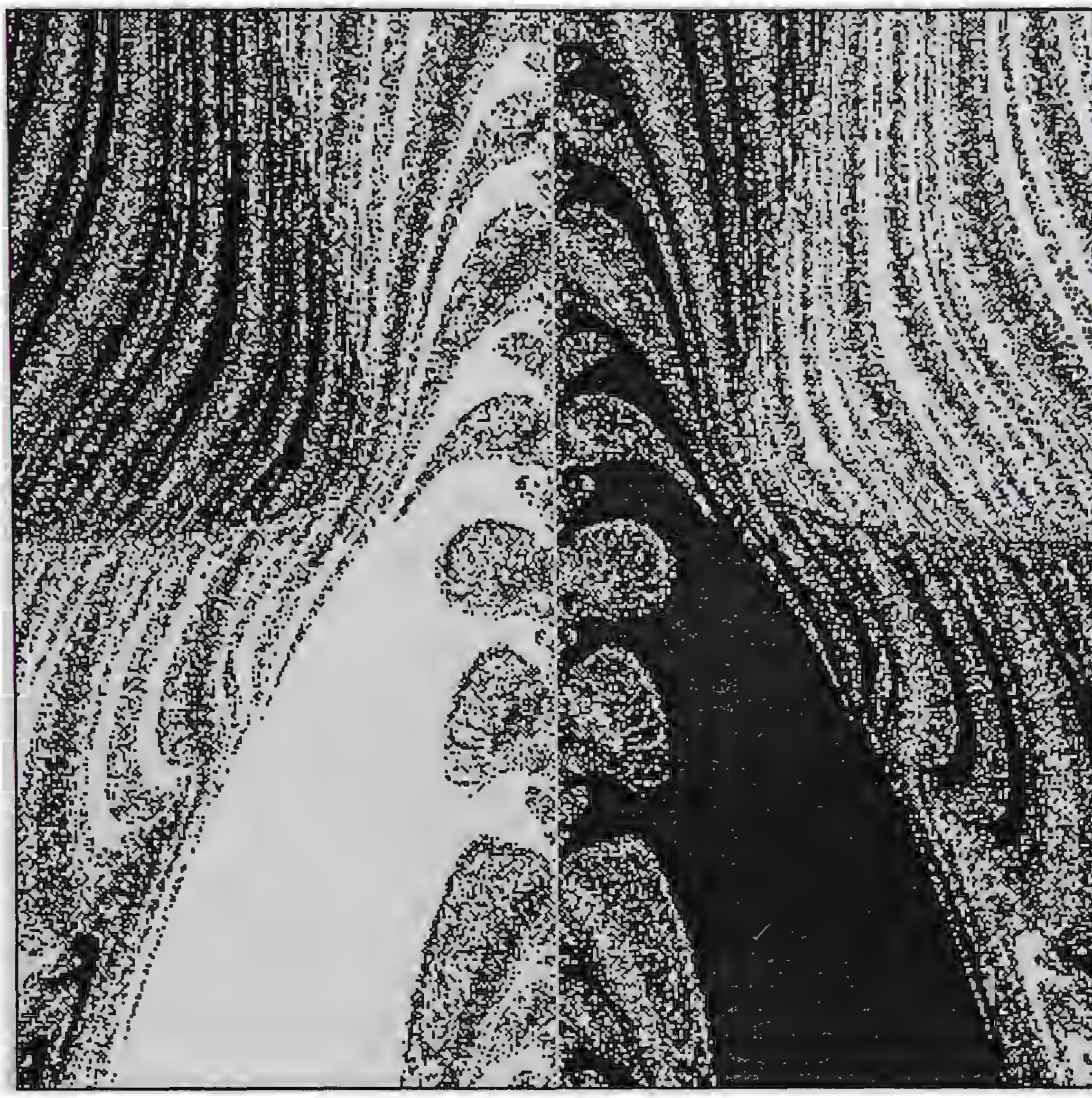
Polynomial : $x^4 + x^2 + 1$

Factors/ Fixed points : $x^2 - x + 1$ / (1,-1)*; $x^2 + x + 1$ / (-1,-1)

Window : $-4 < r < +4$, $-4 < s < +4$

Comments :

The basin for (1,-1) is shown in black. Like examples 6 and 7 the s-axis does not belong to either basin, successive iterates remaining on the axis without converging.



Example 6.

Polynomial : $x^4 + 1$

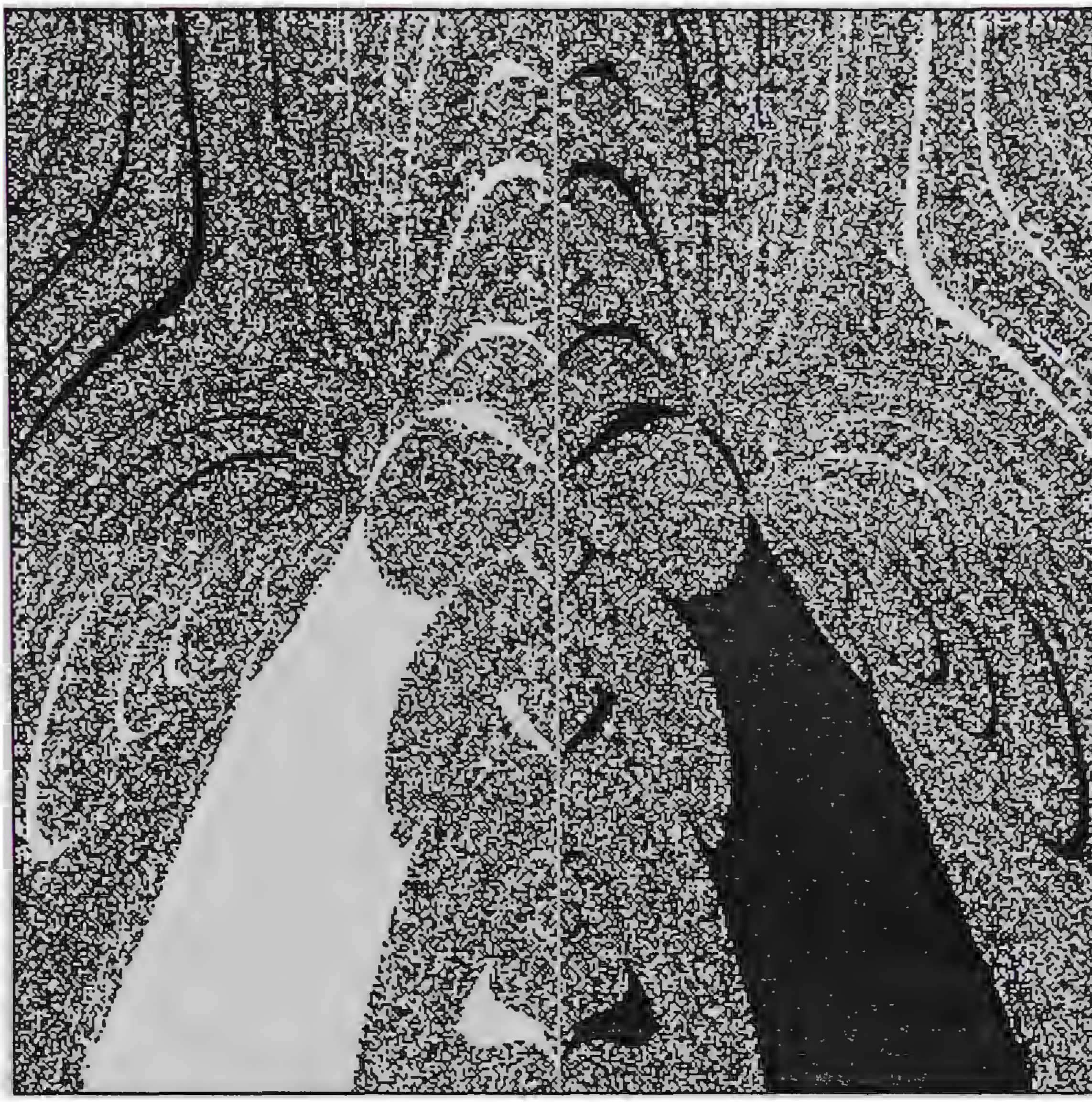
Factors/ Fixed points :

$x^2 - \sqrt{2}x + 1$	/	$(\sqrt{2},-1)^*$
$x^2 + \sqrt{2}x + 1$	/	$(-\sqrt{2},-1)$

Window : $-4 < r < +4$, $-4 < s < +4$

Comments :

The number of iterations required for convergence is generally more than the number required in the previous example, and less than the number required in the next example.



Example 7.

Polynomial : $x^4 - x^2 + 1$

Factors/ Fixed points : $x^2 - \sqrt{3}x + 1$ / $(\sqrt{3},-1)^*$

$x^2 + \sqrt{3}x + 1$ / $(-\sqrt{3},-1)$

Window : $-4 < r < +4$, $-4 < s < +4$

Comments :

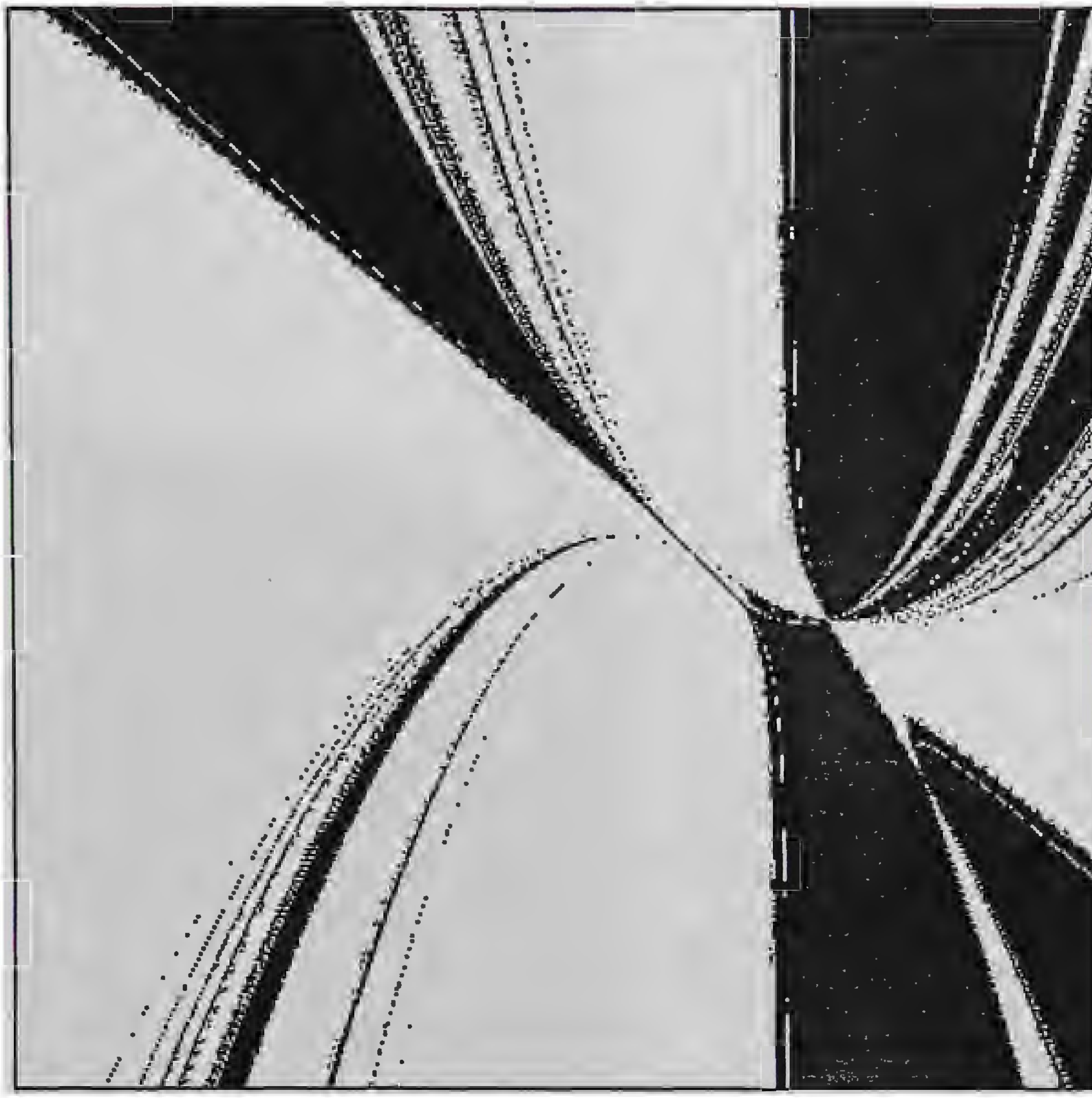
As the fixed points get further from the vertical axis, while remaining on s = -1, the basins become more chaotic. Example 31 has an additional fixed point at (0,-1).

Example 8.

Polynomial :	$x^4 - x$		
Factors/ Fixed points :	$x^2 - x$	/	(1, 0)
	$x^2 + x + 1$	/	(-1,-1)*
Window :	$-3 < r < +1$	,	$-3.5 < s < +0.5$

Comments :

Has some similarities with the green basin in Example 26.

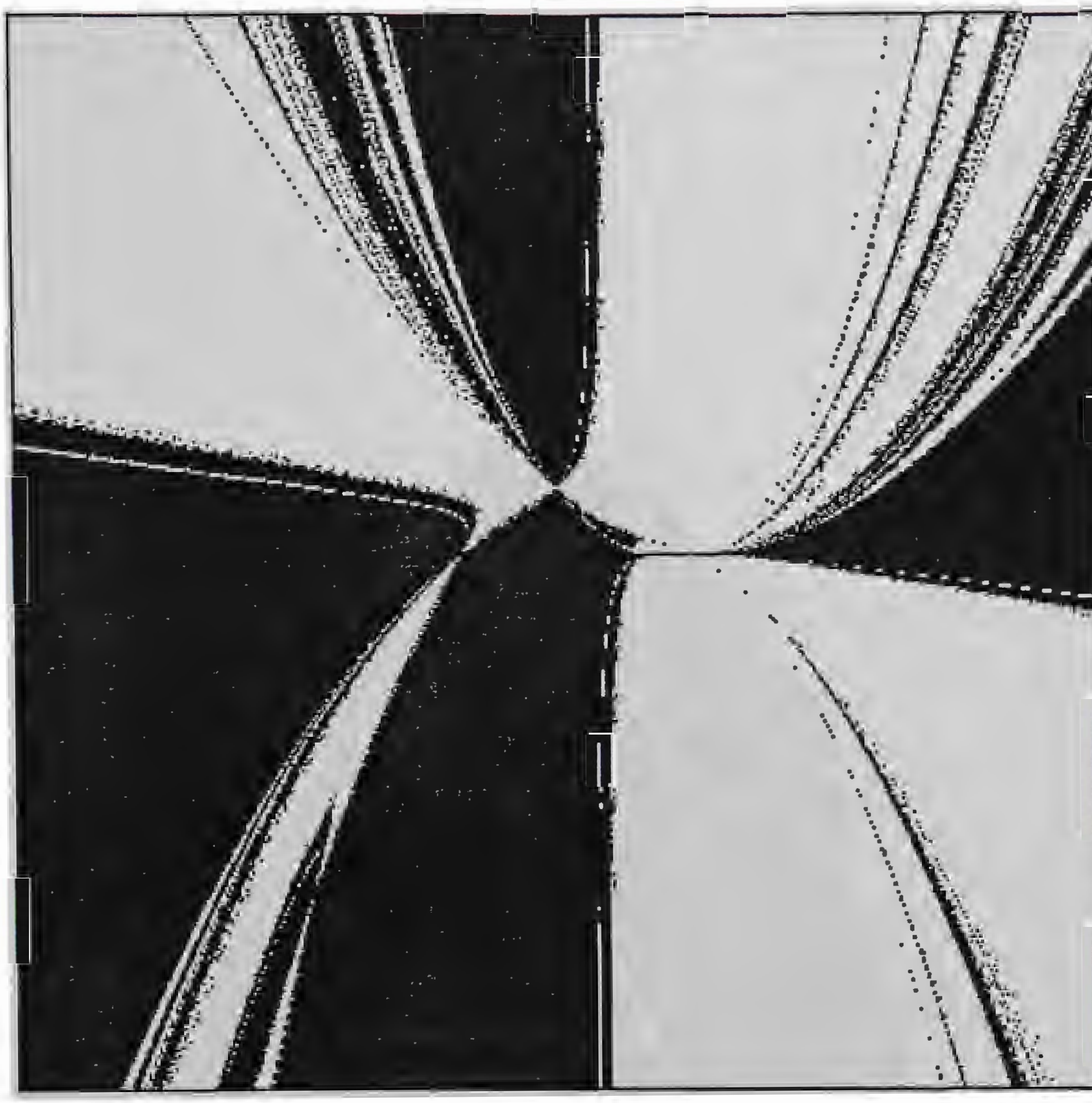


Example 9.

