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INEQUALITIES OF SCHWARZ TYPE FOR n-TUPLES OF VECTORS IN INNER PRODUCT SPACES WITH APPLICATIONS

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Abstract. In this paper some new inequalities of Schwarz and Buzano type for n-tuples of vectors in inner product spaces are given. Applications for norm and numerical radius inequalities for n-tuples of bounded linear operators and for functions of normal operators defined by power series with nonnegative coefficients are also provided.

1. Introduction

Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. The following inequality is well known in literature as the *Schwarz inequality*

$$\|x\| \|y\| \geq |\langle x,y\rangle| \text{ for any } x,y \in H. \tag{1.1}$$

The equality case holds in (1.1) if and only if there exists a constant $\lambda \in \mathbb{K}$ such that $x = \lambda y$.

In 1985 the author [5] (see also [15]) established the following refinement of (1.1):

$$\|x\| \|y\| \geq |\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle| + |\langle x,e\rangle \langle e,y\rangle| \geq |\langle x,y\rangle| \tag{1.2}$$

for any $x,y,e \in H$ with $\|e\| = 1$.

Using the triangle inequality for modulus we have

$$|\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle| \geq |\langle x,e\rangle \langle e,y\rangle| - |\langle x,y\rangle|$$

and by (1.2) we get

$$\begin{aligned} \|x\| \|y\| &\geq |\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle| + |\langle x,e\rangle \langle e,y\rangle| \\ &\geq 2|\langle x,e\rangle \langle e,y\rangle| - |\langle x,y\rangle|, \end{aligned}$$

which implies the Buzano inequality [3]

$$\frac{1}{2}[\|x\| \|y\| + |\langle x,y\rangle|] \geq |\langle x,e\rangle \langle e,y\rangle| \tag{1.3}$$

that holds for any $x,y,e \in H$ with $\|e\| = 1$.

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For a probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathcal{P}_n$, i.e. we recall that $p_i > 0$ for any $i \in \{1, ..., n\}$ and $\sum_{i=1}^{n} p_i = 1$ we define the inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle_p := \sum_{i=1}^{n} p_i \langle x_i, y_i \rangle$$

for n-tuples $\mathbf{x} = (x_1, ..., x_n), \mathbf{y} = (y_1, ..., y_n) \in H^n$. The attached norm is given by

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} p_i \|x_i\|^2 \right)^{1/2}$$

for $\mathbf{x} = (x_1, ..., x_n) \in H^n$.

Let $\mathbf{e} = (e_1, ..., e_n) \in H^n$ with $\sum_{i=1}^{n} p_i \|e_i\|^2 = 1$. Making use of (1.2) and (1.3) for the inner product $\langle \cdot, \cdot \rangle_p$ we have the inequalities

$$\begin{aligned} &\left(\sum_{i=1}^{n} p_i \|x_i\|^2 \right)^{1/2} \left(\sum_{i=1}^{n} p_i \|y_i\|^2 \right)^{1/2} \qquad (1.4) \\ &\geq \left| \sum_{i=1}^{n} p_i \langle x_i, y_i \rangle - \sum_{i=1}^{n} p_i \langle x_i, e_i \rangle \sum_{i=1}^{n} p_i \langle e_i, y_i \rangle \right| + \left| \sum_{i=1}^{n} p_i \langle x_i, e_i \rangle \sum_{i=1}^{n} p_i \langle e_i, y_i \rangle \right| \\ &\geq \left| \sum_{i=1}^{n} p_i \langle x_i, y_i \rangle \right| \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{2} \left[\left(\sum_{i=1}^{n} p_i \|x_i\|^2 \right)^{1/2} \left(\sum_{i=1}^{n} p_i \|y_i\|^2 \right)^{1/2} + \left| \sum_{i=1}^{n} p_i \langle x_i, y_i \rangle \right| \right] \qquad (1.5) \\ &\geq \left| \sum_{i=1}^{n} p_i \langle x_i, e_i \rangle \sum_{i=1}^{n} p_i \langle e_i, y_i \rangle \right|, \end{aligned}$$

for any $\mathbf{x} = (x_1, ..., x_n), \mathbf{y} = (y_1, ..., y_n) \in H^n$.

If we take $\mathbf{e} = (e, ..., e) \in H^n$ with $\|e\| = 1$, then we get from (1.4) and (1.5) the inequalities

$$\begin{aligned} &\left(\sum_{i=1}^{n} p_i \|x_i\|^2 \right)^{1/2} \left(\sum_{i=1}^{n} p_i \|y_i\|^2 \right)^{1/2} \qquad (1.6) \\ &\geq \left| \sum_{i=1}^{n} p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^{n} p_i x_i, e \right\rangle \left\langle e, \sum_{i=1}^{n} p_i y_i \right\rangle \right| + \left| \left\langle \sum_{i=1}^{n} p_i x_i, e \right\rangle \left\langle e, \sum_{i=1}^{n} p_i y_i \right\rangle \right| \\ &\geq \left| \sum_{i=1}^{n} p_i \langle x_i, y_i \rangle \right| \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{2} \left[\left(\sum_{i=1}^{n} p_i \|x_i\|^2 \right)^{1/2} \left(\sum_{i=1}^{n} p_i \|y_i\|^2 \right)^{1/2} + \left| \sum_{i=1}^{n} p_i \langle x_i, y_i \rangle \right| \right] \qquad (1.7) \\ &\geq \left| \left\langle \sum_{i=1}^{n} p_i x_i, e \right\rangle \left\langle e, \sum_{i=1}^{n} p_i y_i \right\rangle \right|, \end{aligned}$$

for any $\mathbf{x} = (x_1, ..., x_n), \mathbf{y} = (y_1, ..., y_n) \in H^n$.

For other Schwarz related inequalities in inner product spaces, see [1], [6]-[10], [13], [14], [18], [19], [20], [21], [22], [23], [24], [26] and the monographs [11] and [12].

Motivated by the above results, we establish in this paper some new inequalities of Schwarz and Buzano type for n-tuples of vectors in inner product spaces. Applications for norm and numerical radius inequalities for n-tuples of bounded linear operators and for functions of normal operators defined by power series with nonnegative coefficients are also provided.

