Some vector inequalities for two operators in Hilbert spaces with applications

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Some vector inequalities for two operators in Hilbert spaces with applications

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Abstract. In this paper we establish some vector inequalities for two operators related to Schwarz and Buzano results. We show amongst others that in a Hilbert space H we have the inequality

$$\frac{1}{2}\left[\left\langle \frac{|A|^2+|B|^2}{2}x,x\right\rangle^{1/2}\left\langle \frac{|A|^2+|B|^2}{2}y,y\right\rangle^{1/2}+\left|\left\langle \frac{|A|^2+|B|^2}{2}x,y\right\rangle\right|\right] \geq |\langle \mathrm{Re}\,(B^*A)\,x,y\rangle|$$

for A, B two bounded linear operators on H such that $\mathrm{Re}\,(B^*A)$ is a nonnegative operator and any vectors $x, y \in H$.

Applications for norm and numerical radius inequalities are given as well.

1 Introduction

Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. The following inequality is well known in literature as the *Schwarz inequality*

$$\|x\|\,\|y\| \geq |\langle x, y\rangle| \text{ for any } x, y \in H. \tag{1}$$

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The equality case holds in (1) if and only if there exists a constant $\lambda \in \mathbb{K}$ such that $x = \lambda y$.

In 1985 the author [5] (see also [24]) established the following refinement of (1):

$$\|x\| \|y\| \geq |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| + |\langle x, e\rangle \langle e, y\rangle| \geq |\langle x, y\rangle| \tag{2}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

Using the triangle inequality for modulus we have

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \geq |\langle x, e\rangle \langle e, y\rangle| - |\langle x, y\rangle|$$

and by (2) we get

$$\begin{aligned} \|x\| \|y\| &\geq |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| + |\langle x, e\rangle \langle e, y\rangle| \\ &\geq 2 |\langle x, e\rangle \langle e, y\rangle| - |\langle x, y\rangle|, \end{aligned}$$

which implies the *Buzano inequality* [2]

$$\frac{1}{2}\left[\, \|x\| \|y\| + |\langle x, y\rangle| \,\right] \geq |\langle x, e\rangle \langle e, y\rangle| \tag{3}$$

that holds for any $x, y, e \in H$ with $\|e\| = 1$.

A family $\{e_j\}_{j\in J}$ of vectors in H is called *orthonormal* if

$$e_j \perp e_k \text{ for any } j, k \in J \text{ with } j \neq k \text{ and } \|e_j\| = 1 \text{ for any } j, k \in J.$$

If the *linear span* of the family $\{e_j\}_{j\in J}$ is *dense* in H, then we call it an *orthonormal basis* in H.

It is well known that for any orthonormal family $\{e_j\}_{j\in J}$ we have *Bessel's inequality*

$$\sum_{j\in J} |\langle x, e_j\rangle|^2 \leq \|x\|^2 \text{ for any } x \in H.$$

This becomes *Parseval's identity*

$$\sum_{j\in J} |\langle x, e_j\rangle|^2 = \|x\|^2 \text{ for any } x \in H,$$

when $\{e_j\}_{j\in J}$ an othonormal basis in H.

For an othonormal family $\mathcal{E} = \{e_j\}_{j\in J}$ we define the operator $P_{\mathcal{E}} : H \to H$ by

$$P_{\mathcal{E}} x := \sum_{j\in J} \langle x, e_j\rangle e_j\,, \; x \in H. \tag{4}$$

We know that $P_{\mathcal{E}}$ is an *orthogonal projection* and

$$\langle P_{\mathcal{E}}x, y\rangle = \sum_{j\in J} \langle x, e_j\rangle \langle e_j, y\rangle, \; x, y \in H \text{ and } \langle P_{\mathcal{E}}x, x\rangle = \sum_{j\in J} |\langle x, e_j\rangle|^2, \; x \in H.$$

The particular case when the family reduces to one vector, namely $\mathcal{E} = \{e\}$, $\|e\| = 1$, is of interest since in this case $P_e x := \langle x, e\rangle e$, $x \in H$,

$$\langle P_e x, y\rangle = \langle x, e\rangle \langle e, y\rangle, \; x, y \in H \tag{5}$$

and Buzano's inequality can be written as

$$\frac{1}{2}\left[\|x\| \|y\| + |\langle x, y\rangle| \right] \geq |\langle P_e x, y\rangle| \tag{6}$$

that holds for any $x, y, e \in H$ with $\|e\| = 1$.

In an effort to generalize the inequality (6) for general projection, in [21] we obtained the following result

$$\frac{1}{2}\left[\|x\| \|y\| + |\langle x, y\rangle| \right] \geq |\langle Px, y\rangle| \tag{7}$$

for any $x, y \in H$ and $P : H \to H$ a projection on H.

