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Research Article

Robust l_2-l_∞ Filtering for Discrete-Time Delay Systems

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The problem of robust l_2-l_∞ filtering for discrete-time system with interval time-varying delay and uncertainty is investigated, where the time delay and uncertainty considered are varying in a given interval and norm-bounded, respectively. The filtering problem based on the l_2-l_∞ performance is to design a filter such that the filtering error system is asymptotically stable with minimizing the peak value of the estimation error for all possible bounded energy disturbances. Firstly, sufficient l_2-l_∞ performance analysis condition is established in terms of linear matrix inequalities (LMIs) for discrete-time delay systems by utilizing reciprocally convex approach. Then a less conservative result is obtained by introducing some variables to decouple the Lyapunov matrices and the filtering error system matrices. Moreover, the robust l_2-l_∞ filter is designed for systems with time-varying delay and uncertainty. Finally, a numerical example is given to demonstrate the effectiveness of the filter design method.

1. Introduction

The uncertainty is unavoidable in practical engineering due to the parameter drafting, modeling error, and component aging. The controllers or filtering obtained based on nominal systems cannot be employed to get the desired performance. Therefore, more and more researchers are devoted to robust control or robust filtering problems; see, for instance, [1–4]. On the other hand, time-delay often exists in the practical engineering systems and is the main reason of the instability and poor performance of the systems. Time-delay systems have been widely studied during the past two decades [5–7]. In order to get less conservative results, more and more approaches have been proposed to develop delay-dependent conditions for discrete-time system with time-varying delay. For examples, Jensen's inequality is proposed in [8]; delay-partitioning method is utilized in [9]; improved results are obtained by using convex combination approach in [10].

In some practical applications, the peak value of the estimation error is required to be within a certain range and the aim of the l_2-l_∞ (energy-to-peak) filtering is to minimize the peak values of the filtering error for any bounded energy disturbance, which has received many attention. By using a parameter-dependent approach, the robust energy-to-peak filtering problem is considered in [11]. An improved robust energy-to-peak filtering condition is proposed by increasing the flexible dimensions in the solution space in [12]. The robust L_2-L_∞ filtering for stochastic systems and the exponential L_2-L_∞ filtering for Markovian jump systems are investigated in [13, 14], respectively. Compared with the corresponding continuous-time systems, discrete-time systems with time-varying delay have more stronger application background [15]. For discrete-time Markovian jumping systems, the reduced-order filter is designed in [16] such that the filtering error system satisfies an energy-to-peak performance. When time-delay appears, the robust energy-to-peak filtering problem for networked systems is tackled in [17]. For discrete-time switched systems with time-varying delay, an improved robust energy-to-peak filtering design method is proposed in [18].

In this paper we consider the problem of robust l_2-l_∞ filtering for uncertain discrete-time systems with time-varying delay. The filter is designed by employing the reciprocally convex approach proposed in [19] such that the filtering error system is asymptotically stable with an l_2-l_∞ performance. Firstly, a sufficient condition of the l_2-l_∞ performance analysis for nominal systems is obtained in terms of LMIs for systems with time-varying delay and uncertainty. Based on this criterion, by introducing some slack matrices, a less conservative result is obtained. Moreover, the desired filter for nominal systems with time-varying delay is obtained by solving a set of LMIs. Then the result is extended to the uncertain systems. A numerical example is given to illustrate the effectiveness of the presented results.

Notation. The notation used throughout the paper is given as follows. $\mathbb{R}^n$ is the n-dimensional Euclidean space and $P > 0$ (≥ 0) denotes that matrix P is real symmetric and positive definite (semidefinite); I and 0 present the identity matrix and zero matrix with compatible dimensions, respectively; $\star$ means the symmetric terms in a symmetric matrix and $\mathrm{sym}(A)$ stands for $A + A^T$; l_2 means the space of square summable infinite vector sequences; for any real function $x \in l_2$, we define $\|x\|_\infty = \sup_k \sqrt{x^T(k)x(k)}$ and $\|x\|_2 = \sqrt{\sum_{k=0}^{\infty} x^T(k)x(k)}$; $\|\cdot\|$ refer to the Euclidean vector norm. Matrices are assumed to be compatible for algebraic operations if their dimensions are not explicitly stated.

2. Problem Statement

Consider a class of uncertain discrete-time systems with time-varying delay described by

$$\begin{aligned} x(k+1) &= A(\sigma)x(k) + A_d(\sigma)x(k-d(k)) + B(\sigma)w(k), \\ y(k) &= C(\sigma)x(k) + C_d(\sigma)x(k-d(k)) + D(\sigma)w(k), \\ z(k) &= L(\sigma)x(k) + L_d(\sigma)x(k-d(k)) + G(\sigma)w(k), \\ x(k) &= \phi(k), \quad k = -d_2, -d_2+1, \ldots, 0, \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector; $y(k) \in \mathbb{R}^m$ is the measured output; $z(k) \in \mathbb{R}^p$ represents the signal to be estimated; $w(k) \in \mathbb{R}^l$ is assumed to be an arbitrary noise belonging to l_2 and $\phi(k)$ is a given initial condition sequence; $d(k)$ is a time-varying delay satisfying

$$1 \leq d_1 \leq d(k) \leq d_2 < \infty, \quad k = 1, 2, \ldots \tag{2}$$

$A(\sigma)$, $A_d(\sigma)$, $B(\sigma)$, $C(\sigma)$, $C_d(\sigma)$, $D(\sigma)$, $L(\sigma)$, $L_d(\sigma)$, and $G(\sigma)$ are system matrices and satisfy

$$\begin{gathered} A(\sigma) = A + \Delta A(\sigma), \qquad A_d(\sigma) = A_d + \Delta A_d(\sigma), \\ B(\sigma) = B + \Delta B(\sigma), \\ C(\sigma) = C + \Delta C(\sigma), \qquad C_d(\sigma) = C_d + \Delta C_d(\sigma), \\ D(\sigma) = D + \Delta D(\sigma), \\ L(\sigma) = L + \Delta L(\sigma), \qquad L_d(\sigma) = L_d + \Delta L_d(\sigma), \\ G(\sigma) = G + \Delta G(\sigma). \end{gathered} \tag{3}$$

