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n-Points Inequalities of Hermite-Hadamard Type for h-Convex Functions on Linear Spaces

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Abstract. Some n-points inequalities of Hermite-Hadamard type for h-convex functions defined on convex subsets in real or complex linear spaces are given. Applications for norm inequalities are provided as well.

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1 Introduction

We recall here some concepts of convexity that are well known in the literature.

Let I be an interval in $\mathbb{R}$.

Definition 1 ([26]) *We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0,1)$ we have*

$$f(tx + (1-t)y) \leq \frac{1}{t} f(x) + \frac{1}{1-t} f(y). \tag{1}$$

Some further properties of this class of functions can be found in [20], [21], [23], [32], [35] and [36]. Among others, its has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

The above concept can be extended for functions $f : C \subseteq X \to [0, \infty)$ where C is a convex subset of the real or complex linear space X and the inequality (1) is satisfied for any vectors $x, y \in C$ and $t \in (0,1)$. If the function $f : C \subseteq X \to \mathbb{R}$ is non-negative and convex, then is of Godunova-Levin type.

Definition 2 ([23]) *We say that a function $f : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0,1]$ we have*

$$f(tx + (1-t)y) \leq f(x) + f(y). \tag{2}$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contains all nonnegative monotone, convex and *quasi convex functions,* i. e. nonnegative functions satisfying

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\} \tag{3}$$

for all $x, y \in I$ and $t \in [0,1]$.

For some results on P-functions see [23] and [33] while for quasi convex functions, the reader can consult [22].

If $f : C \subseteq X \to [0, \infty)$, where C is a convex subset of the real or complex linear space X, then we say that it is of P-type (or quasi-convex) if the inequality (2) (or (3)) holds true for $x, y \in C$ and $t \in [0,1]$.

Definition 3 ([7]) *Let $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$.

For some properties of this class of functions see [1], [2], [7], [8], [18], [19], [27], [29] and [38].

The concept of Breckner s-convexity can be similarly extended for functions defined on convex subsets of linear spaces.

It is well known that if $(X, \|\cdot\|)$ is a normed linear space, then the function $f(x) = \|x\|^p$, $p \geq 1$ is convex on X.

Utilising the elementary inequality $(a+b)^s \leq a^s + b^s$ that holds for any $a, b \geq 0$ and $s \in (0,1]$, we have for the function $g(x) = \|x\|^s$ that

$$\begin{aligned} g(tx + (1-t)y) &= \|tx + (1-t)y\|^s \leq (t\|x\| + (1-t)\|y\|)^s \\ &\leq (t\|x\|)^s + [(1-t)\|y\|]^s \\ &= t^s g(x) + (1-t)^s g(y) \end{aligned}$$

for any $x, y \in X$ and $t \in [0,1]$, which shows that g is Breckner s-convex on X.

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}, (0,1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 4 ([41]) *Let $h : J \to [0, \infty)$ with h not identical to 0. We say that $f : I \to [0, \infty)$ is an h-convex function if for all $x, y \in I$ we have*

$$f(tx + (1-t)y) \le h(t) f(x) + h(1-t) f(y) \tag{4}$$

for all $t \in (0, 1)$.

For some results concerning this class of functions see [41], [6], [30], [39], [37] and [40].

This concept can be extended for functions defined on convex subsets of linear spaces in the same way as above replacing the interval I be the corresponding convex subset C of the linear space X.

We can introduce now another class of functions.

Definition 5 *We say that the function $f : C \subseteq X \to [0, \infty)$ is of s-Godunova-Levin type, with $s \in [0, 1]$, if*

$$f(tx + (1-t)y) \le \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y), \tag{5}$$

for all $t \in (0, 1)$ and $x, y \in C$.

We observe that for $s = 0$ we obtain the class of P-functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(C)$ the class of s-Godunova-Levin functions defined on C, then we obviously have

$$P(C) = Q_0(C) \subseteq Q_{s_1}(C) \subseteq Q_{s_2}(C) \subseteq Q_1(C) = Q(C)$$

for $0 \le s_1 \le s_2 \le 1$.

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$(b-a) f\left(\frac{a+b}{2}\right) < \int_a^b f(x)dx < (b-a)\frac{f(a)+f(b)}{2}, \quad a, b \in \mathbb{R}. \tag{6}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [31]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [5]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [31]. Since (6) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality.

For related results, see [10]-[13], [24] and [34].

We can state the following generalization of the Hermite-Hadamard inequality for h-convex functions defined on convex subsets of linear spaces [17].