In particular, we then have the inequality

$$\frac{1}{2}\left[\|x\| \|y\| + |\langle x, y\rangle| \right] \geq \left| \left\langle \sum_{j\in J} \langle x, e_j\rangle \langle e_j, y\rangle \right\rangle \right| \tag{8}$$

for any orthonormal family $\{e_j\}_{j\in J}$ and any $x, y \in H$.

Motivated by the above results we establish in this paper some vector inequalities for two operators A, B for which the operator $\mathrm{Re}(B^*A)$ is nonnegative in the operator order that are related to the inequality (6). Applications for norm and numerical radius inequalities are provided as well.

For other Schwarz and Buzano related inequalities in inner product spaces, see [1]-[4], [5]-[14], [22]-[26], [30]-[39], and the monographs [16], [17] and [18].

2 Vector inequalities for two operators

For a bounded linear operator T we use the concepts of *absolute value* and *real part* of T defined as

$$|T| = (T^*T)^{1/2} \text{ and } \mathrm{Re}(T) = \frac{T + T^*}{2}. \tag{9}$$

We have the following vector inequality:

Theorem 1 *Let* A, B *two bounded linear operators on* H *such that* $\operatorname{Re}(B^*A)$ *is a nonnegative operator. Then for any* $x, y \in H$ *we have the inequality*

$$\begin{aligned}&\left\langle \frac{|A|^2+|B|^2}{2}x, x\right\rangle^{1/2}\left\langle \frac{|A|^2+|B|^2}{2}y, y\right\rangle^{1/2}\\&\geq \langle \operatorname{Re}(B^*A)x, x\rangle^{1/2}\langle \operatorname{Re}(B^*A)y, y\rangle^{1/2}\\&+\left|\left\langle \frac{|A|^2+|B|^2}{2}x, y\right\rangle - \langle \operatorname{Re}(B^*A)x, y\rangle\right|.\end{aligned} \tag{10}$$

Proof. Using Schwarz inequality we have

$$\|Ax-Bx\|^2\|Ay-By\|^2 \geq |\langle Ax-Bx, Ay-By\rangle|^2 \tag{11}$$

for any $x, y \in H$.

Observe that

$$\begin{aligned}\|Ax-Bx\|^2 &= \langle Ax, Ax\rangle - \langle Ax, Bx\rangle - \langle Bx, Ax\rangle + \langle Bx, Bx\rangle\\&= \langle A^*Ax, x\rangle - \langle B^*Ax, x\rangle - \langle A^*Bx, x\rangle + \langle B^*Bx, x\rangle\\&= \langle |A|^2 x, x\rangle + \langle |B|^2 x, x\rangle - \langle (B^*A+A^*B)x, x\rangle\\&= 2\left[\left\langle \frac{|A|^2+|B|^2}{2}x, x\right\rangle - \langle \operatorname{Re}(B^*A)x, x\rangle\right] \geq 0\end{aligned} \tag{12}$$

and, similarly,

$$\|Ay-By\|^2 = 2\left[\left\langle \frac{|A|^2+|B|^2}{2}y, y\right\rangle - \langle \operatorname{Re}(B^*A)y, y\rangle\right] \geq 0 \tag{13}$$

for any $x, y \in H$.

We also have

$$\langle Ax-Bx, Ay-By\rangle = 2\left[\left\langle \frac{|A|^2+|B|^2}{2}x, y\right\rangle - \langle \operatorname{Re}(B^*A)x, y\rangle\right] \tag{14}$$

for any $x, y \in H$.

Using the inequality (11) and the equalities (12)-(14) we get

$$\begin{aligned}&\left[\left\langle \frac{|A|^2+|B|^2}{2}x, x\right\rangle - \langle \operatorname{Re}(B^*A)x, x\rangle\right]\\&\times\left[\left\langle \frac{|A|^2+|B|^2}{2}y, y\right\rangle - \langle \operatorname{Re}(B^*A)y, y\rangle\right]\\&\geq \left|\left\langle \frac{|A|^2+|B|^2}{2}x, y\right\rangle - \langle \operatorname{Re}(B^*A)x, y\rangle\right|^2\end{aligned} \tag{15}$$

for any $x, y \in H$.

Since $\operatorname{Re}(B^*A) \geq 0$, then we have

$$\left\langle \frac{|A|^2 + |B|^2}{2} x, x \right\rangle \geq \langle \operatorname{Re}(B^*A)\, x, x\rangle \geq 0$$

and

$$\left\langle \frac{|A|^2 + |B|^2}{2} y, y \right\rangle \geq \langle \operatorname{Re}(B^*A)\, y, y\rangle \geq 0$$

for any $x, y \in H$.

