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Article · April 2016

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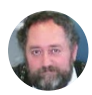

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DOI 10.7603/s40956-016-0002-4

Moroccan J. Pure and Appl. Anal.(MJPAA) RESEARCH ARTICLE
Volume 2(1), 2016, Pages 12–32
ISSN: 2351-8227

Generalization of Some Inequalities for Differentiable Co-ordinated Convex Functions With Applications

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Abstract. In this paper, a new weighted identity for functions defined on a rectangle from the plane is established. By using the obtained identity and analysis, some new weighted integral inequalities for the classes of co-ordinated convex, co-ordinated wright-convex and co-ordinated quasi-convex functions on the rectangle from the plane are established which provide weighted generalization of some recent results proved for co-ordinated convex functions. Some applications of our results to random variables and $2D$ weighted quadrature formula are given as well.

2000 Mathematics Subject Classification. 26D15, 26D20, 26D07.

Key words and phrases. Hermite-Hadamard's inequality, co-ordinated convex function, co-ordinated wright-convex function, co-ordinated quasi-convex function, Hölder's integral inequality, quadrature formula.

1. Introduction

The following definition is well known in mathematical analysis: A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if the inequality

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y),$$

Received February 16 , 2016 - Accepted March 30, 2016.

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Published online: 18 April 2016

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

A number of results have been established for the class of convex functions but the most famous is the Hermite-Hadamard's inequality (see for instance [7]). This double inequality is stated as:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1}$$

where $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$ a convex function, $a, b \in I$ with $a < b$. The inequalities in (1) are reversed if f is a concave function.

The inequalities (1) have become an important cornerstone in mathematical analysis and optimization and many uses of these inequalities have been discovered in a variety of settings. Moreover, many inequalities of special means can be obtained for a particular choice of the function f. Due to the rich geometrical significance of Hermite-Hadamard's inequality (1), there is growing literature providing its new proofs, extensions, refinements and generalizations, see for example [2, 4, 5, 6, 9, 21, 22] and the references therein.

Let us consider now a bidimensional interval $[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A mapping $f : [a,b]\times[c,d] \to \mathbb{R}$ is said to be convex on $[a,b]\times[c,d]$ if the inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda)f(z,w)$$

holds for all $(x,y),(z,w) \in [a,b]\times[c,d]$ and $\lambda \in [0,1]$.

A modification for convex functions on $[a,b]\times[c,d]$, which are also known as co-ordinated convex functions, was initiated by Dragomir [4, 6] as follows:

A function $f : [a,b]\times[c,d] \to \mathbb{R}$ is said to be convex on the co-ordinates on $[a,b]\times[c,d]$ if the partial mappings $f_y : [a,b] \to \mathbb{R}, f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}, f_x(v) = f(x,v)$ are convex where defined for all $x \in [a,b], y \in [c,d]$.

A formal definition for co-ordinated convex functions may be stated as follows:

Definition 1.1. [13] *A function $f : [a,b]\times[c,d] \to \mathbb{R}$ is said to be convex on the co-ordinates on $[a,b]\times[c,d]$ if the inequality*

$$\begin{aligned} &f(tx + (1-t)y, sz + (1-s)w) \\ &\leq tsf(x,z) + t(1-s)f(x,w) + s(1-t)f(y,z) + (1-t)(1-s)f(y,w) \end{aligned}$$

holds for all $(t,s) \in [0,1]\times[0,1]$ and $(x,z),(y,w) \in [a,b]\times[c,d]$.

It has been proved in [4] that every convex mapping $f : [a,b]\times[c,d] \to \mathbb{R}$ is convex on the co-ordinates. Furthermore, there exists co-ordinated convex function which is not convex, (see for example [4, 6]).

The following Hermite-Hadamard type inequality for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$ was also proved in [4]:

Theorem 1.1. [4] *Suppose that $f : [a,b]\times[c,d] \to \mathbb{R}$ is co-ordinated convex on $[a,b]\times[c,d]$. Then one has the inequalities:*

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$$

$$\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]$$
$$\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx$$
$$\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.$$
$$\left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]$$
$$\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \quad (2)$$

The above inequalities are sharp.

Sarikaya et al. [23], proved the following Hermite-Hadamard type inequalities.

Theorem 1.2. [23] *Let $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$, $c<d$. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ is convex on the co-ordinates on $[a,b]\times[c,d]$, then one has the inequalities:*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.$$
$$-\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.$$
$$\left.\left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\right|$$
$$\leq \frac{(b-a)(d-c)}{16}\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right]. \quad (3)$$

The next two results from [23] involve powers of the absolute value of $\frac{\partial^2 f}{\partial s\partial t}$.

Theorem 1.3. [23] *Let $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$, $c<d$. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$, $q\geq 1$, is convex on the co-ordinates on $[a,b]\times[c,d]$, then one has the inequalities:*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.$$
$$-\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.$$
$$\left.\left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\right|$$

$$\leq \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\left[\frac{\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q}{4}\right]^{\frac{1}{q}}, \quad (4)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1.4. [23] *Let* $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ *be a partial differentiable mapping on* $[a,b]\times[c,d]$ *in* $\mathbb{R}^2$ *with* $a<b$, $c<d$. *If* $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$, $q>1$, *is convex on the co-ordinates on* $[a,b]\times[c,d]$, *then one has the inequalities:*

$$\Bigg|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}$$
$$-\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.$$
$$\left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\Bigg|$$
$$\leq \frac{(b-a)(d-c)}{16}\left[\frac{\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q}{4}\right]^{\frac{1}{q}}. \quad (5)$$

In a recent paper [22], M. E. Özdemir et al. give the notion of co-ordinated quasi-convex functions which generalize the notion of co-ordinated convex functions.

Definition 1.2. [20] *A function* $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ *is said to be quasi-convex on* $[a,b]\times[c,d]$ *if the inequality*

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\leq\max\{f(x,y),f(z,w)\}$$

holds for all $(x,y),(z,w)\in[a,b]\times[c,d]$ *and* $\lambda\in[0,1]$.

A function $f:[a,b]\times[c,d]\to\mathbb{R}$ is said to be quasi-convex on the co-ordinates on $[a,b]\times[c,d]$ if the partial mappings $f_y:[a,b]\to\mathbb{R}, f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}, f_x(v)=f(x,v)$ are quasi-convex where defined for all $x\in[a,b], y\in[c,d]$.

Another way of describing the definition of co-ordinated quasi-convex functions is given below.

Definition 1.3. [16] *A function* $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ *is said to be quasi-convex on the co-ordinates on* $[a,b]\times[c,d]$ *if*

$$f(tx+(1-t)z,sy+(1-s)w)\leq\max\{f(x,y),f(x,w),f(z,y),f(z,w)\}$$

for all $(x,y),(z,w)\in[a,b]\times[c,d]$ *and* $(s,t)\in[0,1]\times[0,1]$.

The class of co-ordinated quasi-convex functions on $[a,b]\times[c,d]$ is denoted by $QC([a,b]\times[c,d])$. It has also been proved in [20] that every quasi-convex functions on

$[a,b]\times[c,d]$ is quasi-convex on the co-ordinates on $[a,b]\times[c,d]$. The following example reveals that there exists quasi-convex function on the co-ordinates which is not quasi-convex.