Polynomial :	$x^3 - 2x^2 + x$		
Factors/ Fixed points :	$x^2 - x$ $x^2 - 2x + 1$	/ /	(1, 0) (2,-1)*
Window :	$-4 < r < +4$	,	$-4 < s < +4$

Comments :

Taking the initial point as $(1 - \varepsilon, \varepsilon)$, arbitrarily close to the fixed point (1,0), gives as successive iterates $(2, 2\varepsilon + \varepsilon^2)$ and (2,-1), i.e., convergence to the other fixed point.



Example 10.

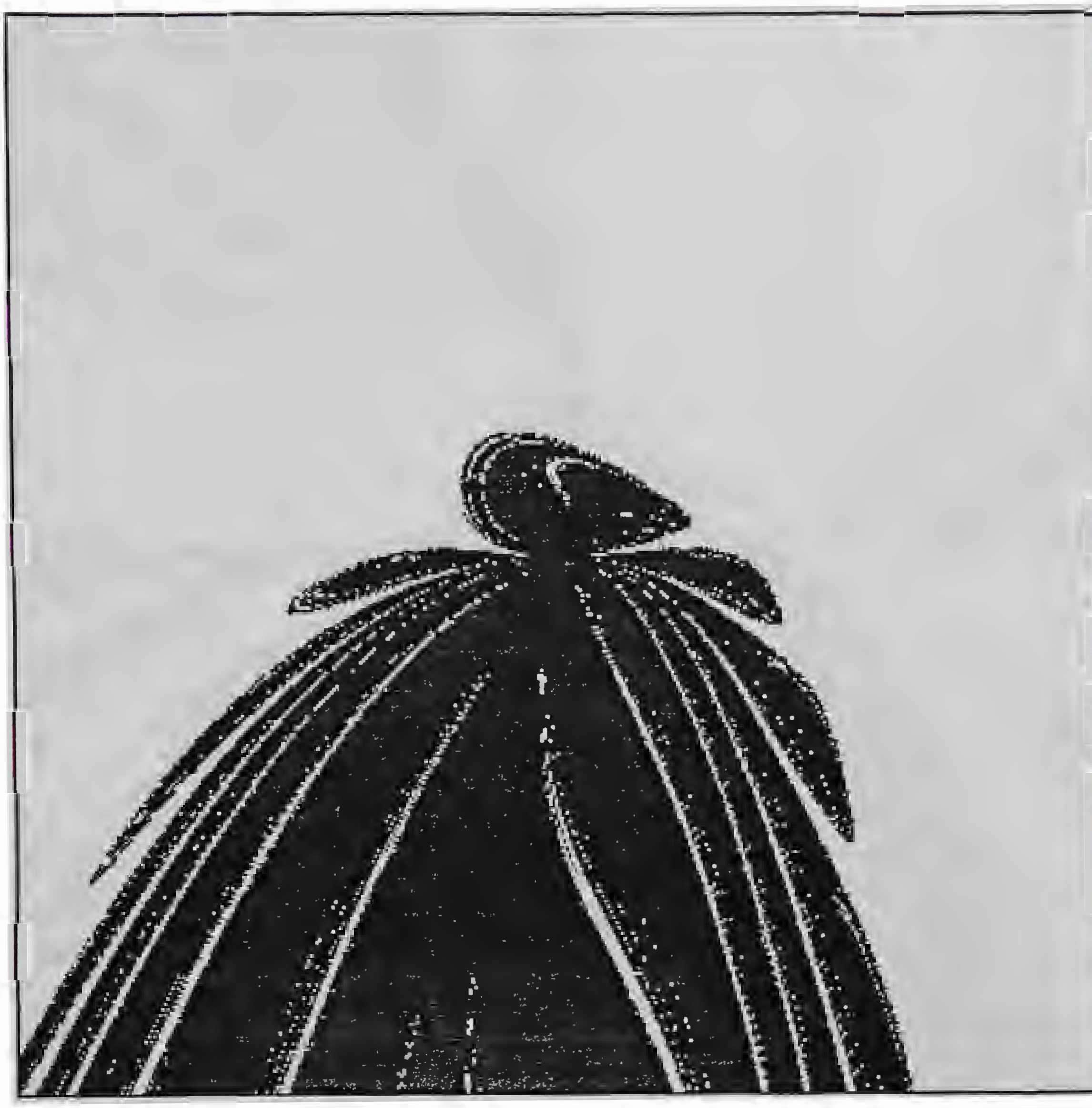
Polynomial : $x^3 - x^2$

Factors/	x^2	/	(0, 0)*
Fixed points :	$x^2 - x$	/	(1, 0)

Window : $-4 < r < +4$, $-4 < s < +4$

Comments :

Examples 11-13 have the same two fixed points as here but the basins show significant differences as the multiplicity of the linear x-factor increases. Similar to example 9 in that starting arbitrarily close to (1,0), this time at $(1 + \varepsilon, 0)$, convergence to the other fixed point occurs in two steps.

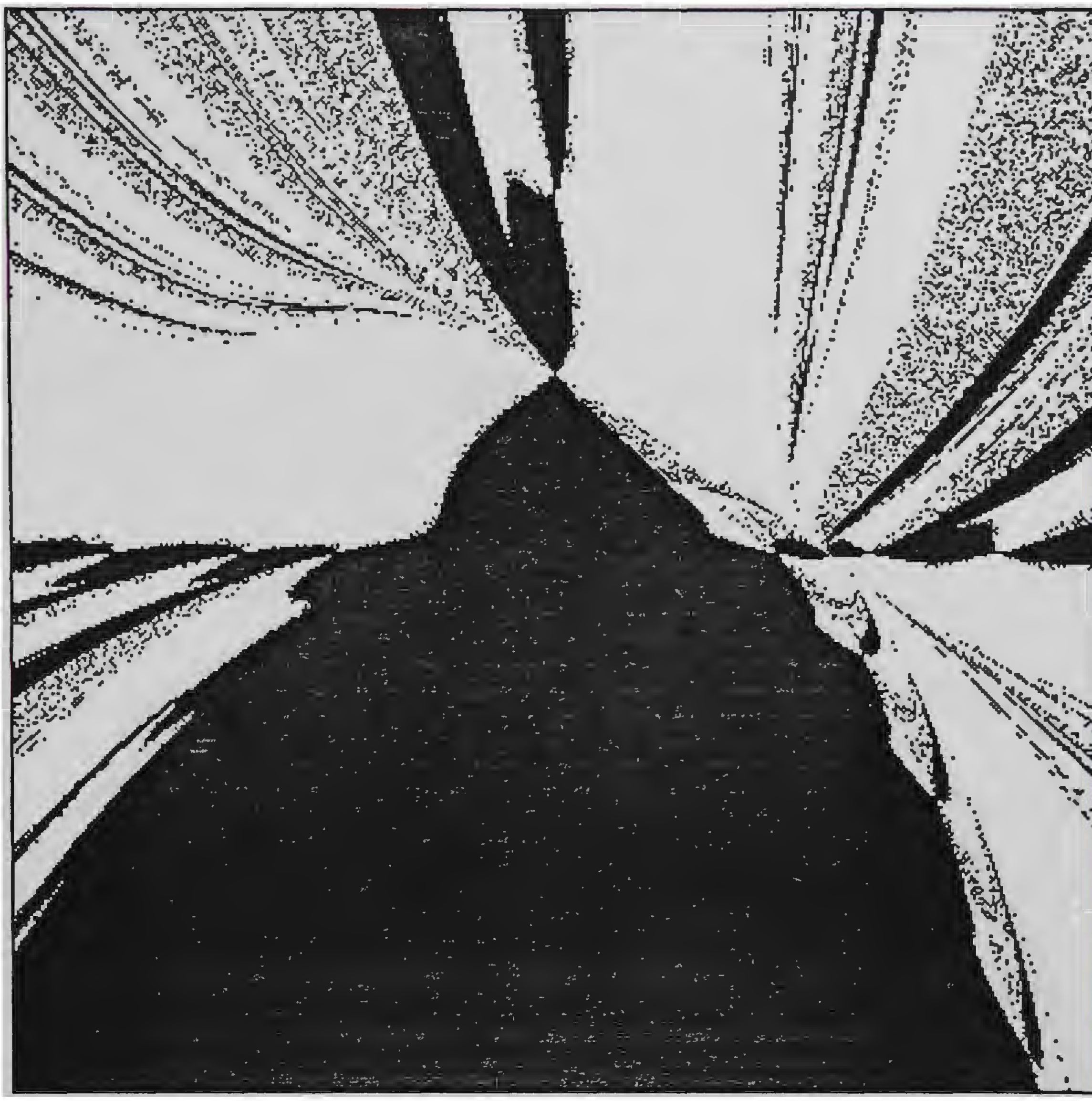


Example 11.

Polynomial :	$x^4 - x^3$		
Factors/ Fixed points :	x^2	/	(0, 0)*
	$x^2 - x$	/	(1, 0)
Window :	$-2 < r < +2$	,	$-2 < s < +2$

Comments :

A similar shape occurs in example 29.



Example 12.

Polynomial :	$x^5 - x^4$		
Factors/ Fixed points :	x^2	/	(0, 0)*
	$x^2 - x$	/	(1, 0)
Window :	$-2 < r < +2$	,	$-2 < s < +2$

Comments :

The mountain shaped part of the basin for x^2 is similar, although not symmetric, to the basin for x^2 in example 17. In both examples the polynomial has x^4 as a factor.

Example 13.

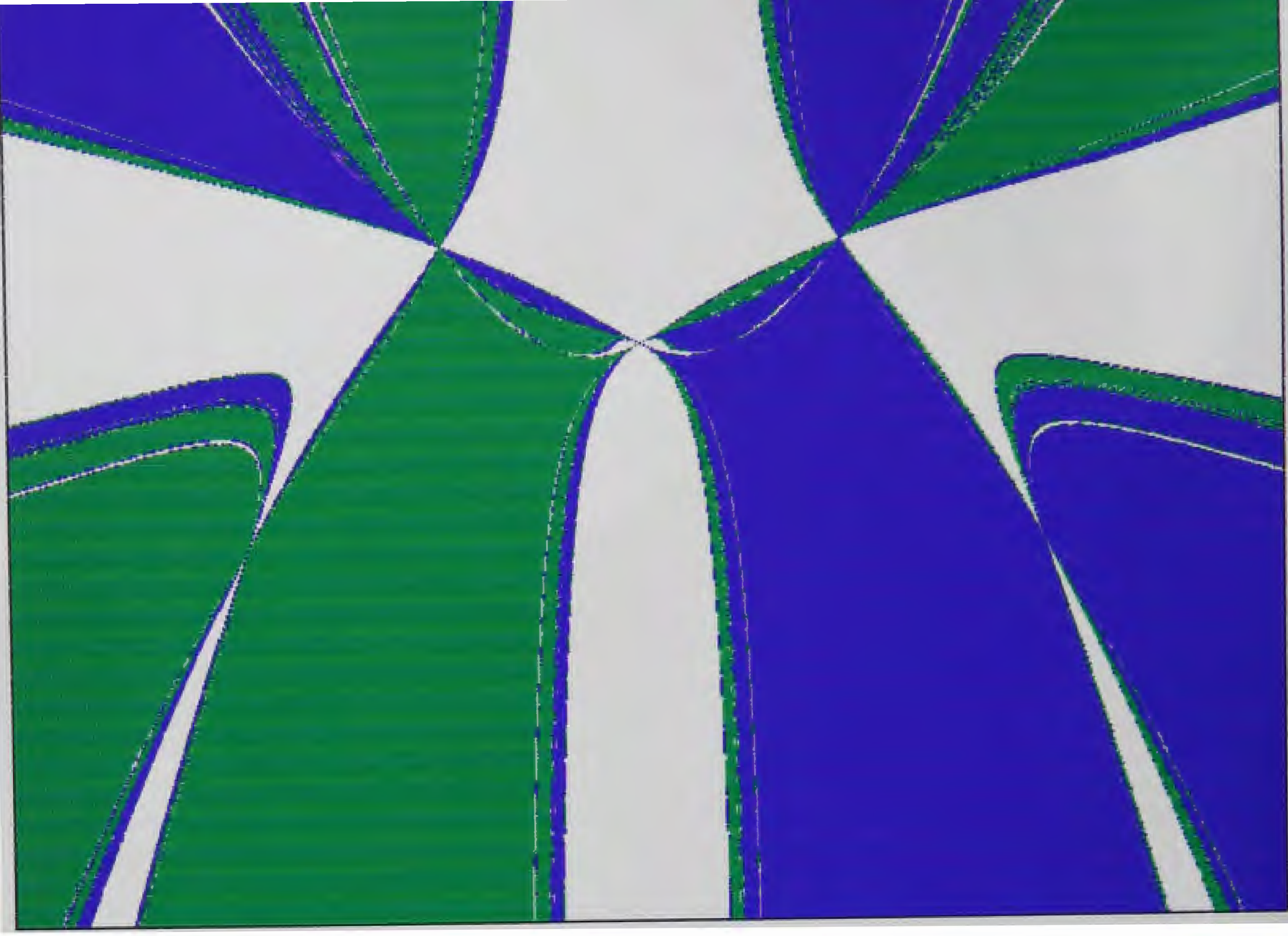
Polynomial : $x^6 - x^5$

Factors/	x^2	/	(0, 0)*
Fixed points :	$x^2 - x$	/	(1, 0)

Window : $-2 < r < +2$, $-2 < s < +2$

Comments :

The same fixed points as in the previous two examples.



Example 14.

Polynomial :	$x^3 - x$		
Factors/ Fixed points/ Colors :	$x^2 - 1$ /	(0, 1)	/ white
	$x^2 - x$ /	(1, 0)	/ blue
	$x^2 + x$ /	(-1, 0)	/ green

Window : $-3.2 < r < +3.2$, $-2.4 < s < +2.4$

Comments :

Typical of odd degree polynomials with fixed points symmetric about the vertical axis, see also examples 15 and 16, the off axis fixed point basins extend indefinitely upwards and down. Bairstow's algorithm fails on the parabola $r^2 + 1 = 2s$, where D = 0, but this curve only becomes visible at points which coincide with the exact pixel coordinates. These are colored black.