Using the elementary inequality that holds for any real numbers a, b, c, d

$$(ac - bd)^2 \geq (a^2 - b^2)(c^2 - d^2),$$

we have

$$\begin{aligned} &\left(\left\langle \frac{|A|^2 + |B|^2}{2} x, x \right\rangle^{1/2} \left\langle \frac{|A|^2 + |B|^2}{2} y, y \right\rangle^{1/2} \right. \\ &\left. - \langle \operatorname{Re}(B^*A)x, x\rangle^{1/2} \langle \operatorname{Re}(B^*A)y, y\rangle^{1/2} \right)^2 \\ &\geq \left[\left\langle \frac{|A|^2 + |B|^2}{2} x, x \right\rangle - \langle \operatorname{Re}(B^*A)x, x\rangle \right] \\ &\times \left[\left\langle \frac{|A|^2 + |B|^2}{2} y, y \right\rangle - \langle \operatorname{Re}(B^*A)y, y\rangle \right] \end{aligned} \tag{16}$$

for any $x, y \in H$.

Making use of (15) and (16) we get

$$\begin{aligned} &\left(\left\langle \frac{|A|^2 + |B|^2}{2} x, x \right\rangle^{1/2} \left\langle \frac{|A|^2 + |B|^2}{2} y, y \right\rangle^{1/2} \right. \\ &\left. - \langle \operatorname{Re}(B^*Ax, x\rangle^{1/2} \langle \operatorname{Re}(B^*A)y, y\rangle^{1/2} \right)^2 \\ &\geq \left| \left\langle \frac{|A|^2 + |B|^2}{2} x, y \right\rangle - \langle \operatorname{Re}(B^*A)\, x, y\rangle \right|^2 \end{aligned} \tag{17}$$

for any $x, y \in H$.

Since

$$\left\langle \frac{|A|^2 + |B|^2}{2} x, x \right\rangle^{1/2} \left\langle \frac{|A|^2 + |B|^2}{2} y, y \right\rangle^{1/2} \geq \langle \operatorname{Re}(B^*A)x, x\rangle^{1/2} \langle \operatorname{Re}(B^*A)y, y\rangle^{1/2}$$

for any $x, y \in H$, then by taking the square root in (17) we get the desired result from (10). □

Corollary 1 *With the assumptions in Theorem 1 we have*

$$\begin{aligned}&\left\langle \frac{|A|^2+|B|^2}{2}x,x\right\rangle^{1/2}\left\langle \frac{|A|^2+|B|^2}{2}y,y\right\rangle^{1/2}-\left|\left\langle \frac{|A|^2+|B|^2}{2}x,y\right\rangle\right| \\ &\geq \langle \operatorname{Re}(B^*A)x,x\rangle^{1/2}\langle \operatorname{Re}(B^*A)y,y\rangle^{1/2}-|\langle \operatorname{Re}(B^*A)x,y\rangle|\geq 0\end{aligned} \tag{18}$$

and

$$\begin{aligned}&\left\langle \frac{|A|^2+|B|^2}{2}x,x\right\rangle^{1/2}\left\langle \frac{|A|^2+|B|^2}{2}y,y\right\rangle^{1/2}+\left|\left\langle \frac{|A|^2+|B|^2}{2}x,y\right\rangle\right| \\ &\geq \langle \operatorname{Re}(B^*A)x,x\rangle^{1/2}\langle \operatorname{Re}(B^*A)y,y\rangle^{1/2}+|\langle \operatorname{Re}(B^*A)x,y\rangle|\end{aligned} \tag{19}$$

for any $x,y\in H$.

Proof. From the triangle inequality we have

$$\left|\left\langle \frac{|A|^2+|B|^2}{2}x,y\right\rangle-\langle \operatorname{Re}(B^*A)x,y\rangle\right|\geq\left|\left\langle \frac{|A|^2+|B|^2}{2}x,y\right\rangle\right|-|\langle \operatorname{Re}(B^*A)x,y\rangle|$$

and

$$\left|\left\langle \frac{|A|^2+|B|^2}{2}x,y\right\rangle-\langle \operatorname{Re}(B^*A)x,y\rangle\right|\geq|\langle \operatorname{Re}(B^*A)x,y\rangle|-\left|\left\langle \frac{|A|^2+|B|^2}{2}x,y\right\rangle\right|$$

for any $x,y\in H$, which together with (10) produce the inequalities (18) and (19). □

Remark 1 *With the assumptions in Theorem 1 we have*

$$\begin{aligned}&\frac{1}{2}\left[\left\langle \frac{|A|^2+|B|^2}{2}x,x\right\rangle^{1/2}\left\langle \frac{|A|^2+|B|^2}{2}y,y\right\rangle^{1/2}\right. \\ &\left.+\left|\left\langle \frac{|A|^2+|B|^2}{2}x,y\right\rangle\right|\right]\geq|\langle \operatorname{Re}(B^*A)x,y\rangle|\end{aligned} \tag{20}$$

for any $x,y\in H$.