Matrices $\Delta A(\sigma)$, $\Delta A_d(\sigma)$, $\Delta B(\sigma)$, $\Delta C(\sigma)$, $\Delta C_d(\sigma)$, $\Delta D(\sigma)$, $\Delta L(\sigma)$, $\Delta L_d(\sigma)$, and $\Delta G(\sigma)$ are unknown time-invariant matrix representing the uncertainty of the system satisfying the following conditions:

$$\begin{aligned} &\left[\Delta A(\sigma) \;\; \Delta A_d(\sigma) \;\; \Delta B(\sigma)\right] = M_1 \Delta_1(\sigma) \left[N_A \;\; N_{Ad} \;\; N_B\right], \\ &\qquad \Delta_1^T(\sigma)\Delta_1(\sigma) \leq I, \\ &\left[\Delta C(\sigma) \;\; \Delta C_d(\sigma) \;\; \Delta D(\sigma)\right] = M_2 \Delta_2(\sigma) \left[N_C \;\; N_{Cd} \;\; N_D\right], \\ &\qquad \Delta_2^T(\sigma)\Delta_2(\sigma) \leq I, \\ &\left[\Delta L(\sigma) \;\; \Delta L_d(\sigma) \;\; \Delta G(\sigma)\right] = M_3 \Delta_3(\sigma) \left[N_L \;\; N_{Ld} \;\; N_G\right], \\ &\qquad \Delta_3^T(\sigma)\Delta_3(\sigma) \leq I, \end{aligned} \tag{4}$$

where $\sigma \in \Theta$ and Θ is a compact set in $\mathbb{R}$. The system in (1) is assumed to be asymptotically stable. Our purpose is to design a full order linear filter for the estimate of $z(k)$:

$$\begin{aligned} \hat{x}(k+1) &= A_f \hat{x}(k) + B_f y(k), \quad \hat{x}(0) = 0, \\ \hat{z}(k) &= C_f \hat{x}(k) + D_f y(k), \end{aligned} \tag{5}$$

where A_f, B_f, C_f, and D_f are filter gains to be determined.

Let the augmented state vector $\tilde{x}(k) = \left[x^T(k) \;\; \hat{x}^T(k)\right]^T$ and $\tilde{z}(k) = z(k) - \hat{z}(k)$. Then the filtering error system is described as

$$\begin{aligned} \tilde{x}(k+1) &= \tilde{A}(\sigma)\tilde{x}(k) + \tilde{A}_d(\sigma)\Phi\tilde{x}(k-d(k)) + \tilde{B}(\sigma)w(k) \\ \tilde{z}(k) &= \tilde{L}(\sigma)\tilde{x}(k) + \tilde{L}_d(\sigma)\Phi\tilde{x}(k-d(k)) + \tilde{G}(\sigma)w(k) \\ \tilde{x}(k) &= \left[\phi^T \;\; (k) \;\; 0\right]^T, \quad k = -d_2, -d_2+1, \ldots, 0, \end{aligned} \tag{6}$$

where $\Phi = [I \;\; 0]$ and

$$\begin{gathered} \tilde{A}(\sigma) = \begin{bmatrix} A(\sigma) & 0 \\ B_f C(\sigma) & A_f \end{bmatrix}, \qquad \tilde{A}_d(\sigma) = \begin{bmatrix} A_d(\sigma) \\ B_f C_d(\sigma) \end{bmatrix}, \\ \tilde{B} = \begin{bmatrix} B(\sigma) \\ B_f D(\sigma) \end{bmatrix}, \\ \tilde{L}(k) = \left[L(\sigma) - D_f C(\sigma) \;\; -C_f\right], \\ \tilde{L}_d(\sigma) = L_d(\sigma) - D_f C_d(\sigma), \qquad \tilde{G}(\sigma) = G(\sigma) - D_f D(\sigma). \end{gathered} \tag{7}$$

The nominal system of (6) is system (6) without uncertainty; that is, $\Delta A(\sigma) = 0$, $\Delta A_d(\sigma) = 0$, $\Delta B(\sigma) = 0$, $\Delta C(\sigma) = 0$, $\Delta C_d(\sigma) = 0$, $\Delta D(\sigma) = 0$, $\Delta L(\sigma) = 0$, $\Delta L_d(\sigma) = 0$, and $\Delta G(\sigma) = 0$.

The following lemmas and definition will be utilized in the derivation of the main results.

Lemma 1 (see [20]). *For any matrices U and $V > 0$, the following inequality holds:*

$$UV^{-1}U^T \geq U + U^T - V. \tag{8}$$

Lemma 2 (see [19]). *Let $f_1, f_2, \dots, f_N : \mathbb{R}^m \to \mathbb{R}$ have positive values in a subset D of $\mathbb{R}^m$. Then, the reciprocally convex combination of f_i over D satisfies*

$$\min_{\{\alpha_i|\alpha_i>0,\sum_i \alpha_i=1\}} \sum_i \frac{1}{\alpha_i} f_i(k) = \sum_i f_i(k) + \max_{g_{i,j}(k)} \sum_{i\neq j} g_{i,j}(k) \tag{9}$$

subject to

$$\left\{ g_{i,j} : \mathbb{R}^m \longrightarrow \mathbb{R}, g_{j,i}(k) = g_{i,j}(k), \begin{bmatrix} f_i(k) & g_{i,j}(k) \\ g_{j,i}(k) & f_j(k) \end{bmatrix} \geq 0 \right\}. \tag{10}$$

Lemma 3. *For any constant matrix $M > 0$, integers $a \leq b$, vector function w: $\{a, a+1, \dots, b\} \to \mathbb{R}^n$, then*

$$-(b-a+1)\sum_{i=a}^{b} w^T(i) M w(i) \leq -\left(\sum_{i=a}^{b} w(i)\right)^T M \left(\sum_{i=a}^{b} w(i)\right). \tag{11}$$

Lemma 4. *Given a symmetric matrix Q and matrices H, E with appropriate dimensions, then*

$$Q + \operatorname{sym}(HFE) < 0, \tag{12}$$

for all $F^T F \leq I$, if and only if there exists a scalar $\varepsilon > 0$ such that

$$Q + \varepsilon E^T E + \varepsilon^{-1} H H^T < 0. \tag{13}$$

Definition 5. Given a scalar $\gamma > 0$, the filtering error $\tilde{z}(k)$ in (6) is said to satisfy the l_2-l_∞ disturbance attenuation level γ under zero initial state, and the following condition is satisfied:

$$\|\tilde{z}\|_\infty < \gamma \|w\|_2. \tag{14}$$

Our aim is to design a filter in the form of (5) such that the filtering error system in (6) is asymptotically stable and satisfies the l_2-l_∞ performance defined in Definition 5.

3. Main Results

In this section, the sufficient l_2-l_∞ performance analysis condition is first derived for nominal filtering error system of (6). Then an equivalent result is obtained by introducing three slack matrices. Based on these results, a desired filter is designed to render the nominal system of (6) asymptotically stable with an l_2-l_∞ performance. Then the result is extended to the uncertain system in (6).

3.1. l_2-l_∞ Performance Analysis

In this subsection, we first give the result of l_2-l_∞ performance analysis for nominal system of (6).

