On New Refinements and Reverses of Young's Operator Inequality

ON NEW REFINEMENTS AND REVERSES OF YOUNG'S OPERATOR INEQUALITY

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Abstract. In this paper we obtain some new refinements and reverses of Young's operator inequality. Extensions for convex functions of operators are also provided.

1. Introduction

Throughout this paper A and B are positive operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. We use the following notations for operators

$$A \nabla_\nu B := (1-\nu) A + \nu B, \text{ the } \textit{weighted arithmetic mean}$$

and

$$A \sharp_\nu B := A^{1/2} \left(A^{-1/2} B A^{-1/2} \right)^\nu A^{1/2}, \text{ the } \textit{weighted geometric mean.}$$

When $\nu = \frac{1}{2}$ we write $A \nabla B$ and $A \sharp B$ for brevity, respectively.

The famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu} b^\nu \leq (1-\nu) a + \nu b \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.1) is also called *ν-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [9]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e \ln \left(h^{\frac{1}{h-1}} \right)} \text{ if } h \in (0,1) \cup (1, \infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.2}$$

It is well known that $\lim_{h \to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1, \infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b} \right)^r \right) a^{1-\nu} b^\nu \leq (1-\nu) a + \nu b \leq S\left(\frac{a}{b} \right) a^{1-\nu} b^\nu, \tag{1.3}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$.

The second inequality in (1.3) is due to Tominaga [10] while the first one is due to Furuichi [2].

The operator version is as follows [2], [10] :

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Theorem 1. *For two positive operators A, B and positive real numbers m, m', M, M' satisfying the following conditions (i) or (ii):*
(i) $0 < m'I \leq A \leq mI < MI \leq B \leq M'I$;
(ii) $0 < m'I \leq B \leq mI < MI \leq A \leq M'I$;
we have

$$S\left(h^{r}\right) A\sharp_{\nu}B \leq A\nabla_{\nu}B \leq S\left(h\right) A\sharp_{\nu}B \tag{1.4}$$

where $h := \frac{M}{m}$, $h' := \frac{M'}{m'}$ *and* $\nu \in [0,1]$.

We consider the *Kantorovich's constant* defined by

$$K\left(h\right) := \frac{\left(h+1\right)^{2}}{4h},\ \ h > 0. \tag{1.5}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$K^{r}\left(\frac{a}{b}\right) a^{1-\nu}b^{\nu} \leq \left(1-\nu\right) a + \nu b \leq K^{R}\left(\frac{a}{b}\right) a^{1-\nu}b^{\nu} \tag{1.6}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu,\nu\}$ and $R = \max\{1-\nu,\nu\}$.

The first inequality in (1.6) was obtained by Zou et al. in [11] while the second by Liao et al. [8].

The operator version is as follows [11], [8]:

Theorem 2. *For two positive operators A, B and positive real numbers m, m', M, M' satisfying the following conditions (i) or (ii):*
(i) $0 < m'I \leq A \leq mI < MI \leq B \leq M'I$;
(ii) $0 < m'I \leq B \leq mI < MI \leq A \leq M'I$;
we have

$$K^{r}\left(h\right) A\sharp_{\nu}B \leq A\nabla_{\nu}B \leq K^{R}\left(h\right) A\sharp_{\nu}B \tag{1.7}$$

where $h := \frac{M}{m}$, $h' := \frac{M'}{m'}$, $\nu \in [0,1]$ $r = \min\{1-\nu,\nu\}$ *and* $R = \max\{1-\nu,\nu\}$.

Kittaneh and Manasrah [5], [6] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a}-\sqrt{b}\right)^{2} \leq \left(1-\nu\right) a + \nu b - a^{1-\nu}b^{\nu} \leq R\left(\sqrt{a}-\sqrt{b}\right)^{2} \tag{1.8}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu,\nu\}$ and $R = \max\{1-\nu,\nu\}$. The case $\nu = \frac{1}{2}$ reduces (1.8) to an identity.

For some operator versions of (1.8) see [5] and [6].

In the recent paper [1] we obtained the following reverses of Young's inequality as well:

$$0 \leq \left(1-\nu\right) a + \nu b - a^{1-\nu}b^{\nu} \leq \nu\left(1-\nu\right)\left(a-b\right)\left(\ln a - \ln b\right) \tag{1.9}$$

and

$$1 \leq \frac{\left(1-\nu\right) a + \nu b}{a^{1-\nu}b^{\nu}} \leq \exp\left[4\nu\left(1-\nu\right)\left(K\left(\frac{a}{b}\right)-1\right)\right], \tag{1.10}$$

where a, $b > 0$, $\nu \in [0,1]$.