If we assume that A is a bounded linear operator such that $\operatorname{Re}\left(A^2\right)\geq 0$, then by taking $B=A^*$ above, we have the inequalities

$$\begin{aligned}&\left\langle \frac{|A|^2+|A^*|^2}{2}x,x\right\rangle^{1/2}\left\langle \frac{|A|^2+|A^*|^2}{2}y,y\right\rangle^{1/2} \\ &\geq\langle \operatorname{Re}(A^2)x,x\rangle^{1/2}\langle \operatorname{Re}(A^2)y,y\rangle^{1/2} \\ &+\left|\left\langle \frac{|A|^2+|A^*|^2}{2}x,y\right\rangle-\langle \operatorname{Re}(A^2)x,y\rangle\right|,\end{aligned} \tag{21}$$

$$\left\langle \frac{|A|^2+|A^*|^2}{2}x,x\right\rangle^{1/2}\left\langle \frac{|A|^2+|A^*|^2}{2}y,y\right\rangle^{1/2}-\left|\left\langle \frac{|A|^2+|A^*|^2}{2}x,y\right\rangle\right| \\ \geq \langle \mathrm{Re}(A^2)x,x\rangle^{1/2}\langle \mathrm{Re}(A^2)y,y\rangle^{1/2}-|\langle \mathrm{Re}(A^2)x,y\rangle|\geq 0, \tag{22}$$

$$\left\langle \frac{|A|^2+|A^*|^2}{2}x,x\right\rangle^{1/2}\left\langle \frac{|A|^2+|A^*|^2}{2}y,y\right\rangle^{1/2}+\left|\left\langle \frac{|A|^2+|B|^2}{2}x,y\right\rangle\right| \\ \geq \langle \mathrm{Re}(A^2)x,x\rangle^{1/2}\langle \mathrm{Re}(A^2)y,y\rangle^{1/2}+|\langle \mathrm{Re}(A^2)x,y\rangle| \tag{23}$$

and

$$\frac{1}{2}\left[\left\langle \frac{|A|^2+|A^*|^2}{2}x,x\right\rangle^{1/2}\left\langle \frac{|A|^2+|A^*|^2}{2}y,y\right\rangle^{1/2} \right. \\ \left. +\left|\left\langle \frac{|A|^2+|A^*|^2}{2}x,y\right\rangle\right|\right]\geq |\langle \mathrm{Re}(A^2)x,y\rangle| \tag{24}$$

for any $x,y\in H$.

Assume that A is invertible, then by selecting $B=(A^{-1})^*$ above and taking into account that

$$|B|^2=B^*B=A^{-1}(A^{-1})^*=A^{-1}(A^*)^{-1}=(A^*A)^{-1}=|A|^{-2}$$

then from the above we get the inequalities

$$\left\langle \frac{|A|^2+|A|^{-2}}{2}x,x\right\rangle^{1/2}\left\langle \frac{|A|^2+|A|^{-2}}{2}y,y\right\rangle^{1/2} \\ \geq \|x\|\,\|y\|+\left|\left\langle \frac{|A|^2+|A|^{-2}}{2}x,y\right\rangle-\langle x,y\rangle\right|, \tag{25}$$

$$\left\langle \frac{|A|^2+|A|^{-2}}{2}x,x\right\rangle^{1/2}\left\langle \frac{|A|^2+|A|^{-2}}{2}y,y\right\rangle^{1/2} \\ -\left|\left\langle \frac{|A|^2+|A|^{-2}}{2}x,y\right\rangle\right|\geq \|x\|\,\|y\|-|\langle x,y\rangle|\geq 0, \tag{26}$$

$$\left\langle \frac{|A|^2+|A|^{-2}}{2}x,x\right\rangle^{1/2}\left\langle \frac{|A|^2+|A|^{-2}}{2}y,y\right\rangle^{1/2} \\ +\left|\left\langle \frac{|A|^2+|A|^{-2}}{2}x,y\right\rangle\right|\geq \|x\|\,\|y\|+|\langle x,y\rangle| \tag{27}$$

and

$$\frac{1}{2}\left[\left\langle\frac{|A|^2+|A|^{-2}}{2}x,x\right\rangle^{1/2}\left\langle\frac{|A|^2+|A|^{-2}}{2}y,y\right\rangle^{1/2} + \left|\left\langle\frac{|A|^2+|A|^{-2}}{2}x,y\right\rangle\right|\right] \geq |\langle x,y\rangle| \tag{28}$$

for any $x, y \in H$.