2. Main Results

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. For an n-tuple $\mathbf{x} = (x_1, ..., x_n) \in H^n$ and a probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathcal{P}_n$, we define the *average vector*

$$\overline{x}_p := \sum_{j=1}^{n} p_j x_j \in H.$$

In particular, for the *uniform probability* $\mathbf{p} = \left(\frac{1}{n}, ..., \frac{1}{n}\right)$ we define

$$\overline{x} := \frac{1}{n} \sum_{j=1}^{n} x_j \in H.$$

We have the following result:

Theorem 2.1. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$, $\mathbf{x} = (x_1, ..., x_n), \mathbf{y} = (y_1, ..., y_n) \in H^n$ and $\mathbf{p} = (p_1, ..., p_n) \in \mathcal{P}_n$. Then we have the inequalities*

$$\begin{aligned}
&\left(\sum_{i=1}^{n} p_i \|x_i\|^2\right)^{1/2} \left(\sum_{i=1}^{n} p_i \|y_i\|^2\right)^{1/2} \qquad (2.1)\\
&\geq \max\left\{\sum_{i=1}^{n} p_i \left|\langle x_i, y_i \rangle - 2\langle \overline{x}_p, y_i \rangle\right|, \sum_{i=1}^{n} p_i \left|\langle x_i, y_i \rangle - 2\langle x_i, \overline{y}_p \rangle\right|\right\}\\
&\geq \sum_{i=1}^{n} p_i \left[\left|\frac{1}{2}\langle x_i, y_i \rangle - \langle \overline{x}_p, y_i \rangle\right| + \left|\frac{1}{2}\langle x_i, y_i \rangle - \langle x_i, \overline{y}_p \rangle\right|\right]\\
&\geq \begin{cases} \sum_{i=1}^{n} p_i \left|\langle x_i, y_i \rangle - \langle \overline{x}_p, y_i \rangle - \langle x_i, \overline{y}_p \rangle\right|, \\ \sum_{i=1}^{n} p_i \left|\langle x_i, \overline{y}_p \rangle - \langle \overline{x}_p, y_i \rangle\right|. \end{cases}
\end{aligned}$$

In particular, we have

$$\begin{aligned}
&\left(\sum_{i=1}^{n}\|x_i\|^2\right)^{1/2}\left(\sum_{i=1}^{n}\|y_i\|^2\right)^{1/2} \qquad (2.2)\\
&\geq \max\left\{\sum_{i=1}^{n}\left|\langle x_i,y_i\rangle - 2\langle \overline{x},y_i\rangle\right|,\sum_{i=1}^{n}\left|\langle x_i,y_i\rangle - 2\langle x_i,\overline{y}\rangle\right|\right\}\\
&\geq \sum_{i=1}^{n}\left[\left|\frac{1}{2}\langle x_i,y_i\rangle - \langle \overline{x},y_i\rangle\right| + \left|\frac{1}{2}\langle x_i,y_i\rangle - \langle x_i,\overline{y}\rangle\right|\right]\\
&\geq \begin{cases} \sum_{i=1}^{n}\left|\langle x_i,y_i\rangle - \langle \overline{x},y_i\rangle - \langle x_i,\overline{y}\rangle\right|,\\ \\ \sum_{i=1}^{n}\left|\langle x_i,\overline{y}\rangle - \langle \overline{x},y_i\rangle\right|.\end{cases}
\end{aligned}$$

Proof. Observe that, by the properties of inner product, we have

$$\begin{aligned}
\sum_{i=1}^{n}p_i||x_i-2\overline{x}_p||^2 &= \sum_{i=1}^{n}p_i\left[||x_i||^2 - 4Re\langle x_i,\overline{x}_p\rangle + 4||\overline{x}_p||^2\right]\\
&= \sum_{i=1}^{n}p_i||x_i||^2 - 4Re\left\langle \sum_{i=1}^{n}p_ix_i,\overline{x}_p\right\rangle + 4||\overline{x_p}||^2\\
&= \sum_{i=1}^{n}p_i||x_i||^2 - 4||\overline{x}_p||^2 + 4||\overline{x_p}||^2 = \sum_{i=1}^{n}p_i||x_i||^2. \qquad (2.3)
\end{aligned}$$

By the Cauchy-Buniyakovsky-Schwarz inequality for sequences of real numbers, we have

$$\begin{aligned}
\left(\sum_{i=1}^{n}p_i\|x_i\|^2\sum_{i=1}^{n}p_i\|y_i\|^2\right)^{1/2} &= \left(\sum_{i=1}^{n}p_i\|x_i-2\overline{x}_p\|^2\sum_{i=1}^{n}p_i\|y_i\|^2\right)^{1/2}\\
&\geq \sum_{i=1}^{n}p_i\|x_i-2\overline{x}_p\|\,\|y_i\|.
\end{aligned}$$

By the Schwarz inequality in $(H,\langle\cdot,\cdot\rangle)$ we have for each $i\in\{1,...,n\}$

$$\|x_i-2\overline{x}_p\|\,\|y_i\| \geq |\langle x_i-2\overline{x}_p,y_i\rangle| = |\langle x_i,y_i\rangle - 2\langle\overline{x}_p,y_i\rangle|$$

and then

$$\sum_{i=1}^{n}p_i\|x_i-2\overline{x}_p\|\,\|y_i\| \geq \sum_{i=1}^{n}p_i|\langle x_i,y_i\rangle - 2\langle\overline{x}_p,y_i\rangle|.$$

Therefore

$$\left(\sum_{i=1}^{n}p_i\|x_i\|^2\sum_{i=1}^{n}p_i\|y_i\|^2\right)^{1/2} \geq \sum_{i=1}^{n}p_i|\langle x_i,y_i\rangle - 2\langle\overline{x}_p,y_i\rangle| \qquad (2.4)$$

and, similarly

$$\left(\sum_{i=1}^{n}p_i\|x_i\|^2\sum_{i=1}^{n}p_i\|y_i\|^2\right)^{1/2} \geq \sum_{i=1}^{n}p_i\left|\langle x_i,y_i\rangle - 2\langle x_i,\overline{y}_p\rangle\right|. \qquad (2.5)$$

Making use of (2.4) and (2.5) we get the first inequality in (2.1).