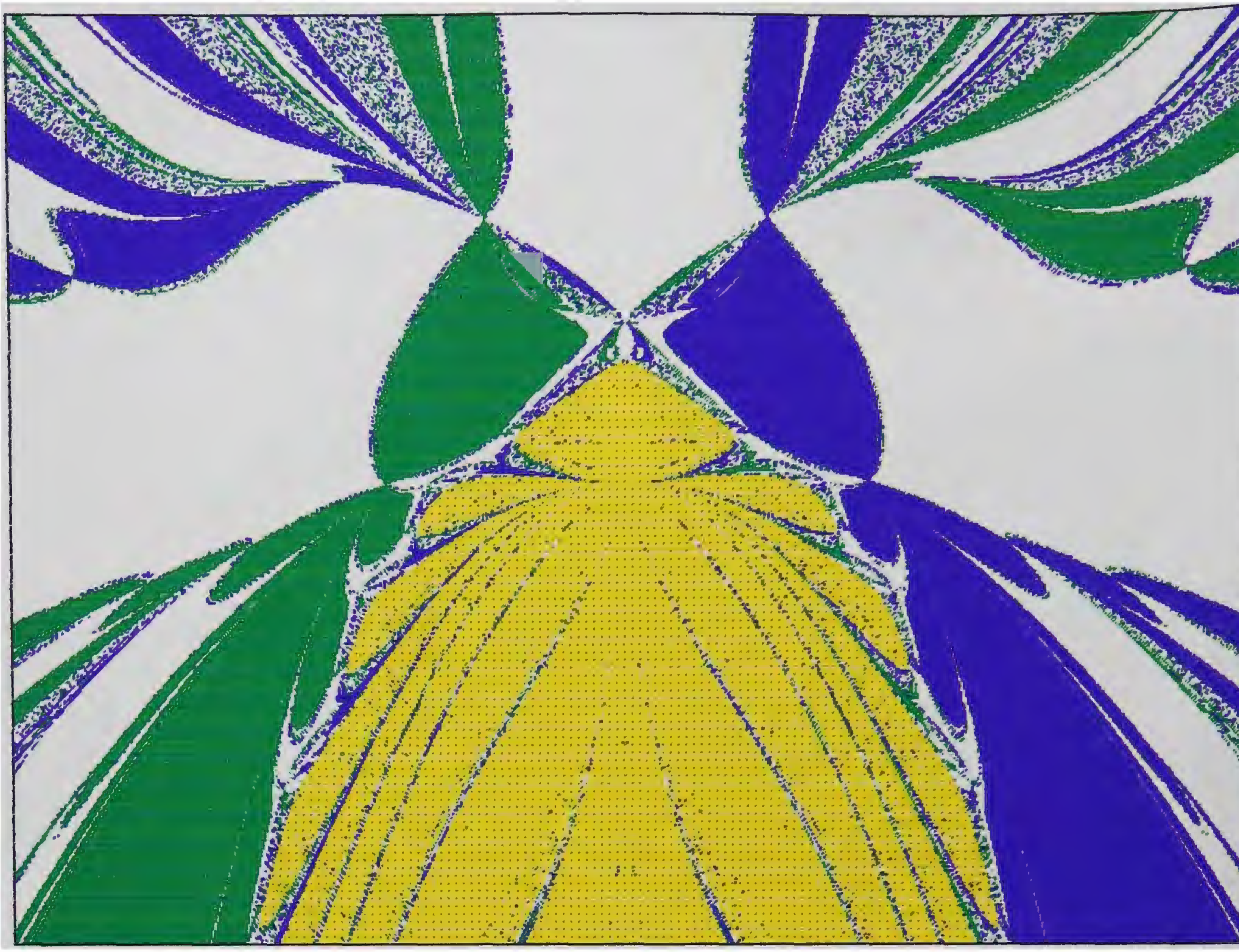
Example 15.

Polynomial :	$x^5 - x$		
Factors/	$x^2 - 1$	/	(0, 1) / white
Fixed points/	$x^2 - x$	/	(1, 0) / blue
Colors :	$x^2 + x$	/	(-1, 0) / green
	$x^2 + 1$	/	(0,-1) / red

Window : $-3.2 < r < +3.2$, $-2.4 < s < +2.4$

Comments :

The presence of another fixed point, in addition to the same three as in example 14, shows the usual effect of having two fixed points on the vertical axis, regardless of the parity of the degree of the polynomial. The basin for the lower one only extends downwards, and does not appear in outer regions unlike the irregular and chaotic nature of the basins for the off axis fixed points.



Example 16.

Polynomial :	$x^5 - x^3$	
Factors/	$x^2 - 1$ /	(0, 0) / white
Fixed points/	$x^2 - x$ /	(1, 0) / blue
Colors :	$x^2 + x$ /	(-1, 0) / green
	x^2 /	(0, 0) / yellow

Window : $-2.56 < r < +2.56$, $-1.92 < s < +1.92$

Comments :

The same three fixed points as example 14 and an additional one at the origin gives a pattern with similar features to example 15, but the lower axial basin having narrow intrusions of regions involving the other basins. Similar intrusions can also be seen in examples 11, 24 and 29.

Example 17.

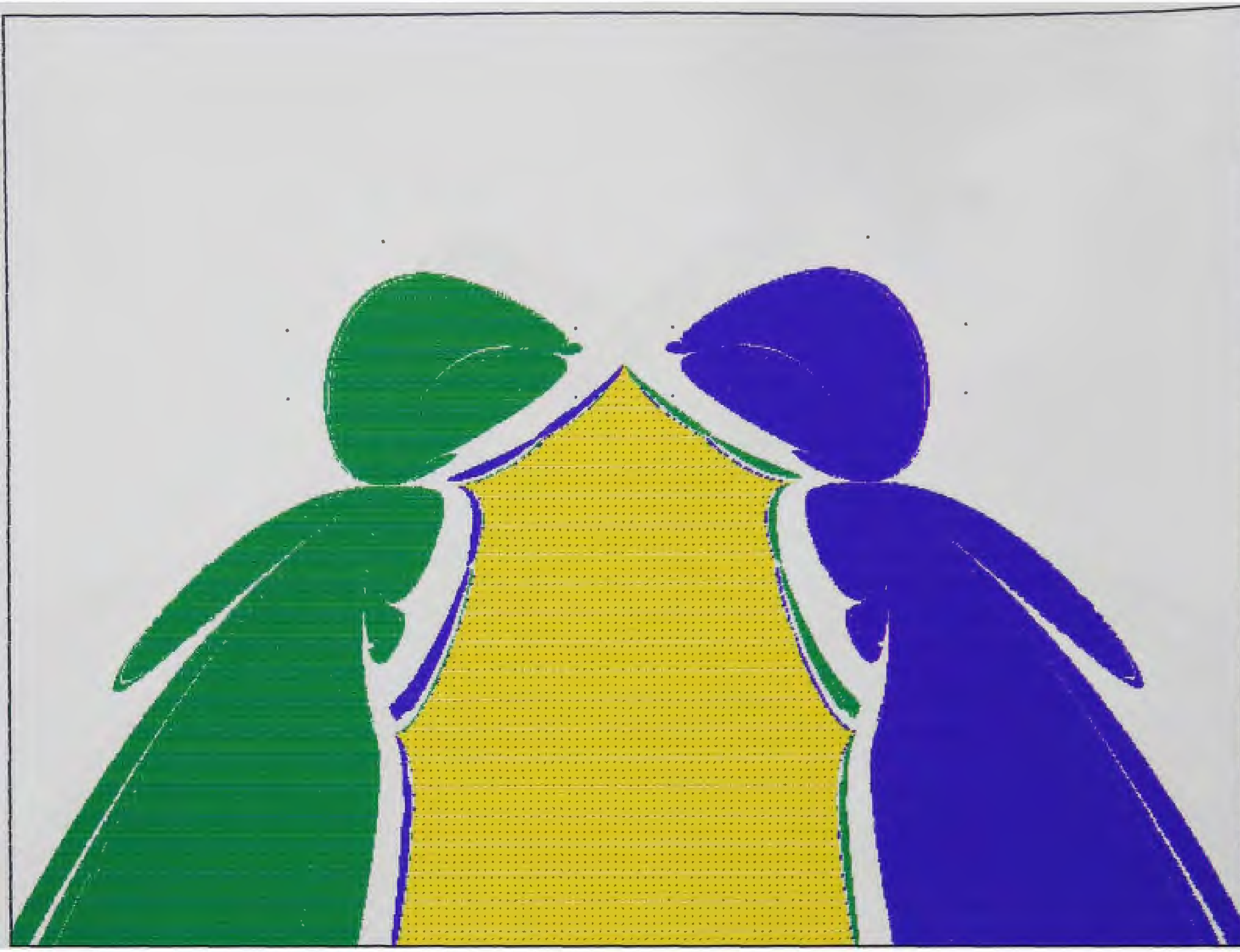
Polynomial : $x^6 - x^4$

Factors/ Fixed points/ Colors :	$x^2 - 1$	/	(0, 1) / white
	$x^2 - x$	/	(1, 0) / blue
	$x^2 + x$	/	(-1, 0) / green
	x^2	/	(0, 0) / yellow

Window : $-2.56 < r < +2.56$, $-1.92 < s < +1.92$

Comments :

The same fixed points as examples 16 and 18 but shows the features common to even degree polynomials with two fixed points on the vertical axis and symmetrically located off axis points. The basin for the lower point still extends downwards but the basin for the upper point becomes the "sky" surrounding the other basins, and the off axis basins, unlike for odd degrees, also only extend downwards



Example 18.

Polynomial :	$x^4 - x^2$	
Factors/ Fixed points/ Colors :	$x^2 - 1$ /	(0, 1) / white
	$x^2 - x$ /	(1, 0) / blue
	$x^2 + x$ /	(-1, 0) / green
	x^2 /	(0, 0) / yellow

Window : $-2.56 < r < +2.56$, $-1.92 < s < +1.92$

Comments :

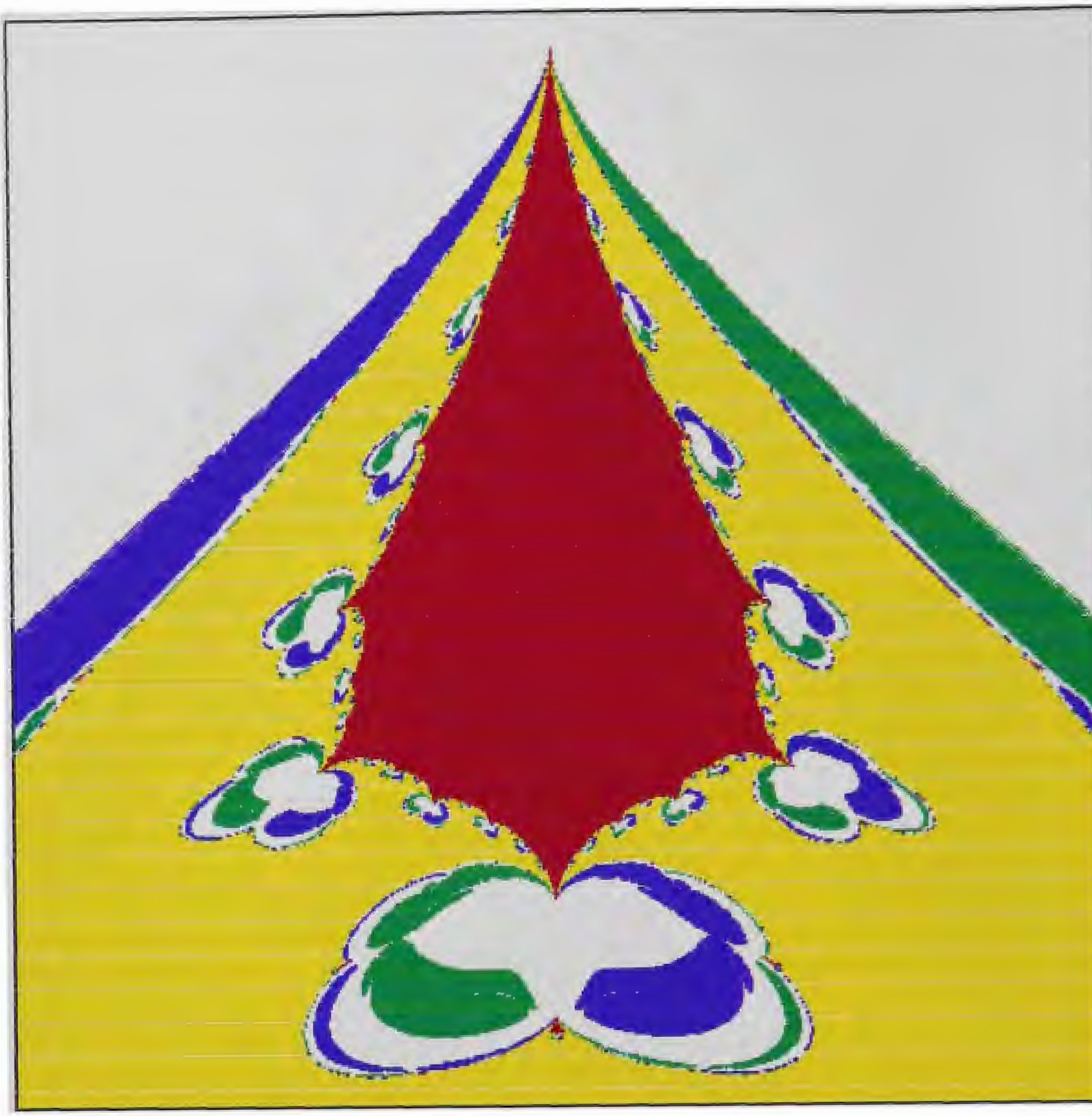
The same fixed points as examples 16 and 17 and showing the even degree characteristics previously described. Also similar to example 17, but more noticeable here, the "necks" of the off axis basins correspond to the fixed points. Starting Bairstow's algorithm with a quadratic arbitrarily close to one of these fixed points, but still on the horizontal axis, will lead to convergence to a different fixed point (the upper one). Similar behavior was noted for example 10 and also can be seen in example 19.

Example 19.

Polynomial :	$x^6 - x^2$		
Factors/ Fixed points/ Colors :	$x^2 - 1$	/	(0, 1) / white
	$x^2 - x$	/	(1, 0) / blue
	$x^2 + x$	/	(-1, 0) / green
	x^2	/	(0, 0) / yellow
	$x^2 + 1$	/	(0, -1) / red
Window :	$- 2.56 < r < +2.56$	,	$- 1.92 < s < +1.92$

Comments :

Progressing from example 18 there are now three fixed points on the vertical axis. The basin for the top one giving the "sky" and the lower one intruding into the middle point's basin with "nodules" attached to it's cusps. The off axis basins retain their general shape with their fixed point "necks". Alternatively, comparison with example 15 shows the changed nature of the image when the additional fixed point is due to the transition from a polynomial of odd degree to even.



Example 20.

Polynomial : $x^6 - x^2$

Factors/
Fixed points/
Colors : Same as example 19.

Window : $-0.072 < r < +0.072$, $+0.438 < s < +0.582$

Comments :

The window for this image is about a 30x magnification of the region at the apex of the yellow basin in example 19. In the cusp of the lowest nodule attached to the above large red basin,and on the vertical axis, a smaller red region can be observed. Magnification of this region would produce an image almost identical to the above, also including a small red region in the analogous location which when enlarged produces a similar image with a small red region ...

Example 21.