Theorem 1 *Assume that the function $f : C \subseteq X \to [0, \infty)$ is a h-convex function with $h \in L[0,1]$. Let $y, x \in C$ with $y \neq x$ and assume that the mapping $[0,1] \ni t \mapsto f[(1-t)x + ty]$ is Lebesgue integrable on $[0,1]$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x+ty]\,dt \leq [f(x)+f(y)] \int_0^1 h(t)\,dt. \tag{7}$$

Remark 1 *If $f : I \to [0,\infty)$ is a h-convex function on an interval I of real numbers with $h \in L[0,1]$ and $f \in L[a,b]$ with $a, b \in I, a < b$, then from (7) we get the Hermite-Hadamard type inequality obtained by Sarikaya et al. in [37]*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(u)\,du \leq [f(a)+f(b)] \int_0^1 h(t)\,dt.$$

If we write (7) for $h(t) = t$, then we get the classical Hermite-Hadamard inequality for convex functions

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x+ty]\,dt \leq \frac{f(x)+f(y)}{2}. \tag{8}$$

If we write (7) for the case of P-type functions $f : C \to [0,\infty)$, i.e., $h(t) = 1, t \in [0,1]$, then we get the inequality

$$\frac{1}{2} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x+ty]\,dt \leq f(x)+f(y), \tag{9}$$

that has been obtained for functions of real variable in [23].

If f is Breckner s-convex on C, for $s \in (0,1)$, then by taking $h(t) = t^s$ in (7) we get

$$2^{s-1} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x+ty]\,dt \leq \frac{f(x)+f(y)}{s+1}, \tag{10}$$

that was obtained for functions of a real variable in [18].

Since the function $g(x) = \|x\|^s$ is Breckner s-convex on on the normed linear space X, $s \in (0,1)$, then for any $x, y \in X$ we have

$$\frac{1}{2}\|x+y\|^s \leq \int_0^1 \|(1-t)x+ty\|^s\,dt \leq \frac{\|x\|^s + \|x\|^s}{s+1}. \tag{11}$$

If $f : C \to [0,\infty)$ is of s-Godunova-Levin type, with $s \in [0,1)$, then

$$\frac{1}{2^{s+1}} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x+ty]\,dt \leq \frac{f(x)+f(y)}{1-s}. \tag{12}$$

We notice that for $s=1$ the first inequality in (12) still holds, i.e.

$$\frac{1}{4}f\left(\frac{x+y}{2}\right)\le\int_0^1 f\left[(1-t)\,x+ty\right]dt. \tag{13}$$

The case of functions of real variables was obtained for the first time in [23].

Motivated by the above results, in this paper some n-points inequalities of Hermite-Hadamard type for h-convex functions defined on convex subsets in real or complex linear spaces are given. Applications for norm inequalities are provided as well.

2 Some New Results

In [17] we also obtained the following result:

Theorem 2 *Assume that the function $f:C\subseteq X\to[0,\infty)$ is an h-convex function with $h\in L[0,1]$. Let $y,x\in C$ with $y\neq x$ and assume that the mapping $[0,1]\ni t\mapsto f[(1-t)\,x+ty]$ is Lebesgue integrable on $[0,1]$. Then for any $\lambda\in[0,1]$ we have the inequalities*

$$\begin{aligned}
&\frac{1}{2h\left(\frac{1}{2}\right)}\left\{(1-\lambda)\,f\left[\frac{(1-\lambda)\,x+(\lambda+1)\,y}{2}\right]+\lambda f\left[\frac{(2-\lambda)\,x+\lambda y}{2}\right]\right\} \qquad (14)\\
&\le\int_0^1 f\left[(1-t)\,x+ty\right]dt\\
&\le\left[f\left((1-\lambda)\,x+\lambda y\right)+(1-\lambda)\,f(y)+\lambda f(x)\right]\int_0^1 h(t)\,dt\\
&\le\left\{\left[h(1-\lambda)+\lambda\right]f(x)+\left[h(\lambda)+1-\lambda\right]f(y)\right\}\int_0^1 h(t)\,dt.
\end{aligned}$$

We can state the following new corollary as well:

Corollary 1 *With the assumptions of Theorem 2 we have*

$$\begin{aligned}
&\frac{1}{2h\left(\frac{1}{2}\right)} \qquad (15)\\
&\times\int_0^1(1-\lambda)\left\{f\left[\frac{(1-\lambda)\,x+(\lambda+1)\,y}{2}\right]+f\left[\frac{(1-\lambda)\,y+(\lambda+1)\,x}{2}\right]\right\}d\lambda\\
&\le\int_0^1 f\left[(1-t)\,x+ty\right]dt\\
&\le\left[\int_0^1 f\left((1-\lambda)\,x+\lambda y\right)d\lambda+\frac{f(y)+f(x)}{2}\right]\int_0^1 h(t)\,dt\\
&\le\left[f(x)+f(y)\right]\left[\int_0^1 h(\lambda)\,d\lambda+\frac{1}{2}\right]\int_0^1 h(t)\,dt.
\end{aligned}$$

Proof. The proof follows by integrating the inequality (14) over λ and by using the equality

$$\int_0^1 \lambda f\left[\frac{(2-\lambda)x+\lambda y}{2}\right]d\lambda = \int_0^1 (1-\mu) f\left[\frac{(1+\mu)x+(1-\mu)y}{2}\right]d\mu.$$

□

The following result for double integral also holds:

Corollary 2 *With the assumptions of Theorem 2 we have*

$$\begin{aligned}
&\frac{1}{2h\left(\frac{1}{2}\right)(b-a)^2} \qquad (16)\\
&\times \int_a^b\int_a^b \frac{\alpha}{\alpha+\beta}\left\{f\left[\frac{\alpha x+(2\beta+\alpha)y}{2(\alpha+\beta)}\right]+f\left[\frac{(2\beta+\alpha)x+\alpha y}{2(\alpha+\beta)}\right]\right\}d\alpha d\beta\\
&\leq \int_0^1 f\left[(1-t)x+ty\right]dt\\
&\leq \left[\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)d\alpha d\beta+\frac{f(y)+f(x)}{2}\right]\int_0^1 h(t)\,dt\\
&\leq \left[\frac{1}{(b-a)^2}\int_a^b\int_a^b h\left(\frac{\beta}{\alpha+\beta}\right)d\alpha d\beta+\frac{1}{2}\right]\left[f(x)+f(y)\right]\int_0^1 h(t)\,dt,
\end{aligned}$$

for any $b>a\geq 0$.

Proof. If we take $\lambda=\frac{\alpha}{\alpha+\beta}$ we have

$$\begin{aligned}
&\frac{1}{2h\left(\frac{1}{2}\right)} \qquad (17)\\
&\times\left\{\frac{\beta}{\alpha+\beta}f\left[\frac{\beta x+(2\alpha+\beta)y}{2(\alpha+\beta)}\right]+\frac{\alpha}{\alpha+\beta}f\left[\frac{(2\beta+\alpha)x+\alpha y}{2(\alpha+\beta)}\right]\right\}\\
&\leq \int_0^1 f\left[(1-t)x+ty\right]dt\\
&\leq \left[f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)+\frac{\beta}{\alpha+\beta}f(y)+\frac{\alpha}{\alpha+\beta}f(x)\right]\int_0^1 h(t)\,dt\\
&\leq \left\{\left[h\left(\frac{\beta}{\alpha+\beta}\right)+\frac{\alpha}{\alpha+\beta}\right]f(x)+\left[h\left(\frac{\alpha}{\alpha+\beta}\right)+\frac{\beta}{\alpha+\beta}\right]f(y)\right\}\\
&\times\int_0^1 h(t)\,dt,
\end{aligned}$$

for any $\alpha,\beta\geq 0$ with $\alpha+\beta>0$.

Since the mapping $[0,1] \ni t \mapsto f[(1-t)x+ty]$ is Lebesgue integrable on $[0,1]$, then the double integral $\int_a^b \int_a^b f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right) d\alpha d\beta$ exists for any $b > a \geq 0$. The same holds for the other integrals in (16).

Integrating the inequality (17) on the square $[a,b]^2$ over (α,β) we have

$$\begin{aligned}
&\frac{1}{2h\left(\frac{1}{2}\right)(b-a)^2} \\
&\times \int_a^b \int_a^b \left\{ \frac{\beta}{\alpha+\beta} f\left[\frac{\beta x+(2\alpha+\beta)y}{2(\alpha+\beta)}\right] + \frac{\alpha}{\alpha+\beta} f\left[\frac{(2\beta+\alpha)x+\alpha y}{2(\alpha+\beta)}\right] \right\} d\alpha d\beta \\
&\qquad \leq \int_0^1 f[(1-t)x+ty]\,dt \\
&\leq \int_a^b \int_a^b \left[f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right) + \frac{\beta}{\alpha+\beta} f(y) + \frac{\alpha}{\alpha+\beta} f(x) \right] d\alpha d\beta \int_0^1 h(t)\,dt \\
&\leq \frac{1}{(b-a)^2} \int_0^1 h(t)\,dt \times \int_a^b \int_a^b \left\{ \left[h\left(\frac{\beta}{\alpha+\beta}\right) + \frac{\alpha}{\alpha+\beta} \right] f(x) + \right. \\
&\qquad \left. + \left[h\left(\frac{\alpha}{\alpha+\beta}\right) + \frac{\beta}{\alpha+\beta} \right] f(y) \right\} d\alpha d\beta. \quad (18)
\end{aligned}$$

Observe that

$$\begin{aligned}
&\int_a^b \int_a^b \frac{\beta}{\alpha+\beta} f\left[\frac{\beta x+(2\alpha+\beta)y}{2(\alpha+\beta)}\right] d\alpha d\beta \\
&= \int_a^b \int_a^b \frac{\alpha}{\alpha+\beta} f\left[\frac{\alpha x+(2\beta+\alpha)y}{2(\alpha+\beta)}\right] d\alpha d\beta
\end{aligned}$$

and then

$$\begin{aligned}
&\int_a^b \int_a^b \left\{ \frac{\beta}{\alpha+\beta} f\left[\frac{\beta x+(2\alpha+\beta)y}{2(\alpha+\beta)}\right] + \frac{\alpha}{\alpha+\beta} f\left[\frac{(2\beta+\alpha)x+\alpha y}{2(\alpha+\beta)}\right] \right\} d\alpha d\beta \\
&= \int_a^b \int_a^b \frac{\alpha}{\alpha+\beta} \left\{ f\left[\frac{\alpha x+(2\beta+\alpha)y}{2(\alpha+\beta)}\right] + f\left[\frac{(2\beta+\alpha)x+\alpha y}{2(\alpha+\beta)}\right] \right\} d\alpha d\beta.
\end{aligned}$$