Since $\max\{a,b\} \geq \frac{1}{2}(a+b)$ for positive numbers a and b,we get the second inequality in (2.1). The last part of (2.1) follows by the triangle inequality. □

Remark. *Using the generalized triangle inequality we have*

$$\sum_{i=1}^{n} p_i \left|\langle x_i, y_i\rangle - \langle \overline{x}_p, y_i\rangle - \left\langle x_i, \overline{y}_p\right\rangle\right| \geq \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle - 2\left\langle \overline{x}_p, \overline{y}_p\right\rangle\right|$$

and by (2.1) we get the following string of inequalities

$$\begin{aligned} &\left(\sum_{i=1}^{n} p_i \|x_i\|^2\right)^{1/2} \left(\sum_{i=1}^{n} p_i \|y_i\|^2\right)^{1/2} \qquad (2.6)\\ &\geq \max\left\{\sum_{i=1}^{n} p_i \left|\langle x_i, y_i\rangle - 2\langle \overline{x}_p, y_i\rangle\right|, \sum_{i=1}^{n} p_i \left|\langle x_i, y_i\rangle - 2\left\langle x_i, \overline{y}_p\right\rangle\right|\right\}\\ &\geq \sum_{i=1}^{n} p_i \left[\left|\frac{1}{2}\langle x_i, y_i\rangle - \langle \overline{x}_p, y_i\rangle\right| + \left|\frac{1}{2}\langle x_i, y_i\rangle - \left\langle x_i, \overline{y}_p\right\rangle\right|\right]\\ &\geq \sum_{i=1}^{n} p_i \left|\langle x_i, y_i\rangle - \langle \overline{x}_p, y_i\rangle - \left\langle x_i, \overline{y}_p\right\rangle\right|\\ &\geq \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle - 2\left\langle \overline{x}_p, \overline{y}_p\right\rangle\right|. \end{aligned}$$

The unweighted case is as follows

$$\begin{aligned} &\left(\sum_{i=1}^{n} \|x_i\|^2\right)^{1/2} \left(\sum_{i=1}^{n} \|y_i\|^2\right)^{1/2} \qquad (2.7)\\ &\geq \max\left\{\sum_{i=1}^{n} \left|\langle x_i, y_i\rangle - 2\langle \overline{x}, y_i\rangle\right|, \sum_{i=1}^{n} \left|\langle x_i, y_i\rangle - 2\langle x_i, \overline{y}\rangle\right|\right\}\\ &\geq \sum_{i=1}^{n} \left[\left|\frac{1}{2}\langle x_i, y_i\rangle - \langle \overline{x}, y_i\rangle\right| + \left|\frac{1}{2}\langle x_i, y_i\rangle - \langle x_i, \overline{y}\rangle\right|\right]\\ &\geq \sum_{i=1}^{n} \left|\langle x_i, y_i\rangle - \langle \overline{x}, y_i\rangle - \langle x_i, \overline{y}\rangle\right| \geq \left|\sum_{i=1}^{n} \langle x_i, y_i\rangle - 2\langle \overline{x}, \overline{y}\rangle\right|. \end{aligned}$$

The inequality between the first and 4th and 5th terms in (2.7) was obtained in [25].

Corollary 2.2. *With the assumptions of Theorem 2.1 we have*

$$\frac{1}{2}\left[\left(\sum_{i=1}^{n} p_i \|x_i\|^2\right)^{1/2} \left(\sum_{i=1}^{n} p_i \|y_i\|^2\right)^{1/2} + \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle\right|\right] \geq \left|\left\langle \overline{x}_p, \overline{y}_p\right\rangle\right| \qquad (2.8)$$

and, in particular

$$\frac{1}{2n}\left[\left(\sum_{i=1}^{n} \|x_i\|^2\right)^{1/2} \left(\sum_{i=1}^{n} \|y_i\|^2\right)^{1/2} + \left|\sum_{i=1}^{n} \langle x_i, y_i\rangle\right|\right] \geq \left|\langle \overline{x}, \overline{y}\rangle\right|. \qquad (2.9)$$

Proof. By the triangle inequality we have

$$\left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle - 2\left\langle \overline{x}_p, \overline{y}_p\right\rangle\right| \geq 2\left|\left\langle \overline{x}_p, \overline{y}_p\right\rangle\right| - \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle\right|$$

and from (2.6) we get

$$\left(\sum_{i=1}^{n} p_i \|x_i\|^2\right)^{1/2}\left(\sum_{i=1}^{n} p_i \|y_i\|^2\right)^{1/2} \geq 2\left|\left\langle \overline{x}_p, \overline{y}_p\right\rangle\right| - \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle\right|,$$

i.e.

$$\left(\sum_{i=1}^{n} p_i \|x_i\|^2\right)^{1/2}\left(\sum_{i=1}^{n} p_i \|y_i\|^2\right)^{1/2} + \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle\right| \geq 2\left|\left\langle \overline{x}_p, \overline{y}_p\right\rangle\right|,$$

and the inequality (2.8) is proved. □

The following result also holds:

Theorem 2.3. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$, $\mathbf{x} = (x_1, ..., x_n), \mathbf{y} = (y_1, ..., y_n) \in H^n$ and $\mathbf{p} = (p_1, ..., p_n) \in \mathcal{P}_n$. Then we have the inequalities*

$$\begin{aligned} &\left(\sum_{i=1}^{n} p_i \|x_i\|^2 \sum_{i=1}^{n} p_i \|y_i\|^2\right)^{1/2} \\ &\geq \sum_{i=1}^{n} p_i \left|\langle x_i, y_i\rangle - 2\langle \overline{x}_p, y_i\rangle - 2\left\langle x_i, \overline{y}_p\right\rangle + 4\left\langle \overline{x}_p, \overline{y}_p\right\rangle\right| \\ &\geq \sum_{i=1}^{n} p_i \left|\langle x_i, y_i\rangle\right| \geq \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle\right| \end{aligned} \tag{2.10}$$

and, in particular

$$\begin{aligned} &\left(\sum_{i=1}^{n} \|x_i\|^2 \sum_{i=1}^{n} \|y_i\|^2\right)^{1/2} \\ &\geq \sum_{i=1}^{n} \left|\langle x_i, y_i\rangle - 2\langle \overline{x}, y_i\rangle - 2\langle x_i, \overline{y}\rangle + 4\langle \overline{x}, \overline{y}\rangle\right| \\ &\geq \sum_{i=1}^{n} \left|\langle x_i, y_i\rangle\right| \geq \left|\sum_{i=1}^{n} \langle x_i, y_i\rangle\right|. \end{aligned} \tag{2.11}$$

Proof. By the Cauchy-Buniyakovsky-Schwarz inequality for sequences of real numbers, we have

$$\begin{aligned} &\left(\sum_{i=1}^{n} p_i \|x_i\|^2 \sum_{i=1}^{n} p_i \|y_i\|^2\right)^{1/2} \\ &= \left(\sum_{i=1}^{n} p_i \|x_i - 2\overline{x}_p\|^2 \sum_{i=1}^{n} p_i \left\|y_i - 2\overline{y}_p\right\|^2\right)^{1/2} \\ &\geq \sum_{i=1}^{n} p_i \|x_i - 2\overline{x}_p\| \left\|y_i - 2\overline{y}_p\right\|. \end{aligned} \tag{2.12}$$

By the Schwarz inequality in $(H, \langle\cdot,\cdot\rangle)$ we have

$$\|x_i - 2\overline{x}_p\| \left\|y_i - 2\overline{y}_p\right\| \geq \left|\left\langle x_i - 2\overline{x}_p, y_i - 2\overline{y}_p\right\rangle\right| = \left|\langle x_i, y_i\rangle - 2\langle \overline{x}_p, y_i\rangle - 2\left\langle x_i, \overline{y}_p\right\rangle + 4\left\langle \overline{x}_p, \overline{y}_p\right\rangle\right|$$

for any $i \in \{1, ..., n\}$.