Example 1.1. [16] *The function* $f:[-2,2]^2\to\mathbb{R}$, *defined by* $f(x,y)=\lfloor x\rfloor\lfloor y\rfloor$, *where* $\lfloor . \rfloor$ *is the floor function. This function is quasi-convex on the co-ordinates on* $[-2,2]^2$ *but is not quasi-convex on* $[0,1]^2$.
For example, take $(x,y)=(-2,1)$, $(z,w)=(1,-1)$ *and* $\lambda=\frac{1}{2}$, *then*

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)=f\left(-\frac{1}{2},0\right)=0,$$

on the other hand

$$\max\{f(x,y),f(z,w)\}=\max\{f(-2,1),f(1,-1)\}=-1,$$

which shows that $f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)>\max\{f(x,y),f(z,w)\}$.

Another generalization of the notion of the co-ordinated convex functions is the concept of wright-convex functions which is given in the definition below.

Definition 1.4. [20] *A function* $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ *is said to be wright-convex on* $[a,b]\times[c,d]$ *if the inequality*

$$\begin{aligned}&f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)+f((1-\lambda)x+\lambda z,(1-\lambda)y+\lambda w)\\&\leq\max\{f(x,z),f(y,w)\},\end{aligned}$$

holds for all $(x,z),(y,w)\in[a,b]\times[c,d]$ *and* $\lambda\in[0,1]$.

A function $f:[a,b]\times[c,d]\to\mathbb{R}$ is said to be wright-convex on the co-ordinates on $[a,b]\times[c,d]$ if the partial mappings $f_y:[a,b]\to\mathbb{R}, f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}, f_x(v)=f(x,v)$ are wright-convex where defined for all $x\in[a,b], y\in[c,d]$.

The above definition of wright-convex functions on the co-ordinates can be reformulated as follows.

Definition 1.5. [20] *A function* $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ *is said to be wright-convex on the co-ordinates on* $[a,b]\times[c,d]$ *if*

$$\begin{aligned}&f(tx+(1-t)z,sy+(1-s)w)+f((1-t)x+tz,(1-s)y+sw)\\&\leq f(x,y)+f(z,y)+f(x,w)+f(z,w)\end{aligned}$$

for all $(x,z),(y,w)\in[a,b]\times[c,d]$ *and* $(s,t)\in[0,1]\times[0,1]$.

The class of co-ordinated wright-convex functions on $[a,b]\times[c,d]$ is represented by $W([a,b]\times[c,d])$. It has also been proved in [20] that every wright-convex functions on $[a,b]\times[c,d]$ is wright-convex on the co-ordinates on $[a,b]\times[c,d]$.

For more recent results on co-ordinated convex, co-ordinated quasi-convex, co-ordinated m-convex, co-ordinated (α,m)-convex and co-ordinated s-convex functions on a rectangle $[a,b]\times[c,d]$ from the plane $\mathbb{R}^2$, we refer the readers to [1, 5, 8], [10]-[20].

In the present paper, we establish a new weighted identity for differentiable mappings defined on a rectangle $[a,b]\times[c,d]$ from the plane $\mathbb{R}^2$ and by using the obtained identity and analysis, some new weighted integral inequalities for differentiable co-ordinated convex, co-ordinated wright-convex and co-ordinated quasi convex functions are proved. The results proved in the paper provide a weighted generalization of the results given in Theorem 1.2, Theorem 1.3 and Theorem 1.4. Applications of our results to random variables and $2D$ weighted quadrature formula are provided as well.

2. Main Results

We need the following lemma to prove our results. Moreover, the following notions will be used throughout in the section

$$U_1(a,b,t)=U_1(t)=\frac{1-t}{2}a+\frac{1+t}{2}b, L_1(a,b,t)=L_1(t)=\frac{1+t}{2}a+\frac{1-t}{2}b,$$
$$U_2(c,d,s)=U_2(s)=\frac{1-s}{2}c+\frac{1+s}{2}d, L_2(c,d,s)=L_2(s)=\frac{1+s}{2}c+\frac{1-s}{2}d,$$
$$\Psi(a,b,c,d;|f_{ts}|)=\frac{|f_{ts}(a,c)|+|f_{ts}(a,d)|+|f_{ts}(b,c)|+|f_{ts}(b,d)|}{4},$$

$$\lambda_1\left(b,d,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|\right)$$
$$=\max\left\{|f_{ts}(b,d)|,\left|f_{ts}\left(b,\frac{c+d}{2}\right)\right|,\left|f_{ts}\left(\frac{a+b}{2},d\right)\right|,\left|f_{ts}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\},$$

$$\lambda_2\left(a,d,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|\right)$$
$$=\max\left\{|f_{ts}(a,d)|,\left|f_{ts}\left(a,\frac{c+d}{2}\right)\right|,\left|f_{ts}\left(\frac{a+b}{2},d\right)\right|,\left|f_{ts}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\},$$

$$\lambda_3\left(b,c,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|\right)$$
$$=\max\left\{|f_{ts}(b,c)|,\left|f_{ts}\left(b,\frac{c+d}{2}\right)\right|,\left|f_{ts}\left(\frac{a+b}{2},c\right)\right|,\left|f_{ts}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}$$

and

$$\lambda_4\left(a,c,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|\right)$$
$$=\max\left\{|f_{ts}(a,c)|,\left|f_{ts}\left(a,\frac{c+d}{2}\right)\right|,\left|f_{ts}\left(\frac{a+b}{2},c\right)\right|,\left|f_{ts}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}.$$

Lemma 2.1. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a twice partially differentiable mapping on Δ° and $p:[a,b]\times[c,d]\to[0,\infty)$ be continuous and symmetric to $\frac{a+b}{2}$ and $\frac{c+d}{2}$ for $[a,b]\times[c,d]\subset\Delta^\circ$ with $a<b$, $c<d$. If $f_{ts}\in L([a,b]\times[c,d])$, then*

$$\Phi(a,b,c,d;p,f)=\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_c^d\int_a^b p(x,y)\,dxdy$$

$$-\frac{1}{2}\int_c^d\int_a^b\left[f\left(x,c\right)+f\left(x,d\right)\right]p\left(x,y\right)dxdy$$
$$-\frac{1}{2}\int_c^d\int_a^b\left[f\left(a,y\right)+f\left(b,y\right)\right]p\left(x,y\right)dxdy+\int_c^d\int_a^b f\left(x,y\right)p\left(x,y\right)dxdy$$
$$=\frac{\left(b-a\right)\left(d-c\right)}{16}\int_0^1\int_0^1\left[\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)}p\left(x,y\right)dxdy\right]\left[f_{ts}\left(U_1\left(t\right),U_2\left(s\right)\right)\right.$$
$$\left.-f_{ts}\left(U_1\left(t\right),L_2\left(s\right)\right)-f_{ts}\left(L_1\left(t\right),U_2\left(s\right)\right)+f_{ts}\left(L_1\left(t\right),L_2\left(s\right)\right)\right]dsdt. \quad (6)$$