Theorem 6. *Given a scalar $\gamma > 0$, the nominal system of (6) is asymptotically stable with an l_2-l_∞ performance if there exist matrices $P > 0$, $Q_3 > 0$, $\overline{Q}_i > 0$, $i = 1, 2$, $S_j > 0$, $j = 1, 2$, and M such that the following LMIs hold:*

$$\begin{bmatrix} S_2 & M \\ \star & S_2 \end{bmatrix} \geq 0, \tag{15}$$

$$\begin{bmatrix} P & 0 & 0 & \widetilde{L}^T \\ \star & Q_3 & 0 & \widetilde{L}_d^T \\ \star & \star & I & \widetilde{G}^T \\ \star & \star & \star & \gamma^2 I \end{bmatrix} > 0, \tag{16}$$

$$\widetilde{\Pi} = \begin{bmatrix} \widetilde{\Pi}_{11} & \Phi^T S_1 & 0 & 0 & 0 & \widetilde{\Pi}_{16} S_1 & \widetilde{\Pi}_{17} S_2 & \widetilde{A}^T P \\ \star & \widetilde{\Pi}_{22} & \widetilde{\Pi}_{23} & M^T & 0 & 0 & 0 & 0 \\ \star & \star & \widetilde{\Pi}_{33} & \widetilde{\Pi}_{34} & 0 & d_1 A_d^T S_1 & \tilde{d} A_d^T S_2 & \widetilde{A}_d^T P \\ \star & \star & \star & \widetilde{\Pi}_{44} & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -I & d_1 B^T S_1 & \tilde{d} B^T S_2 & \widetilde{B}^T P \\ \star & \star & \star & \star & \star & -S_1 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -S_2 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -P \end{bmatrix} < 0, \tag{17}$$

where

$$\begin{gathered} \widetilde{\Pi}_{11} = -P + Q_1 + Q_2 + \left(\tilde{d} + 1\right) \Phi^T Q_3 \Phi - \Phi^T S_1 \Phi, \\ \widetilde{\Pi}_{22} = -S_2 - \overline{Q}_1 - S_1, \qquad \widetilde{\Pi}_{23} = S_2 - M^T, \\ \widetilde{\Pi}_{33} = -Q_3 - 2S_2 + \operatorname{sym}(M), \qquad \widetilde{\Pi}_{34} = S_2 - M^T, \\ \widetilde{\Pi}_{44} = -\overline{Q}_2 - S_2, \\ \widetilde{\Pi}_{16} = d_1 \Phi^T (A - I)^T, \qquad \widetilde{\Pi}_{17} = \tilde{d} \Phi^T (A - I)^T, \\ Q_i = \operatorname{diag}\left\{\overline{Q}_i, \overline{Q}_i\right\}, \quad i = 1, 2, \ \tilde{d} = d_2 - d_1. \end{gathered} \tag{18}$$

Proof. First, the asymptotic stability of the nominal system of (6) is proved. We denote $\tilde{\eta}(k) = \tilde{x}(k+1) - \tilde{x}(k)$ and the following Lyapunov functional is chosen:

$$V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k) + V_5(k), \tag{19}$$

where

$$\begin{gathered} V_1(k) = \tilde{x}^T(k) P \tilde{x}(k), \\ V_2(k) = \sum_{j=1}^{2} \sum_{i=k-d_j}^{k-1} \tilde{x}^T(i) Q_i \tilde{x}(i), \end{gathered}$$

$$V_3(k)=\sum_{i=k-d(k)}^{k-1}\tilde{x}^T(i)\Phi^T Q_3\Phi\tilde{x}(i)+\sum_{j=-d_2+1}^{-d_1}\sum_{i=k+j}^{k-1}\tilde{x}^T(i)\Phi^T Q_3\Phi\tilde{x}(i),$$

$$V_4(k)=\sum_{j=-d_1}^{-1}\sum_{i=k+j}^{k-1}d_1\tilde{\eta}^T(i)\Phi^T S_1\Phi\tilde{\eta}(i),$$

$$V_5(k)=\sum_{j=-d_2}^{-d_1-1}\sum_{i=k+j}^{k-1}\tilde{d}\tilde{\eta}^T(i)\Phi^T S_2\Phi\tilde{\eta}(i). \tag{20}$$

Calculating the forward difference of $V(k)$ along the trajectories of filtering error system (6) with $w(k)=0$ yields

$$\begin{aligned}\Delta V_1(k)&=\tilde{x}^T(k+1)P\tilde{x}(k+1)-\tilde{x}^T(k)P\tilde{x}(k)\\&=\left(\widetilde{A}\tilde{x}(k)+\widetilde{A}_d\Phi\tilde{x}(k-d(k))\right)^T\\&\quad\times P\left(\widetilde{A}\tilde{x}(k)+\widetilde{A}_d\Phi\tilde{x}(k-d(k))\right)\\&\quad-\tilde{x}^T(k)P\tilde{x}(k),\end{aligned} \tag{21}$$

$$\begin{aligned}\Delta V_2(k)&=\sum_{j=1}^{2}\tilde{x}^T(k)Q_j\tilde{x}(k)-\sum_{j=1}^{2}\tilde{x}\left(k-d_j\right)Q_j\tilde{x}\left(k-d_j\right)\\&\le\sum_{j=1}^{2}\tilde{x}^T(k)Q_j\tilde{x}(k)-\sum_{j=1}^{2}\tilde{x}\left(k-d_j\right)\Phi^T\overline{Q}_j\Phi\tilde{x}\left(k-d_j\right),\end{aligned} \tag{22}$$

$$\begin{aligned}\Delta V_3(k)&=\left(\tilde{d}+1\right)\tilde{x}^T(k)\Phi^T Q_3\Phi\tilde{x}(k)\\&\quad+\sum_{i=k+1-d(k+1)}^{k-1}\tilde{x}^T(i)\Phi^T Q_3\Phi\tilde{x}(i)\\&\quad-\sum_{i=k+1-d(k)}^{k-1}\tilde{x}^T(i)\Phi^T Q_3\Phi\tilde{x}(i)\\&\quad-\tilde{x}^T(k-d(k))\Phi^T Q_3\Phi\tilde{x}(k-d(k))\\&\quad-\sum_{i=k-d_2+1}^{k-d_1}\tilde{x}^T(i)\Phi^T Q_3\Phi\tilde{x}(i)\\&=\left(\tilde{d}+1\right)\tilde{x}^T(k)\Phi^T Q_3\Phi\tilde{x}(k)\\&\quad+\sum_{i=k+1-d_1}^{k-1}\tilde{x}^T(i)\Phi^T Q_3\Phi\tilde{x}(i)\\&\quad+\sum_{i=k+1-d(k+1)}^{k-d_1}\tilde{x}^T(i)\Phi^T Q_3\Phi\tilde{x}(i)\\&\quad-\tilde{x}^T(k-d(k))\Phi^T Q_3\Phi\tilde{x}(k-d(k))\\&\quad-\sum_{i=k+1-d(k)}^{k-1}\tilde{x}^T(i)\Phi^T Q_3\Phi\tilde{x}(i)\\&\quad-\sum_{i=k-d_2+1}^{k-d_1}\tilde{x}^T(i)\Phi^T Q_3\Phi\tilde{x}(i)\\&\le\left(\tilde{d}+1\right)\tilde{x}^T(k)\Phi^T Q_3\Phi\tilde{x}(k)\\&\quad-\tilde{x}^T(k-d(k))\Phi^T Q_3\Phi\tilde{x}(k-d(k)).\end{aligned} \tag{23}$$