Therefore

$$\begin{aligned} &\sum_{i=1}^{n} p_i \|x_i - 2\overline{x}_p\| \left\|y_i - 2\overline{y}_p\right\| \qquad (2.13)\\ &\geq \sum_{i=1}^{n} p_i \left|\langle x_i, y_i\rangle - 2\langle \overline{x}_p, y_i\rangle - 2\left\langle x_i, \overline{y}_p\right\rangle + 4\left\langle \overline{x}_p, \overline{y}_p\right\rangle\right| \\ &\geq \left|\sum_{i=1}^{n} p_i \left[\langle x_i, y_i\rangle - 2\langle \overline{x}_p, y_i\rangle - 2\left\langle x_i, \overline{y}_p\right\rangle + 4\left\langle \overline{x}_p, \overline{y}_p\right\rangle\right]\right| \\ &= \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle - 2\left\langle \overline{x}_p, \sum_{i=1}^{n} p_i y_i\right\rangle - 2\left\langle \sum_{i=1}^{n} p_i x_i, \overline{y}_p\right\rangle + 4\left\langle \overline{x}_p, \overline{y}_p\right\rangle\right| \\ &= \sum_{i=1}^{n} p_i \left|\langle x_i, y_i\rangle\right|. \end{aligned}$$

Making use of (2.12) and (2.13) we get the desired result (2.10). □

The following corollary holds:

Corollary 2.4. *With the assumptions of Theorem 2.3 we have*

$$\begin{aligned} &\frac{1}{4}\left[\left(\sum_{i=1}^{n} p_i \|x_i\|^2 \sum_{i=1}^{n} p_i \|y_i\|^2\right)^{1/2} + \sum_{i=1}^{n} p_i \left|\langle x_i, y_i\rangle\right|\right] \qquad (2.14)\\ &\geq \sum_{i=1}^{n} p_i \left|\left\langle \overline{x}_p, \overline{y}_p\right\rangle - \frac{1}{2}\left[\langle \overline{x}_p, y_i\rangle + \left\langle x_i, \overline{y}_p\right\rangle\right]\right|. \end{aligned}$$

Proof. If we add to the first inequality in (2.10) the quantity $\sum_{i=1}^{n} p_i \left|\langle x_i, y_i\rangle\right|$ we have

$$\begin{aligned} &\left(\sum_{i=1}^{n} p_i \|x_i\|^2 \sum_{i=1}^{n} p_i \|y_i\|^2\right)^{1/2} + \sum_{i=1}^{n} p_i \left|\langle x_i, y_i\rangle\right| \qquad (2.15)\\ &\geq \sum_{i=1}^{n} p_i \left[\left|\langle x_i, y_i\rangle - 2\langle \overline{x}_p, y_i\rangle - 2\left\langle x_i, \overline{y}_p\right\rangle + 4\left\langle \overline{x}_p, \overline{y}_p\right\rangle\right| + \left|\langle x_i, y_i\rangle\right|\right] \\ &\geq \sum_{i=1}^{n} p_i \left|4\left\langle \overline{x}_p, \overline{y}_p\right\rangle - 2\langle \overline{x}_p, y_i\rangle - 2\left\langle x_i, \overline{y}_p\right\rangle\right| \\ &= 4\sum_{i=1}^{n} p_i \left|\left\langle \overline{x}_p, \overline{y}_p\right\rangle - \frac{1}{2}\left[\langle \overline{x}_p, y_i\rangle + \left\langle x_i, \overline{y}_p\right\rangle\right]\right|. \end{aligned}$$

Dividing by 4 we get the desired result (2.14). □

We observe that, if we take $H=\mathbb{C}$ with the inner product $\langle z,w\rangle = z\overline{w}$ then by taking above $x_i=a_i\in\mathbb{C}$ and $y_i=\overline{b}_i, i\in\{1,...,n\}$, then from (2.1) we get the inequality

$$\begin{aligned}
&\left(\sum_{i=1}^{n} p_i\left|a_i\right|^2\right)^{1/2}\left(\sum_{i=1}^{n} p_i\left|b_i\right|^2\right)^{1/2} \qquad (2.16)\\
&\geq \max\left\{\sum_{i=1}^{n} p_i\left|a_ib_i-2b_i\overline{a}_p\right|, \sum_{i=1}^{n} p_i\left|a_ib_i-2a_i\overline{b}_p\right|\right\}\\
&\geq \sum_{i=1}^{n} p_i\left[\left|\frac{1}{2}a_ib_i-b_i\overline{a}_p\right|+\left|\frac{1}{2}a_ib_i-a_i\overline{b}_p\right|\right]\\
&\geq \begin{cases} \sum_{i=1}^{n} p_i\left|a_ib_i-b_i\overline{a}_p-a_i\overline{b}_p\right|, \\ \\ \sum_{i=1}^{n} p_i\left|a_i\overline{b}_p-b_i\overline{a}_p\right|, \end{cases}
\end{aligned}$$

where $\mathbf{p}=(p_1,...,p_n)\in\mathcal{P}_n$ and

$$\overline{a}_p=\sum_{i=1}^{n} p_ia_i.$$

Utilising the inequality (2.8) we also have

$$\begin{aligned}
&\frac{1}{2}\left[\left(\sum_{i=1}^{n} p_i\left|a_i\right|^2\right)^{1/2}\left(\sum_{i=1}^{n} p_i\left|b_i\right|^2\right)^{1/2}+\left|\sum_{i=1}^{n} p_ia_ib_i\right|\right] \qquad (2.17)\\
&\geq \left|\sum_{i=1}^{n} p_ia_i\right|\left|\sum_{i=1}^{n} p_ib_i\right|
\end{aligned}$$

for any $\mathbf{p}=(p_1,...,p_n)\in\mathcal{P}_n$ and $a_i,b_i\in\mathbb{C}$, $i\in\{1,...,n\}$.