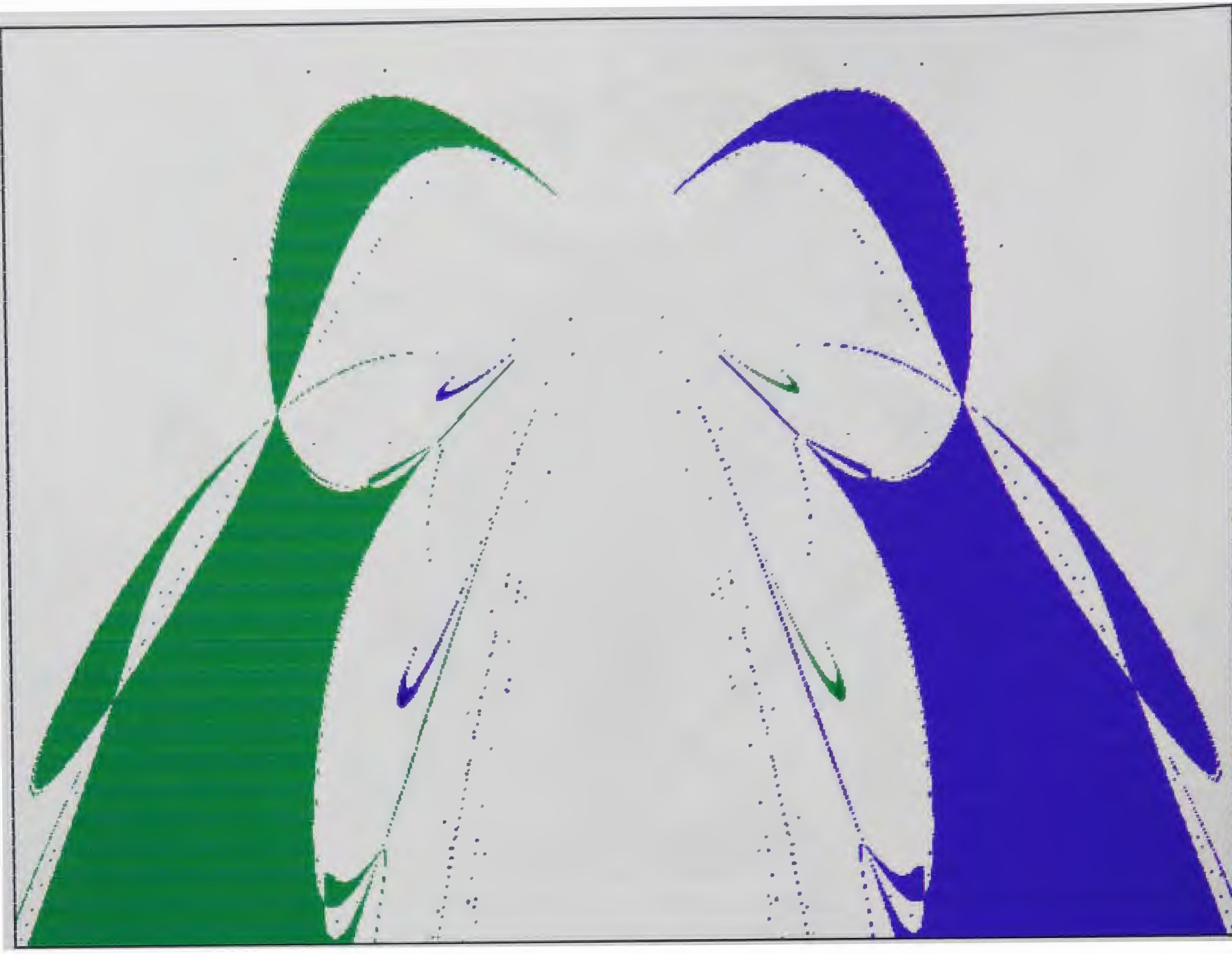
Polynomial : $x^6 - x^2$

Factors/
Fixed points/
Colors : Same as example 19.

Window : $-0.0108 < r < +0.0108$, $-0.4672 < s < -0.4456$

Comments :

Similar to example 20 exept that the region in example 19 that has been magnified by a factor of about 200x is located in the cusp of the nodule attached to the apex of the large red basin. Successive magnifications of the small red region near the top of the above image will again produce a sequence of apparently self -similar images.

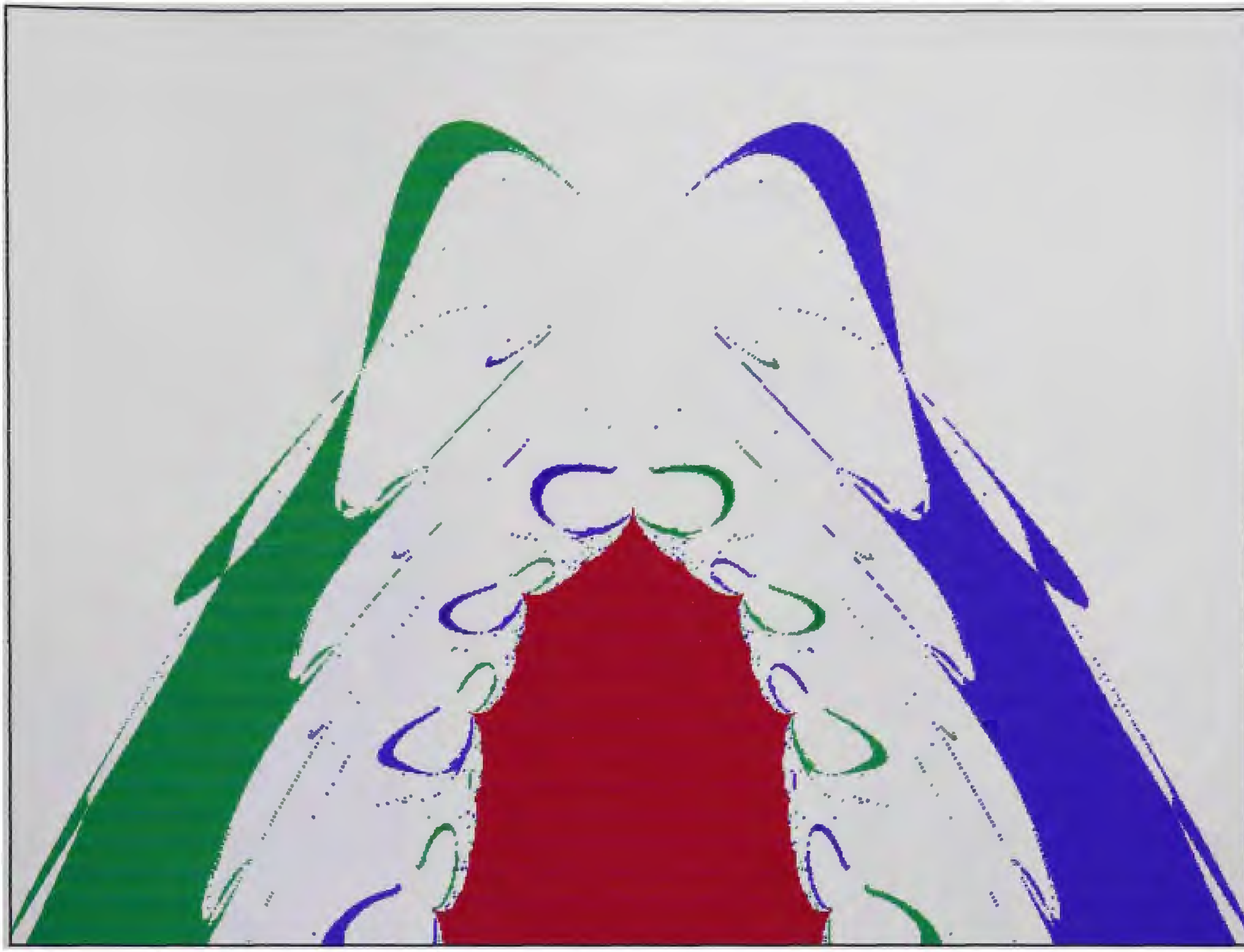


Example 22.

Polynomial :	$x^4 - 2x^2 + 1$		
Factors/	$x^2 - 1$	/	(0, 1) / white
Fixed points/	$x^2 - 2x + 1$	/	(2,-1) / blue
Colors :	$x^2 + 2x + 1$	/	(-2,-1) / green
Window :	$-3.2 < r < +3.2$	,	$-2.6 < s < +2.2$

Comments :

This polynomial can be considered as generating the image that is the transition from the chaotic two basin shapes like example 7 to the six basin shapes like example 35. Magnification of the central region would show narrow bands of blue and green extending towards (0,1). A small number of black points are visible when the pixel coordinates correspond exactly to points on the curves $r^2 = 2[\, s \pm \sqrt{(2s - 1)}]$ where Bairstow's method breaks down because D = 0 on these curves.

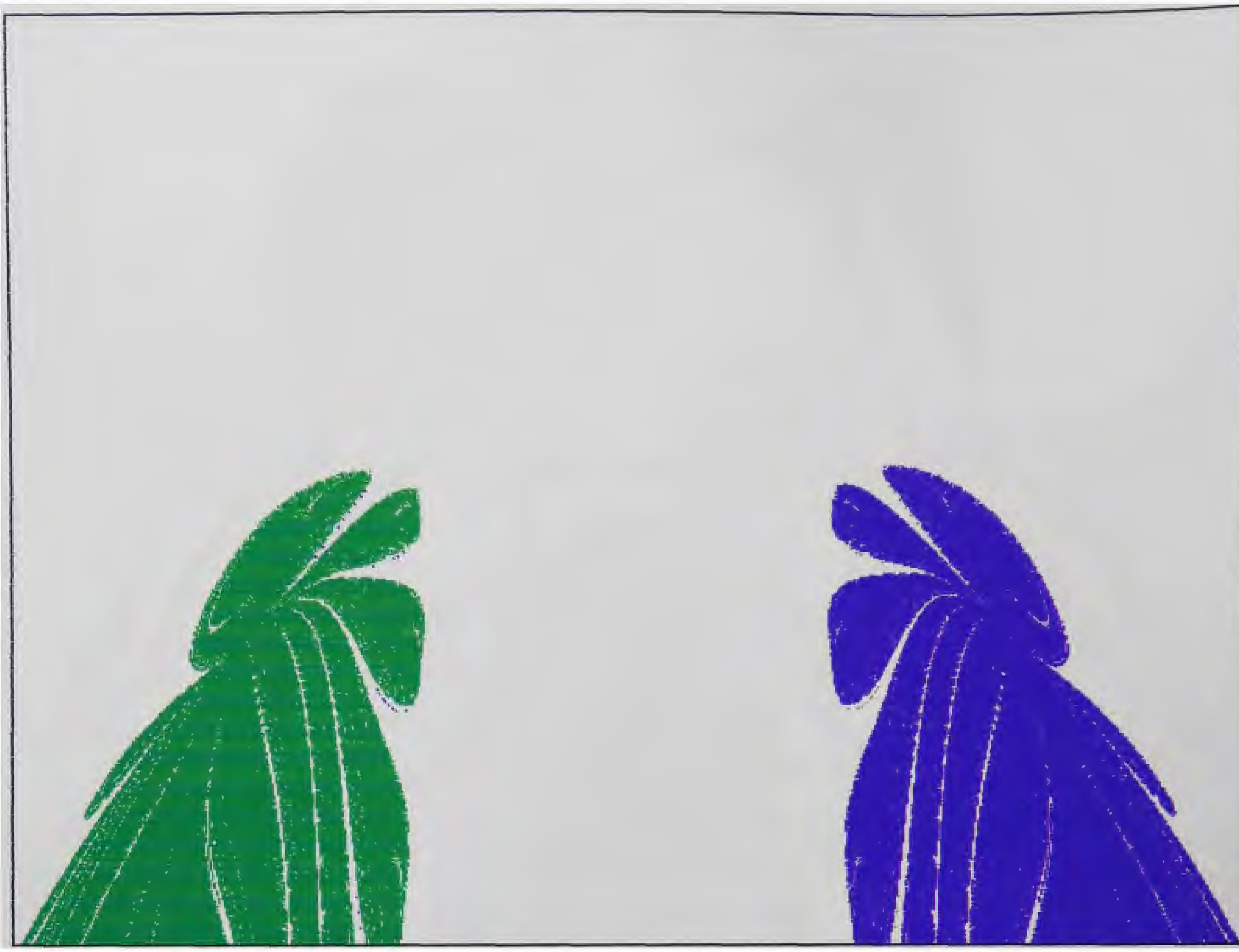


Example 23.

Polynomial :	$x^6 - x^4 - x^2 + 1$		
Factors/ Fixed points/ Colors :	$x^2 - 1$	/	(0, 1) / white
	$x^2 - 2x + 1$	/	(2,-1) / blue
	$x^2 + 2x + 1$	/	(-2,-1) / green
	$x^2 + 1$	/	(0,-1) / red
Window :	$-3.2 < r < +3.2$	,	$-2.6 < s < +2.2$

Comments :

The additional fixed point on the vertical axis on the same horizontal line s = -1 as two other fixed points produces the red basin extending downwards. This image contrasts with example 31 which also has three horizontally aligned fixed points, but no upper axial fixed point.



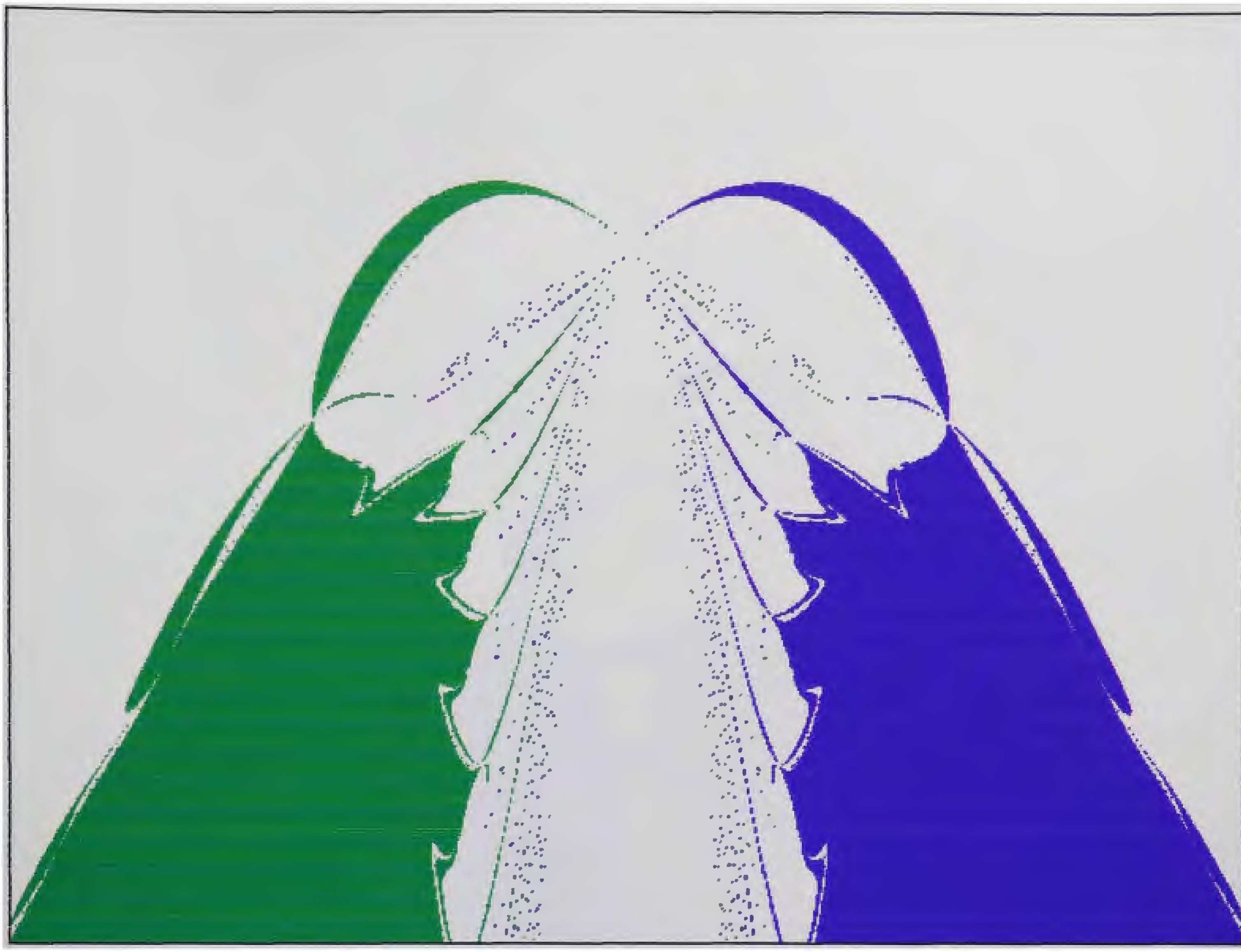
Example 24.