By using Lemma 3, we have

$$\begin{aligned}\Delta V_4(k)&=d_1^2\tilde{\eta}^T(k)\Phi^T S_1\Phi\tilde{\eta}(k)-d_1\sum_{i=k-d_1}^{k-1}\tilde{\eta}^T(i)\Phi^T S_1\Phi\tilde{\eta}(i)\\&\le d_1^2\left((A-I)\Phi\tilde{x}(k)+A_d\Phi\tilde{x}(k-d(k))\right)^T\\&\quad\times S_1\left((A-I)\Phi\tilde{x}(k)+A_d\Phi\tilde{x}(k-d(k))\right)\\&\quad-\left(\Phi\tilde{x}(k)-\Phi\tilde{x}\left(k-d_1\right)\right)^T\\&\quad\times S_1\left(\Phi\tilde{x}(k)-\Phi\tilde{x}\left(k-d_1\right)\right).\end{aligned} \tag{24}$$

Since $\begin{bmatrix} S_2 & M \\ \star & S_2 \end{bmatrix}\ge 0$, the following inequality holds:

$$\begin{bmatrix}\sqrt{\frac{\alpha_1}{\alpha_2}}\left(x(k-d(k))-x\left(k-d_2\right)\right)\\-\sqrt{\frac{\alpha_2}{\alpha_1}}\left(x\left(k-d_1\right)-x(k-d(k))\right)\end{bmatrix}^T\times\begin{bmatrix} S_2 & M \\ \star & S_2 \end{bmatrix}\times\begin{bmatrix}\sqrt{\frac{\alpha_1}{\alpha_2}}\left(x(k-d(k))-x\left(k-d_2\right)\right)\\-\sqrt{\frac{\alpha_2}{\alpha_1}}\left(x\left(k-d_1\right)-x(k-d(k))\right)\end{bmatrix}\ge 0, \tag{25}$$

where $\alpha_1 = (d_2 - d(k))/\tilde{d}$ and $\alpha_2 = (d(k) - d_1)/\tilde{d}$. Then employing Lemma 2, for $d_1 < d(k) < d_2$, we have

$$\begin{aligned}
\Delta V_5(k) &= \tilde{d}^2 \tilde{\eta}^T(k) \Phi^T S_2 \Phi \tilde{\eta}(k) \\
&\quad - \tilde{d} \sum_{i=k-d_2}^{k-d(k)-1} \tilde{\eta}^T(i) \Phi^T S_2 \Phi \tilde{\eta}(i) \\
&\quad - \tilde{d} \sum_{i=k-d(k)}^{k-d_1-1} \tilde{\eta}^T(i) \Phi^T S_2 \Phi \tilde{\eta}(i) \\
&\le \tilde{d}^2 \tilde{\eta}^T(k) \Phi^T S_2 \Phi \tilde{\eta}(k) \\
&\quad - \frac{\tilde{d}}{d_2 - d(k)} \left(\sum_{i=k-d_2}^{k-d(k)-1} \Phi \tilde{\eta}(i) \right)^T S_2 \left(\sum_{i=k-d_2}^{k-d(k)-1} \Phi \tilde{\eta}(i) \right) \\
&\quad - \frac{\tilde{d}}{d(k) - d_1} \left(\sum_{i=k-d(k)}^{k-d_1-1} \Phi \tilde{\eta}(i) \right)^T S_2 \left(\sum_{i=k-d(k)}^{k-d_1-1} \Phi \tilde{\eta}(i) \right) \\
&\le \tilde{d}^2 \left((A - I) \Phi \tilde{x}(k) + A_d \Phi \tilde{x}(k - d(k)) \right)^T \\
&\quad \times S_2 \left((A - I) \Phi \tilde{x}(k) + A_d \Phi \tilde{x}(k - d(k)) \right) \\
&\quad - \begin{bmatrix} x(k - d(k)) - x(k - d_2) \\ x(k - d_1) - x(k - d(k)) \end{bmatrix}^T \begin{bmatrix} S_2 & M \\ \star & S_2 \end{bmatrix} \\
&\quad \times \begin{bmatrix} x(k - d(k)) - x(k - d_2) \\ x(k - d_1) - x(k - d(k)) \end{bmatrix}.
\end{aligned} \tag{26}$$

Note that when $d(k) = d_1$ or $d(k) = d_2$, it yields $x(k - d_1) - x(k - d(k)) = 0$ or $x(k - d(k)) - x(k - d_2) = 0$. Hence, the inequality in (24) still holds. Combining the conditions from (21) to (24), we have

$$\Delta V(k) = \zeta^T(k) \Pi_s \zeta(k), \tag{27}$$

where

$$\begin{aligned}
&\zeta(k) \\
&= \left[\tilde{x}^T(k) \quad \tilde{x}^T(k - d_1) \Phi^T \quad \tilde{x}^T(k - d(k)) \Phi^T \quad \tilde{x}^T(k - d_2) \Phi^T \right]^T, \\
&\Pi_s = \begin{bmatrix} \tilde{\Pi}_{11} & \Phi^T S_1 & 0 & 0 \\ \star & \tilde{\Pi}_{22} & \tilde{\Pi}_{23} & M^T \\ \star & \star & \tilde{\Pi}_{33} & \tilde{\Pi}_{34} \\ \star & \star & \star & \tilde{\Pi}_{44} \end{bmatrix} + \begin{bmatrix} \tilde{\Pi}_{16} \\ 0 \\ d_1 A_d^T \\ 0 \end{bmatrix} S_1 \begin{bmatrix} \tilde{\Pi}_{16} \\ 0 \\ d_1 A_d^T \\ 0 \end{bmatrix}^T \\
&\quad + \begin{bmatrix} \tilde{\Pi}_{17} \\ 0 \\ \tilde{d} A_d^T \\ 0 \end{bmatrix} S_2 \begin{bmatrix} \tilde{\Pi}_{17} \\ 0 \\ \tilde{d} A_d^T \\ 0 \end{bmatrix}^T + \begin{bmatrix} \tilde{A}^T \\ 0 \\ \tilde{A}_d^T \\ 0 \end{bmatrix} P \begin{bmatrix} \tilde{A}^T \\ 0 \\ \tilde{A}_d^T \\ 0 \end{bmatrix}^T.
\end{aligned} \tag{28}$$

On the other hand, the following inequality can be obtained from (17):

$$\Pi_{s1} = \begin{bmatrix} \tilde{\Pi}_{11} & \Phi^T S_1 & 0 & 0 & \tilde{\Pi}_{16} S_1 & \tilde{\Pi}_{17} S_2 & \tilde{A}^T P \\ \star & \tilde{\Pi}_{22} & \tilde{\Pi}_{23} & M^T & 0 & 0 & 0 \\ \star & \star & \tilde{\Pi}_{33} & \tilde{\Pi}_{34} & d_1 A_d^T S_1 & \tilde{d} A_d^T S_2 & \tilde{A}_d^T P \\ \star & \star & \star & \tilde{\Pi}_{44} & 0 & 0 & 0 \\ \star & \star & \star & \star & -S_1 & 0 & 0 \\ \star & \star & \star & \star & \star & -S_2 & 0 \\ \star & \star & \star & \star & \star & \star & -P \end{bmatrix} < 0 \tag{29}$$

which is equivalent to $\Pi_s < 0$. Hence, $\Delta V(k) < 0$ which implies that the filtering error system in (6) with $w(k) = 0$ is asymptotically stable.