It has been shown in [1] that there is no ordering for the upper bounds of the quantity $\left(1-\nu\right) a + \nu b - a^{1-\nu}b^{\nu}$ as provided by the inequalities (1.8) and (1.9). The same conclusion is true for the upper bounds of the quantity $\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}$ incorporated in the inequalities (1.3), (1.6) and (1.10).

By the use of two new refinements and reverses of Young's inequality we establish in this paper several other operators inequalities that are similar to those from above. Extensions for convex functions of operators with some examples are also provided.

2. Some Preliminary Results

We have the following result:

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interval $\mathring{I}$, the interior of I. If there exists the constants d, D such that*

$$d \le f''(t) \le D \text{ for any } t \in \mathring{I}, \tag{2.1}$$

then

$$\frac{1}{2}\nu(1-\nu)d(b-a)^2 \le (1-\nu)f(a) + \nu f(b) - f((1-\nu)a+\nu b) \le \frac{1}{2}\nu(1-\nu)D(b-a)^2 \tag{2.2}$$

for any a, $b \in \mathring{I}$ and $\nu \in [0,1]$.

In particular, we have

$$\frac{1}{8}(b-a)^2 d \le \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) \le \frac{1}{8}(b-a)^2 D, \tag{2.3}$$

for any a, $b \in \mathring{I}$.

The constant $\frac{1}{8}$ is best possible in both inequalities in (2.3).

Proof. We consider the auxiliary function $f_D : I \subset \mathbb{R} \to \mathbb{R}$ defined by $f_D(x) = \frac{1}{2}Dx^2 - f(x)$. The function f_D is differentiable on $\mathring{I}$ and $f_D''(x) = D - f''(x) \ge 0$, showing that f_D is a convex function on $\mathring{I}$.

By the convexity of f_D we have for any a, $b \in \mathring{I}$ and $\nu \in [0,1]$ that

$$\begin{aligned}
0 &\le (1-\nu)f_D(a) + \nu f_D(b) - f_D((1-\nu)a+\nu b) \\
&= (1-\nu)\left(\frac{1}{2}Da^2 - f(a)\right) + \nu\left(\frac{1}{2}Db^2 - f(b)\right) \\
&- \left(\frac{1}{2}D((1-\nu)a+\nu b)^2 - f_D((1-\nu)a+\nu b)\right) \\
&= \frac{1}{2}D\left[(1-\nu)a^2 + \nu b^2 - ((1-\nu)a+\nu b)^2\right] \\
&- (1-\nu)f(a) - \nu f(b) + f_D((1-\nu)a+\nu b) \\
&= \frac{1}{2}\nu(1-\nu)D(b-a)^2 - (1-\nu)f(a) - \nu f(b) + f_D((1-\nu)a+\nu b),
\end{aligned}$$

which implies the second inequality in (2.20).

The first inequality follows in a similar way by considering the auxiliary function $f_d : I \subset \mathbb{R} \to \mathbb{R}$ defined by $f_D(x) = f(x) - \frac{1}{2}dx^2$ that is twice differentiable and convex on $\mathring{I}$.

If we take $f(x) = x^2$, then (2.1) holds with equality for $d = D = 2$ and (2.3) reduces to an equality as well. □

If $D>0$, the second inequality in (2.2) is better than the corresponding inequality obtained by Furuichi and Minculete in [4] by applying Lagrange's theorem two times. They had instead of $\frac{1}{2}$ the constant 1. Our method also allowed to obtain, for $d>0$, a lower bound that can not be established by Lagrange's theorem method employed in [4].

We have:

Theorem 3. *For any $a,\ b>0$ and $\nu\in[0,1]$ we have*

$$\frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^2\min\{a,b\}\le(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\le\frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^2\max\{a,b\}\tag{2.4}$$

and

$$\exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max^2\{a,b\}}\right]\le\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}\le\exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min^2\{a,b\}}\right].\tag{2.5}$$

Proof. If write the inequality (2.2) for the convex function $f:\mathbb{R}\to(0,\infty)$, $f(x)=\exp(x)$, then we have

$$\begin{aligned}&\frac{1}{2}\nu(1-\nu)(x-y)^2\min\{\exp x,\exp y\}\\&\le(1-\nu)\exp(x)+\nu\exp(y)-\exp((1-\nu)x+\nu y)\\&\le\frac{1}{2}\nu(1-\nu)(x-y)^2\max\{\exp x,\exp y\}\end{aligned}\tag{2.6}$$

for any $x,\ y\in\mathbb{R}$ and $\nu\in[0,1]$.