Polynomial :	$x^6 - 3x^4 + 3x^2 - 1$		
Factors/	$x^2 - 1$	/	(0, 1) / white
Fixed points/	$x^2 - 2x + 1$	/	(2,-1) / blue
Colors :	$x^2 +2x + 1$	/	(-2,-1) / green

Window : $-3.2 < r < +3.2$, $-2.6 < s < +2.2$

Comments :

The polynomial being $(x^2 - 1)^3$ has the same quadratic factors as examples 22 and 25. Other examples which show narrow intrusions of other basins into colored basins are 11,16, and 29. The common feature of the polynomials associated with these images is that they all have linear factors of multiplicity three.



Example 25.

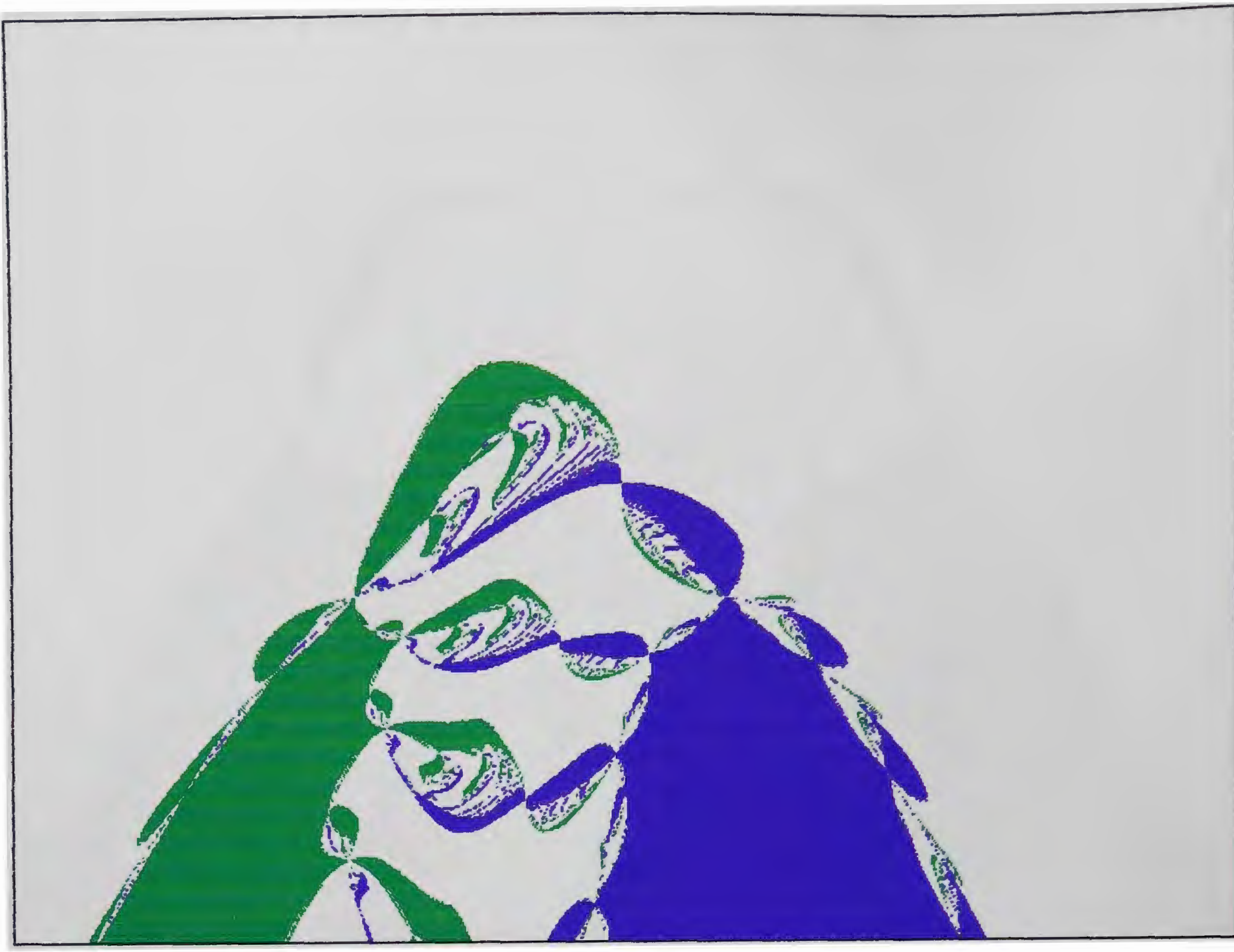
Polynomial : $x^8 - 4x^6 + 6x^2 - 4x^2 + 1$

Factors/	$x^2 - 1$	/	(0, 1) / white
Fixed points/	$x^2 - 2x + 1$	/	(2,-1) / blue
Colors :	$x^2 + 2x + 1$	/	(-2,-1) / green

Window : $-3.2 < r < +3.2$, $-2.6 < s < +2.2$

Comments :

Again the same fixed points as examples 22 and 24 but being the expansion of $(x^2 - 1)^4$, an even power, the image is more like the former. Also, the presence of narrow bands of color associated with the off axis fixed points can be seen extending towards the axial fixed point. This indicates that an initial quadratic in Bairstow's algorithm may be close to the factor $x^2 - 1$, but the sequence of subsequent quadratics converges to one of the other factors.



Example 26.

Polynomial :	$x^6 - x$		
Factors/	$x^2 - x$	/ (1, 0)	/ white
Fixed points/	$x^2 +0.5(1 - \sqrt{5})x + 1$	/ (0.616,-1)	/ blue
Colors :	$x^2 +0.5(1 +\sqrt{5})x + 1$	/ (-1.616,-1)	/ green

Window : $-3.2 < r < +3.2$, $-2.4 < s < +2.4$

Comments :

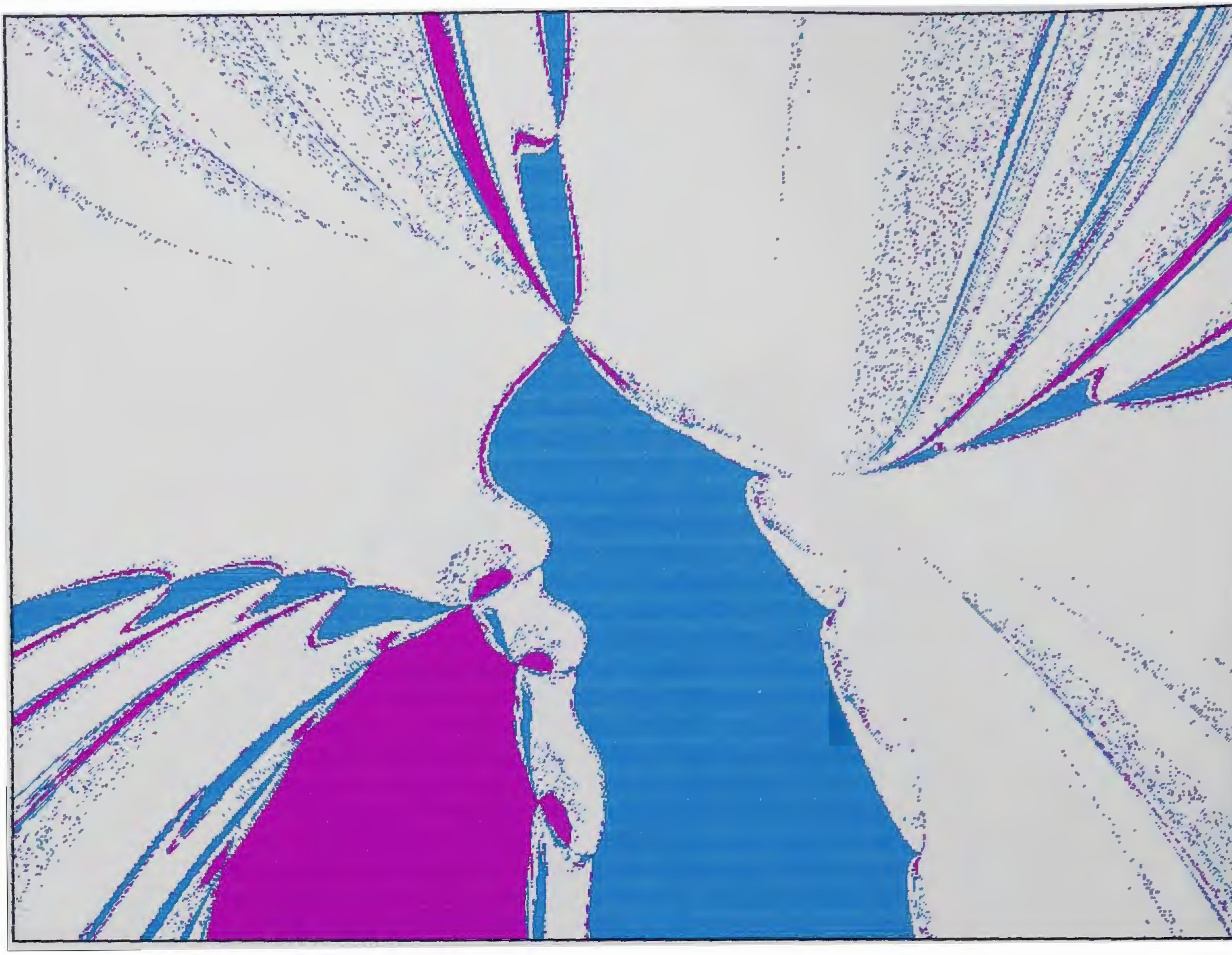
The location of the three fixed points has no obvious symmetries, hence the distortion of the colored basin shapes. Two of the fixed points also occur in examples 3 and 27, the latter having a different third fixed point. A title for this image could be "Battle of the Bugs".

Example 27.

Polynomial :	$x^6 + x^5 - x - 1$
Factors/ Fixed points/ Colors :	$x^2 - 1$ / (0, 1) / white
	$x^2 + 0.5(1 - \sqrt{5})x + 1$ / (0.616,-1) / blue
	$x^2 + 0.5(1 + \sqrt{5})x + 1$ / (-1.616,-1) / green
Window :	$-3.2 < r < +3.2$, $-2.4 < s < +2.4$

Comments :

Has two fixed points in common with example 26, but the third is now located on the vertical axis rather than the horizontal axis. If the previous example is the battle then perhaps this image is "after the battle".

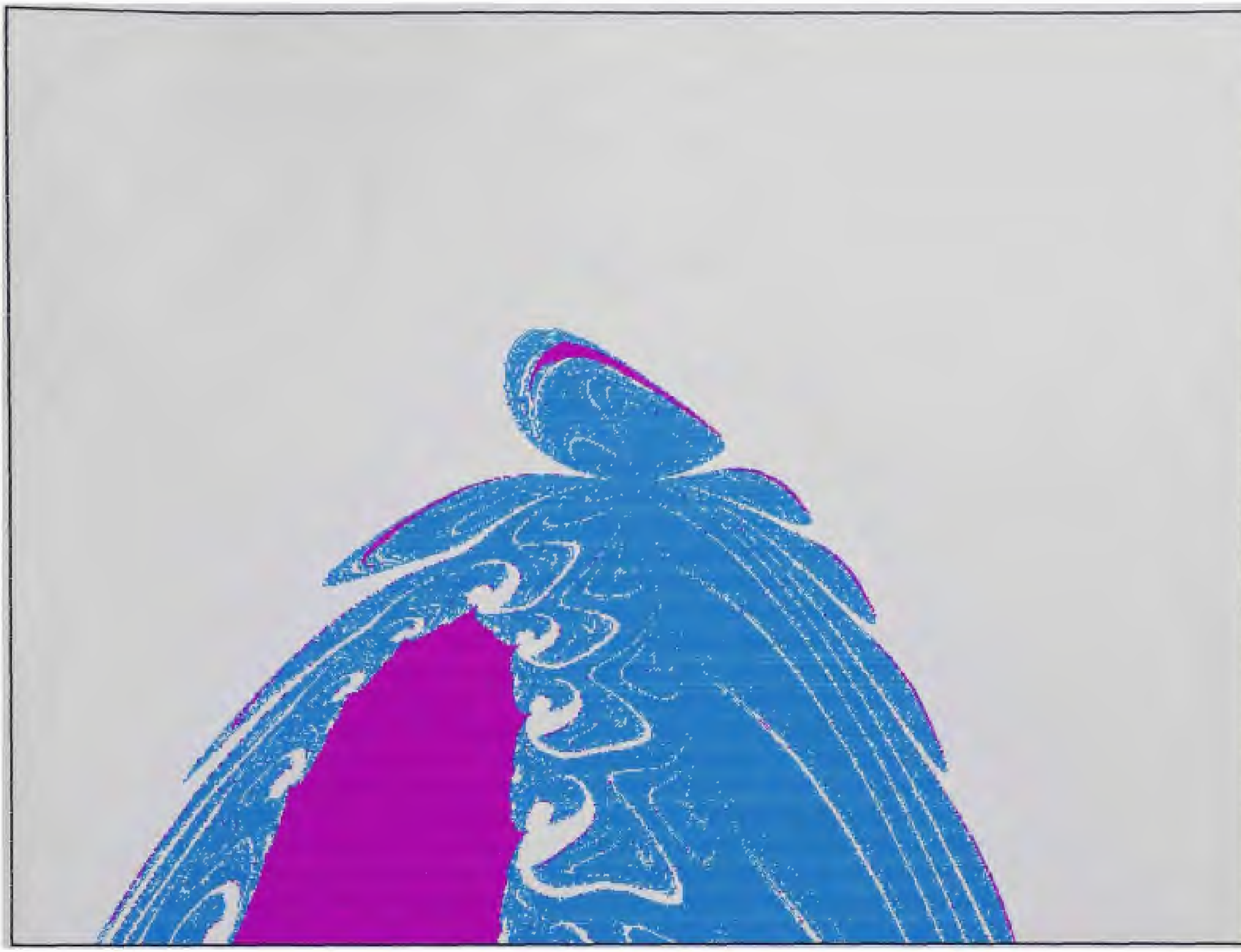


Example 28.

Polynomial :	$x^5 - x^2$		
Factors/	$x^2 - x$	/	(1, 0) / white
Fixed points/	x^2	/	(0, 0) / light blue
Colors :	$x^2 + x + 1$	/	(-1,-1) / purple
Window :	$-3.2 < r < +3.2$	,	$-2.4 < s < +2.4$

Comments :

The same fixed points as example 29, but shows the usual characteristics of basins associated with odd degree polynomials and asymmetric location of these points.