If in (2.17) we take $p_i=\frac{q_i}{\sum_{k=1}^{n} q_k}$ with $q_i\geq 0, i\in\{1,...,n\}$ and $\sum_{k=1}^{n} q_k>0$, then we get

$$\begin{aligned}
&\frac{1}{2}\left[\left(\sum_{i=1}^{n} q_i\left|a_i\right|^2\right)^{1/2}\left(\sum_{i=1}^{n} q_i\left|b_i\right|^2\right)^{1/2}+\left|\sum_{i=1}^{n} q_ia_ib_i\right|\right]\sum_{k=1}^{n} q_k \qquad (2.18)\\
&\geq \left|\sum_{i=1}^{n} q_ia_i\right|\left|\sum_{i=1}^{n} q_ib_i\right|.
\end{aligned}$$

Moreover, by taking $b_i=a_i$, $i\in\{1,...,n\}$ in (2.18) we get

$$\frac{1}{2}\left[\sum_{i=1}^{n} q_i\left|a_i\right|^2+\left|\sum_{i=1}^{n} q_ia_i^2\right|\right]\sum_{k=1}^{n} q_k\geq\left|\sum_{i=1}^{n} q_ia_i\right|^2. \qquad (2.19)$$

If $r_i\in\mathbb{R}\backslash\{0\}$ and $z_i\in\mathbb{C}$, $i\in\{1,...,n\}$, then by taking $q_i=r_i^2$ and $a_i=\frac{z_i}{r_i}$, $i\in\{1,...,n\}$ we get the well known *de Bruijn inequality* [2]

$$\frac{1}{2}\left[\sum_{i=1}^{n}\left|z_i\right|^2+\left|\sum_{i=1}^{n} z_i^2\right|\right]\sum_{i=1}^{n} r_i^2\geq\left|\sum_{i=1}^{n} r_ia_i\right|^2. \qquad (2.20)$$

The other vector inequalities from above have similar versions for complex numbers. However the details are not presented here.

3. Applications for n-Tuples of Operators

If in (2.8) we take $p_i = \frac{q_i}{\sum_{k=1}^{n} q_k}$ with $q_i \geq 0, i \in \{1, ..., n\}$ and $\sum_{k=1}^{n} q_k > 0$, then we get

$$\frac{1}{2}\left[\left(\sum_{i=1}^{n} q_i \|x_i\|^2\right)^{1/2}\left(\sum_{i=1}^{n} q_i \|y_i\|^2\right)^{1/2} + \left|\sum_{i=1}^{n} q_i \langle x_i, y_i\rangle\right|\right]\sum_{k=1}^{n} q_k \geq \left|\left\langle \sum_{i=1}^{n} q_i x_i, \sum_{i=1}^{n} q_i y_i\right\rangle\right| \tag{3.1}$$

for any $\mathbf{x} = (x_1, ..., x_n), \mathbf{y} = (y_1, ..., y_n) \in H^n$.

Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [17, p. 1]:

$$W(T) = \{\langle Tx, x\rangle,\ x \in H,\ \|x\| = 1\}.$$

The *numerical radius* $w(T)$ of an operator T on H is given by [17, p. 8]:

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{3.2}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators on the Hilbert space H. This norm is equivalent with the operator norm. In fact, the following more precise result holds [17, p. 9]:

Theorem 3.1 (Equivalent norm)**.** *For any $T \in B(H)$ one has*

$$w(T) \leq \|T\| \leq 2w(T). \tag{3.3}$$

The following result holds:

Theorem 3.2. *Let $(A_1, ..., A_n)$, $(B_1, ..., B_n)$ be two n-tuple of bounded linear operators on the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ and $q_i \geq 0, i \in \{1, ..., n\}$ with $\sum_{k=1}^{n} q_k > 0$. Then*

$$\left\|\sum_{i=1}^{n} q_i B_i^* \sum_{i=1}^{n} q_i A_i\right\| \leq \frac{1}{2}\left[\left\|\sum_{i=1}^{n} q_i |A_i|^2\right\|^{1/2}\left\|\sum_{i=1}^{n} q_i |B_i|^2\right\|^{1/2} + \left\|\sum_{i=1}^{n} q_i B_i^* A_i\right\|\right]\sum_{k=1}^{n} q_k \tag{3.4}$$

and

$$w\left(\sum_{i=1}^{n} q_i B_i^* \sum_{i=1}^{n} q_i A_i\right) \leq \frac{1}{2}\left[\left\|\sum_{i=1}^{n} q_i |A_i|^2\right\|^{1/2}\left\|\sum_{i=1}^{n} q_i |B_i|^2\right\|^{1/2} + w\left(\sum_{i=1}^{n} q_i B_i^* A_i\right)\right]\sum_{k=1}^{n} q_k. \tag{3.5}$$

Proof. We take in inequality (3.1) $x_i = A_i x$, $y_i = B_i y$, $i \in \{1, ..., n\}$ to get

$$\begin{aligned}&\frac{1}{2}\left[\left(\sum_{i=1}^{n} q_i \|A_i x\|^2\right)^{1/2}\left(\sum_{i=1}^{n} q_i \|B_i y\|^2\right)^{1/2} + \left|\sum_{i=1}^{n} q_i \langle A_i x, B_i y\rangle\right|\right]\sum_{k=1}^{n} q_k \\ &\geq \left|\left\langle \sum_{i=1}^{n} q_i A_i x, \sum_{i=1}^{n} q_i B_i y\right\rangle\right|,\end{aligned}$$

for any $x, y \in H$.

Since

$$\begin{aligned}\sum_{i=1}^{n} q_i \|A_i x\|^2 &= \sum_{i=1}^{n} q_i \langle A_i x, A_i x\rangle = \sum_{i=1}^{n} q_i \langle A_i^* A_i x, x\rangle \\ &= \sum_{i=1}^{n} q_i \left\langle |A_i|^2 x, x\right\rangle = \left\langle \sum_{i=1}^{n} q_i |A_i|^2 x, x\right\rangle,\end{aligned}$$

$$\sum_{i=1}^{n} q_i \|B_i y\|^2 = \left\langle \sum_{i=1}^{n} q_i |B_i|^2 y, y\right\rangle,$$

$$\sum_{i=1}^{n} q_i \langle A_i x, B_i y\rangle = \left\langle \sum_{i=1}^{n} q_i B_i^* A_i x, y\right\rangle$$

and

$$\left\langle \sum_{i=1}^{n} q_i A_i x, \sum_{i=1}^{n} q_i B_i y\right\rangle = \left\langle \sum_{i=1}^{n} q_i B_i^* \sum_{i=1}^{n} q_i A_i x, y\right\rangle,$$

then we have

$$\begin{aligned}&\left|\left\langle \sum_{i=1}^{n} q_i B_i^* \sum_{i=1}^{n} q_i A_i x, y\right\rangle\right| \\ &\leq \frac{1}{2}\left[\left\langle \sum_{i=1}^{n} q_i |A_i|^2 x, x\right\rangle^{1/2}\left\langle \sum_{i=1}^{n} q_i |B_i|^2 y, y\right\rangle^{1/2} \right. \\ &\left. + \left|\left\langle \sum_{i=1}^{n} q_i B_i^* A_i x, y\right\rangle\right|\right]\sum_{k=1}^{n} q_k\end{aligned} \tag{3.6}$$

for any $x, y \in H$.