Let $a,\ b>0$. If we take $x=\ln a$, $y=\ln b$ in (2.6), then we get the desired inequality (2.4).

Now, if we write the inequality (2.2) for the convex function $f:(0,\infty)\to\mathbb{R}$, $f(x)=-\ln x$, then we get for any $a,\ b>0$ and $\nu\in[0,1]$ that

$$\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max^2\{a,b\}}\le\ln((1-\nu)a+\nu b)-(1-\nu)\ln a-\nu\ln b\le\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min^2\{a,b\}}.\tag{2.7}$$

□

The second inequalities in (2.4) and (2.5) are better than the corresponding results obtained by Furuichi and Minculete in [4] where instead of constant $\frac{1}{2}$ they had the constant 1.

Now, since

$$\frac{(b-a)^2}{\min^2\{a,b\}}=\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\text{ and }\frac{(b-a)^2}{\max^2\{a,b\}}=\left(\frac{\min\{a,b\}}{\max\{a,b\}}-1\right)^2,$$

then (2.5) can also be written as:

$$\begin{aligned} &\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right] \\ &\le \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}} \\ &\le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\right] \end{aligned} \tag{2.8}$$

for any $a,\ b>0$ and $\nu\in[0,1]$.

Remark 1. *For $\nu=\frac{1}{2}$ we get the following inequalities of interest*

$$\frac{1}{8}(\ln a-\ln b)^2\min\{a,b\}\le\frac{a+b}{2}-\sqrt{ab}\le\frac{1}{8}(\ln a-\ln b)^2\max\{a,b\} \tag{2.9}$$

and

$$\exp\left[\frac{1}{8}\frac{(b-a)^2}{\max^2\{a,b\}}\right]\le\frac{\frac{a+b}{2}}{\sqrt{ab}}\le\exp\left[\frac{1}{8}\frac{(b-a)^2}{\min^2\{a,b\}}\right], \tag{2.10}$$

for any $a,b>0$.

Consider the functions

$$P_1(\nu,x):=\nu(1-\nu)(x-1)\ln x$$

and

$$P_2(\nu,x):=\frac{1}{2}\nu(1-\nu)(\ln x)^2\max\{x,1\}$$

for $\nu\in[0,1]$ and $x>0$. A $3D$ plot for $\nu\in(0,1)$ and $x\in(0,2)$ reveals that the difference $P_2(\nu,x)-P_1(\nu,x)$ takes both positive and negative values showing that there is no ordering between the upper bounds of the quantity $(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}$ provided by (1.9) and (2.4) respectively.

Also, if we consider the functions

$$Q_1(\nu,x):=\exp\left[\nu(1-\nu)\frac{(x-1)^2}{x}\right]$$

and

$$Q_2(\nu,x):=\exp\left[\frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\min^2\{x,1\}}\right]$$

for $\nu\in[0,1]$ and $x>0$, then a $3D$ plot for $\nu\in(0,1)$ and $x\in(0,10)$ reveals that the difference $P_2(\nu,x)-P_1(\nu,x)$ takes also both positive and negative values showing that there is no ordering between the upper bounds of the quantity $\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}$ provided by (1.10) and (2.5).

3. Operator Inequalities

Let A be a positive operator and B a selfadjoint operator. Assume that the spectrum of $A^{-1/2}BA^{-1/2}$, $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right)$ is included in I, an interval of real numbers and $f:I\to\mathbb{R}$ a continuous function on I. Using the functional calculus for continuous functions we can consider the selfadjoint operator

$$A\sharp_f B:=A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}. \tag{3.1}$$

If $f(x)=x^{\nu}$ with $\nu\in[0,1]$ then by (3.1) we recapture the concept of weighted geometric mean of two operators.