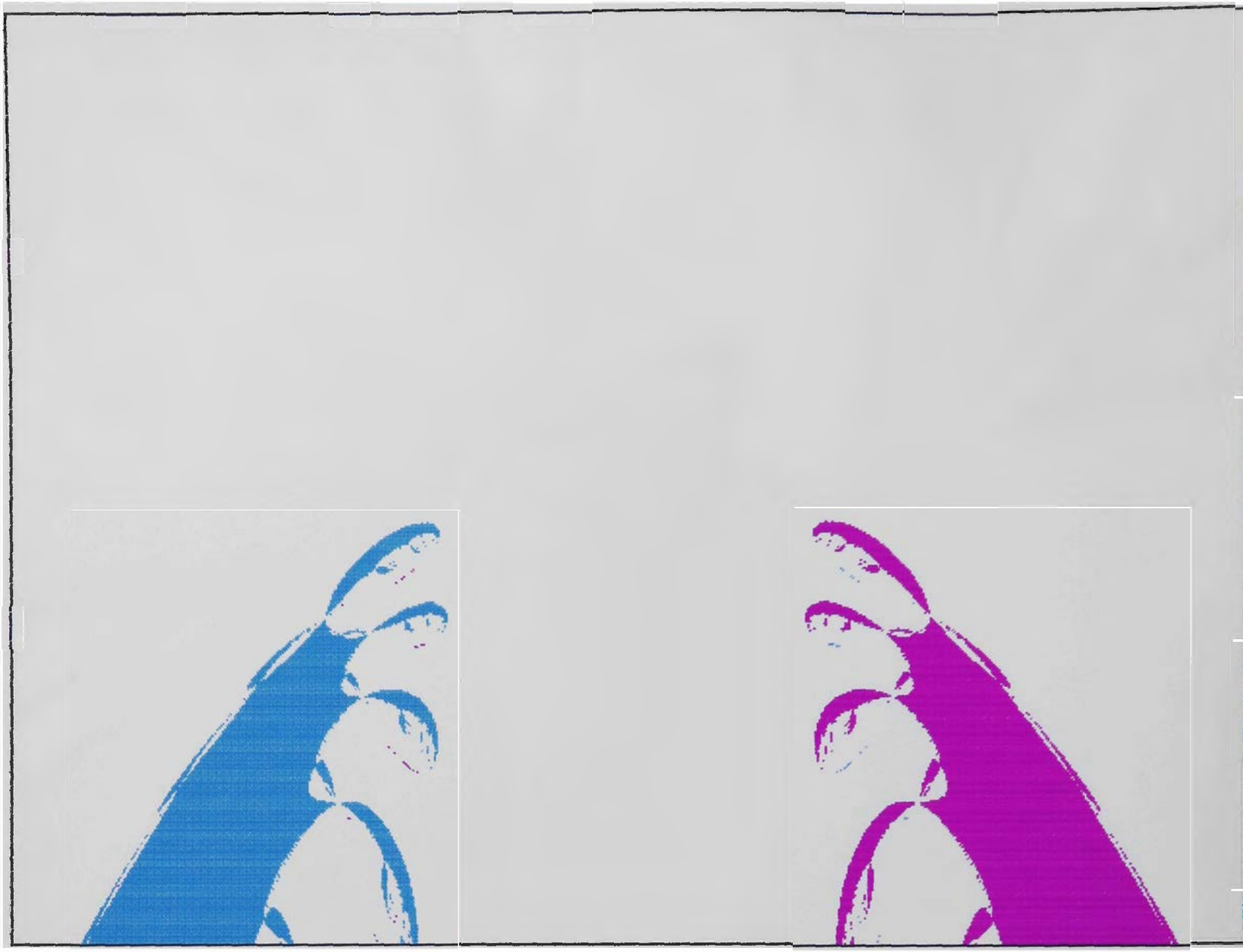


Example 29.

Polynomial :	$x^6 - x^3$	
Factors/	$x^2 - x$ /	(1, 0) / white
Fixed points/	x^2 /	(0, 0) / light blue
Colors :	$x^2 + x + 1$ /	(-1,-1) / purple
Window :	$-3.2 < r < +3.2$,	$-2.4 < s < +2.4$

Comments :

An even degree polynomial with the same quadratic factors as example 28 but now the image has intrusions common to cases where there is a linear factor of multiplicity three.



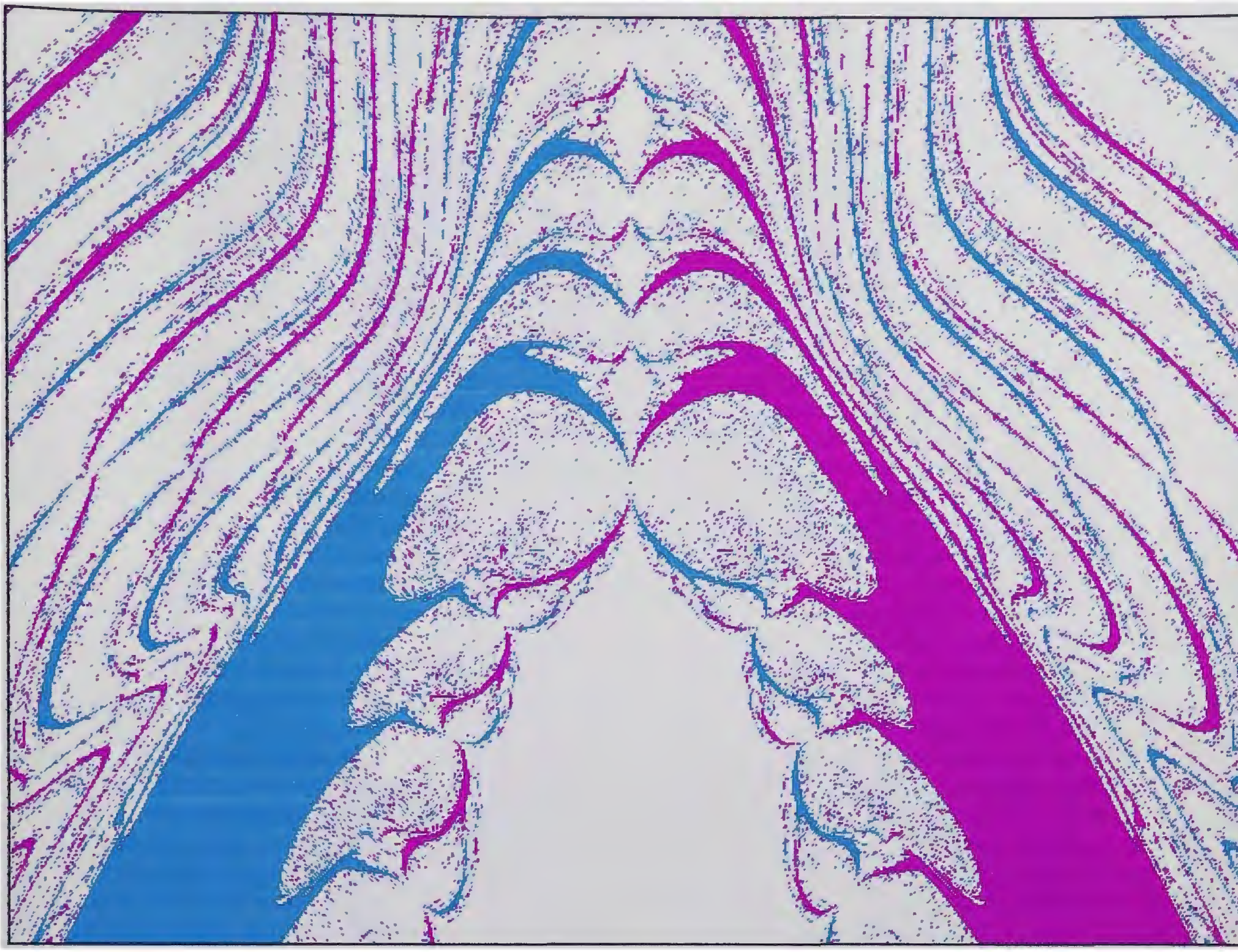
Example 30.

Polynomial :	$x^6 - x^4 + x^2$		
Factors/	x^2	/ (0, 0)	/ white
Fixed points/	$x^2 - \sqrt{3}x + 1$	/ ($\sqrt{3}$,-1)	/ purple
Colors :	$x^2 + \sqrt{3}x + 1$	/ (-$\sqrt{3}$,-1)	/ light blue

Window : $-3.2 < r < +3.2$, $-2.4 < s < +2.4$

Comments :

Examples 7 and 31 have the same two lower fixed points on the line $s = -1$, but the presence of the higher fixed point "stabilises" their basins.



Example 31.

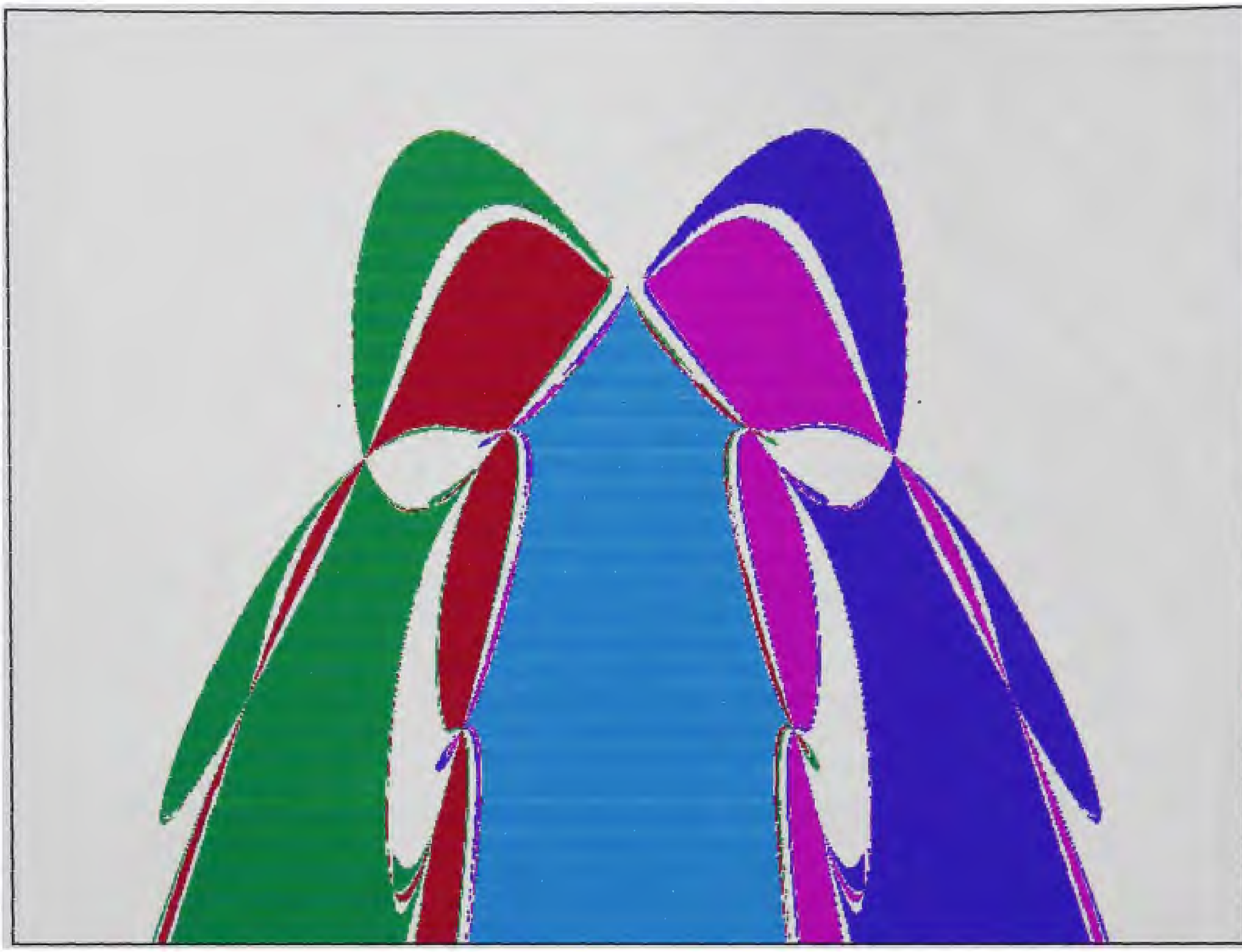
Polynomial : $x^6 + 1$

Factors/	$x^2 + 1$	/ (0, -1)	/ white
Fixed points/	$x^2 - \sqrt{3}x + 1$	/ ($\sqrt{3}$,-1)	/ purple
Colors :	$x^2 + \sqrt{3}x + 1$	/ (-$\sqrt{3}$,-1)	/ light blue

Window : $-3.2 < r < +3.2$, $-2.4 < s < +2.4$

Comments :

All three fixed points are on the same horizontal line and the "sky" effect of the basin for the axial point is not present. A similar image is obtained in example 7 (the above polynomial with the $x^2 + 1$ factor divided out) except the above white basin points are replaced by a chaotic random looking arrangement of the remaining two basins.



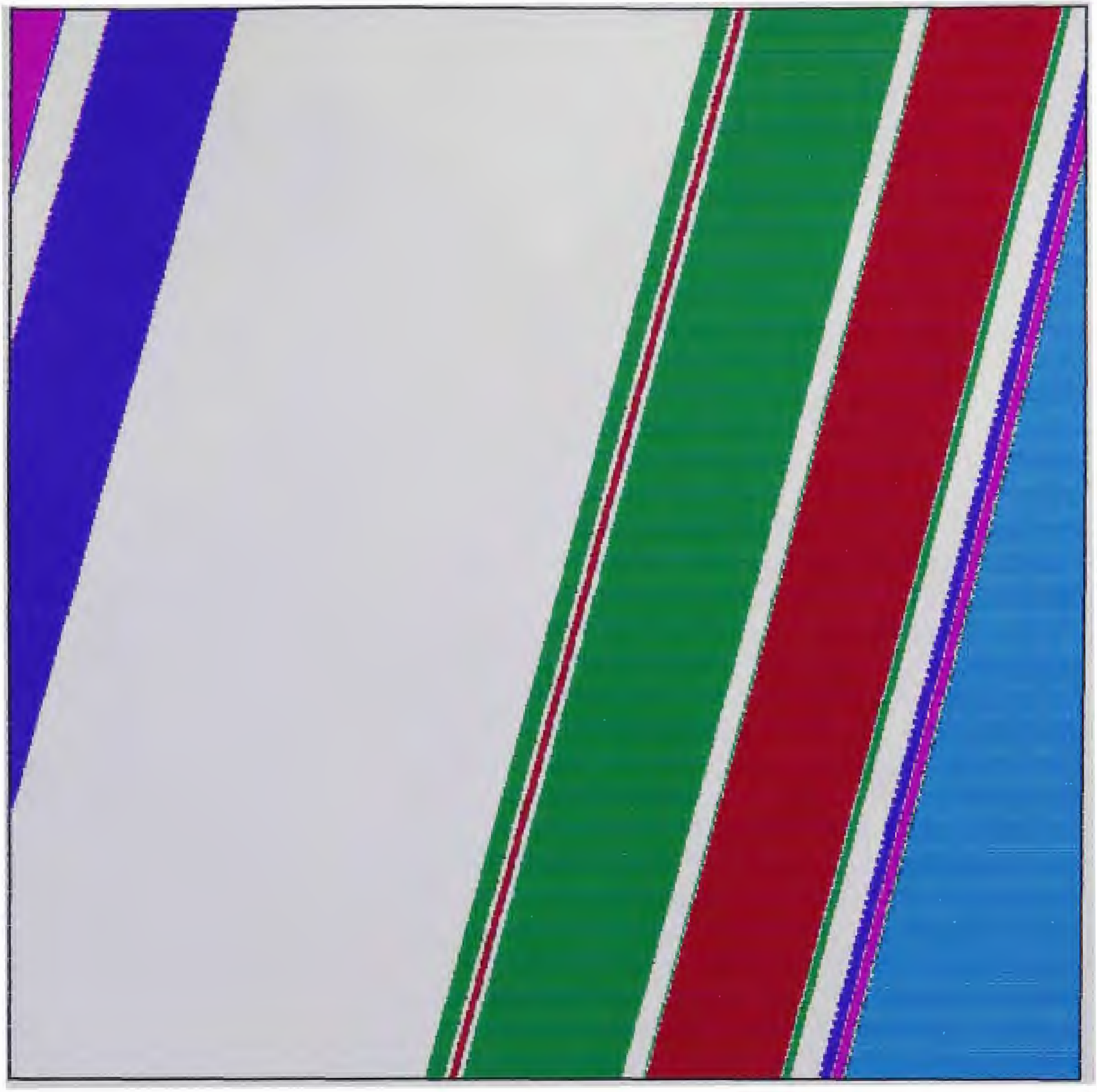
Example 32.