Also

$$\int_a^b \int_a^b \frac{\alpha}{\alpha+\beta} d\alpha d\beta = \int_a^b \int_a^b \frac{\beta}{\alpha+\beta} d\alpha d\beta$$

and since

$$\int_a^b \int_a^b \frac{\alpha}{\alpha+\beta} d\alpha d\beta + \int_a^b \int_a^b \frac{\beta}{\alpha+\beta} d\alpha d\beta = \int_a^b \int_a^b \frac{\alpha+\beta}{\alpha+\beta} d\alpha d\beta = (b-a)^2,$$

then we have

$$\int_a^b \int_a^b \frac{\alpha}{\alpha+\beta} d\alpha d\beta = \frac{1}{2}(b-a)^2.$$

Moreover, we have

$$\int_a^b \int_a^b h\left(\frac{\alpha}{\alpha+\beta}\right) d\alpha d\beta = \int_a^b \int_a^b h\left(\frac{\beta}{\alpha+\beta}\right) d\alpha d\beta.$$

Utilising (18), we get the desired result (16). □

Remark 2 *Let $f : C \subseteq X \to \mathbb{C}$ be a convex function on the convex subset C of a real or complex linear space X. Then for any $x, y \in C$ and $b > a \geq 0$ we have*

$$\begin{aligned} & f\left(\frac{x+y}{2}\right) \qquad (19) \\ & \leq \frac{1}{(b-a)^2} \\ & \times \int_a^b \int_a^b \frac{\alpha}{\alpha+\beta}\left\{f\left[\frac{\alpha x+(2\beta+\alpha) y}{2(\alpha+\beta)}\right]+f\left[\frac{(2\beta+\alpha) x+\alpha y}{2(\alpha+\beta)}\right]\right\} d\alpha d\beta \\ & \leq \int_0^1 f[(1-t) x+t y] d t \\ & \leq \frac{1}{2}\left[\frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right) d\alpha d\beta+\frac{f(y)+f(x)}{2}\right] \\ & \leq \frac{f(y)+f(x)}{2}. \end{aligned}$$

The second and third inequalities are obvious from (16) for $h(t) = t$.

By the convexity of f we have

$$\begin{aligned} & \frac{1}{2}\left\{f\left[\frac{\alpha x+(2\beta+\alpha) y}{2(\alpha+\beta)}\right]+f\left[\frac{(2\beta+\alpha) x+\alpha y}{2(\alpha+\beta)}\right]\right\} \\ & \geq f\left[\frac{1}{2}\left\{\left[\frac{\alpha x+(2\beta+\alpha) y}{2(\alpha+\beta)}\right]+\left[\frac{(2\beta+\alpha) x+\alpha y}{2(\alpha+\beta)}\right]\right\}\right] \\ & = f\left(\frac{x+y}{2}\right) \end{aligned}$$

for any $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$.

If we multiply this inequality by $\frac{2\alpha}{\alpha+\beta} \geq 0$ and integrate on the square $[a, b]^2$ we get

$$\begin{aligned} & \int_a^b \int_a^b \frac{\alpha}{\alpha+\beta}\left\{f\left[\frac{\alpha x+(2\beta+\alpha) y}{2(\alpha+\beta)}\right]+f\left[\frac{(2\beta+\alpha) x+\alpha y}{2(\alpha+\beta)}\right]\right\} d\alpha d\beta \\ & \geq 2 f\left(\frac{x+y}{2}\right) \int_a^b \int_a^b \frac{\alpha}{\alpha+\beta} d\alpha d\beta=(b-a)^2 f\left(\frac{x+y}{2}\right), \end{aligned}$$

since we know that

$$\int_a^b \int_a^b \frac{\alpha}{\alpha+\beta} d\alpha d\beta = \frac{1}{2}(b-a)^2 .$$

This proves the first inequality in (19).

By the convexity of f we also have

$$f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right) \le \frac{\beta}{\alpha+\beta} f(x) + \frac{\alpha}{\alpha+\beta} f(y)$$

for any $\alpha, \beta \ge 0$ with $\alpha+\beta > 0$. Integrating on the square $[a,b]^2$ we get

$$\begin{aligned}
&\int_a^b \int_a^b f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right) d\alpha d\beta \\
&\le f(x) \int_a^b \int_a^b \frac{\beta}{\alpha+\beta} d\alpha d\beta + f(y) \int_a^b \int_a^b \frac{\alpha}{\alpha+\beta} d\alpha d\beta \\
&= \frac{1}{2}(b-a)^2 [f(y)+f(x)],
\end{aligned}$$

which proves the last inequality in (19).