Proof. Let

$$I=\frac{\left(b-a\right)\left(d-c\right)}{16}\int_0^1\int_0^1\left[\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)}p\left(x,y\right)dxdy\right]\left[f_{ts}\left(U_1\left(t\right),U_2\left(s\right)\right)\right.$$
$$\left.-f_{ts}\left(U_1\left(t\right),L_2\left(s\right)\right)-f_{ts}\left(L_1\left(t\right),U_2\left(s\right)\right)+f_{ts}\left(L_1\left(t\right),L_2\left(s\right)\right)\right]dsdt$$

and

$$\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)}p\left(x,y\right)dxdy=q\left(t,s\right).$$

then

$$I=\frac{\left(b-a\right)\left(d-c\right)}{16}\int_0^1\int_0^1 q\left(t,s\right)\left[f_{ts}\left(U_1\left(t\right),U_2\left(s\right)\right)-f_{ts}\left(U_1\left(t\right),L_2\left(s\right)\right)\right.$$
$$\left.-f_{ts}\left(L_1\left(t\right),U_2\left(s\right)\right)+f_{ts}\left(L_1\left(t\right),L_2\left(s\right)\right)\right]dsdt.$$

Now by integration by parts and by using the symmetry of $p\left(x,y\right)$ about $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$, we have

$$\frac{\left(b-a\right)\left(d-c\right)}{16}\int_0^1\int_0^1 q\left(t,s\right)f_{ts}\left(U_1\left(t\right),U_2\left(s\right)\right)dsdt$$
$$=\frac{\left(b-a\right)\left(d-c\right)}{16}\int_0^1\left[\int_0^1 q\left(t,s\right)f_{ts}\left(U_1\left(t\right),U_2\left(s\right)\right)ds\right]dt$$
$$=\frac{\left(b-a\right)\left(d-c\right)}{16}\int_0^1\left[\frac{2}{d-c}q\left(t,s\right)f_t\left(U_1\left(t\right),U_2\left(s\right)\right)\Big|_0^1\right.$$
$$\left.-\frac{2}{d-c}\int_0^1 q_s\left(t,s\right)f_t\left(U_1\left(t\right),U_2\left(s\right)\right)ds\right]dt$$
$$=\frac{\left(b-a\right)}{8}\int_0^1\left[f_t\left(U_1\left(t\right),d\right)\left(\int_c^d\int_{L_1(t)}^{U_1(t)}p\left(x,y\right)dxdy\right)\right.$$
$$\left.-\left(d-c\right)\int_0^1\left(\int_{L_1(t)}^{U_1(t)}p\left(x,U_2\left(s\right)\right)dx\right)f_t\left(U_1\left(t\right),U_2\left(s\right)\right)ds\right]dt$$
$$=\frac{\left(b-a\right)}{8}\int_0^1 f_t\left(U_1\left(t\right),d\right)\left(\int_c^d\int_{L_1(t)}^{U_1(t)}p\left(x,y\right)dxdy\right)dt$$

$$
\begin{aligned}
&-\frac{(b-a)}{4}\int_{\frac{c+d}{2}}^{d}\int_{0}^{1}\left(\int_{L_1(t)}^{U_1(t)}p(x,y)\,dx\right)f_t\left(U_1(t),y\right)dtdy\\
&=\frac{1}{4}f(b,d)\int_c^d\int_a^b p(x,y)\,dxdy-\frac{1}{2}\int_c^d\int_{\frac{a+b}{2}}^{b}p(x,y)\,f(x,d)\,dxdy\\
&\quad-\frac{1}{2}\int_{\frac{c+d}{2}}^{d}\int_a^b p(x,y)\,f(b,y)\,dxdy+\int_{\frac{c+d}{2}}^{d}\int_{\frac{a+b}{2}}^{b}p(x,y)\,f(x,y)\,dxdy. \qquad (7)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&-\frac{(b-a)(d-c)}{16}\int_0^1\int_0^1 q(t,s)\,f_{ts}\left(U_1(t),L_2(s)\right)dsdt\\
&=\frac{1}{4}f(b,c)\int_c^d\int_a^b p(x,y)\,dxdy-\frac{1}{2}\int_c^d\int_{\frac{a+b}{2}}^{b}p(x,y)\,f(x,c)\,dxdy\\
&\quad-\frac{1}{2}\int_c^{\frac{c+d}{2}}\int_a^b p(x,y)\,f(b,y)\,dxdy+\int_c^{\frac{c+d}{2}}\int_{\frac{a+b}{2}}^{b}p(x,y)\,f(x,y)\,dxdy, \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{(b-a)(d-c)}{16}\int_0^1\int_0^1 q(t,s)\,f_{ts}\left(L_1(t),U_2(s)\right)dsdt\\
&=\frac{1}{4}f(a,d)\int_c^d\int_a^b p(x,y)\,dxdy-\frac{1}{2}\int_c^d\int_a^{\frac{a+b}{2}}p(x,y)\,f(x,d)\,dxdy\\
&\quad-\frac{1}{2}\int_{\frac{c+d}{2}}^{d}\int_a^b p(x,y)\,f(a,y)\,dxdy+\int_{\frac{c+d}{2}}^{d}\int_a^{\frac{a+b}{2}}p(x,y)\,f(x,y)\,dxdy \qquad (9)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{(b-a)(d-c)}{16}\int_0^1\int_0^1 q(t,s)\,f_{ts}\left(L_1(t),L_2(s)\right)dsdt\\
&=\frac{1}{4}f(a,c)\int_c^d\int_a^b p(x,y)\,dxdy-\frac{1}{2}\int_c^d\int_a^{\frac{a+b}{2}}p(x,y)\,f(x,c)\,dxdy\\
&\quad-\frac{1}{2}\int_c^{\frac{c+d}{2}}\int_a^b p(x,y)\,f(a,y)\,dxdy+\int_c^{\frac{c+d}{2}}\int_a^{\frac{a+b}{2}}p(x,y)\,f(x,y)\,dxdy. \qquad (10)
\end{aligned}
$$

Adding (7)-(10), we get the desired result. □

Remark 2.1. *If we take* $p(x,y)=\frac{1}{(b-a)(d-c)}$ *for all* $(x,y)\in[a,b]\times[c,d]$ *in Lemma 2.1, we get Lemma 1 from* [23, page 139].

Now by using lemma 2.1, we present the main results of this section.