We have the following result:

Theorem 4. *Let A, B be two positive operators. Then we have*

$$\frac{1}{4}\nu(1-\nu)A\sharp_{f_{\min}}B\leq A\nabla_{\nu}B-A\sharp_{\nu}B\leq\frac{1}{4}\nu(1-\nu)A\sharp_{f_{\max}}B \tag{3.2}$$

for any $\nu\in[0,1]$, where $f_{\min}$, $f_{\max}:(0,\infty)\to\mathbb{R}$ are defined by

$$f_{\min}(x)=(x+1-|x-1|)\ln^2 x,\ \ f_{\max}(x)=(x+1+|x-1|)\ln^2 x. \tag{3.3}$$

In particular, we have

$$\frac{1}{16}A\sharp_{f_{\min}}B\leq A\nabla B-A\sharp B\leq\frac{1}{16}A\sharp_{f_{\max}}B. \tag{3.4}$$

Proof. From the inequality (2.4) we have

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)\min\{1,x\}\ln^2 x&\leq 1-\nu+\nu x-x^{\nu}\\&\leq\frac{1}{2}\nu(1-\nu)\max\{1,x\}\ln^2 x\end{aligned} \tag{3.5}$$

for any $x>0$ and $\nu\in[0,1]$.

Since $\min\{1,x\}=\frac{1}{2}(x+1-|x-1|)$ and $\max\{1,x\}=\frac{1}{2}(x+1+|x-1|)$ then (3.5) can be written as

$$\begin{aligned}&\frac{1}{4}\nu(1-\nu)(x+1-|x-1|)\ln^2 x\\&\leq 1-\nu+\nu x-x^{\nu}\\&\leq\frac{1}{4}\nu(1-\nu)(x+1+|x-1|)\ln^2 x\end{aligned} \tag{3.6}$$

for any $x>0$ and $\nu\in[0,1]$.

Using the functional calculus for continuous functions we have for any positive X that

$$\begin{aligned}&\frac{1}{4}\nu(1-\nu)(X+I-|X-I|)\ln^2 X\\&\leq(1-\nu)I+\nu X-X^{\nu}\\&\leq\frac{1}{4}\nu(1-\nu)(X+I+|X-I|)\ln^2 X\end{aligned}$$

where I is the identity operator.

Substituting $A^{-1/2}BA^{-1/2}$ for X we have

$$\begin{aligned}&\frac{1}{4}\nu(1-\nu)\left(A^{-1/2}BA^{-1/2}+I-\left|A^{-1/2}BA^{-1/2}-I\right|\right)\\&\times\ln^2\left(A^{-1/2}BA^{-1/2}\right)\\&\leq(1-\nu)I+\nu A^{-1/2}BA^{-1/2}-\left(A^{-1/2}BA^{-1/2}\right)^{\nu}\\&\leq\frac{1}{4}\nu(1-\nu)\left(A^{-1/2}BA^{-1/2}+I-\left|A^{-1/2}BA^{-1/2}-I\right|\right)\\&\times\ln^2\left(A^{-1/2}BA^{-1/2}\right)\end{aligned} \tag{3.7}$$

for any $\nu\in[0,1]$.

Multiplying both sides of (3.7) by $A^{1/2}$ we get the desired result (3.2). □

The following result provides simpler lower and upper bounds for the difference between the weighted arithmetic and geometric operator means.

Theorem 5. *Let A, B be two positive operators such that*

$$0 < kI \leq A^{-1/2}BA^{-1/2} \leq KI,$$

for some constants k, K. Then we have

$$\frac{1}{4}\nu(1-\nu)\min_{x\in[k,K]} f_{\min}(x)\, A \leq A\nabla_\nu B - A\sharp_\nu B \leq \frac{1}{4}\nu(1-\nu)\max_{x\in[k,K]} f_{\max}(x)\, A \tag{3.8}$$

for any $\nu \in [0,1]$, where $f_{\min}$, $f_{\max}$ are defined by (3.3).

Proof. From the inequality (3.6) we have

$$\begin{aligned}\frac{1}{4}\nu(1-\nu)\min_{x\in[k,K]} f_{\min}(x) &\leq 1-\nu+\nu x - x^\nu \\ &\leq \frac{1}{4}\nu(1-\nu)\max_{x\in[k,K]} f_{\max}(x)\end{aligned} \tag{3.9}$$

for any $x \in [k,K]$ and $\nu \in [0,1]$.

If X is a selfadjoint operator with $\operatorname{Sp}(X) \subset [k,K]$, then by (3.9) we have

$$\begin{aligned}\frac{1}{4}\nu(1-\nu)\min_{x\in[k,K]} f_{\min}(x)\, I &\leq (1-\nu)I+\nu X - X^\nu \\ &\leq \frac{1}{4}\nu(1-\nu)\max_{x\in[k,K]} f_{\max}(x)\, I.\end{aligned} \tag{3.10}$$

for any $\nu \in [0,1]$.