Polynomial :	$x^4 - 5x^2 + 4$		
Factors/	$x^2 - 4$	/	(0, 4) / white
Fixed points/	$x^2 - 1$	/	(0, 1) / light blue
Colors :	$x^2 - x - 2$	/	(1, 2) / purple
	$x^2 + x - 2$	/	(-1, 2) / red
	$x^2 - 3x + 2$	/	(3,-2) / green
	$x^2 + 3x + 2$	/	(-3,-2) / blue

Window : $-6.4 < r < +6.4$, $-6.0 < s < +6.0$

Comments :

Two axial and two pairs of off axial fixed points give similarities with previous examples and example 34 which also has six fixed points.



Example 33.

Polynomial : $x^4 - 5x^2 + 4$

Factors/
Fixed points/
Colors : Same as example 32.

Window : $-0.00790 < r < -0.00726$, $+2.4394 < s < +2.4400$

Comments :

A magnification factor of 20,000x has been used on a region adjacent to the left edge of the light blue basin in example 32. This image illustrates a property that can also be observed in some of the previous examples under magnification - that the boundaries of some basins have a large number of narrow bands of color in close proximity.

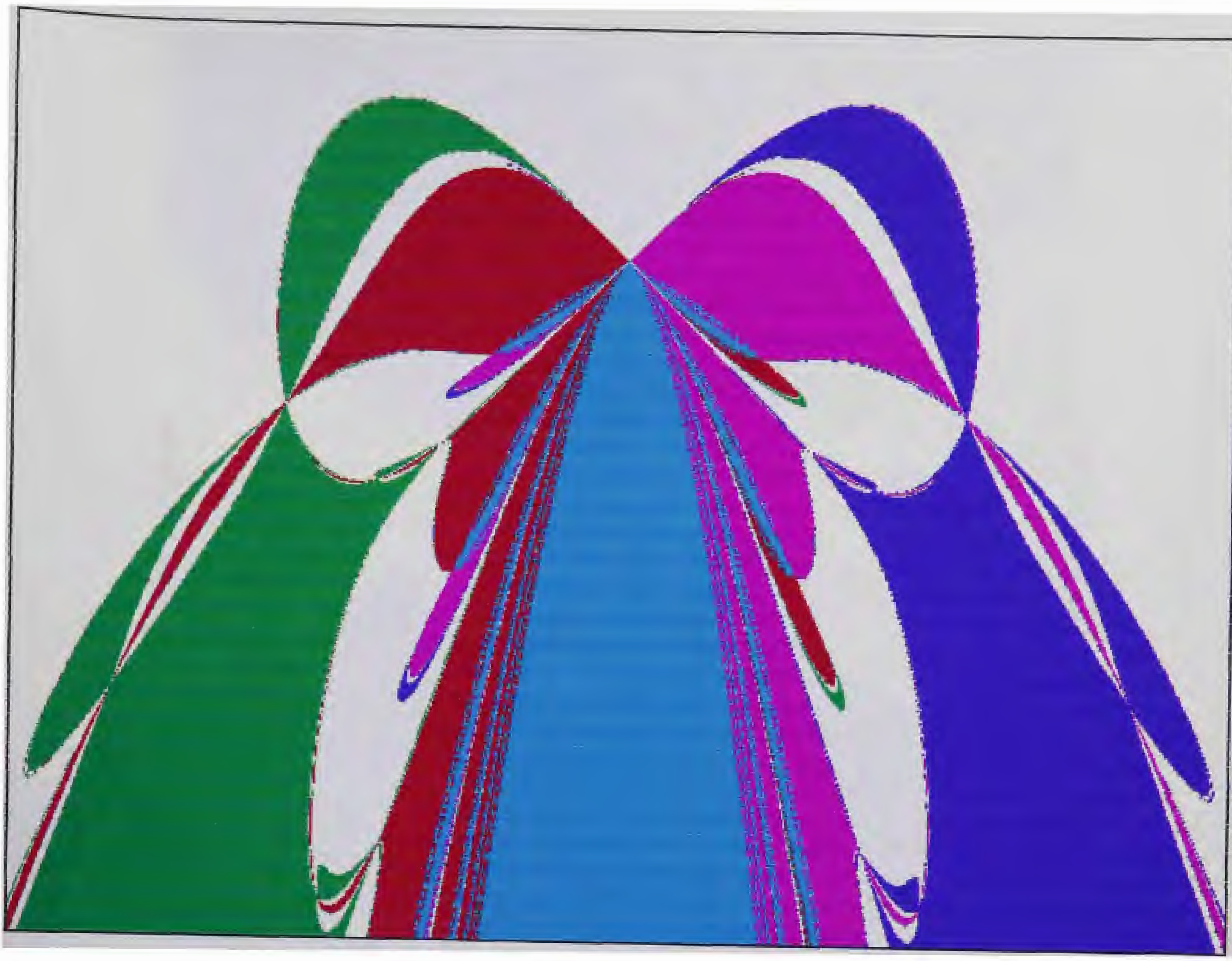
Example 34.

Polynomial :	$x^6 - 2x^4 + x^2$		
Factors/ Fixed points/ Colors :	$x^2 - 1$	/	(0, 1) / white
	x^2	/	(0, 0) / light blue
	$x^2 - x$	/	(1, 0) / purple
	$x^2 + x$	/	(-1, 0) / red
	$x^2 - 2x + 1$	/	(2,-1) / green
	$x^2 + 2x + 1$	/	(-2,-1) / blue

Window : $-2.56 < r < +2.56$, $-1.92 < s < +1.92$

Comments :

. The red and purple basins at their fixed points show two directions in which arbitrarily close points belong to the basin of the upper axial fixed point, one direction being horizontal as in examples 18 and 19, the other along the line of fixed points.



Example 35.

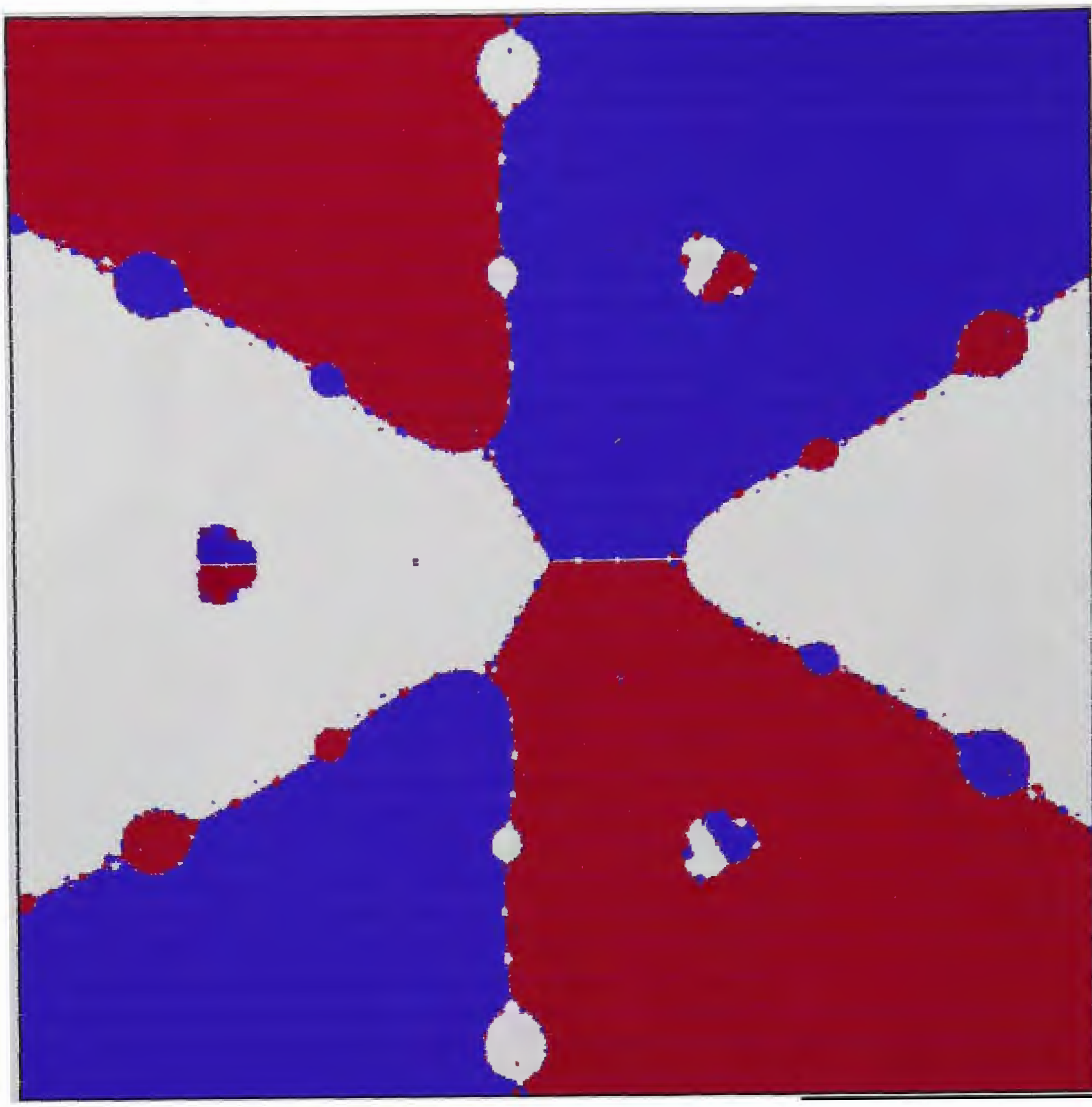
Polynomial : $x^4 - (2 + 4\varepsilon + \varepsilon^2)x^2 + 1$

Factors/ Fixed points/ Colors :			
	$x^2 - (2 + \varepsilon)x + 1$	$(2 + \varepsilon, -1)$	blue
	$x^2 + (2 + \varepsilon)x + 1$	$(-2 - \varepsilon, -1)$	green
	$x^2 - (1 + 2\sqrt{\varepsilon})$*	$(0, 1 + 2\sqrt{\varepsilon})$	white
	$x^2 - (1 - 2\sqrt{\varepsilon})$*	$(0, 1 - 2\sqrt{\varepsilon})$	light blue
	$x^2 + 2\sqrt{\varepsilon}x - 1$*	$(-2\sqrt{\varepsilon}, 1)$	red
	$x^2 - 2\sqrt{\varepsilon}x - 1$*	$(+2\sqrt{\varepsilon}, 1)$	purple

Window : $-3.2 < r < +3.2$, $-2.6 < s < +2.2$

Comments :

The four factors marked by an asterisk are approximations for small ε. The corresponding fixed points are arbitrarily close to (0,1). The above image has been generated using $\varepsilon = 10^{-8}$. Comparing with example 22, the green and blue basins are not visibly different, whereas the red, purple and light blue basins slot in between the narrow green and blue bands referred to in that example. Larger values of ε give images similar to example 32.



Example 36.

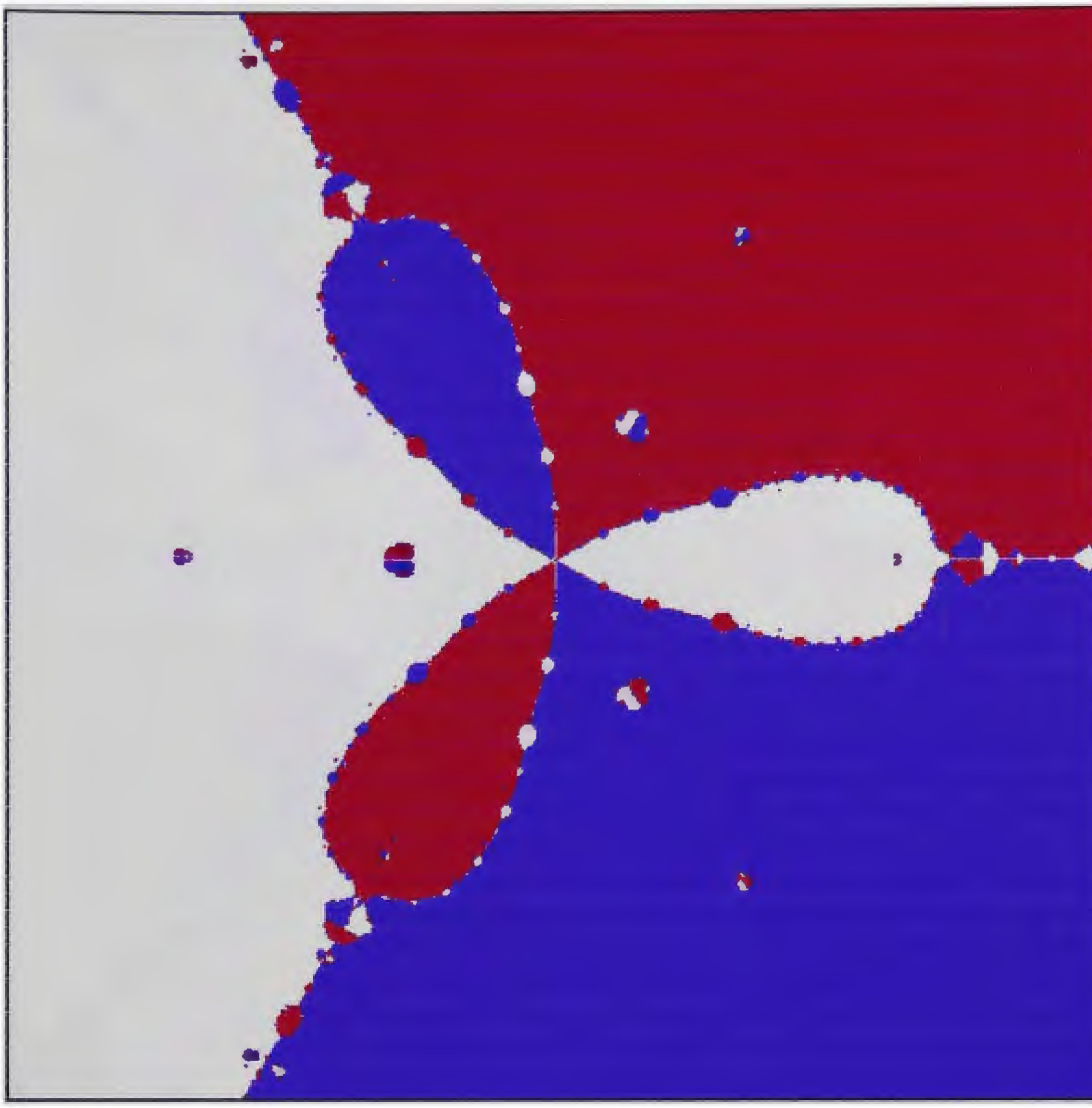
Polynomial : $x^3 - 1$

Factors/ Colors :	$x^2 + x + 1$	/ white
	$x^2 - 0.5(1 + i\sqrt{3})x - 0.5(1 - i\sqrt{3})$	/ blue
	$x^2 - 0.5(1 - i\sqrt{3})x - 0.5(1 + i\sqrt{3})$	/ red

Window : $s = 0$, $-2.4 < Re(r) < +2.4$, $-2.4 < Im(r) < +2.4$

Comments :

In the field of real numbers there is only one quadratic factor of $x^3 - 1$, but allowing for complex coefficients there are three factors. Starting with an initial quadratic $x^2 - rx$, the basins in the complex r-plane show a different character to all the previous examples.