Theorem 2.1. *Let* $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ *be a twice partially differentiable mapping on* Δ° *and* $p:[a,b]\times[c,d]\to[0,\infty)$ *be continuous and symmetric to* $\frac{a+b}{2}$ *and* $\frac{c+d}{2}$ *for*

$[a,b]\times[c,d]\subset\Delta^{\circ}$ with $a<b$, $c<d$. If $f_{ts}\in L([a,b]\times[c,d])$ and $|f_{ts}|$ is convex on the co-ordinates on $[a,b]\times[c,d]$, then

$$\begin{aligned}&|\Phi(a,b,c,d;p,f)|\\&\qquad\leq\frac{(b-a)(d-c)}{4}\Psi(a,b,c,d;|f_{ts}|)\int_0^1\int_0^1\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)}p(x,y)\,dxdydtds.\quad(11)\end{aligned}$$

Proof. Taking absolute value on both sides of (6) and using the properties of absolute value, we have

$$\begin{aligned}&|\Phi(a,b,c,d;p,f)|\\&\quad\leq\frac{(b-a)(d-c)}{16}\int_0^1\int_0^1\left[\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)}p(x,y)\,dxdy\right][|f_{ts}(U_1(t),U_2(s))|\\&\qquad+|f_{ts}(U_1(t),L_2(s))|+|f_{ts}(L_1(t),U_2(s))|+|f_{ts}(L_1(t),L_2(s))|]\,dsdt.\quad(12)\end{aligned}$$

By the convexity of $|f_{ts}|$ on the co-ordinates on $[a,b]\times[c,d]$, we have

$$\begin{aligned}&|f_{ts}(U_1(t),U_2(s))|\\&\quad\leq\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)|f_{ts}(a,c)|+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)|f_{ts}(a,d)|\\&\qquad+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)|f_{ts}(b,c)|+\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)|f_{ts}(b,d)|,\quad(13)\end{aligned}$$

$$\begin{aligned}&|f_{ts}(U_1(t),L_2(s))|\\&\quad\leq\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)|f_{ts}(a,c)|+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)|f_{ts}(a,d)|\\&\qquad+\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)|f_{ts}(b,c)|+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)|f_{ts}(b,d)|,\quad(14)\end{aligned}$$

$$\begin{aligned}&|f_{ts}(L_1(t),U_2(s))|\\&\quad\leq\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)|f_{ts}(a,c)|+\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)|f_{ts}(a,d)|\\&\qquad+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)|f_{ts}(b,c)|+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)|f_{ts}(b,d)|,\quad(15)\end{aligned}$$

and

$$\begin{aligned}&|f_{ts}(L_1(t),L_2(s))|\\&\quad\leq\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)|f_{ts}(a,c)|+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)|f_{ts}(a,d)|\\&\qquad+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)|f_{ts}(b,c)|+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)|f_{ts}(b,d)|.\quad(16)\end{aligned}$$

Using (13)-(16) in (12), we get (11). □

Remark 2.2. *If we take* $p(x,y) = \frac{1}{(b-a)(d-c)}$ *for all* $(x,y) \in [a,b] \times [c,d]$ *in Theorem 2.1, we get Theorem 1.2 from* [23].

A more general result is given in the following theorem.

Theorem 2.2. *Let* $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ *be a twice partially differentiable mapping on* Δ° *and* $p : [a,b] \times [c,d] \to [0,\infty)$ *be continuous and symmetric to* $\frac{a+b}{2}$ *and* $\frac{c+d}{2}$ *for* $[a,b] \times [c,d] \subset \Delta^\circ$ *with* $a < b$, $c < d$. *If* $f_{ts} \in L([a,b] \times [c,d])$ *and* $|f_{ts}|^q$ *is convex on the co-ordinates on* $[a,b] \times [c,d]$ *for* $q \geq 1$, *then*

$$|\Phi(a,b,c,d;p,f)| \leq \frac{(b-a)(d-c)}{4} \left[\Psi\left(a,b,c,d;|f_{ts}|^q\right)\right]^{\frac{1}{q}} \int_0^1 \int_0^1 \int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdydsdt. \quad (17)$$

Proof. Taking absolute value on both sides of (6), by using the properties of absolute value and the Hölder inequality, we have

$$\begin{aligned}
&|\Phi(a,b,c,d;p,f)| \\
&\leq \frac{(b-a)(d-c)}{16} \left(\int_0^1 \int_0^1 \left[\int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdy \right] dsdt \right)^{1-\frac{1}{q}} \\
&\times \left[\left(\int_0^1 \int_0^1 \left[\int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdy \right] |f_{ts}(U_1(t), U_2(s))|^q\, dsdt \right)^{\frac{1}{q}} \right. \\
&+ \left(\int_0^1 \int_0^1 \left[\int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdy \right] |f_{ts}(U_1(t), L_2(s))|^q\, dsdt \right)^{\frac{1}{q}} \\
&+ \left(\int_0^1 \int_0^1 \left[\int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdy \right] |f_{ts}(L_1(t), U_2(s))|^q\, dsdt \right)^{\frac{1}{q}} \\
&\left. + \left(\int_0^1 \int_0^1 \left[\int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdy \right] |f_{ts}(L_1(t), L_2(s))|^q\, dsdt \right)^{\frac{1}{q}} \right]. \quad (18)
\end{aligned}$$

By the power-mean inequality ($a_1^r + a_2^r + a_3^r + a_4^r \leq 4^{1-r}(a_1+a_2+a_3+a_4)^r$ for a_1, a_2, a_3, $a_4 > 0$ and $r < 1$) and using the convexity of $|f_{ts}|^q$ on the co-ordinates on $[a,b] \times [c,d]$ for $q \geq 1$, we have

$$\begin{aligned}
&\left(\int_0^1 \int_0^1 \left[\int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdy \right] |f_{ts}(U_1(t), U_2(s))|^q\, dsdt \right)^{\frac{1}{q}} \\
&+ \left(\int_0^1 \int_0^1 \left[\int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdy \right] |f_{ts}(U_1(t), L_2(s))|^q\, dsdt \right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\int_0^1\int_0^1\left[\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)}p\left(x,y\right)dxdy\right]\left|f_{ts}\left(L_1\left(t\right),U_2\left(s\right)\right)\right|^q dsdt\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_0^1\int_0^1\left[\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)}p\left(x,y\right)dxdy\right]\left|f_{ts}\left(L_1\left(t\right),L_2\left(s\right)\right)\right|^q dsdt\right)^{\frac{1}{q}}\right]$$
$$\leq 4^{1-\frac{1}{q}}\left[\left|f_{ts}\left(a,c\right)\right|+\left|f_{ts}\left(a,d\right)\right|+\left|f_{ts}\left(b,c\right)\right|+\left|f_{ts}\left(b,d\right)\right|\right]^{\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)}p\left(x,y\right)dxdydsdt\right)^{\frac{1}{q}}. \quad (19)$$

A usage of (19) in (18) yields the desired result. □

Remark 2.3. *If we take* $p\left(x,y\right)=\frac{1}{(b-a)(d-c)}$ *for all* $(x,y)\in[a,b]\times[c,d]$ *in Theorem 2.2, we get Theorem 1.4.*

A different approach leads to the following result.