This inequality is of interest in itself.

We know that for any bounded operator T we have

$$\|T\| = \sup_{\|x\|=\|y\|=1} |\langle Tx, y\rangle|.$$

Taking the supremum in (3.6) over $\|x\| = \|y\| = 1$ we have

$$\begin{aligned}
&\left\|\sum_{i=1}^{n} q_i B_i^* \sum_{i=1}^{n} q_i A_i\right\| \qquad (3.7)\\
&= \sup_{\|x\|=\|y\|=1} \left|\left\langle \sum_{i=1}^{n} q_i B_i^* \sum_{i=1}^{n} q_i A_i x, y\right\rangle\right| \sum_{k=1}^{n} q_k\\
&\leq \frac{1}{2} \sup_{\|x\|=\|y\|=1} \left[\left\langle \sum_{i=1}^{n} q_i \left|A_i\right|^2 x, x\right\rangle^{1/2} \left\langle \sum_{i=1}^{n} q_i \left|B_i\right|^2 y, y\right\rangle^{1/2}\right.\\
&\left. + \left|\left\langle \sum_{i=1}^{n} q_i B_i^* A_i x, y\right\rangle\right|\right] \sum_{k=1}^{n} q_k\\
&\leq \frac{1}{2} \left[\sup_{\|x\|=1} \left[\left\langle \sum_{i=1}^{n} q_i \left|A_i\right|^2 x, x\right\rangle^{1/2} \sup_{\|y\|=1} \left\langle \sum_{i=1}^{n} q_i \left|B_i\right|^2 x, x\right\rangle^{1/2}\right]\right.\\
&\left. + \sup_{\|x\|=\|y\|=1} \left|\left\langle \sum_{i=1}^{n} q_i B_i^* A_i x, y\right\rangle\right|\right] \sum_{k=1}^{n} q_k\\
&= \frac{1}{2} \left[\left\|\sum_{i=1}^{n} q_i \left|A_i\right|^2\right\|^{1/2} \left\|\sum_{i=1}^{n} q_i \left|B_i\right|^2\right\|^{1/2} + \left\|\sum_{i=1}^{n} q_i B_i^* A_i\right\|\right] \sum_{k=1}^{n} q_k,
\end{aligned}$$

because $\sum_{i=1}^{n} q_i \left|A_i\right|^2$ and $\sum_{i=1}^{n} q_i \left|B_i\right|^2$ are positive selfadjoint operators.

This proves (3.4).

If we put in (3.6) $y = x$ and then take the supremum over $\|x\| = 1$, we get

$$\begin{aligned}
&w\left(\sum_{i=1}^{n} q_i B_i^* \sum_{i=1}^{n} q_i A_i\right) \qquad (3.8)\\
&= \sup_{\|x\|=1} \left|\left\langle \sum_{i=1}^{n} q_i B_i^* \sum_{i=1}^{n} q_i A_i x, x\right\rangle\right|\\
&\leq \frac{1}{2} \left[\sup_{\|x\|=1} \left[\left(\left\langle \sum_{i=1}^{n} q_i \left|A_i\right|^2 x, x\right\rangle\right)^{1/2} \left(\left\langle \sum_{i=1}^{n} q_i \left|B_i\right|^2 x, x\right\rangle\right)^{1/2}\right]\right.\\
&\left. + \sup_{\|x\|=1} \left|\left\langle \sum_{i=1}^{n} q_i B_i^* A_i x, x\right\rangle\right|\right] \sum_{k=1}^{n} q_k.
\end{aligned}$$

Since

$$\sup_{\|x\|=1}\left[\left\langle \sum_{i=1}^{n} q_i |A_i|^2 x, x\right\rangle^{1/2} \left\langle \sum_{i=1}^{n} q_i |B_i|^2 x, x\right\rangle^{1/2}\right] \tag{3.9}$$
$$\leq \sup_{\|x\|=1}\left\langle \sum_{i=1}^{n} q_i |A_i|^2 x, x\right\rangle^{1/2} \sup_{\|x\|=1}\left\langle \sum_{i=1}^{n} q_i |B_i|^2 x, x\right\rangle^{1/2}$$
$$= \left\|\sum_{i=1}^{n} q_i |A_i|^2\right\|^{1/2} \left\|\sum_{i=1}^{n} q_i |B_i|^2\right\|^{1/2},$$

we get from (3.8) and (3.9) the desired result (3.5). □

Corollary 3.3. *Let $(A_1, ..., A_n)$ be an n-tuple of bounded linear operators on the Hilbert space $(H; \langle\cdot,\cdot\rangle)$ and $q_i \geq 0, i \in \{1, ..., n\}$ with $\sum_{k=1}^{n} q_k > 0$. Then*

$$\left\|\left(\sum_{i=1}^{n} q_i A_i\right)^2\right\| \tag{3.10}$$
$$\leq \frac{1}{2}\left[\left\|\sum_{i=1}^{n} q_i |A_i|^2\right\|^{1/2} \left\|\sum_{i=1}^{n} q_i |A_i^*|^2\right\|^{1/2} + \left\|\sum_{i=1}^{n} q_i A_i^2\right\|\right]\sum_{k=1}^{n} q_k$$

and

$$w\left(\left(\sum_{i=1}^{n} q_i A_i\right)^2\right) \tag{3.11}$$
$$\leq \frac{1}{2}\left[\left\|\sum_{i=1}^{n} q_i |A_i|^2\right\|^{1/2} \left\|\sum_{i=1}^{n} q_i |A_i^*|^2\right\|^{1/2} + w\left(\sum_{i=1}^{n} q_i A_i^2\right)\right]\sum_{k=1}^{n} q_k.$$

Remark. *If we take $n = 2$ and $q_1 = q_2 = 1$ in (3.4), then we get*

$$\|(B_1^* + B_2^*)(A_1 + A_2)\| \tag{3.12}$$
$$\leq \left\||A_1|^2 + |A_2|^2\right\|^{1/2} \left\||B_1|^2 + |B_2|^2\right\|^{1/2} + \|B_1^* A_1 + B_2^* A_2\|$$

for any operators A_1, A_2, B_1, B_2.