Now, if we take in (3.10) $X = A^{-1/2}BA^{-1/2}$, then we get

$$\begin{aligned}&\frac{1}{4}\nu(1-\nu)\min_{x\in[k,K]} f_{\min}(x)\, I \\ &\leq (1-\nu)I+\nu A^{-1/2}BA^{-1/2} - \left(A^{-1/2}BA^{-1/2}\right)^\nu \\ &\leq \frac{1}{4}\nu(1-\nu)\max_{x\in[k,K]} f_{\max}(x)\, I.\end{aligned} \tag{3.11}$$

Multiplying both sides of (3.2) by $A^{1/2}$ we get the desired result (3.8). □

Remark 2. *If $0 < m'I \leq A \leq mI < MI \leq B \leq M'I$ for positive real numbers m, m', M, M' then by putting $h := \frac{M}{m}$, $h' := \frac{M'}{m'}$ we have*

$$0 < h'I \leq A^{-1/2}BA^{-1/2} \leq hI.$$

Therefore we can take in Theorem 5 $k = h'$ and $K = h$.

If $0 < m'I \leq B \leq mI < MI \leq A \leq M'I$, then

$$0 < \frac{1}{h}I \leq A^{-1/2}BA^{-1/2} \leq \frac{1}{h'}I$$

and we can take in Theorem 5 $k = \frac{1}{h}$ and $K = \frac{1}{h'}$.

The following multiplicative refinement and reverse of Young's operator inequality also holds.

Theorem 6. *Let A, B be two positive operators such that*

$$0 < kI \leq A^{-1/2}BA^{-1/2} \leq KI,$$

for some constants k, K. Then we have

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{1,K\}}{\max\{1,k\}}\right)^2\right]A\sharp_\nu B \leq A\nabla_\nu B \leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{1,K\}}{\min\{1,k\}}-1\right)^2\right]A\sharp_\nu B \tag{3.12}$$

any $\nu \in [0,1]$.

Proof. From the inequality (2.8) we have

$$\begin{aligned} &\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{1,x\}}{\max\{1,x\}}\right)^2\right]x^\nu \\ &\leq 1-\nu+\nu x \\ &\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{1,x\}}{\min\{1,x\}}-1\right)^2\right]x^\nu \end{aligned} \tag{3.13}$$

for any $x > 0$ and any $\nu \in [0,1]$.

If $x \in [k,K] \subset (0,\infty)$ then

$$0 \leq \frac{\max\{1,x\}}{\min\{1,x\}} - 1 \leq \frac{\max\{1,K\}}{\min\{1,k\}} - 1$$

and

$$0 \leq 1 - \frac{\min\{1,K\}}{\max\{1,k\}} \leq 1 - \frac{\min\{1,x\}}{\max\{1,x\}},$$

which implies that

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{1,x\}}{\min\{1,x\}}-1\right)^2\right] \leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{1,K\}}{\min\{1,k\}}-1\right)^2\right]$$

and

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{1,K\}}{\max\{1,k\}}\right)^2\right] \leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{1,x\}}{\max\{1,x\}}\right)^2\right].$$

By (3.13) we then have

$$\begin{aligned} &\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{1,K\}}{\max\{1,k\}}\right)^2\right]x^\nu \\ &\leq 1-\nu+\nu x \\ &\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{1,K\}}{\min\{1,k\}}-1\right)^2\right]x^\nu \end{aligned} \tag{3.14}$$

for any $x \in [k,K]$ and any $\nu \in [0,1]$.

If X is a selfadjoint operator with $\operatorname{Sp}(X) \subset [k, K]$, then by (3.14) we have

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{1,K\}}{\max\{1,k\}}\right)^2\right]X^\nu \tag{3.15}$$
$$\leq (1-\nu)I + \nu X$$
$$\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{1,K\}}{\min\{1,k\}}-1\right)^2\right]X^\nu.$$

Now, if we take in (3.15) $X = A^{-1/2}BA^{-1/2}$, then we get

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{1,K\}}{\max\{1,k\}}\right)^2\right]\left(A^{-1/2}BA^{-1/2}\right)^\nu \tag{3.16}$$
$$\leq (1-\nu)I + \nu A^{-1/2}BA^{-1/2}$$
$$\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{1,K\}}{\min\{1,k\}}-1\right)^2\right]\left(A^{-1/2}BA^{-1/2}\right)^\nu.$$

By multiplying both sides of (3.7) by $A^{1/2}$ we get the desired result (3.12). □