Next, we show the l_2-l_∞ performance of system (6). To this end, we define

$$J(k) = V(k) - \sum_{i=0}^{k-1} w^T(i) w(i). \tag{30}$$

Then under the zero initial condition, that is, $x(k) = 0$, $k = -d_2, -d_2 + 1, \ldots, 0$, it can be shown that for any nonzero $w(k) \in l_2[0, \infty)$

$$\begin{aligned}
J(k) &= \sum_{i=0}^{k-1} \left[\Delta V(i) - w^T(i) w(i) \right] \\
&= \sum_{i=0}^{k-1} \xi^T(i) \left(\overline{\Pi} + \overline{\Pi}_1^T P \overline{\Pi}_1 + d_1^2 \overline{\Pi}_2^T S_1 \overline{\Pi}_2 \right. \\
&\quad \left. + d_{12}^2 \overline{\Pi}_2^T S_2 \overline{\Pi}_2 \right) \xi(i),
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
&\xi(i) \\
&= \left[\tilde{x}^T(i) \quad \tilde{x}^T(i - d_1) \Phi^T \quad \tilde{x}^T(i - d(i)) \Phi^T \quad \tilde{x}^T(i - d_2) \Phi^T \quad w(i) \right]^T, \\
&\overline{\Pi} = \begin{bmatrix} \tilde{\Pi}_{11} & \Phi^T S_1 & 0 & 0 & \tilde{\Pi}_{15} \\ \star & \tilde{\Pi}_{22} & \tilde{\Pi}_{23} & M^T & 0 \\ \star & \star & \tilde{\Pi}_{33} & \tilde{\Pi}_{34} & -\tilde{L}_d^T S \\ \star & \star & \star & \tilde{\Pi}_{44} & 0 \\ \star & \star & \star & \star & \tilde{\Pi}_{55} \end{bmatrix}, \\
&\overline{\Pi}_1 = \left[\tilde{A} \quad 0 \quad \tilde{A}_d \quad 0 \quad \tilde{B} \right], \\
&\overline{\Pi}_2 = \left[(A - I) \Phi \quad 0 \quad A_d \quad 0 \quad B \right], \\
&\overline{\Pi}_3 = \left[\tilde{L} \quad 0 \quad \tilde{L}_d \quad 0 \quad \tilde{G} \right].
\end{aligned} \tag{32}$$

By using Schur complement equivalence, the inequality in (17) is equivalent to $\overline{\Pi} + \overline{\Pi}_1^T P \overline{\Pi}_1 + d_1^2 \overline{\Pi}_2^T S_1 \overline{\Pi}_2 + d_{12}^2 \overline{\Pi}_2^T S_2 \overline{\Pi}_2 < 0$. Then we have $J(k) < 0$; that is,

$$V(k) < \sum_{i=0}^{k-1} w^T(i) w(i). \tag{33}$$

On the other hand, it yields from (16) and (33) that

$$\begin{aligned}\tilde{z}^T(k)\tilde{z}(k) &= \eta^T(k)\begin{bmatrix}\tilde{L} & \tilde{L}_d & \tilde{G}\end{bmatrix}^T\begin{bmatrix}\tilde{L} & \tilde{L}_d & \tilde{G}\end{bmatrix}\eta(k)\\ &\le \gamma^2\eta^T(k)\begin{bmatrix}P & 0 & 0\\ \star & Q_3 & 0\\ \star & \star & I\end{bmatrix}\eta(k)\\ &\le \gamma^2\left(V(k)+w^T(k)w(k)\right)\\ &\le \gamma^2\sum_{i=0}^{k}w^T(i)w(i)\le\gamma^2\sum_{i=0}^{\infty}w^T(i)w(i),\end{aligned}\tag{34}$$

where

$$\eta(k)=\begin{bmatrix}\tilde{x}(k)\\ \Phi\tilde{x}(k-d(k))\\ w(k)\end{bmatrix}.\tag{35}$$

Then, we have $\|\tilde{z}\|_\infty < \gamma\|w\|_2$ by taking the supremum over time $k > 0$. By Definition 5, the filtering error $\tilde{z}(k)$ satisfies a given l_2-l_∞ disturbance attenuation level. This completes the proof. □

Remark 7. The advantage of the results benefits from utilizing the reciprocally convex combination approach proposed in [19]. For the extensively used Jensen inequality [8], the integral term

$$\begin{aligned}&-\sum_{i=k-d_2}^{k-d_1-1}(d_2-d_1)\eta^T(i)S\eta(i)\\ &=-\sum_{i=k-d_2}^{k-d(k)-1}(d_2-d_1)\eta^T(i)S\eta(i)\\ &\quad-\sum_{i=k-d(k)}^{k-d_1-1}(d_2-d_1)\eta^T(i)S\eta(i)\end{aligned}\tag{36}$$

with $d_1 \le d(k) \le d_2$, $\eta(i) = x(i+1) - x(i)$ by the term

$$\begin{aligned}&-\left[x(k-d_1)-x(k-d(k))\right]^T S\left[x(k-d_1)-x(k-d(k))\right]\\ &-\left[x(k-d(k))-x(k-d_2)\right]^T S\left[x(k-d(k))-x(k-d_2)\right]\end{aligned}\tag{37}$$

which is a special case of the following term with $M = 0$

$$\begin{aligned}&-\begin{bmatrix}x(k-d(k))-x(k-d_1)\\ x(k-d(k))-x(k-d_2)\end{bmatrix}^T\\ &\times\begin{bmatrix}S & M\\ \star & S\end{bmatrix}\begin{bmatrix}x(k-d(k))-x(k-d_1)\\ x(k-d(t))-x(k-d_2)\end{bmatrix}\end{aligned}\tag{38}$$

with $\left[\begin{smallmatrix}S & M\\ \star & S\end{smallmatrix}\right] \ge 0$, which is one of the advantages of reciprocally convex combination approach. On the other hand, the delay partitioning method is widely applied to reduce the conservatism of the results [9, 21, 22]. Also, the method can be extended to the problem considered in this paper. However, it will rise significant computation cost with the partitioning number increasing. Therefore, the reciprocally convex method needs less decision variables and can be seen as a tradeoff between the conservatism and the computation cost.

Then, an equivalent condition of LMI (17) is obtained by introducing three slack matrices H_1, H_2, and T, which is presented in the following theorem.