Let $(X, \|\cdot\|)$ be a normed linear space over the real or complex number fields. Then for any $x, y \in X$, $p \ge 1$ and $b > a \ge 0$ we have:

$$\begin{aligned}
&\left\|\frac{x+y}{2}\right\|^p \\
&\le \frac{1}{(b-a)^2} \\
&\times \int_a^b \int_a^b \frac{\alpha}{\alpha+\beta} \left\{ \left\|\frac{\alpha x+(2\beta+\alpha) y}{2(\alpha+\beta)}\right\|^p + \left\|\frac{(2\beta+\alpha) x+\alpha y}{2(\alpha+\beta)}\right\|^p \right\} d\alpha d\beta \\
&\le \int_0^1 \|(1-t) x + ty\|^p dt \\
&\le \frac{1}{2}\left[\frac{1}{(b-a)^2} \int_a^b \int_a^b \left\|\frac{\beta x+\alpha y}{\alpha+\beta}\right\|^p d\alpha d\beta + \frac{\|y\|^p+\|x\|^p}{2}\right] \\
&\le \frac{\|y\|^p+\|x\|^p}{2}.
\end{aligned} \tag{20}$$

The case of Breckner s-convexity is as follows:

Remark 3 *Assume that the function $f : C \subseteq X \to [0,\infty)$ is a Breckner s-convex function with $s \in (0,1)$. Let $y, x \in C$ with $y \neq x$ and assume that* *the mapping* $[0,1] \ni t \mapsto f[(1-t)x+ty]$ *is Lebesgue integrable on* $[0,1]$. *Then for any* $b > a \geq 0$ *we have*

$$\begin{aligned}
&\frac{2^{s-1}}{(b-a)^2} \qquad (21)\\
&\times \int_a^b \int_a^b \frac{\alpha}{\alpha+\beta}\left\{f\left[\frac{\alpha x+(2\beta+\alpha)y}{2(\alpha+\beta)}\right]+f\left[\frac{(2\beta+\alpha)x+\alpha y}{2(\alpha+\beta)}\right]\right\}d\alpha d\beta\\
&\leq \int_0^1 f[(1-t)x+ty]\,dt\\
&\leq \frac{1}{s+1}\left[\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)d\alpha d\beta+\frac{f(y)+f(x)}{2}\right].
\end{aligned}$$

We also have the norm inequalities:

$$\begin{aligned}
&\frac{2^{s-1}}{(b-a)^2} \qquad (22)\\
&\times \int_a^b \int_a^b \frac{\alpha}{\alpha+\beta}\left\{\left\|\frac{\alpha x+(2\beta+\alpha)y}{2(\alpha+\beta)}\right\|^s+\left\|\frac{(2\beta+\alpha)x+\alpha y}{2(\alpha+\beta)}\right\|^s\right\}d\alpha d\beta\\
&\leq \int_0^1 \|(1-t)x+ty\|^s\,dt\\
&\leq \frac{1}{2}\left[\frac{1}{(b-a)^2}\int_a^b\int_a^b \left\|\frac{\beta x+\alpha y}{\alpha+\beta}\right\|^s d\alpha d\beta+\frac{\|y\|^s+\|x\|^s}{2}\right],
\end{aligned}$$

for any $x, y \in X$, *a normed linear space.*

3 Inequalities for n-Points

In order to extend the above results for n-points, we need the following representation of the integral that is of interest in itself.

Theorem 3 *Let* $f : C \subseteq X \to \mathbb{C}$ *be defined on the convex subset* C *of a real or complex linear space* X. *Assume that for* $x, y \in C$ *with* $x \neq y$ *the mapping* $[0,1] \mapsto f((1-t)x+ty) \in \mathbb{C}$ *is Lebesgue integrable on* $[0,1]$. *Then for any partition*

$$0 = \lambda_0 < \lambda_1 < ... < \lambda_{n-1} < \lambda_n = 1 \text{ with } n \geq 1,$$

we have the representation

$$\begin{aligned}
\int_0^1 f((1-t)x+ty)\,dt &= \sum_{j=0}^{n-1}(\lambda_{j+1}-\lambda_j)\cdot\\
&\cdot\int_0^1 f\{(1-u)[(1-\lambda_j)x+\lambda_j y]+u[(1-\lambda_{j+1})x+\lambda_{j+1}y]\}\,du. \qquad (23)
\end{aligned}$$

Proof. We have

$$\int_0^1 f((1-t)x+ty)\,dt=\sum_{j=0}^{n-1}\int_{\lambda_j}^{\lambda_{j+1}} f((1-t)x+ty)\,dt. \tag{24}$$

In the integral

$$\int_{\lambda_j}^{\lambda_{j+1}} f((1-t)x+ty)\,dt,\ \ j\in\{0,...,n-1\},$$

consider the change of variable

$$u:=\frac{1}{\lambda_{j+1}-\lambda_j}(t-\lambda_j),t\in[\lambda_j,\lambda_{j+1}].$$

Then

$$du=\frac{1}{\lambda_{j+1}-\lambda_j}dt,$$

$u=0$ for $t=\lambda_j$, $u=1$ for $t=\lambda_{j+1}$, $t=(1-u)\lambda_j+u\lambda_{j+1}$ and

$$\begin{aligned}
&\int_{\lambda_j}^{\lambda_{j+1}} f((1-t)x+ty)\,dt \qquad (25)\\
&=(\lambda_{j+1}-\lambda_j)\\
&\times\int_0^1 f[(1-(1-u)\lambda_j-u\lambda_{j+1})x+((1-u)\lambda_j+u\lambda_{j+1})y]\,du\\
&=(\lambda_{j+1}-\lambda_j)\\
&\times\int_0^1 f[(1-u+u-(1-u)\lambda_j-u\lambda_{j+1})x+((1-u)\lambda_j+u\lambda_{j+1})y]\,du\\
&=(\lambda_{j+1}-\lambda_j)\\
&\times\int_0^1 f[((1-u)(1-\lambda_j)+u(1-\lambda_{j+1}))x+((1-u)\lambda_j+u\lambda_{j+1})y]\,du\\
&=\int_0^1 f\{(1-u)[(1-\lambda_j)x+\lambda_j y]+u[(1-\lambda_{j+1})x+\lambda_{j+1}y]\}\,du
\end{aligned}$$

for any $j\in\{0,...,n-1\}$.