If $A, B \geq 0$ with $AB = BA$, then from (10) we have

$$\left\langle\frac{A^2+B^2}{2}x,x\right\rangle^{1/2}\left\langle\frac{A^2+B^2}{2}y,y\right\rangle^{1/2} \geq \langle ABx,x\rangle^{1/2}\langle ABy,y\rangle^{1/2} + \left|\left\langle\frac{A^2+B^2}{2}x,y\right\rangle - \langle ABx,y\rangle\right|, \tag{29}$$

$$\left\langle\frac{A^2+B^2}{2}x,x\right\rangle^{1/2}\left\langle\frac{A^2+B^2}{2}y,y\right\rangle^{1/2} - \left|\left\langle\frac{A^2+B^2}{2}x,y\right\rangle\right| \geq \langle ABx,x\rangle^{1/2}\langle ABy,y\rangle^{1/2} - |\langle ABx,y\rangle| \geq 0, \tag{30}$$

$$\left\langle\frac{A^2+B^2}{2}x,x\right\rangle^{1/2}\left\langle\frac{A^2+B^2}{2}y,y\right\rangle^{1/2} + \left|\left\langle\frac{A^2+B^2}{2}x,y\right\rangle\right| \geq \langle ABx,x\rangle^{1/2}\langle ABy,y\rangle^{1/2} + |\langle ABx,y\rangle| \tag{31}$$

and

$$\frac{1}{2}\left[\left\langle\frac{A^2+B^2}{2}x,x\right\rangle^{1/2}\left\langle\frac{A^2+B^2}{2}y,y\right\rangle^{1/2} + \left|\left\langle\frac{A^2+B^2}{2}x,y\right\rangle\right|\right] \geq |\langle ABx,y\rangle| \tag{32}$$

for any $x, y \in H$.

We observe that if $A = 1_H$ and $B = P$, with P a projection on H, then we obtain from (32)

$$\frac{1}{2}\left[\left\langle\frac{1_H+P}{2}x,x\right\rangle^{1/2}\left\langle\frac{1_H+P}{2}y,y\right\rangle^{1/2} + \left|\left\langle\frac{1_H+P}{2}x,y\right\rangle\right|\right] \geq |\langle Px,y\rangle| \tag{33}$$

for any $x, y \in H$.

If $e \in H$, $\|e\| = 1$ then by taking $P = P_e$ defined in the introduction, we get the inequality

$$\frac{1}{4}\left[\left[\|x\|^2 + |\langle x,e\rangle|^2\right]^{1/2}\left[\|y\|^2 + |\langle y,e\rangle|^2\right]^{1/2} + |\langle x,y\rangle + \langle x,e\rangle\langle e,y\rangle|\right] \geq |\langle x,e\rangle\langle e,y\rangle| \tag{34}$$

for any $x, y \in H$.

Since

$$|\langle x,y\rangle + \langle x,e\rangle\langle e,y\rangle| \le |\langle x,y\rangle| + |\langle x,e\rangle\langle e,y\rangle|$$

then by (34) we have

$$\frac{1}{4}\Big[\big[\,\|x\|^2+|\langle x,e\rangle|^2\,\big]^{1/2}\big[\,\|y\|^2+|\langle y,e\rangle|^2\,\big]^{1/2}+|\langle x,y\rangle|+|\langle x,e\rangle\langle e,y\rangle|\Big] \ge |\langle x,e\rangle\langle e,y\rangle|,$$

which implies that

$$\frac{1}{3}\Big(\big[\|x\|^2 + |\langle x,e\rangle|^2\big]^{1/2}\big[\|y\|^2 + |\langle y,e\rangle|^2\big]^{1/2} + |\langle x,y\rangle|\Big) \ge |\langle x,e\rangle\langle e,y\rangle| \tag{35}$$

for any $x, y \in H$.

We recall that $U : H \to H$ is a *unitary operator* if $U^*U = UU^* = 1_H$. If U and V are unitary operators with $\mathrm{Re}\,(V^*U) \ge 0$, then by (20) we have

$$\frac{1}{2}\big[\,\|x\|\,\|y\| + |\langle x,y\rangle|\,\big] \ge \big|\langle \mathrm{Re}\,(V^*U)\,x,y\rangle\big| \tag{36}$$

for any $x, y \in H$.

In particular, if U is a unitary operator with $\mathrm{Re}\,(U) \ge 0$ then by taking $V = 1_H$ in (36) we get

$$\frac{1}{2}\big[\|x\|\|y\| + |\langle x,y\rangle|\big] \ge \big|\langle \mathrm{Re}(U)x,y\rangle\big| \tag{37}$$

for any $x, y \in H$.