Theorem 2.3. *Let* $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ *be a twice differentiable mapping on* Δ° *and* $p:[a,b]\times[c,d]\to[0,\infty)$ *be continuous and symmetric to* $\frac{a+b}{2}$ *and* $\frac{c+d}{2}$ *for* $[a,b]\times[c,d]\subset\Delta^\circ$ *with* $a<b$, $c<d$. *If* $f_{ts}\in L\left([a,b]\times[c,d]\right)$ *and* $\left|f_{ts}\right|^q$ *is convex on the co-ordinates on* $[a,b]\times[c,d]$ *for* $q>1$, *then*

$$\left|\Phi\left(a,b,c,d;p,f\right)\right|$$
$$\leq\frac{\left(b-a\right)\left(d-c\right)}{4}\left[\Psi\left(a,b,c,d;\left|f_{ts}\right|^q\right)\right]^{\frac{1}{q}}\left(\int_0^1\int_0^1\left[\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)}p\left(x,y\right)dxdy\right]^p dsdt\right)^{\frac{1}{p}}, \quad (20)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1 and the Hölder inequality, we have

$$\left|\Phi\left(a,b,c,d;p,f\right)\right|\leq\frac{\left(b-a\right)\left(d-c\right)}{16}\left(\int_0^1\int_0^1\left[\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)}p\left(x,y\right)dxdy\right]^p dsdt\right)^{\frac{1}{p}}$$
$$\times\left[\left(\int_0^1\int_0^1\left|f_{ts}\left(U_1\left(t\right),U_2\left(s\right)\right)\right|^q dsdt\right)^{\frac{1}{q}}+\left(\int_0^1\int_0^1\left|f_{ts}\left(U_1\left(t\right),L_2\left(s\right)\right)\right|^q dsdt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\int_0^1\left|f_{ts}\left(L_1\left(t\right),U_2\left(s\right)\right)\right|^q dsdt\right)^{\frac{1}{q}}+\left(\int_0^1\int_0^1\left|f_{ts}\left(L_1\left(t\right),L_2\left(s\right)\right)\right|^q dsdt\right)^{\frac{1}{q}}\right]. \quad (21)$$

By the power-mean inequality ($a_1^r+a_2^r+a_3^r+a_4^r\leq 4^{1-r}(a_1+a_2+a_3+a_4)^r$ for a_1, a_2, a_3, $a_4>0$ and $r<1$) and using the convexity of $\left|f_{ts}\right|^q$ on the co-ordinates on $[a,b]\times[c,d]$ for $q>1$, we have

$$\left(\int_0^1\int_0^1 |f_{ts}(U_1(t),U_2(s))|^q\,dsdt\right)^{\frac{1}{q}} + \left(\int_0^1\int_0^1 |f_{ts}(U_1(t),L_2(s))|^q\,dsdt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\int_0^1 |f_{ts}(L_1(t),U_2(s))|^q\,dsdt\right)^{\frac{1}{q}} + \left(\int_0^1\int_0^1 |f_{ts}(L_1(t),L_2(s))|^q\,dsdt\right)^{\frac{1}{q}}$$
$$\leq 4^{1-\frac{1}{q}}\left[\int_0^1\int_0^1 |f_{ts}(U_1(t),U_2(s))|^q\,dsdt + \int_0^1\int_0^1 |f_{ts}(U_1(t),L_2(s))|^q\,dsdt\right.$$
$$\left.+\int_0^1\int_0^1 |f_{ts}(L_1(t),U_2(s))|^q\,dsdt + \int_0^1\int_0^1 |f_{ts}(L_1(t),L_2(s))|^q\,dsdt\right]^{\frac{1}{q}}$$
$$\leq 4\left[\frac{|f_{ts}(a,c)|^q + |f_{ts}(a,d)|^q + |f_{ts}(b,c)|^q + |f_{ts}(b,d)|^q}{4}\right]^{\frac{1}{q}}. \quad (22)$$

From (21) and (22), we get (20). □

Remark 2.4. *If we take* $p(x,y) = \frac{1}{(b-a)(d-c)}$ *for all* $(x,y) \in [a,b]\times[c,d]$ *in Theorem 2.3, we get Theorem 1.3.*

Remark 2.5. *Theorem 2.1-Theorem 2.3 continue to hold true if in their statements we replace the condition "convex on the co-ordinates" with the condition "wright-convex on the co-ordinates". However, the details are left to the interested reader.*

In what follows we give our results for the quasi-convex mappings on the co-ordinates on $[a,b]\times[c,d]$.

Theorem 2.4. *Suppose the assumptions of Theorem 2.1 are satisfied. If the mapping* $|f_{ts}|$ *is quasi-convex on the co-ordinates on* $[a,b]\times[c,d]$*, then the following inequality holds*

$$|\Phi(a,b,c,d;p,f)| \leq \frac{(b-a)(d-c)}{16}\left[\lambda_1\left(b,d,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|\right)\right.$$
$$+\lambda_2\left(a,d,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|\right) + \lambda_3\left(b,c,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|\right)$$
$$\left.+\lambda_4\left(a,c,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|\right)\right]\int_0^1\int_0^1\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdydtds. \quad (23)$$

Proof. We continue inequality (12) in the proof of Theorem 2.1. Now, by the quasi-convexity on the co-ordinates of $|f_{ts}|$ on $[a,b]\times[c,d]$, we obtain

$$|f_{ts}(U_1(t),U_2(s))|$$
$$\leq \max\left\{|f_{ts}(b,d)|, \left|f_{ts}\left(b,\frac{c+d}{2}\right)\right|, \left|f_{ts}\left(\frac{a+b}{2},d\right)\right|, \left|f_{ts}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}, \quad (24)$$

$$|f_{ts}(L_1(t),U_2(s))|$$
$$\leq \max\left\{|f_{ts}(a,d)|, \left|f_{ts}\left(a,\frac{c+d}{2}\right)\right|, \left|f_{ts}\left(\frac{a+b}{2},d\right)\right|, \left|f_{ts}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}, \quad (25)$$

$$|f_{ts}(U_1(t), L_2(s))| \leq \max\left\{|f_{ts}(b,c)|, \left|f_{ts}\left(b, \frac{c+d}{2}\right)\right|, \left|f_{ts}\left(\frac{a+b}{2}, c\right)\right|, \left|f_{ts}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|\right\}, \quad (26)$$

and

$$|f_{ts}(L_1(t), L_2(s))| \leq \max\left\{|f_{ts}(a,c)|, \left|f_{ts}\left(a, \frac{c+d}{2}\right)\right|, \left|f_{ts}\left(\frac{a+b}{2}, c\right)\right|, \left|f_{ts}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|\right\}, \quad (27)$$

for all $(t,s) \in [0,1] \times [0,1]$. A combination of (24)-(27) and (12)gives the required inequality (23). □

Corollary 2.1. *Suppose the assumptions of Theorem 2.4 are fulfilled and if* $p(x,y) = \frac{1}{(b-a)(d-c)}$ *for all* $(x,y) \in [a,b] \times [c,d]$*, then the following inequality holds valid*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - \frac{1}{2(b-a)}\int_a^b [f(x,c)+f(x,d)]\,dx - \frac{1}{2(d-c)}\int_c^d [f(a,y)+f(b,y)]\,dy + \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,dxdy\right| \leq \frac{(b-a)(d-c)}{16} \times \left[\lambda_1\left(b,d,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|\right) + \lambda_2\left(a,d,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|\right) + \lambda_3\left(b,c,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|\right) + \lambda_4\left(a,c,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|\right)\right]. \quad (28)$$

Corollary 2.2. *Suppose the assumptions of Theorem 2.4 are satisfied and additionally*

(1) *If* $|f_{ts}|$ *is non-decreasing on the co-ordinates on* $[a,b] \times [c,d]$*, then the following inequality holds true*

$$|\Phi(a,b,c,d;p,f)| \leq \frac{(b-a)(d-c)}{16}\left[|f_{ts}(b,d)| + \left|f_{ts}\left(\frac{a+b}{2}, d\right)\right| + \left|f_{ts}\left(b, \frac{c+d}{2}\right)\right| + \left|f_{ts}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|\right]\int_0^1\int_0^1\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdydtds. \quad (29)$$