Assume that T, V are bounded linear operators and consider the Cartesian decomposition

$$T = A + iB,\ V = C + iD$$

with the selfadjoint operators A, B, C, D given by

$$A = \frac{1}{2}(T^* + T),\ B = \frac{1}{2i}(T - T^*)$$

and

$$C = \frac{1}{2}(V^* + V),\ D = \frac{1}{2i}(V - V^*).$$

Take $A_1 = A$, $A_2 = iB$, $B_1 = C$ and $B_2 = -iD$. Then

$$(B_1^* + B_2^*)(A_1 + A_2) = VT,$$
$$|A_1|^2 + |A_2|^2 = A^2 + B^2 = \frac{1}{2}\left(|T|^2 + |T^*|^2\right),$$

$$|B_1|^2 + |B_2|^2 = C^2 + D^2 = \frac{1}{2}\left(|V|^2 + |V^*|^2\right)$$

and

$$\begin{aligned} B_1^* A_1 + B_2^* A_2 &= CA + (-iD)^* (iB) = CA - DB \\ &= \frac{1}{2}(V^* + V)\frac{1}{2}(T^* + T) - \frac{1}{2i}(V - V^*)\frac{1}{2i}(T - T^*) \\ &= \frac{1}{4}\left[(V^* + V)(T^* + T) + (V - V^*)(T - T^*)\right] \\ &= \frac{1}{2}(VT + V^*T^*). \end{aligned}$$

Then by (3.12) we get

$$\|VT\| \le \frac{1}{2}\left[\left\||V|^2 + |V^*|^2\right\|^{1/2}\left\||T|^2 + |T^*|^2\right\|^{1/2} + \|VT + V^*T^*\|\right]. \tag{3.13}$$

If we replace in this inequality V with V^* and T with T^*, then we get the dual inequality

$$\|V^*T^*\| \le \frac{1}{2}\left[\left\||V|^2 + |V^*|^2\right\|^{1/2}\left\||T|^2 + |T^*|^2\right\|^{1/2} + \|VT + V^*T^*\|\right]. \tag{3.14}$$

Adding these inequalities give us

$$\|VT\| + \|V^*T^*\| \le \left\||V|^2 + |V^*|^2\right\|^{1/2}\left\||T|^2 + |T^*|^2\right\|^{1/2} + \|VT + V^*T^*\|, \tag{3.15}$$

which implies that

$$\begin{aligned} 0 &\le \|VT\| + \|V^*T^*\| - \|VT + V^*T^*\| \\ &\le \left\||V|^2 + |V^*|^2\right\|^{1/2}\left\||T|^2 + |T^*|^2\right\|^{1/2}, \end{aligned} \tag{3.16}$$

for any bounded linear operators V, T.

Using the inequality (3.5) for $n = 2$ we can derive in a similar way the numerical radius inequality

$$\begin{aligned} 0 &\le w(VT) + w(V^*T^*) - w(VT + V^*T^*) \\ &\le \left\||V|^2 + |V^*|^2\right\|^{1/2}\left\||T|^2 + |T^*|^2\right\|^{1/2}, \end{aligned} \tag{3.17}$$

for any bounded linear operators V, T.

If we take in (3.16) and (3.17) $V = T$, then we get

$$0 \le \left\|V^2\right\| + \left\|(V^*)^2\right\| - \left\|V^2 + (V^*)^2\right\| \le \left\||V|^2 + |V^*|^2\right\|, \tag{3.18}$$

and

$$0 \le w\left(V^2\right) + w\left((V^*)^2\right) - w\left(V^2 + (V^*)^2\right) \le \left\||V|^2 + |V^*|^2\right\| \tag{3.19}$$

for any bounded linear operators V.

4. Applications for Functions of Normal Operators

Recall some examples of power series with nonnegative coefficients

$$\frac{1}{1-\lambda} = \sum_{n=0}^{\infty} \lambda^n, \ \lambda \in D(0,1); \tag{4.1}$$
$$\ln \frac{1}{1-\lambda} = \sum_{n=1}^{\infty} \frac{1}{n}\lambda^n, \ \lambda \in D(0,1);$$
$$\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n, \ \lambda \in \mathbb{C};$$
$$\sinh \lambda = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\lambda^{2n+1}, \ \lambda \in \mathbb{C};$$
$$\cosh \lambda = \sum_{n=0}^{\infty} \frac{1}{(2n)!}\lambda^{2n}, \ \lambda \in \mathbb{C}.$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \ \lambda \in D(0,1); \tag{4.2}$$
$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,(2n+1)\,n!}\lambda^{2n+1}, \ \lambda \in D(0,1);$$
$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \ \lambda \in D(0,1);$$
$${}_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\,\Gamma(n+\beta)\,\Gamma(\gamma)}{n!\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(n+\gamma)}\lambda^n, \alpha,\beta,\gamma>0,$$
$$\lambda \in D(0,1)$$

where Γ is *Gamma function*.

The following result for power series with nonnegative coefficients holds:

Theorem 4.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients $a_n \geq 0$ for $n \in \mathbb{N}$ and having the radius of convergence $R > 0$ or $R = \infty$. If U, V are normal operators on the Hilbert space $(H; \langle \cdot,\cdot \rangle)$, $U^*V = VU^*$ and $\alpha > 0$ such that $\alpha < R$ and $\|U\|, \|V\| \leq 1$, then*

$$|\langle f(\alpha U^*)\, f(\alpha V)\, x, y\rangle| \tag{4.3}$$
$$\leq \frac{1}{2} f(\alpha)\left[\left\langle f\left(\alpha |V|^2\right) x, x\right\rangle^{1/2} \left\langle f\left(\alpha |U|^2\right) y, y\right\rangle^{1/2} + |\langle f(\alpha U^* V)\, x, y\rangle|\right]$$

for any $x, y \in H$.

Proof. Using the inequality (3.6) we can state that

$$\begin{aligned}&\left|\left\langle \sum_{i=0}^{n} a_i\alpha^i (U^*)^i \sum_{i=0}^{n} a_i\alpha^i V^i x, y\right\rangle\right| \\ &\leq \frac{1}{2}\left[\left\langle \sum_{i=0}^{n} a_i\alpha^i \left|V^i\right|^2 x, x\right\rangle^{1/2} \left\langle \sum_{i=0}^{n} a_i\alpha^i \left|(U^*)^i\right|^2 y, y\right\rangle^{1/2}\right. \\ &\left.+\left|\left\langle \sum_{i=0}^{n} a_i\alpha^i (U^*)^i V^i x, y\right\rangle\right|\right] \sum_{i=0}^{n} a_i\alpha^i\end{aligned} \tag{4.4}$$

for any $x, y \in H$ and $n \geq 1$.