Corollary 1. *If either $0 < m'I \leq A \leq mI < MI \leq B \leq M'I$ for positive real numbers m, m', M, M' or $0 < m'I \leq B \leq mI < MI \leq A \leq M'I$, then by putting $h := \frac{M}{m}$, $h' := \frac{M'}{m'}$ we have*

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{h'-1}{h'}\right)^2\right]A\sharp_\nu B \leq A\nabla_\nu B \tag{3.17}$$
$$\leq \exp\left[\frac{1}{2}\nu(1-\nu)(h-1)^2\right]A\sharp_\nu B.$$

Proof. If $0 < m'I \leq A \leq mI < MI \leq B \leq M'I$, then we have

$$0 < h'I \leq A^{-1/2}BA^{-1/2} \leq hI.$$

Using (3.12) for $k = h'$ and $K = h$ we get

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{1}{h'}\right)^2\right]A\sharp_\nu B \leq A\nabla_\nu B$$
$$\leq \exp\left[\frac{1}{2}\nu(1-\nu)(h-1)^2\right]A\sharp_\nu B$$

and the inequality (3.17) is proved.

If $0 < m'I \leq B \leq mI < MI \leq A \leq M'I$, then we have

$$0 < \frac{1}{h}I \leq A^{-1/2}BA^{-1/2} \leq \frac{1}{h'}I.$$

Using (3.12) for $k = \frac{1}{h}$ and $K = \frac{1}{h'}$ we get

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{1}{h'}\right)^2\right]A\sharp_\nu B \leq A\nabla_\nu B$$
$$\leq \exp\left[\frac{1}{2}\nu(1-\nu)(h-1)^2\right]A\sharp_\nu B$$

and the inequality (3.17) is also obtained. □

4. Some Extension for Functions

We can extend some of the above results for functions of operators as follows.

Theorem 7. *Let $f : J \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interval $\mathring{J}$, the interior of J. Suppose that there exists the constants d, D such that*

$$d \leq f''(t) \leq D \text{ for any } t \in \mathring{J}. \tag{4.1}$$

If A is a positive operator and B a selfadjoint operator such that

$$\gamma I \leq A^{-1/2} B A^{-1/2} \leq \Gamma I, \tag{4.2}$$

with $[\gamma, \Gamma] \subset \mathring{J}$, then we have

$$\begin{aligned} &\frac{1}{2}\left(\Gamma A^{1/2} - BA^{-1/2}\right)\left(A^{-1/2}B - A^{1/2}\gamma\right) d \\ &\leq \frac{1}{\Gamma - \gamma}\left[(\Gamma A - B) f(\gamma) + (B - A\gamma) f(\Gamma)\right] - A \sharp_f B \\ &\leq \frac{1}{2}\left(\Gamma A^{1/2} - BA^{-1/2}\right)\left(A^{-1/2}B - A^{1/2}\gamma\right) D. \end{aligned} \tag{4.3}$$

Proof. From Lemma 1 we have

$$\begin{aligned} \frac{1}{2}\nu(1-\nu) d (\Gamma - \gamma)^2 &\leq (1-\nu) f(\gamma) + \nu f(\Gamma) - f((1-\nu)\gamma + \nu\Gamma) \\ &\leq \frac{1}{2}\nu(1-\nu) D (\Gamma - \gamma)^2 \end{aligned} \tag{4.4}$$

for any $\nu \in [0, 1]$.

If we take in (4.4) $\nu = \frac{x-\gamma}{\Gamma-\gamma} \in [0, 1]$ for $x \in [\gamma, \Gamma]$, then we get

$$\begin{aligned} \frac{1}{2}(\Gamma - x)(x - \gamma) d &\leq \frac{1}{\Gamma - \gamma}\left[(\Gamma - x) f(\gamma) + (x - \gamma) f(\Gamma)\right] - f(x) \\ &\leq \frac{1}{2}(\Gamma - x)(x - \gamma) D \end{aligned}$$

for any $x \in [\gamma, \Gamma]$.

Using the functional calculus for continuous functions we have

$$\begin{aligned} &\frac{1}{2}(\Gamma I - X)(X - \gamma I) d \\ &\leq \frac{1}{\Gamma - \gamma}\left[(\Gamma I - X) f(\gamma) + (X - \gamma I) f(\Gamma)\right] - f(X) \\ &\leq \frac{1}{2}(\Gamma I - X)(X - \gamma I) D \end{aligned} \tag{4.5}$$

for any selfadjoint operator X with $\mathrm{Sp}(X) \subset [\gamma, \Gamma]$.