(2) *If* $|f_{ts}|$ *is non-increasing on the co-ordinates on* $[a,b] \times [c,d]$*, then the following inequality holds true*

$$|\Phi(a,b,c,d;p,f)| \leq \frac{(b-a)(d-c)}{16}\left[|f_{ts}(a,c)| + \left|f_{ts}\left(a, \frac{c+d}{2}\right)\right| + \left|f_{ts}\left(\frac{a+b}{2}, c\right)\right|\right.$$

$$+\left|f_{ts}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\Bigg]\int_0^1\int_0^1\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)}p\left(x,y\right)dxdydtds. \quad (30)$$

Corollary 2.3. *If we take* $p\left(x,y\right)=\frac{1}{(b-a)(d-c)}$ *for all* $(x,y)\in[a,b]\times[c,d]$ *in Corollary 2.2 and additionally*

(1) *If* $|f_{ts}|$ *is non-decreasing on the co-ordinates on* $[a,b]\times[c,d]$*, then the following inequality holds true*

$$\Bigg|\frac{f\left(a,c\right)+f\left(a,d\right)+f\left(b,c\right)+f\left(b,d\right)}{4}$$
$$-\frac{1}{2\left(b-a\right)}\int_a^b\left[f\left(x,c\right)+f\left(x,d\right)\right]dx-\frac{1}{2\left(d-c\right)}\int_c^d\left[f\left(a,y\right)+f\left(b,y\right)\right]dy$$
$$+\frac{1}{\left(b-a\right)\left(d-c\right)}\int_c^d\int_a^b f\left(x,y\right)dxdy\Bigg|\leq\frac{\left(b-a\right)\left(d-c\right)}{16}$$
$$\times\left[\left|f_{ts}\left(b,d\right)\right|+\left|f_{ts}\left(\frac{a+b}{2},d\right)\right|+\left|f_{ts}\left(b,\frac{c+d}{2}\right)\right|+\left|f_{ts}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right]. \quad (31)$$

(2) *If* $|f_{ts}|$ *is non-increasing on the co-ordinates on* $[a,b]\times[c,d]$*, then the following inequality holds true*

$$\Bigg|\frac{f\left(a,c\right)+f\left(a,d\right)+f\left(b,c\right)+f\left(b,d\right)}{4}$$
$$-\frac{1}{2\left(b-a\right)}\int_a^b\left[f\left(x,c\right)+f\left(x,d\right)\right]dx-\frac{1}{2\left(d-c\right)}\int_c^d\left[f\left(a,y\right)+f\left(b,y\right)\right]dy$$
$$+\frac{1}{\left(b-a\right)\left(d-c\right)}\int_c^d\int_a^b f\left(x,y\right)dxdy\Bigg|\leq\frac{\left(b-a\right)\left(d-c\right)}{16}$$
$$\times\left[\left|f_{ts}\left(a,c\right)\right|+\left|f_{ts}\left(a,\frac{c+d}{2}\right)\right|+\left|f_{ts}\left(\frac{a+b}{2},c\right)\right|+\left|f_{ts}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right]. \quad (32)$$

Theorem 2.5. *Suppose the assumptions of Theorem 2.1 are satisfied. If the mapping* $|f_{ts}|^q$ *is quasi-convex on the co-ordinates on* $[a,b]\times[c,d]$ *for* $q\geq 1$*, then the following inequality holds*

$$\left|\Phi\left(a,b,c,d;p,f\right)\right|\leq\frac{\left(b-a\right)\left(d-c\right)}{16}\Bigg\{\left[\lambda_1\left(b,d,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|^q\right)\right]^{\frac{1}{q}}$$
$$+\left[\lambda_2\left(a,d,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|^q\right)\right]^{\frac{1}{q}}+\left[\lambda_3\left(b,c,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|^q\right)\right]^{\frac{1}{q}}$$
$$+\left[\lambda_4\left(a,c,\frac{a+b}{2},\frac{c+d}{2};|f_{ts}|^q\right)\right]^{\frac{1}{q}}\Bigg]\int_0^1\int_0^1\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)}p\left(x,y\right)dxdydtds. \quad (33)$$

Proof. We continue inequality (18) in the proof of Theorem 2.2. Now, by the quasi-convexity on the co-ordinates of $|f_{ts}|^q$ on $[a,b]\times[c,d]$ for $q\geq 1$ and the power-mean inequality, we obtain

$$\begin{aligned}&\left|f_{ts}\left(U_1\left(t\right),U_2\left(s\right)\right)\right|^q\\&\quad\leq\max\left\{\left|f_{ts}\left(b,d\right)\right|^q,\left|f_{ts}\left(b,\frac{c+d}{2}\right)\right|^q,\left|f_{ts}\left(\frac{a+b}{2},d\right)\right|^q,\left|f_{ts}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\},\end{aligned}\tag{34}$$

$$\begin{aligned}&\left|f_{ts}\left(L_1\left(t\right),U_2\left(s\right)\right)\right|^q\\&\quad\leq\max\left\{\left|f_{ts}\left(a,d\right)\right|^q,\left|f_{ts}\left(a,\frac{c+d}{2}\right)\right|^q,\left|f_{ts}\left(\frac{a+b}{2},d\right)\right|^q,\left|f_{ts}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\},\end{aligned}\tag{35}$$

$$\begin{aligned}&\left|f_{ts}\left(U_1\left(t\right),L_2\left(s\right)\right)\right|^q\\&\quad\leq\max\left\{\left|f_{ts}\left(b,c\right)\right|^q,\left|f_{ts}\left(b,\frac{c+d}{2}\right)\right|^q,\left|f_{ts}\left(\frac{a+b}{2},c\right)\right|^q,\left|f_{ts}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\},\end{aligned}\tag{36}$$

and

$$\begin{aligned}&\left|f_{ts}\left(L_1\left(t\right),L_2\left(s\right)\right)\right|^q\\&\quad\leq\max\left\{\left|f_{ts}\left(a,c\right)\right|^q,\left|f_{ts}\left(a,\frac{c+d}{2}\right)\right|^q,\left|f_{ts}\left(\frac{a+b}{2},c\right)\right|^q,\left|f_{ts}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\},\end{aligned}\tag{37}$$

for all $(t,s)\in[0,1]\times[0,1]$. Using (34)-(37) in (18) we get the desired result. □