Theorem 8. *Given a scalar $\gamma > 0$, the nominal system of (6) is asymptotically stable with an l_2-l_∞ performance if there exist matrices $P > 0$, $\overline{Q}_i > 0$, $i = 1, 2, 3$, $S_j > 0$, H_j, $j = 1, 2$, T, and M, such that the following LMIs hold:*

$$\begin{bmatrix}S_2 & M\\ \star & S_2\end{bmatrix}\ge 0,\tag{39}$$

$$\begin{bmatrix}P & 0 & 0 & \tilde{L}^T\\ \star & Q_3 & 0 & \tilde{L}_d^T\\ \star & \star & I & \tilde{G}^T\\ \star & \star & \star & \gamma^2 I\end{bmatrix}>0,\tag{40}$$

$$\tilde{\Omega}=\begin{bmatrix}\tilde{\Pi}_{11} & \Phi^T S_1 & 0 & 0 & 0 & \tilde{\Pi}_{16}H_1^T & \tilde{\Pi}_{17}H_2^T & \tilde{A}^T T^T\\ \star & \tilde{\Pi}_{22} & \tilde{\Pi}_{23} & M^T & 0 & 0 & 0 & 0\\ \star & \star & \tilde{\Pi}_{33} & \tilde{\Pi}_{34} & 0 & d_1 A_d^T H_1^T & \tilde{d}A_d^T H_2^T & \tilde{A}_d^T T^T\\ \star & \star & \star & \tilde{\Pi}_{44} & 0 & 0 & 0 & 0\\ \star & \star & \star & \star & -I & d_1 B^T H_1^T & \tilde{d}B^T H_2^T & \tilde{B}^T T^T\\ \star & \star & \star & \star & \star & S_1 - H_1^T - H_1 & 0 & 0\\ \star & \star & \star & \star & \star & \star & S_2 - H_2^T - H_2 & 0\\ \star & \star & \star & \star & \star & \star & \star & P - T^T - T\end{bmatrix}<0,\tag{41}$$

where $\widetilde{\Pi}_{ii}$, $i = 1, \ldots, 4$, $\widetilde{\Pi}_{16}$, $\widetilde{\Pi}_{17}$, $\widetilde{\Pi}_{23}$, *and* $\widetilde{\Pi}_{34}$ *are defined in* (17).

Proof. On one hand, if (17) holds, then there exist $H_j = H_j^T = S_j$, $j = 1, 2$, and $T = T^T = P$ such that (41) holds. On the other hand, if (41) holds, we have the following inequality based on Lemma 1:

$$\begin{bmatrix} \widetilde{\Pi}_{11} & \Phi^T S_1 & 0 & 0 & 0 & \widetilde{\Pi}_{16} H_1^T & \widetilde{\Pi}_{17} H_2^T & \widetilde{A}^T T^T \\ \star & \widetilde{\Pi}_{22} & \widetilde{\Pi}_{23} & M^T & 0 & 0 & 0 & 0 \\ \star & \star & \widetilde{\Pi}_{33} & \widetilde{\Pi}_{34} & 0 & d_1 A_d^T H_1^T & \tilde{d} A_d^T H_2^T & \widetilde{A}_d^T T^T \\ \star & \star & \star & \widetilde{\Pi}_{44} & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -I & d_1 B^T H_1^T & \tilde{d} B^T H_2^T & \widetilde{B}^T T^T \\ \star & \star & \star & \star & \star & -H_1 S_1^{-1} H_1^T & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -H_2 S_2^{-1} H_2^T & 0 \\ \star & \star & \star & \star & \star & \star & \star & -T P^{-1} T^T \end{bmatrix} < 0. \tag{42}$$

In addition, matrices H_j, $j = 1, 2$, and T are nonsingular due to $S_j - H_j^T - H_j < 0$, $j = 1, 2$, and $P - T^T - T < 0$. Then, pre- and promultiplying (42) by $\operatorname{diag}\{I, I, I, I, I, S_1 H_1^{-1}, S_2 H_2^{-1}, P T^{-1}\}$ and its transpose yields (17). Therefore, the equivalence between (41) and (17) is proved. □

3.2. Robust Filter Design. In this subsection, the filter in the form of (5) is firstly designed such that the nominal filtering error system of (6) is asymptotically stable with an l_2-l_∞ performance. Then the robust filtering problem is solved. Based on the result of Theorem 8, the filter design method for nominal system of (1) is presented in the following theorem.

Theorem 9. *Given a scalar* $\gamma > 0$, *the nominal system of* (6) *is asymptotically stable with an* l_2-l_∞ *performance if there exist matrices* $\left[\begin{smallmatrix} P_1 & P_2 \\ \star & P_3 \end{smallmatrix}\right] > 0$, $Q_3 > 0$, $\widetilde{Q}_l > 0$, $\overline{Q}_l > 0$, $l = 1, 2$, $S_j > 0$, H_j, F_j, $j = 1, 2$, *diagonal matrix* $N > 0$, T_1, *and* M *such that the following set of LMIs hold:*

$$\begin{bmatrix} S_2 & M \\ \star & S_2 \end{bmatrix} \geq 0 \tag{43}$$

$$\Omega = \begin{bmatrix} \Xi & \Gamma \\ \star & \Lambda \end{bmatrix} < 0 \tag{44}$$

$$\Upsilon = \begin{bmatrix} P_1 & P_2 & 0 & 0 & \left(L - \overline{D}_f C\right)^T \\ \star & P_3 & 0 & 0 & -\overline{C}_f^T \\ \star & \star & Q_3 & 0 & \left(L_d - \overline{D}_f C_d\right)^T \\ \star & \star & \star & I & \left(G - \overline{D}_f C_d\right)^T \\ \star & \star & \star & \star & \gamma^2 I \end{bmatrix} > 0, \tag{45}$$

where

$$\Xi = \begin{bmatrix} \Xi_{11} & -P_2 & S_1 & C^T N C_d & 0 & 0 \\ \star & \Xi_{22} & 0 & 0 & 0 & 0 \\ \star & \star & \Xi_{33} & S_2 - M^T & M^T & 0 \\ \star & \star & \star & \Xi_{44} & S_2 - M^T & 0 \\ \star & \star & \star & \star & \Xi_{55} & 0 \\ \star & \star & \star & \star & \star & -I \end{bmatrix},$$

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & A^T T_1^T + C^T \mathscr{A}_{s0} \overline{B}_f^T & A^T F_1^T + C^T \mathscr{A}_{s0} \overline{B}_f^T \\ 0 & 0 & \overline{A}_f^T & \overline{A}_f^T \\ 0 & 0 & 0 & 0 \\ d_1 A_d^T H_1^T & \tilde{d} A_d^T H_2^T & A_d^T T_1^T + C_d^T \mathscr{A}_{s0} \overline{B}_f^T & A_d^T F_1^T + C_{di}^T \mathscr{A}_{s0} \overline{B}_f^T \\ 0 & 0 & 0 & 0 \\ d_1 B^T H_1^T & \tilde{d} B^T H_2^T & B^T T_1^T + D^T \mathscr{A}_{s0} \overline{B}_f^T & B^T F_1^T + D^T \mathscr{A}_{s0} \overline{B}_f^T \end{bmatrix},$$

$$\Lambda = \begin{bmatrix} S_1 - H_1 - H_1^T & 0 & 0 & 0 \\ \star & S_2 - H_2 - H_2^T & 0 & 0 \\ \star & \star & P_1 - T_1 - T_1^T & P_2 - F_2 - F_1^T \\ \star & \star & \star & P_3 - F_2 - F_2^T \end{bmatrix},$$

$$\begin{gathered} \Xi_{11} = -\left(P_1 + S_1\right) + \overline{Q}_1 + \overline{Q}_2 + \left(\tilde{d} + 1\right) Q_3, \\ \Xi_{22} = -P_3 + \widetilde{Q}_1 + \widetilde{Q}_2, \\ \Xi_{33} = -S_2 - \overline{Q}_1 - S_1, \\ \Xi_{44} = -Q_3 + \operatorname{sym}\left(M_i - S_{2i}\right), \\ \Xi_{55} = -\overline{Q}_2 - S_2, \\ \Gamma_{11} = d_1 (A - I)^T H_1^T, \qquad \Gamma_{12} = \tilde{d} (A - I)^T H_2^T. \end{gathered} \tag{46}$$