Now, if we write the inequality (4.5) for $X = A^{-1/2}BA^{-1/2}$ then we get

$$\begin{aligned}(4.6)\qquad &\frac{1}{2}\left(\Gamma I - A^{-1/2}BA^{-1/2}\right)\left(A^{-1/2}BA^{-1/2} - \gamma I\right) d \\ &\leq \frac{1}{\Gamma - \gamma}\left[\left(\Gamma I - A^{-1/2}BA^{-1/2}\right) f(\gamma) + \left(A^{-1/2}BA^{-1/2} - \gamma I\right) f(\Gamma)\right] \\ &- f\left(A^{-1/2}BA^{-1/2}\right) \\ &\leq \frac{1}{2}\left(\Gamma I - A^{-1/2}BA^{-1/2}\right)\left(A^{-1/2}BA^{-1/2} - \gamma I\right) D.\end{aligned}$$

By multiplying both sides of (4.6) by $A^{1/2}$ we get (4.3). □

If $\gamma > 0$ in (4.2) and take $f(t) = t^p$, $t > 0$ where $p \in (0, \infty) \cup (1, \infty)$ then

$$f''(t) = p(p-1)t^{p-2}, t > 0$$

and if we take

$$d = d_p := p(p-1)\begin{cases}\gamma^{p-2}, \ p \geq 2 \\ \\ \Gamma^{p-2}, p \in (0, \infty) \cup (1, 2)\end{cases}$$

and

$$D = D_p := p(p-1)\begin{cases}\Gamma^{p-2}, \ p \geq 2 \\ \\ \gamma^{p-2}, p \in (0, \infty) \cup (1, 2)\end{cases}$$

we have from (3.14) that

$$\begin{aligned}(4.7)\qquad &\frac{1}{2}\left(\Gamma A^{1/2} - BA^{-1/2}\right)\left(A^{-1/2}B - A^{1/2}\gamma\right) d_p \\ &\leq \frac{\Gamma A - B}{\Gamma - \gamma}\gamma^p + \frac{B - A\gamma}{\Gamma - \gamma}\Gamma^p - A\sharp_p B \\ &\leq \frac{1}{2}\left(\Gamma A^{1/2} - BA^{-1/2}\right)\left(A^{-1/2}B - A^{1/2}\gamma\right) D_p\end{aligned}$$

where $A\sharp_p B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^p A^{1/2}$, $p \in (0, \infty) \cup (1, \infty)$ and A, $B > 0$ satisfy the condition (4.2).

If $p \in (0, 1)$ and A, $B > 0$ satisfy the condition (4.2) with $\gamma > 0$ then by taking $f(t) = -t^p$, $t > 0$ we have $f''(t) = p(1-p)t^{p-2}$, $t > 0$ giving that

$$p(1-p)\Gamma^{p-2} \leq f''(t) \leq p(1-p)\gamma^{p-2} \text{ for } t \in [\gamma, \Gamma].$$

Therefore, by choosing $d = p(1-p)\Gamma^{p-2}$ and $D = p(1-p)\gamma^{p-2}$ in (3.14) we get

$$\begin{aligned}(4.8)\qquad &p(1-p)\Gamma^{p-2}\left(\Gamma A^{1/2} - BA^{-1/2}\right)\left(A^{-1/2}B - A^{1/2}\gamma\right) \\ &\leq A\sharp_p B - \frac{\Gamma A - B}{\Gamma - \gamma}\gamma^p - \frac{B - A\gamma}{\Gamma - \gamma}\Gamma^p \\ &\leq p(1-p)\gamma^{p-2}\left(\Gamma A^{1/2} - BA^{-1/2}\right)\left(A^{-1/2}B - A^{1/2}\gamma\right)\end{aligned}$$

provided $p \in (0, 1)$ and A, $B > 0$ satisfy the condition (4.2) with $\gamma > 0$.

Theorem 8. *Let $f : J \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interval $\mathring{J}$, the interior of J. Suppose that there exists the constants d, D such that (4.1) is* *valid. If A is a positive operator and B a selfadjoint operator such that the condition (4.2) is valid with $[\gamma,\Gamma]\subset \mathring{J}$ and $\gamma<1<\Gamma$ then we have*

$$\begin{aligned}\tag{4.9} &\frac{1}{2}\nu(1-\nu)\, dA^{1/2}\left(A^{-1/2}BA^{-1/2}-I\right)^2 A^{1/2}\\ &\quad\leq (1-\nu) f(1) A+\nu A^{1/2} f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}\\ &\quad\quad - A^{1/2} f\left((1-\nu) I+\nu A^{-1/2}BA^{-1/2}\right)A^{1/2}\\ &\quad\quad\quad\leq \frac{1}{2}\nu(1-\nu)\, DA^{1/2}\left(A^{-1/2}BA^{-1/2}-I\right)^2 A^{1/2}\end{aligned}$$

for any $\nu\in[0,1]$.