3 Inequalities for norm and numerical radius

Let $\big(H;\langle\cdot,\cdot\rangle\big)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [27, p. 1]:

$$W(T) = \big\{\langle Tx,x\rangle,\ x \in H,\ \|x\| = 1\big\}.$$

The *numerical radius* $w\,(T)$ of an operator T on H is defined by [27, p. 8]:

$$w\,(T) = \sup\big\{\,|\lambda|\,,\lambda \in W\,(T)\,\big\} = \sup\big\{|\langle Tx,x\rangle|, \|x\| = 1\big\}.$$

It is well known that $w\,(\cdot)$ is a norm on the Banach algebra $B\,(H)$ and the following inequality holds true

$$w\,(T) \le \|T\| \le 2w\,(T)\,,\ \text{ for any } T \in B\,(H)\,.$$

Utilising Buzano's inequality (3) we obtained the following inequality for the numerical radius [13] or [14]:

Theorem 2 *Let* $(H; \langle \cdot, \cdot \rangle)$ *be a Hilbert space and* $T : H \to H$ *a bounded linear operator on* H. *Then*

$$w^2(T) \leq \frac{1}{2}\left[w(T^2 + \|T\|^2\right]. \tag{38}$$

The constant $\frac{1}{2}$ *is best possible in (38).*

The following general result for the product of two operators holds [27, p. 37]:

Theorem 3 *If* U, V *are two bounded linear operators on the Hilbert space* $(H, \langle \cdot, \cdot \rangle)$, *then* $w(UV) \leq 4w(U)w(V)$. *In the case that* $UV = VU$, *then* $w(UV) \leq 2w(U)w(V)$. *The constant* 2 *is best possible here.*

The following results are also well known [27, p. 38].

Theorem 4 *If* U *is a unitary operator that commutes with another operator* V, *then*

$$w(UV) \leq w(V). \tag{39}$$

If U *is an isometry and* $UV = VU$, *then (39) also holds true.*

We say that U and V *double commute* if $UV = VU$ and $UV^* = V^*U$. The following result holds [27, p. 38].

Theorem 5 *If the operators* U *and* V *double commute, then*

$$w(UV) \leq w(V)\|U\|. \tag{40}$$

As a consequence of the above, we have [27, p. 39]:

Corollary 2 *Let* U *be a normal operator commuting with* V. *Then*

$$w(UV) \leq w(U)w(V). \tag{41}$$

A related problem with the inequality (40) is to find the best constant c for which the inequality

$$w(UV) \leq cw(U)\|V\|$$

holds for any two commuting operators $U, V \in B(H)$. It is known that $1.064 < c < 1.169$, see [3], [35] and [36].

In relation to this problem, it has been shown in [25] that:

Theorem 6 *For any* $U, V \in B(H)$ *we have*

$$w\left(\frac{UV + VU}{2}\right) \leq \sqrt{2}w(U)\|V\|. \tag{42}$$

For other numerical radius inequalities see the recent monograph [18] and the references therein.

Theorem 7 *Let* A, B *two bounded linear operators on* H *such that* $\mathrm{Re}(B^*A)$ *is a nonnegative operator. Then for any* $U, V \in B(H)$ *we have*

$$\begin{aligned} \|V\mathrm{Re}(B^*A)U\| \leq{} & \frac{1}{2}\left\|\left(\frac{|A|^2 + |B|^2}{2}\right)^{1/2} U\right\| \left\|V\left(\frac{|A|^2 + |B|^2}{2}\right)^{1/2}\right\| \\ & + \frac{1}{2}\left\|V\left(\frac{|A|^2 + |B|^2}{2}\right)U\right\|, \end{aligned} \tag{43}$$

$$\begin{aligned} w(V\mathrm{Re}(B^*A)U) \leq{} & \frac{1}{2}\left\|\left(\frac{|A|^2 + |B|^2}{2}\right)^{1/2} U\right\| \left\|V\left(\frac{|A|^2 + |B|^2}{2}\right)^{1/2}\right\| \\ & + \frac{1}{2}w\left(V\left(\frac{|A|^2 + |B|^2}{2}\right)U\right) \end{aligned} \tag{44}$$

and

$$\begin{aligned} w(V\mathrm{Re}(B^*A)U) \leq{} & \frac{1}{4}\left\|\left|\left(\frac{|A|^2 + |B|^2}{2}\right)^{1/2} U\right|^2 + \left|\left(\frac{|A|^2 + |B|^2}{2}\right)^{1/2} V^*\right|^2\right\| \\ & + \frac{1}{2}w\left(V\left(\frac{|A|^2 + |B|^2}{2}\right)U\right). \end{aligned} \tag{45}$$

Proof. From the inequality (20) we have

$$\begin{aligned} \left|\langle \mathrm{Re}(B^*A)Ux, V^*y\rangle\right| \leq{} & \frac{1}{2}\left[\left\langle \frac{|A|^2 + |B|^2}{2}Ux, Ux\right\rangle^{1/2} \left\langle \frac{|A|^2 + |B|^2}{2}V^*y, V^*y\right\rangle^{1/2} \right. \\ & \left. + \left|\left\langle \frac{|A|^2 + |B|^2}{2}Ux, V^*y\right\rangle\right|\right] \end{aligned}$$

for any $x, y \in H$, which is equivalent to

$$\begin{aligned} & \left|\langle V\mathrm{Re}(B^*A)Ux, y\rangle\right| \\ & \leq \frac{1}{2}\left[\left\langle U^*\frac{|A|^2 + |B|^2}{2}Ux, x\right\rangle^{1/2} \left\langle V\frac{|A|^2 + |B|^2}{2}V^*y, y\right\rangle^{1/2} \right. \\ & \left. \quad + \left|\left\langle V\frac{|A|^2 + |B|^2}{2}Ux, y\right\rangle\right|\right] \end{aligned} \tag{46}$$

for any $x, y \in H$.