Making use of (24) and (25) we deduce the desired result (23). □

The following particular case is of interest and has been obtained in [17].

Corollary 3 *In the the assumptions of Theorem 3 we have*

$$\begin{aligned}
\int_0^1 f((1-t)x+ty)\,dt &=\lambda\int_0^1 f\{(1-u)x+u[(1-\lambda)x+\lambda y]\}\,du \qquad (26)\\
&+(1-\lambda)\int_0^1 f\{(1-u)[(1-\lambda)x+\lambda y]+uy\}\,du
\end{aligned}$$

for any $\lambda\in[0,1]$.

Proof. Follows from (23) by choosing $0 = \lambda_0 \le \lambda_1 = \lambda \le \lambda_2 = 1.$ $\square$

The following result holds for h-convex functions:

Theorem 4 *Let $f : C \subseteq X \to \mathbb{C}$ be defined on the convex subset C of a real or complex linear space X and f is h-convex on C with $h \in L[0,1]$. Assume that for $x, y \in C$ with $x \ne y$ the mapping $[0,1] \mapsto f((1-t)x + ty) \in \mathbb{R}$ is Lebesgue integrable on $[0,1]$. Then for any partition*

$$0 = \lambda_0 < \lambda_1 < ... < \lambda_{n-1} < \lambda_n = 1 \text{ with } n \ge 1,$$

we have the inequalities

$$\begin{aligned} &\frac{1}{2h\left(\frac{1}{2}\right)} \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) f\left\{\left(1 - \frac{\lambda_j + \lambda_{j+1}}{2}\right)x + \frac{\lambda_j + \lambda_{j+1}}{2} y\right\} \quad (27) \\ &\le \int_0^1 f((1-t)x + ty)\, dt \\ &\le \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) \left[f((1-\lambda_j)x + \lambda_j y) + f((1-\lambda_{j+1})x + \lambda_{j+1} y)\right] \\ &\times \int_0^1 h(u)\, du. \end{aligned}$$

Proof. Since f is h-convex, then

$$\begin{aligned} &f\{(1-u)[(1-\lambda_j)x + \lambda_j y] + u[(1-\lambda_{j+1})x + \lambda_{j+1} y]\} \\ &\le h(1-u) f((1-\lambda_j)x + \lambda_j y) + h(u) f((1-\lambda_{j+1})x + \lambda_{j+1} y) \end{aligned}$$

for any $u \in [0,1]$ and for any $j \in \{0, ..., n-1\}$.

Integrating this inequality over $u \in [0,1]$ we get

$$\begin{aligned} &\int_0^1 f\{(1-u)[(1-\lambda_j)x + \lambda_j y] + u[(1-\lambda_{j+1})x + \lambda_{j+1} y]\}\, du \\ &\le \int_0^1 \{h(1-u) f((1-\lambda_j)x + \lambda_j y) + h(u) f((1-\lambda_{j+1})x + \lambda_{j+1} y)\}\, du \\ &= f((1-\lambda_j)x + \lambda_j y) \int_0^1 h(1-u)\, du + f((1-\lambda_{j+1})x + \lambda_{j+1} y) \int_0^1 h(u)\, du \\ &= [f((1-\lambda_j)x + \lambda_j y) + f((1-\lambda_{j+1})x + \lambda_{j+1} y)] \int_0^1 h(u)\, du, \end{aligned}$$

for any $j \in \{0, ..., n-1\}$.

Multiplying this inequality by $\lambda_{j+1} - \lambda_j \ge 0$ and summing over j from 0 to $n-1$ we get, via the equality (23), the second inequality in (27).

Since f is h-convex, then for any $v, w \in C$ we also have

$$f(v) + f(w) \geq \frac{1}{h\left(\frac{1}{2}\right)} f\left(\frac{v+w}{2}\right).$$