In particular, we have

$$\begin{aligned}\tag{4.10} &\frac{1}{8} dA^{1/2}\left(A^{-1/2}BA^{-1/2}-I\right)^2 A^{1/2}\\ &\quad\leq \frac{1}{2}\left[f(1) A+A^{1/2} f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}\right]\\ &\quad\quad - A^{1/2} f\left(\frac{I+A^{-1/2}BA^{-1/2}}{2}\right)A^{1/2}\\ &\quad\quad\quad\leq \frac{1}{8} DA^{1/2}\left(A^{-1/2}BA^{-1/2}-I\right)^2 A^{1/2}.\end{aligned}$$

Proof. We have from (2.2) that

$$\begin{aligned}\tag{4.11} \frac{1}{2}\nu(1-\nu)\, d(b-a)^2 &\leq (1-\nu) f(a)+\nu f(b)-f((1-\nu) a+\nu b)\\ &\leq \frac{1}{2}\nu(1-\nu)\, D(b-a)^2\end{aligned}$$

for any $a,\, b\in[\gamma,\Gamma]$ and $\nu\in[0,1]$.

If we take $a=1$ and $b=x$ in (4.11) then we get

$$\begin{aligned}\tag{4.12} \frac{1}{2}\nu(1-\nu)\, d(x-1)^2 &\leq (1-\nu) f(1)+\nu f(x)-f((1-\nu) 1+\nu x)\\ &\leq \frac{1}{2}\nu(1-\nu)\, D(x-1)^2\end{aligned}$$

for any $x\in(\gamma,\Gamma)$.

This implies in the operator order that

$$\begin{aligned}\tag{4.13} \frac{1}{2}\nu(1-\nu)\, d(X-I)^2 &\leq (1-\nu) f(1) I+\nu f(X)-f((1-\nu) I+\nu X)\\ &\leq \frac{1}{2}\nu(1-\nu)\, D(X-I)^2\end{aligned}$$

for any selfadjoint operator X with $\operatorname{Sp}(X)\subset[\gamma,\Gamma]$.

If we take in (4.13) $X=A^{-1/2}BA^{-1/2}$ and multiply in both sides with $A^{1/2}$ then we get the desired result (4.9). $\square$

Consider the convex functions $f(t) = -\ln t$, $t > 0$. Then $f''(t) = \frac{1}{t^2}$, $t \in [\gamma, \Gamma]$ and by taking $d = \frac{1}{\Gamma^2}$ and $D = \frac{1}{\gamma^2}$ in (4.9) we get

$$\begin{aligned}(4.14)\quad & \frac{1}{2\Gamma^2}\nu(1-\nu)A^{1/2}\left(A^{-1/2}BA^{-1/2}-I\right)^2 A^{1/2} \\ & \leq A^{1/2}\left[\ln\left((1-\nu)I+\nu A^{-1/2}BA^{-1/2}\right)\right]A^{1/2} \\ & \quad -\nu A^{1/2}\left[\ln\left(A^{-1/2}BA^{-1/2}\right)\right]A^{1/2} \\ & \leq \frac{1}{2\gamma^2}\nu(1-\nu)A^{1/2}\left(A^{-1/2}BA^{-1/2}-I\right)^2 A^{1/2}\end{aligned}$$

provided A, $B > 0$ and satisfy (4.2) while $\nu \in [0,1]$.

In particular, we have

$$\begin{aligned}(4.15)\quad & \frac{1}{8\Gamma^2}A^{1/2}\left(A^{-1/2}BA^{-1/2}-I\right)^2 A^{1/2} \\ & \leq A^{1/2}\left[\ln\left(\frac{I+A^{-1/2}BA^{-1/2}}{2}\right)\right]A^{1/2}-\frac{1}{2}A^{1/2}\left[\ln\left(A^{-1/2}BA^{-1/2}\right)\right]A^{1/2} \\ & \leq \frac{1}{8\gamma^2}A^{1/2}\left(A^{-1/2}BA^{-1/2}-I\right)^2 A^{1/2}.\end{aligned}$$

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