Taking the supremum over $x, y \in H$, $\|x\| = \|y\| = 1$ we have

$$\begin{aligned}
\|V\mathrm{Re}(B^*A)U\| &= \sup_{\|x\|=\|y\|=1} \left|\langle V\mathrm{Re}(B^*A)Ux, y\rangle\right| \\
&\le \frac{1}{2} \sup_{\|x\|=\|y\|=1} \left[\left\langle U^*\frac{|A|^2+|B|^2}{2}Ux, x\right\rangle^{1/2}\left\langle V\frac{|A|^2+|B|^2}{2}V^*y, y\right\rangle^{1/2}\right. \\
&\quad \left. + \left|\left\langle V\frac{|A|^2+|B|^2}{2}Ux, y\right\rangle\right|\right] \\
&\le \frac{1}{2}\left[\sup_{\|x\|=1}\left\langle U^*\frac{|A|^2+|B|^2}{2}Ux, x\right\rangle^{1/2} \sup_{\|y\|=1}\left\langle V\frac{|A|^2+|B|^2}{2}V^*y, y\right\rangle^{1/2}\right. \\
&\quad \left. + \sup_{\|x\|=\|y\|=1}\left|\left\langle V\frac{|A|^2+|B|^2}{2}Ux, y\right\rangle\right|\right] \\
&= \frac{1}{2}\left[\left\|U^*\frac{|A|^2+|B|^2}{2}U\right\|^{1/2}\left\|V\frac{|A|^2+|B|^2}{2}V^*\right\|^{1/2} + \left\|V\frac{|A|^2+|B|^2}{2}U\right\|\right].
\end{aligned} \tag{47}$$

Since

$$U^*\frac{|A|^2+|B|^2}{2}U = \left|\left(\frac{|A|^2+|B|^2}{2}\right)^{1/2}U\right|^2$$

and

$$V\frac{|A|^2+|B|^2}{2}V^* = \left|\left(\frac{|A|^2+|B|^2}{2}\right)^{1/2}V^*\right|^2$$

then

$$\left\|U^*\frac{|A|^2+|B|^2}{2}U\right\|^{1/2} = \left\|\left(\frac{|A|^2+|B|^2}{2}\right)^{1/2}U\right\|$$

and

$$\left\|V\frac{|A|^2+|B|^2}{2}V^*\right\|^{1/2} = \left\|\left(\frac{|A|^2+|B|^2}{2}\right)^{1/2}V^*\right\| = \left\|V\left(\frac{|A|^2+|B|^2}{2}\right)^{1/2}\right\|.$$

Using (46) we also have

$$\begin{aligned}
\left|\langle V\mathrm{Re}(B^*A)Ux, x\rangle\right| \le \frac{1}{2}&\left[\left\langle U^*\frac{|A|^2+|B|^2}{2}Ux, x\right\rangle^{1/2}\left\langle V\frac{|A|^2+|B|^2}{2}V^*x, x\right\rangle^{1/2}\right. \\
&\left. + \left|\left\langle V\frac{|A|^2+|B|^2}{2}Ux, x\right\rangle\right|\right]
\end{aligned} \tag{48}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$ we have

$$
\begin{aligned}
w(V\operatorname{Re}(B^*A)U) &= \sup_{\|x\|=1} \left|\langle V\operatorname{Re}(B^*A)Ux, x\rangle\right| \\
&\le \frac{1}{2}\left[\sup_{\|x\|=1}\left\langle U^*\frac{|A|^2+|B|^2}{2}Ux, x\right\rangle^{1/2}\sup_{\|x\|=1}\left\langle V\frac{|A|^2+|B|^2}{2}V^*x, x\right\rangle^{1/2}\right. \\
&\quad \left.+\sup_{\|x\|=1}\left|\left\langle V\frac{|A|^2+|B|^2}{2}Ux, x\right\rangle\right|\right] \\
&= \frac{1}{2}\left[\left\|\left(\frac{|A|^2+|B|^2}{2}\right)^{1/2}U\right\|\left\|V\left(\frac{|A|^2+|B|^2}{2}\right)^{1/2}\right\| + w\left(V\frac{|A|^2+|B|^2}{2}U\right)\right]
\end{aligned} \tag{49}
$$

and the inequality (44) is proved.