Corollary 2.4. *Suppose the assumptions of Theorem 2.5 are fulfilled and if* $p\left(x,y\right)=\frac{1}{(b-a)(d-c)}$ *for all* $(x,y)\in[a,b]\times[c,d]$*, then the following inequality holds valid*

$$\begin{aligned}&\Bigg|\frac{f\left(a,c\right)+f\left(a,d\right)+f\left(b,c\right)+f\left(b,d\right)}{4}\\&\qquad-\frac{1}{2}\int_a^b\left[f\left(x,c\right)+f\left(x,d\right)\right]dx-\frac{1}{2}\int_c^d\left[f\left(a,y\right)+f\left(b,y\right)\right]dy\\&\quad+\int_c^d\int_a^b f\left(x,y\right)dxdy\Bigg|\leq\frac{\left(b-a\right)\left(d-c\right)}{16}\left\{\left[\lambda_1\left(b,d,\frac{a+b}{2},\frac{c+d}{2};\left|f_{ts}\right|^q\right)\right]^{\frac{1}{q}}\right.\\&\quad+\left[\lambda_2\left(a,d,\frac{a+b}{2},\frac{c+d}{2};\left|f_{ts}\right|^q\right)\right]^{\frac{1}{q}}+\left[\lambda_3\left(b,c,\frac{a+b}{2},\frac{c+d}{2};\left|f_{ts}\right|^q\right)\right]^{\frac{1}{q}}\\&\quad\left.+\left[\lambda_4\left(a,c,\frac{a+b}{2},\frac{c+d}{2};\left|f_{ts}\right|^q\right)\right]^{\frac{1}{q}}\right\}\int_0^1\int_0^1\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)}p\left(x,y\right)dxdydtds.\end{aligned}\tag{38}$$

Remark 2.6. *Suppose the assumptions of Theorem 2.5 are satisfied and additionally*

(1) *If* $|f_{ts}|^q$ *is non-decreasing on the co-ordinates on* $[a,b]\times[c,d]$*, then (29) holds valid.*
(2) *If* $|f_{ts}|^q$ *is non-increasing on the co-ordinates on* $[a,b]\times[c,d]$*, then (30) holds true.*

Remark 2.7. *In Corollary 2.4*
(1) *If* $|f_{ts}|^q$ *is non-decreasing on the co-ordinates on* $[a,b]\times[c,d]$*, then (31) holds valid.*
(2) *If* $|f_{ts}|^q$ *is non-increasing on the co-ordinates on* $[a,b]\times[c,d]$*, then (32) holds true.*

3. Applications to Random Variables

Let $0<a<b$, $0<c<d$, $\alpha,\beta\in\mathbb{R}$ and let X and Y be two independent continuous random variables having the bi-variate continuous probability density function $p:[a,b]\times[c,d]\to[0,\infty)$ which is symmetric to $\frac{a+b}{2}$ and $\frac{c+d}{2}$ the α-moment of X and the β-moment of Y about the origin are respectively defined as follows

$$E_\alpha(X)=\int_c^b t^\alpha p_1(t)\,dt,\, E_\beta(Y)=\int_c^b s^\beta p_2(s)\,ds$$

which are assumed to be finite, here $p_1:[a,b]\to[0,\infty)$ and $p_2:[c,d]\to[0,\infty)$ are the marginal probability density functions of X and Y. Since X and Y are independent random variables, we have

$$p(t,s)=p_1(t)\,p_2(s)$$

for all $(t,s)\in[a,b]\times[c,d]$.

Now we give some applications of our results to random variables.

Theorem 3.1. *The inequality*

$$\left|\left(E_\alpha(X)-\frac{a^\alpha+b^\alpha}{2}\right)\left(E_\beta(Y)-\frac{c^\beta+d^\beta}{2}\right)\right| \le \frac{(b-a)(d-c)}{4}\alpha\beta\left(\frac{a^{\alpha-1}+b^{\alpha-1}}{2}\right)\left(\frac{c^{\beta-1}+d^{\beta-1}}{2}\right). \quad (39)$$

holds holds for $0<a<b$, $0<c<d$ *and* $\alpha,\beta\ge 2$.

Proof. Let $f(t,s)=t^\alpha s^\beta$ on $[a,b]\times[c,d]$ for $\alpha,\beta\ge2$, we observe that $|f_{ts}(t,s)|=\alpha\beta t^{\alpha-1}s^{\beta-1}$ is convex on the co-ordinates on $[a,b]\times[c,d]$. Since

$$\begin{aligned}&|f_{ts}(a,c)|+|f_{ts}(a,d)|+|f_{ts}(b,c)|+|f_{ts}(b,d)|\\&=\alpha\beta\left(a^{\alpha-1}+b^{\alpha-1}\right)\left(c^{\beta-1}+d^{\beta-1}\right),\end{aligned}$$

$$\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)}p(x,y)\,dxdy\le\int_c^d\int_a^b p(x,y)\,dxdy=1$$

and hence

$$\int_0^1 \int_0^1 \int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x,y)\, dxdydtds \le 1.$$

Also

$$\begin{aligned}&\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_c^d \int_a^b p(x,y)\, dxdy \\ &= \frac{a^\alpha c^\beta + a^\alpha d^\beta + b^\alpha c^\beta + b^\alpha d^\beta}{4} = \frac{(a^\alpha + b^\alpha)\left(c^\beta + d^\beta\right)}{4},\end{aligned}$$

$$\begin{aligned}&\frac{1}{2}\int_c^d \int_a^b [f(x,c)+f(x,d)]\, p(x,y)\, dxdy + \frac{1}{2}\int_c^d \int_a^b [f(a,y)+f(b,y)]\, p(x,y)\, dxdy \\ &= \left(\frac{c^\beta + d^\beta}{2}\right) E_\alpha(X) + \left(\frac{a^\alpha + b^\alpha}{2}\right) E_\beta(Y)\end{aligned}$$

and

$$\int_c^d \int_a^b f(x,y)\, p(x,y)\, dxdy = E_\alpha(X)\, E_\beta(Y).$$

The result follows immediately from the inequality (11). □

Theorem 3.2. *The inequality*

$$\begin{aligned}&\left|\left(E_\alpha(X) - \frac{a^\alpha + b^\alpha}{2}\right)\left(E_\beta(Y) - \frac{c^\beta + d^\beta}{2}\right)\right| \\ &\le \frac{(b-a)(d-c)}{16}\alpha\beta\left(b^{\alpha-1} + \left(\frac{a+b}{2}\right)^{\alpha-1}\right)\left(d^{\beta-1} + \left(\frac{c+d}{2}\right)^{\beta-1}\right). \quad (40)\end{aligned}$$

holds holds for $0 < a < b$, $0 < c < d$ and $\alpha, \beta \ge 1$.

Proof. Let $f(t,s) = t^\alpha s^\beta$ on $[a,b] \times [c,d]$ for $\alpha, \beta \ge 1$, we observe that $|f_{ts}(t,s)| = \alpha\beta t^{\alpha-1} s^{\beta-1}$ is non-decreasing and quasi-convex on the co-ordinates on $[a,b] \times [c,d]$. The proof is similar to that of Theorem 3.1 by using the inequality (29) we obtain the required result. □

Remark 3.1. *For $\alpha = \beta = 1$, we have from Theorem 3.2 that*

$$\left|\left(E(X) - \frac{a+b}{2}\right)\left(E(Y) - \frac{c+d}{2}\right)\right| \le \frac{(b-a)(d-c)}{4}, \quad (41)$$

where $E_1(X) = E(X)$ and $E_1(Y) = E(Y)$ are the expectation of the random variables X and Y respectively.

4. Applications to 2D weighted trapezoidal formula

Let $[a,b]\times[c,d]$ be a rectangle from the plane $\mathbb{R}^2$. Suppose d_1 and d_2 are the divisions $a=x_0<x_1<\cdots<x_{n-1}<x_n=b$ and $c=y_0<y_1<\cdots<y_{m-1}<y_m=b$ of the intervals $[a,b]$ and $[c,d]$ respectively and let $\Omega=\{[x_i,x_{i+1}]\times[y_j,y_{j+1}]:0\le i\le n-1,0\le j\le m-1\}$ be a corresponding division of the rectangle $[a,b]\times[c,d]$ from the plane $\mathbb{R}^2$.