If we write this inequality for

$$v = (1-u)\left[(1-\lambda_j)x + \lambda_j y\right] + u\left[(1-\lambda_{j+1})x + \lambda_{j+1}y\right]$$

and

$$w = u\left[(1-\lambda_j)x + \lambda_j y\right] + (1-u)\left[(1-\lambda_{j+1})x + \lambda_{j+1}y\right]$$

and take into account that

$$\begin{aligned}\frac{v+w}{2} &= \frac{1}{2}\left\{\left[(1-\lambda_j)x + \lambda_j y\right] + \left[(1-\lambda_{j+1})x + \lambda_{j+1}y\right]\right\}\\ &= \left(1 - \frac{\lambda_j + \lambda_{j+1}}{2}\right)x + \frac{\lambda_j + \lambda_{j+1}}{2}y,\end{aligned}$$

then we get

$$\begin{aligned}&f\left\{(1-u)\left[(1-\lambda_j)x + \lambda_j y\right] + u\left[(1-\lambda_{j+1})x + \lambda_{j+1}y\right]\right\}\\ &+ f\left\{u\left[(1-\lambda_j)x + \lambda_j y\right] + (1-u)\left[(1-\lambda_{j+1})x + \lambda_{j+1}y\right]\right\}\\ &\geq \frac{1}{h\left(\frac{1}{2}\right)} f\left\{\left(1 - \frac{\lambda_j + \lambda_{j+1}}{2}\right)x + \frac{\lambda_j + \lambda_{j+1}}{2}y\right\}\end{aligned} \tag{28}$$

for any $u \in [0,1]$ and $j \in \{0, ..., n-1\}$.

Integrating the inequality (28) over $u \in [0,1]$ we get

$$\begin{aligned}&\int_0^1 f\left\{(1-u)\left[(1-\lambda_j)x + \lambda_j y\right] + u\left[(1-\lambda_{j+1})x + \lambda_{j+1}y\right]\right\}du\\ &+ \int_0^1 f\left\{u\left[(1-\lambda_j)x + \lambda_j y\right] + (1-u)\left[(1-\lambda_{j+1})x + \lambda_{j+1}y\right]\right\}du\\ &\geq \frac{1}{h\left(\frac{1}{2}\right)} f\left\{\left(1 - \frac{\lambda_j + \lambda_{j+1}}{2}\right)x + \frac{\lambda_j + \lambda_{j+1}}{2}y\right\}\end{aligned} \tag{29}$$

for any $j \in \{0, ..., n-1\}$.

Since

$$\begin{aligned}&\int_0^1 f\left\{(1-u)\left[(1-\lambda_j)x + \lambda_j y\right] + u\left[(1-\lambda_{j+1})x + \lambda_{j+1}y\right]\right\}du\\ &= \int_0^1 f\left\{u\left[(1-\lambda_j)x + \lambda_j y\right] + (1-u)\left[(1-\lambda_{j+1})x + \lambda_{j+1}y\right]\right\}du,\end{aligned}$$

then by (29) we get

$$\begin{aligned}&\int_0^1 f\left\{(1-u)\left[(1-\lambda_j)x+\lambda_j y\right]+u\left[(1-\lambda_{j+1})x+\lambda_{j+1}y\right]\right\}du\\&\geq \frac{1}{2h\left(\frac{1}{2}\right)}f\left\{\left(1-\frac{\lambda_j+\lambda_{j+1}}{2}\right)x+\frac{\lambda_j+\lambda_{j+1}}{2}y\right\}\end{aligned}$$

for any $j\in\{0,...,n-1\}$.

Multiplying this inequality by $\lambda_{j+1}-\lambda_j\geq 0$ and summing over j from 0 to $n-1$ we get, via the equality (23), the first inequality in (27). □

Remark 4 *If we take in (27) $0=\lambda_0\leq\lambda_1=\lambda\leq\lambda_2=1$, then we get the first two inequalities in (14).*

The case of convex functions is as follows:

Corollary 4 *Let $f:C\subseteq X\to\mathbb{R}$ be a convex function on the convex subset C of a real or complex linear space X. Then for any partition*

$$0=\lambda_0<\lambda_1<...<\lambda_{n-1}<\lambda_n=1 \text{ with } n\geq 1,$$

and for any $x,y\in C$ we have the inequalities

$$\begin{aligned}&f\left(\frac{x+y}{2}\right)\\&\leq\sum_{j=0}^{n-1}(\lambda_{j+1}-\lambda_j)f\left\{\left(1-\frac{\lambda_j+\lambda_{j+1}}{2}\right)x+\frac{\lambda_j+\lambda_{j+1}}{2}y\right\}\\&\leq\int_0^1 f((1-t)x+ty)\,dt\\&\leq\frac{1}{2}\sum_{j=0}^{n-1}(\lambda_{j+1}-\lambda_j)\left[f((1-\lambda_j)x+\lambda_j y)+f((1-\lambda_{j+1})x+\lambda_{j+1}y)\right]\\&\leq\frac{f(x)+f(y)}{2}.\end{aligned}\tag{30}$$

Proof. The second and third inequalities in (30) follows from (27) by taking $h(t)=t$.