By the arithmetic mean – geometric mean inequality we have

$$
\begin{aligned}
&\left\langle U^*\frac{|A|^2+|B|^2}{2}Ux, x\right\rangle^{1/2}\left\langle V\frac{|A|^2+|B|^2}{2}V^*x, x\right\rangle^{1/2} \\
&\le \frac{1}{2}\left[\left\langle U^*\frac{|A|^2+|B|^2}{2}Ux, x\right\rangle + \left\langle V\frac{|A|^2+|B|^2}{2}V^*x, x\right\rangle\right] \\
&= \frac{1}{2}\left\langle\left[\left|\left(\frac{|A|^2+|B|^2}{2}\right)^{1/2}U\right|^2 + \left|\left(\frac{|A|^2+|B|^2}{2}\right)^{1/2}V^*\right|^2\right]x, x\right\rangle
\end{aligned} \tag{50}
$$

for any $x \in H$, $\|x\| = 1$.

From (48) we have

$$
\begin{aligned}
&\left|\langle V\operatorname{Re}(B^*A)Ux, x\rangle\right| \\
&\le \frac{1}{4}\left\langle\left[\left|\left(\frac{|A|^2+|B|^2}{2}\right)^{1/2}U\right|^2 + \left|\left(\frac{|A|^2+|B|^2}{2}\right)^{1/2}V^*\right|^2\right]x, x\right\rangle \\
&\quad + \frac{1}{2}\left|\left\langle V\frac{|A|^2+|B|^2}{2}Ux, x\right\rangle\right|
\end{aligned} \tag{51}
$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$ in (51) we get the desired inequality (45). □

Corollary 3 *If* $A, B \geq 0$ *with* $AB = BA$, *then for any* $u, V \in B(H)$ *we have*

$$\|VABu\| \leq \frac{1}{2}\left\| \left(\frac{A^2+B^2}{2}\right)^{1/2} u\right\| \left\| V\left(\frac{A^2+B^2}{2}\right)^{1/2}\right\| + \frac{1}{2}\left\| V\left(\frac{A^2+B^2}{2}\right) u\right\|, \tag{52}$$

$$w(VABu) \leq \frac{1}{2}\left\| \left(\frac{A^2+B^2}{2}\right)^{1/2} u\right\| \left\| V\left(\frac{A^2+B^2}{2}\right)^{1/2}\right\| + \frac{1}{2} w\left(V\left(\frac{A^2+B^2}{2}\right) u\right) \tag{53}$$

and

$$w(VABu) \leq \frac{1}{4}\left\| \left| \left(\frac{A^2+B^2}{2}\right)^{1/2} u\right|^2 + \left| \left(\frac{A^2+B^2}{2}\right)^{1/2} V^*\right|^2 \right\| + \frac{1}{2} w\left(V\left(\frac{A^2+B^2}{2}\right) u\right). \tag{54}$$

Remark 2 *If we take in Corollary 3* $A = 1_H$ *and* $B = P$, *a projection on* H, *then we get*

$$\|VPu\| \leq \frac{1}{2}\left\| \left(\frac{1_H+P}{2}\right)^{1/2} u\right\| \left\| V\left(\frac{1_H+P}{2}\right)^{1/2}\right\| + \frac{1}{2}\left\| V\left(\frac{1_H+P}{2}\right) u\right\|, \tag{55}$$

$$w(VPu) \leq \frac{1}{2}\left\| \left(\frac{1_H+P}{2}\right)^{1/2} u\right\| \left\| V\left(\frac{1_H+P}{2}\right)^{1/2}\right\| + \frac{1}{2} w\left(V\left(\frac{1_H+P}{2}\right) u\right) \tag{56}$$

and

$$wVPu) \leq \frac{1}{4}\left\| \left| \left(\frac{1_H+P}{2}\right)^{1/2} u\right|^2 + \left| \left(\frac{1_H+P}{2}\right)^{1/2} V^*\right|^2 \right\| + \frac{1}{2} w\left(V\left(\frac{1_H+P}{2}\right) u\right). \tag{57}$$

Finally, we have:

Corollary 4 *Let* T *be a unitary operator with* $\mathrm{Re}(T) \geq 0$. *Then for any* $U, V \in B(H)$ *we have*

$$\|V\mathrm{Re}(T)U\| \leq \frac{1}{2}\left[\|U\|\|V\| + \|VU\|\right], \tag{58}$$

$$w(V\mathrm{Re}(T)U) \leq \frac{1}{2}\left[\|U\|\|V\| + w(VU)\right] \tag{59}$$

and

$$w(V\mathrm{Re}(T)U) \leq \frac{1}{4}\left\||U|^2 + |V^*|^2\right\| + \frac{1}{2}w(VU). \tag{60}$$

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