Consider the following $2D$ weighted quadrature formula

$$\int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy=T(f,p,\Omega)+E(f,p,\Omega), \tag{42}$$

where

$$\begin{aligned} &T(f,p,\Omega)\\ &\quad=-\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\Bigg[\frac{f(x_i,y_j)+f(x_i,y_{j+1})+f(x_{i+1},y_j)+f(x_{i+1},y_{j+1})}{4}\\ &\times\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}}p(x,y)\,dxdy+\frac{1}{2}\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}}[f(x,y_j)+f(x,y_{j+1})]\,p(x,y)\,dxdy\\ &\qquad+\frac{1}{2}\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}}[f(x_i,y)+f(x_{i+1},y)]\,p(x,y)\,dxdy\Bigg] \end{aligned} \tag{43}$$

for the trapezoidal version and $E(f,p,\Omega)$ denotes the associated approximation error.

The following results provide some estimates of the remainder term $E(f,p,\Omega)$.

Theorem 4.1. *Suppose the assumptions of Theorem 2.2 are satisfied. If $|f_{ts}|^q$ is convex on the co-ordinates on $[a,b]\times[c,d]$ for $q\ge 1$, then in (42), for every division Ω of the rectangle $[a,b]\times[c,d]$ from the plane $\mathbb{R}^2$, the following holds*

$$\begin{aligned} |E(f,p,\Omega)|&\le\frac{1}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(x_{i+1}-x_i)(y_{j+1}-y_j)\\ &\times[\Psi(x_i,x_{i+1},y_j,y_{j+1};|f_{ts}|^q)]^{\frac{1}{q}}\int_0^1\int_0^1\int_{L_2(y_j,y_{j+1},s)}^{U_2(y_j,y_{j+1},s)}\int_{L_1(x_i,x_{i+1},t)}^{U_1(x_i,x_{i+1},t)}p(x,y)\,dxdydsdt. \end{aligned} \tag{44}$$

Proof. Applying Theorem 2.2 on the rectangles $[x_i,x_{i+1}]\times[y_j,y_{j+1}]$ $(0\le i\le n-1,0\le j\le m-1)$ of the division Ω of the rectangle $[a,b]\times[c,d]$ from the plane $\mathbb{R}^2$, we get

$$\begin{aligned} |\Phi(x_i,x_{i+1},y_j,y_{j+1};p,f)|&\le\frac{(x_{i+1}-x_i)(y_{j+1}-y_j)}{4}\\ &\times\left[\frac{|f_{ts}(x_i,y_j)|^q+|f_{ts}(x_i,y_{j+1})|^q+|f_{ts}(x_{i+1},y_j)|^q+|f_{ts}(x_{i+1},y_{j+1})|^q}{4}\right]^{\frac{1}{q}}\\ &\times\int_0^1\int_0^1\int_{L_2(y_j,y_{j+1},s)}^{U_2(y_j,y_{j+1},s)}\int_{L_1(x_i,x_{i+1},t)}^{U_1(x_i,x_{i+1},t)}p(x,y)\,dxdydsdt. \end{aligned} \tag{45}$$

Summing over i from 0 to $n-1$ and j over 0 to $m-1$, we deduce, by the triangle inequality, that (44) holds. □

Remark 4.1. *The inequality holds if the condition of convexity of $|f_{ts}|^q$ on the co-ordinates on $[a,b]\times[c,d]$ is replaced with the condition of wright-convexity of $|f_{ts}|^q$ on the co-ordinates on $[a,b]\times[c,d]$ for $q\geq 1$.*

Theorem 4.2. *Suppose the assumptions of Theorem 2.2 are satisfied. If $|f_{ts}|^q$ is convex on the co-ordinates on $[a,b]\times[c,d]$ for $q\geq 1$, then in (42), for every division Ω of the rectangle $[a,b]\times[c,d]$ from the plane $\mathbb{R}^2$, the following holds*

$$\begin{aligned}|E(f,p,\Omega)| &\leq \frac{1}{16}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(x_{i+1}-x_i)(y_{j+1}-y_j)\\ &\times\Bigg\{\left[\lambda_1\left(x_{i+1},y_{j+1},\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2};|f_{ts}|^q\right)\right]^{\frac{1}{q}}\\ &+\left[\lambda_2\left(x_i,y_{j+1},\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2};|f_{ts}|^q\right)\right]^{\frac{1}{q}}\\ &+\left[\lambda_3\left(x_{i+1},y_j,\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2};|f_{ts}|^q\right)\right]^{\frac{1}{q}}\\ &+\left[\lambda_4\left(x_i,y_j,\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2};|f_{ts}|^q\right)\right]^{\frac{1}{q}}\Bigg]\\ &\times\int_0^1\int_0^1\int_{L_2(y_j,y_{j+1},s)}^{U_2(y_j,y_{j+1},s)}\int_{L_1(x_i,x_{i+1},t)}^{U_1(x_i,x_{i+1},t)}p(x,y)\,dxdydsdt. \quad (46)\end{aligned}$$

Proof. The proof follows from (33) by using the similar arguments as that of the proof of Theorem 4.1. □

Remark 4.2. *If $|f_{ts}|$ is non-decreasing in Theorem 4.2, then the following inequality holds*

$$\begin{aligned}|E(f,p,\Omega)| &\leq \frac{1}{16}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(x_{i+1}-x_i)(y_{j+1}-y_j)\\ &\times\left[|f_{ts}(x_{i+1},y_{j+1})|+\left|f_{ts}\left(\frac{x_i+x_{i+1}}{2},y_{j+1}\right)\right|+\left|f_{ts}\left(x_{i+1},\frac{y_j+y_{j+1}}{2}\right)\right|\right.\\ &\left.+\left|f_{ts}\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)\right|\right]\int_0^1\int_0^1\int_{L_2(y_j,y_{j+1},s)}^{U_2(y_j,y_{j+1},s)}\int_{L_1(x_i,x_{i+1},t)}^{U_1(x_i,x_{i+1},t)}p(x,y)\,dxdydsdt. \quad (47)\end{aligned}$$

and if $|f_{ts}|$ is non-increasing in Theorem 4.2, then the following inequality holds

$$|E(f,p,\Omega)| \leq \frac{1}{16}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(x_{i+1}-x_i)(y_{j+1}-y_j)$$
$$\times\left[|f_{ts}(x_i,y_j)| + \left|f_{ts}\left(x_i,\frac{y_j+y_{j+1}}{2}\right)\right| + \left|f_{ts}\left(\frac{x_i+x_{i+1}}{2},y_j\right)\right|\right.$$
$$\left.+\left|f_{ts}\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)\right|\right]\int_0^1\int_0^1\int_{L_2(y_j,y_{j+1},s)}^{U_2(y_j,y_{j+1},s)}\int_{L_1(x_i,x_{i+1},t)}^{U_1(x_i,x_{i+1},t)} p(x,y)\,dxdydsdt. \tag{48}$$

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