Example 37.

Polynomial : $x^3 - 1$

Factors/ Colors : Same as example 36.

Window : $r = 0$, $-1.2 < Re(s) < +1.2$, $-1.2 < Im(s) < +1.2$

Comments :

The starting quadratic in this case is $x^2 - s$, where values of s are taken from the window region in the s-plane.

Example 38.

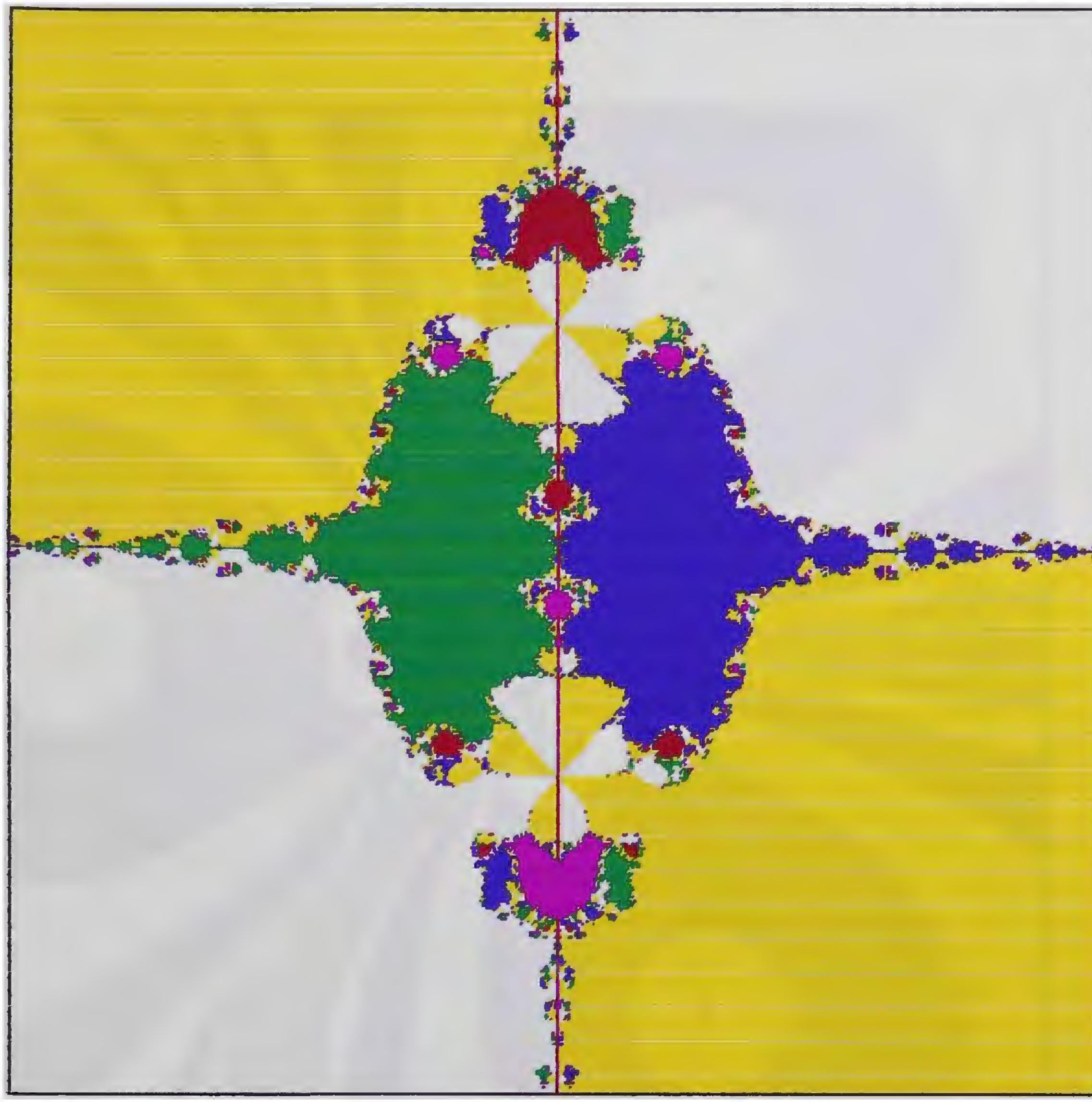
Polynomial : $x^4 - 1$

Factors/ Colors :	$x^2 - 1$	/ white
	$x^2 + 1$	/ yellow
	$x^2 - (1 + i)x + i$	/ blue
	$x^2 + (1 + i)x + i$	/ green
	$x^2 - (1 - i)x - i$	/ red
	$x^2 + (1 - i)x - i$	/ purple

Window : $s = 0$, $-1.92 < Re(r) < +1.92$, $-1.92 < Im(r) < +1.92$

Comments :

Example 1 in the real r-s domain has only the two quadratic factors but in the field of complex numbers there are the above six. Bairstow's algorithm starting with $x^2 - rx$ produces the above image in the complex r-plane.



Example 39.

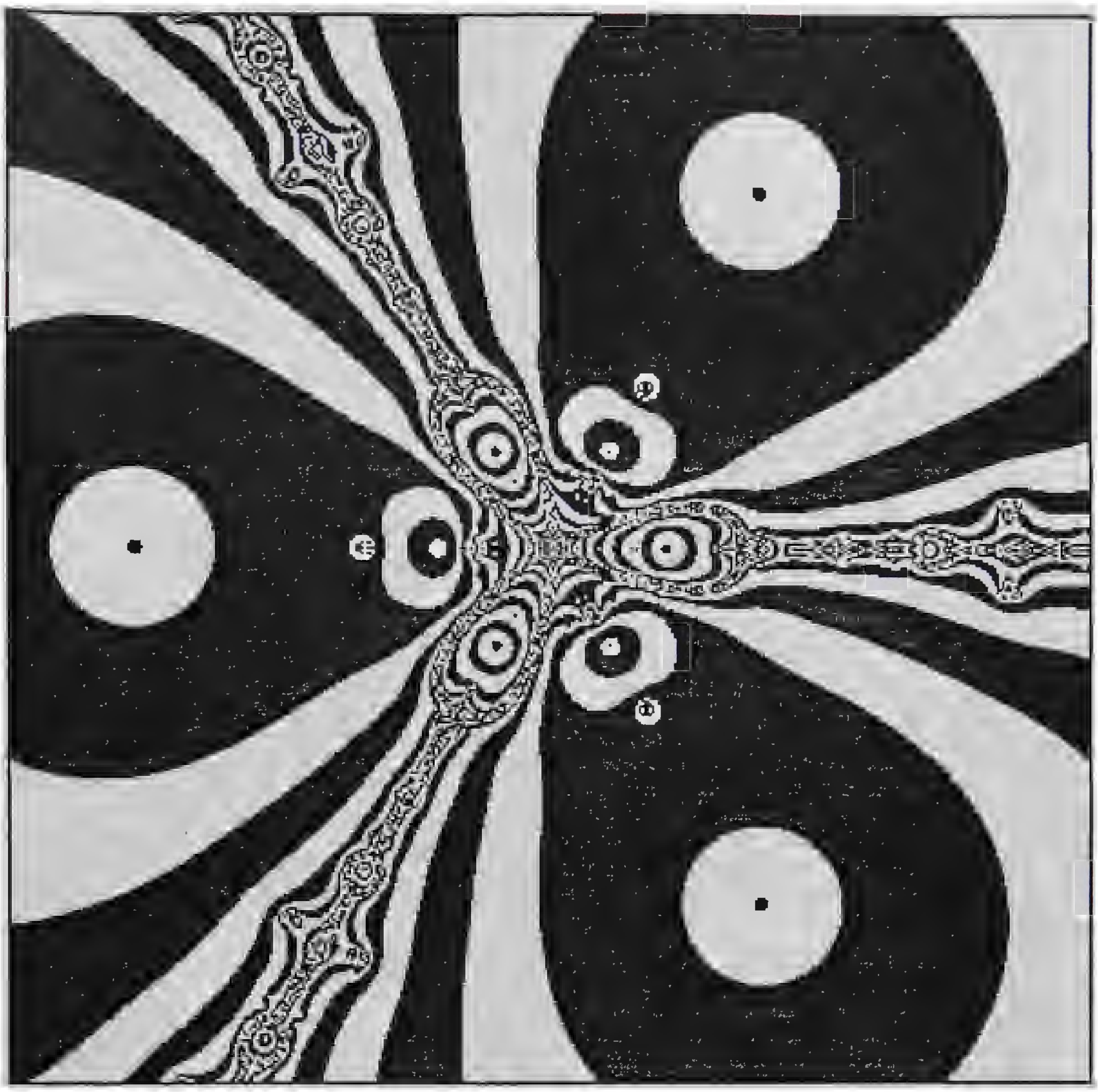
Polynomial : $x^4 + x^2 + 1$

Factors/ Colors :	$x^2 - x + 1$	/ blue
	$x^2 + x + 1$	/ green
	$x^2 + 0.5(1 + i\sqrt{3})$	/ yellow
	$x^2 + 0.5(1 - i\sqrt{3})$	/ white
	$x^2 - i\sqrt{3}x - 1$	/ red
	$x^2 + i\sqrt{3}x - 1$	/ purple

Window : $s = 0$, $-1.92 < Re(r) < +1.92$, $-1.92 < Im(r) < +1.92$

Comments :

The same starting quadratic and complex r-plane domain as example 38. Example 5 shows the real r-s plane basins.



Example 40.

Polynomial : $x^3 - 1$

Factors : Same as example 36.

Window : r = 0 , - 2.4 < Re(s) < +2.4 , - 2.4 < Im(s) < +2.4

Comments :

Variation of the number of iterations required for convergence to any factor.

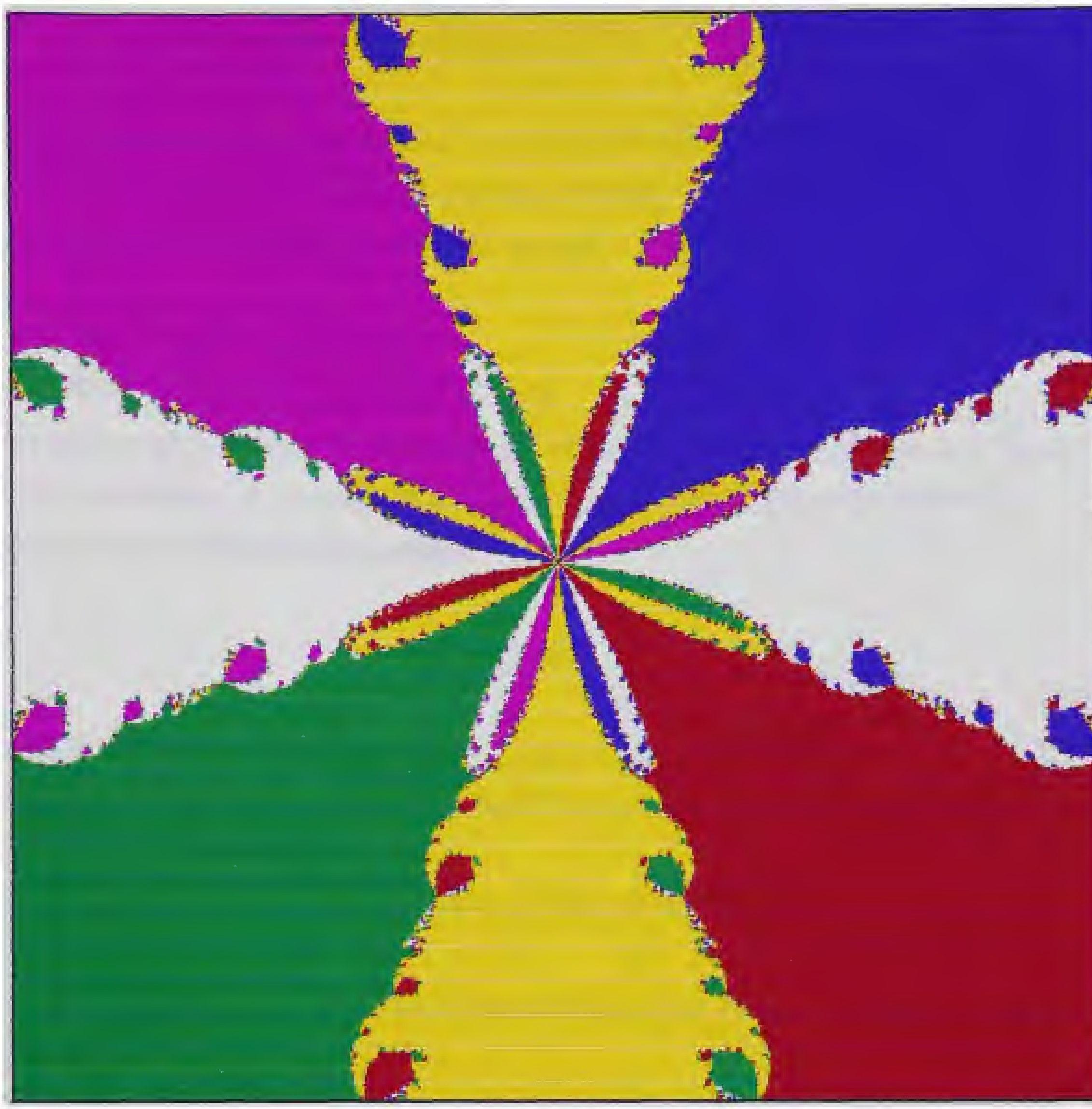
Example 41.

Polynomial : $x^3 - 1$

Factors : Same as example 36.

Window : s = 0 , - 2.4 < Re(r) < +2.4 , - 2.4 < Im(r) < +2.4

Comments :
Variation of the number of iterations required for convergence to any factor.



Example 42.

Polynomial : $x^4 - 1$

Factors/ Colors : Same as example 38

Window : $s = 0$, $-1.92 < Re(r) < +1.92$, $-1.92 < Im(r) < +1.92$

Comments :

An illustration of the different basins when the alternative form of the remainder term is used in Bairstow's method.

7. Concluding example

If, in the division of p(x) by $x^2 - rx - s$, the remainder is taken as $u'x + v'$ instead of $u(x - r) + v$ then, as shown in Fröberg (1972), Bairstow's iterative algorithm still follows the steps described in section 5 but the equations to be solved in (iii) are

$$(c_1 - b_1)\Delta r + c_2\Delta s = - b_0 \quad , \quad c_2\Delta r + c_3\Delta s = - b_1$$

and the algorithm fails when $(c_1 - b_1)c_3 - c_2^2 = 0$.

The basins of attraction for this version of the algorithm are different to those presented in examples 1-41, however example 42 is the only picture included for this alternative form of Bairstow's method.

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