Moreover, a suitable l_2-l_∞ *filter is given by*

$$\begin{gathered} A_f = \overline{A}_f F_2^{-1}, \qquad B_f = \overline{B}_f, \qquad C_f = \overline{C}_f F_2^{-1}, \\ D_f = \overline{D}_f. \end{gathered} \tag{47}$$

Proof. Firstly, we introduce four matrices T_1, T_2, T_3, and T_4 with T_4 invertible and define

$$\begin{gathered} J_1 = \begin{bmatrix} I & 0 \\ 0 & T_2 T_4^{-1} \end{bmatrix}, \qquad F_1 = T_2 T_4^{-1} T_3, \qquad F_2 = T_2 T_4^{-T} T_2^T, \\ \widetilde{Q}_l = T_2 T_4^{-1} \overline{Q}_l T_4^{-T} T_2^T, \qquad J = \operatorname{diag}\{J_1, I, I, I, I, I, I, I, J_1\}, \\ \begin{bmatrix} P_1 & P_2 \\ \star & P_3 \end{bmatrix} = J_1 P J_1^T, \qquad T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}, \\ J_2 = \operatorname{diag}\{J_1, I, I, I\}, \\ \begin{bmatrix} \overline{A}_f & \overline{B}_f \\ \overline{C}_f & \overline{D}_f \end{bmatrix} = \begin{bmatrix} T_2 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix} \begin{bmatrix} T_4^{-T} T_2^T & 0 \\ 0 & I \end{bmatrix}. \end{gathered} \tag{48}$$

From (44), we have $F_2 + F_2^T = T_2 T_4^{-T} T_2^T + T_2 T_4^{-1} T_2^T > 0$ which implies that T_2 is nonsingular. Hence, J and J_2 are nonsingular. The inequality in (45) can be obtained by pre- an promultiplying (33) with J_2 and J_2^T, respectively. Noting that

$$\overline{\Omega} = J\widetilde{\Omega}J^T \tag{49}$$

we have $\widetilde{\Omega} < 0$. On the other hand, because T_2 and T_4 cannot be obtained from (44), we cannot determine the filters from (48). However, we can construct an equivalent filter transfer function from $y(k)$ to $\tilde{z}(k)$:

$$\begin{aligned} T_{\tilde{z}y} &= C_f\left(zI - A_f\right)^{-1}B_f + D_f \\ &= \overline{C}_f T_2^{-T}T_4^T\left(zI - T_2^{-1}\overline{A}_f T_2^{-T}T_4^T\right)^{-1}T_2^{-1}\overline{B}_f + \overline{D}_f \\ &= \overline{C}_f\left(zF_2 - \overline{A}_f\right)^{-1}\overline{B}_f + \overline{D}_f \\ &= \overline{C}_f F_2^{-1}\left(zI - \overline{A}_f F_2^{-1}\right)^{-1}\overline{B}_f + \overline{D}_f. \end{aligned} \tag{50}$$

Therefore, the desired filter can be obtained from (47). This completes the proof. □

Then the filter design result for uncertain system (6) is presented in the following theorem.

Theorem 10. *Given a scalar $\gamma > 0$, the system in (6) with uncertainty is asymptotically stable with an l_2-l_∞ performance if there exist matrices $\left[\begin{smallmatrix} P_1 & P_2 \\ \star & P_3 \end{smallmatrix}\right] > 0$, $Q_3 > 0$, $\widetilde{Q}_l > 0$, $\overline{Q}_l > 0$, $l = 1, 2$, $S_j > 0$, H_j, F_j, $j = 1, 2$, diagonal matrix $N > 0$, T_1, M, and scalars $\varepsilon_i > 0$, $i = 1, \ldots, 4$ such that the following set of LMIs hold:*

$$\begin{bmatrix} S_2 & M \\ \star & S_2 \end{bmatrix} \geq 0, \tag{51}$$

$$\begin{bmatrix} \Omega + \varepsilon_3\Omega_1^T\Omega_1 + \varepsilon_4\Omega_2^T\Omega_2 & \Omega_3 & \Omega_4 \\ \star & -\varepsilon_3 I & 0 \\ \star & \star & -\varepsilon_4 I \end{bmatrix} < 0, \tag{52}$$

$$\begin{bmatrix} \Upsilon_{11} & P_2 & \Upsilon_{13} & \Upsilon_{14} & \left(L - \overline{D}_f C\right)^T & 0 & 0 \\ \star & P_3 & 0 & 0 & -\overline{C}_f^T & 0 & 0 \\ \star & \star & \Upsilon_{33} & \Upsilon_{34} & \left(L_d - \overline{D}_f C_d\right)^T & 0 & 0 \\ \star & \star & \star & \Upsilon_{44} & \left(G - \overline{D}_f C_d\right)^T & 0 & 0 \\ \star & \star & \star & \star & \gamma^2 I & M_3 & 0 \\ \star & \star & \star & \star & \star & \varepsilon_1 I & \overline{D}_f M_2 \\ \star & \star & \star & \star & \star & \star & \varepsilon_2 I \end{bmatrix} > 0, \tag{53}$$

where Ω is defined in (44) and

$$\Omega_1 = \left[N_A \;\; 0 \;\; 0 \;\; N_{Ad} \;\; 0 \;\; N_B \;\; 0 \;\; 0 \;\; 0 \;\; 0\right],$$

$$\Omega_2 = \left[N_C \;\; 0 \;\; 0 \;\; N_{Cd} \;\; 0 \;\; N_D \;\; 0 \;\; 0 \;\; 0 \;\; 0\right],$$

$$\begin{aligned} \Omega_3 &= \left[0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; d_1 M_1^T H_1^T \;\; \tilde{d} M_1^T H_2^T \;\; M_1^T T_1^T \;\; M_1^T F_1^T\right]^T, \\ \Omega_4 &= \left[0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; M_2^T \overline{B}_f^T \;\; M_2^T \overline{B}_f^T\right]^T, \\ \Upsilon_{11} &= P_1 - \varepsilon_1 N_L^T N_L - \varepsilon_2 N_C^T N_C, \\ \Upsilon_{13} &= -\varepsilon_1 N_L^T N_{Ld} - \varepsilon_2 N_C^T N_{Cd}, \\ \Upsilon_{14} &= -\varepsilon_1 N_L^T N_G - \varepsilon_2 N_C^T N_D, \\ \Upsilon_{33} &= Q_3 - \varepsilon_1 N_{Ld}^T N_{Ld} - \varepsilon_2 N_{Cd}^T N_{Cd}, \\ \Upsilon_{34} &= -\varepsilon_1 N_{Ld}^T N_G - \varepsilon_2 N_{Cd}^T N_D, \\ \Upsilon_{44} &= I - \varepsilon_1 N_G^T N_G - \varepsilon_2 N_D^T N_D. \end{aligned} \tag{54}$$