By the Jensen discrete inequality

$$\sum_{j=1}^{m}p_j f(z_j)\geq f\left(\sum_{j=1}^{m}p_j z_j\right),$$

where $p_j \geq 0$, $j \in \{1, ..., m\}$ with $\sum_{j=1}^{m} p_j = 1$ and $z_j \in C$, $j \in \{1, ..., m\}$ we have

$$\begin{aligned}
&\sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) f \left\{ \left(1 - \frac{\lambda_j + \lambda_{j+1}}{2}\right) x + \frac{\lambda_j + \lambda_{j+1}}{2} y \right\} \\
&\geq f \left\{ \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) \left[\left(1 - \frac{\lambda_j + \lambda_{j+1}}{2}\right) x + \frac{\lambda_j + \lambda_{j+1}}{2} y \right] \right\} \\
&= f \left\{ \left(\sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) - \frac{\sum_{j=0}^{n-1} \left(\lambda_{j+1}^2 - \lambda_j^2\right)}{2} \right) x + \frac{\sum_{j=0}^{n-1} \left(\lambda_{j+1}^2 - \lambda_j^2\right)}{2} y \right\} \\
&= f \left\{ \left(1 - \frac{1}{2}\right) x + \frac{1}{2} y \right\} = f \left(\frac{x+y}{2} \right)
\end{aligned}$$

and the first part of (30) is proved.

By the convexity of f we also have

$$\begin{aligned}
&\sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) \left[f\left((1-\lambda_j) x + \lambda_j y\right) + f\left((1-\lambda_{j+1}) x + \lambda_{j+1} y\right) \right] \\
&\leq \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) \left[(1-\lambda_j) f(x) + \lambda_j f(y) + (1-\lambda_{j+1}) f(x) + \lambda_{j+1} f(y) \right] \\
&= \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) \left[(2 - (\lambda_j + \lambda_{j+1})) f(x) + (\lambda_j + \lambda_{j+1}) f(y) \right] \\
&= \left(2 \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) - \sum_{j=0}^{n-1} \left(\lambda_{j+1}^2 - \lambda_j^2\right) \right) f(x) + \sum_{j=0}^{n-1} \left(\lambda_{j+1}^2 - \lambda_j^2\right) f(y) \\
&= f(x) + f(y),
\end{aligned}$$

which proves the last part of (30). $\square$

Remark 5 *Let $(X, \|\cdot\|)$ be a normed linear space over the real or complex number fields. Then for any partition*

$$0 = \lambda_0 < \lambda_1 < ... < \lambda_{n-1} < \lambda_n = 1 \text{ with } n \geq 1,$$

and for any $x, y \in X$ we have the inequalities

$$\begin{aligned}
&\left\|\frac{x+y}{2}\right\|^p \qquad (31)\\
&\leq \sum_{j=0}^{n-1}(\lambda_{j+1}-\lambda_j)\left\|\left(1-\frac{\lambda_j+\lambda_{j+1}}{2}\right)x+\frac{\lambda_j+\lambda_{j+1}}{2}y\right\|^p\\
&\leq \int_0^1 \|(1-t)x+ty\|^p\,dt\\
&\leq \frac{1}{2}\sum_{j=0}^{n-1}(\lambda_{j+1}-\lambda_j)\left[\|(1-\lambda_j)x+\lambda_j y\|^p+\|(1-\lambda_{j+1})x+\lambda_{j+1}y\|^p\right]\\
&\leq \frac{\|x\|^p+\|y\|^p}{2},
\end{aligned}$$

where $p \geq 1$.

Corollary 5 *Let $f : C \subseteq X \to \mathbb{R}$ be defined on a convex subset C of a real or complex linear space X and f is Breckner s-convex on C with $s \in (0,1)$. Assume that for $x, y \in C$ with $x \neq y$ the mapping $[0,1] \mapsto f((1-t)x+ty) \in \mathbb{R}$ is Lebesgue integrable on $[0,1]$. Then for any partition*

$$0=\lambda_0<\lambda_1<...<\lambda_{n-1}<\lambda_n=1 \text{ with } n\geq 1,$$

we have the inequalities

$$\begin{aligned}
&2^{s-1}\sum_{j=0}^{n-1}(\lambda_{j+1}-\lambda_j)f\left\{\left(1-\frac{\lambda_j+\lambda_{j+1}}{2}\right)x+\frac{\lambda_j+\lambda_{j+1}}{2}y\right\} \qquad (32)\\
&\leq \int_0^1 f((1-t)x+ty)\,dt\\
&\leq \frac{1}{s+1}\sum_{j=0}^{n-1}(\lambda_{j+1}-\lambda_j)\left[f((1-\lambda_j)x+\lambda_j y)+f((1-\lambda_{j+1})x+\lambda_{j+1}y)\right].
\end{aligned}$$

Since, for $s \in (0,1)$, the function $f(x) = \|x\|^s$ is Breckner s-convex on the normed linear space X, then by (32) we get for any $x, y \in X$

$$\begin{aligned}
&2^{s-1}\sum_{j=0}^{n-1}(\lambda_{j+1}-\lambda_j)\left\|\left(1-\frac{\lambda_j+\lambda_{j+1}}{2}\right)x+\frac{\lambda_j+\lambda_{j+1}}{2}y\right\|^s \qquad (33)\\
&\leq \int_0^1 \|(1-t)x+ty\|^s\,dt\\
&\leq \frac{1}{s+1}\sum_{j=0}^{n-1}(\lambda_{j+1}-\lambda_j)\left[\|(1-\lambda_j)x+\lambda_j y\|^s+\|(1-\lambda_{j+1})x+\lambda_{j+1}y\|^s\right].
\end{aligned}$$

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