Since U, V are normal operators, then for $i \geq 1$

$$\left|V^i\right|^2 = \left(V^i\right)^* V^i = (V^*)^i V^i = (V^*V)^i = |V|^{2i}$$

and

$$\left|(U^*)^i\right|^2 = |U|^{2i}.$$

Also, since $U^*V = VU^*$, then

$$(U^*)^i V^i = (U^*V)^i$$

for any $i \geq 1$.

Therefore from (4.4) we get

$$\begin{aligned}&\left|\left\langle \sum_{i=0}^{n} a_i\alpha^i (U^*)^i \sum_{i=0}^{n} a_i\alpha^i V^i x, y\right\rangle\right| \\ &\leq \frac{1}{2}\left[\left\langle \sum_{i=0}^{n} a_i\alpha^i |V|^{2i} x, x\right\rangle^{1/2} \left\langle \sum_{i=0}^{n} a_i\alpha^i |U|^{2i} y, y\right\rangle^{1/2}\right. \\ &\left.+\left|\left\langle \sum_{i=0}^{n} a_i\alpha^i (U^*V)^i x, y\right\rangle\right|\right] \sum_{i=0}^{n} a_i\alpha^i,\end{aligned} \tag{4.5}$$

for any $x, y \in H$ and $n \geq 1$.

Since $\left\|\alpha |V|^2\right\| = \alpha \|V\|^2 < R$, $\left\|\alpha |U|^2\right\| = \alpha \|U\|^2 < R$, $\|\alpha U^*V\| \leq \alpha \|U\| \|V\| < R$, $\|\alpha U^*\| < R$ and $\|\alpha V\| < R$, then the series

$$\sum_{i=0}^{\infty} a_i\alpha^i (U^*)^i,\ \sum_{i=0}^{\infty} a_i\alpha^i V^i,\ \sum_{i=0}^{n} a_i\alpha^i |V|^{2i},\ \sum_{i=0}^{n} a_i\alpha^i |U|^{2i},\ \sum_{i=0}^{\infty} a_i\alpha^i (U^*V)^i$$

are convergent in $B(H)$ and $\sum_{i=0}^{\infty} a_i\alpha^i$ is convergent in $\mathbb{R}$.

Taking the limit over $n \to \infty$ in (4.5) we get the desired result (4.3). □

Corollary 4.2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients $a_n \geq 0$ for $n \in \mathbb{N}$ and having the radius of convergence $R > 0$ or $R = \infty$. If V is normal operator on the Hilbert space $(H; \langle\cdot,\cdot\rangle)$ and $\alpha > 0$ such that $\alpha < R$ and $\|V\| \leq 1$, then*

$$\begin{aligned}&\left|\left\langle f^2(\alpha V) x, y\right\rangle\right| \\ &\leq \frac{1}{2} f(\alpha)\left[\left\langle f\left(\alpha |V|^2\right) x, x\right\rangle^{1/2} \left\langle f\left(\alpha |V|^2\right) y, y\right\rangle^{1/2} + \left|\left\langle f\left(\alpha V^2\right) x, y\right\rangle\right|\right]\end{aligned} \tag{4.6}$$

and

$$\left|\left\langle |f(\alpha V)|^2 x, y\right\rangle\right| \tag{4.7}$$
$$\leq \frac{1}{2} f(\alpha)\left[\left\langle f\left(\alpha |V|^2\right) x, x\right\rangle^{1/2}\left\langle f\left(\alpha |V|^2\right) y, y\right\rangle^{1/2} + \left|\left\langle f\left(\alpha |V|^2\right) x, y\right\rangle\right|\right]$$

for any $x, y \in H$.

Proof. It follows by Theorem 4.1 by choosing $U = V^*$ and $U = V$. □

Example: a. If U, V are normal operators on the Hilbert space $(H; \langle \cdot, \cdot \rangle)$, $U^*V = VU^*$ and $\alpha > 0$ then

$$|\langle \exp(\alpha(U^* + V)) x, y\rangle| \tag{4.8}$$
$$\leq \frac{1}{2}\left[\left\langle \exp\left(\alpha |V|^2\right) x, x\right\rangle^{1/2}\left\langle \exp\left(\alpha |U|^2\right) y, y\right\rangle^{1/2}\right.$$
$$\left. + |\langle \exp(\alpha U^* V) x, y\rangle|\right] \exp(\alpha)$$

for any $x, y \in H$.

b. If U, V are normal operators on the Hilbert space $(H; \langle \cdot, \cdot \rangle)$, $U^*V = VU^*$ and $\|U\|, \|V\| < 1$, $\alpha \in (0, 1)$ then

$$\left|\left\langle (1_H - \alpha U^*)^{-1}(1_H - \alpha V)^{-1} x, y\right\rangle\right| \tag{4.9}$$
$$\leq \frac{1}{2}\left[\left\langle \left(1 - \alpha |V|^2\right)^{-1} x, x\right\rangle^{1/2}\left\langle \left(1 - \alpha |U|^2\right)^{-1} y, y\right\rangle^{1/2}\right.$$
$$\left. + \left|\left\langle (1 - \alpha U^* V)^{-1} x, y\right\rangle\right|\right](1 - \alpha)^{-1}$$

for any $x, y \in H$.

By taking the supremum in (4.3) over $\|x\| = \|y\| = 1$ we can state the following norm inequality:

Corollary 4.3. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a power series with nonnegative coefficients* $a_n \geq 0$ *for* $n \in \mathbb{N}$ *and having the radius of convergence* $R > 0$ *or* $R = \infty$. *If* U, V *are normal operators on the Hilbert space* $(H; \langle \cdot, \cdot \rangle)$, $U^*V = VU^*$ *and* $\alpha > 0$ *such that* $\alpha < R$ *and* $\|U\|, \|V\| \leq 1$, *then*

$$\|f(\alpha U^*) f(\alpha V)\| \tag{4.10}$$
$$\leq \frac{1}{2} f(\alpha)\left[\left\|f\left(\alpha |V|^2\right)\right\|^{1/2}\left\|f\left(\alpha |U|^2\right)\right\|^{1/2} + \|f(\alpha U^* V)\|\right].$$

The reader may state various particular inequalities of interest. However the details are omitted here.

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