Moreover, a suitable l_2-l_∞ filter is given by

$$\begin{gathered} A_f = \overline{A}_f F_2^{-1}, \qquad B_f = \overline{B}_f, \qquad C_f = \overline{C}_f F_2^{-1}, \\ D_f = \overline{D}_f. \end{gathered} \tag{55}$$

Proof. Firstly, replace matrices A, A_d, B, C, C_d, and D in (44) with $A+\Delta A$, $A_d+\Delta A_d$, $B+\Delta B$, $C+\Delta C$, $C_d+\Delta C_d$, and $D+\Delta D$, respectively, and the following inequality is obtained:

$$\Omega + \operatorname{sym}\left(\Omega_1^T \Delta_1^T \Omega_3^T\right) + \operatorname{sym}\left(\Omega_2^T \Delta_2^T \Omega_4^T\right) < 0, \tag{56}$$

where Ω_i, $i = 1, \ldots, 4$ are defined in (52). Then by using Lemma 4, the above inequality holds if and only if

$$\Omega + \varepsilon_3\Omega_1^T\Omega_1 + \varepsilon_3^{-1}\Omega_3\Omega_3^T + \varepsilon_4\Omega_2^T\Omega_2 + \varepsilon_4^{-1}\Omega_4\Omega_4^T < 0. \tag{57}$$

Then by using Schur complement equivalence, the inequality in (57) is equivalent to (44). Substituting L, L_d, and G in (45) with $L + \Delta L$, $L_d + \Delta L_d$, and $G + \Delta G$, respectively, we can get

$$\Upsilon + \operatorname{sym}\left(\Upsilon_1^T \Delta_3^T \Upsilon_3^T\right) + \operatorname{sym}\left(\Upsilon_2^T \Delta_2^T \Upsilon_4^T\right) > 0, \tag{58}$$

where

$$\begin{gathered} \Upsilon_1 = \left[N_L \;\; N_{Ld} \;\; N_G \;\; 0\right], \qquad \Upsilon_2 = \left[N_C \;\; N_{Cd} \;\; N_D \;\; 0\right] \\ \Upsilon_3 = \left[0 \;\; 0 \;\; 0 \;\; M_3^T\right], \qquad \Upsilon_4 = \left[0 \;\; 0 \;\; 0 \;\; M_2^T \overline{D}_f^T\right]. \end{gathered} \tag{59}$$

By following the similar line, the equivalence between (58) and (53) can be proved. □

4. Illustrative Example

In this section, the following example is given to demonstrate the effectiveness of the proposed approach.

Example 1. Firstly, consider a nominal discrete-time delay system in (1) with the following parameters:

$$A = \begin{bmatrix} 0.1 & -0.5 \\ 0.2 & 0.5 \end{bmatrix}, \qquad A_d = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix},$$
$$B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} -0.1 & -1 \end{bmatrix}, \qquad C_d = \begin{bmatrix} -0.1 & 0.6 \end{bmatrix},$$
$$L = \begin{bmatrix} 1 & 0.2 \end{bmatrix}, \qquad L_d = \begin{bmatrix} 0.5 & 0.6 \end{bmatrix}, \qquad D = 0.1,$$
$$G = -0.5. \tag{60}$$

For different delay cases, the different minima of γ can be calculated by solving the LMIs in Theorem 9. When the upper bound of the time-varying delay is 5, that is, $d_2 = 5$, the minima of γ for a given d_1 are listed in Table 1.

Moreover, when $d_1 = 2$, $d_2 = 5$, the corresponding l_2-l_∞ filter is given as follows:

$$A_f = \begin{bmatrix} 2.6553 & -1.9535 \\ 2.2783 & -1.5631 \end{bmatrix}, \qquad B_f = \begin{bmatrix} -1.5405 \\ -1.9024 \end{bmatrix},$$
$$C_f = \begin{bmatrix} -1.3230 & 1.0479 \end{bmatrix}, \qquad D_f = 0.2378. \tag{61}$$

When uncertainty appears in the system, Theorem 10 will be used for the desired filter design. The following uncertainty parameters are considered:

$$M_1 = \begin{bmatrix} 0.35 & 0 \\ -0.2 & 0.1 \end{bmatrix}, \qquad N_A = \begin{bmatrix} 0.2 & 0.4 \\ 0 & 0.5 \end{bmatrix},$$
$$N_{Ad} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, \qquad N_B = \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix},$$
$$M_2 = 0.2, \qquad N_C = \begin{bmatrix} 0.15 & -0.22 \end{bmatrix}, \tag{62}$$
$$N_{Cd} = \begin{bmatrix} -0.3 & 0.2 \end{bmatrix}, \qquad N_D = -0.5,$$
$$M_3 = -0.4, \qquad N_L = \begin{bmatrix} -0.25 & -0.2 \end{bmatrix},$$
$$N_{Ld} = \begin{bmatrix} 0.13 & 0.32 \end{bmatrix}, \qquad N_D = 0.2.$$

Similarly, the allowed minimal values of γ can be obtained by solving the LMIs in Theorem 10. For $d_2 = 5$, the different minimum allowed γ are listed in Table 2 for the uncertain system with different d_1.

Moreover, when $d_1 = 2$, $d_2 = 5$, the desired filter is given as follows:

$$A_f = \begin{bmatrix} -0.1261 & 0.2147 \\ -0.3333 & 0.5671 \end{bmatrix}, \qquad B_f = \begin{bmatrix} -0.1725 \\ -0.2622 \end{bmatrix},$$
$$C_f = \begin{bmatrix} -0.0076 & 0.0125 \end{bmatrix}, \qquad D_f = 0.1106. \tag{63}$$

Table 1: Minimum allowed γ for $d_2 = 5$.

Methods	d_1			
	1	2	3	4
Theorem 9	1.8371	1.5652	1.3585	1.1861

Table 2: Minimum allowed γ for $d_2 = 5$.

Methods	d_1			
	1	2	3	4
Theorem 10	3.4157	2.6514	2.1568	1.8405

5. Conclusions

The robust l_2-l_∞ filtering has been studied for uncertain discrete-time systems with time-varying delay in this paper. Based on reciprocally convex approach, the sufficient l_2-l_∞ performance analysis conditions in terms of LMIs have been proposed to render the filtering error systems asymptotically stable with an l_2-l_∞ performance. Then the desired filter has been designed for the filtering error system with time-varying delay. The results presented in this paper are in terms of strict LMIs which make the conditions more tractable. Finally, a numerical example is given to demonstrate the effectiveness of our methods. For future research topic, the results can be extended to the system with actuator/sensor failures which may lead to unsatisfactory performance and has attracted many researchers' attention such as faulty diagnosis [23, 24] and fault tolerant control [25].

Conflict of Interests

None of the authors of the paper has declared any conflict of interests.

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