VICTORIA UNIVERSITY
MELBOURNE AUSTRALIA

Hermite-Hadamard type inequalities for (m, M)-Ψ-convex functions when Ψ = -ln

This is the Published version of the following publication

Dragomir, Sever S and Gomm, Ian (2018) Hermite-Hadamard type inequalities for (m, M)-Ψ-convex functions when Ψ = -ln. Mathematica Moravica, 22 (1). 65 - 79. ISSN 1450-5932

The publisher's official version can be found at
http://www.moravica.ftn.kg.ac.rs/Vol_22-1/06-Dragomir-Gomm.pdf
Note that access to this version may require subscription.

Downloaded from VU Research Repository https://vuir.vu.edu.au/37797/

Mathematica Moravica
Vol. **22**, No. **1** (2018), 65–79 doi:10.5937/MatMor1801065S

Hermite-Hadamard type inequalities for (m,M)-Ψ-convex functions when $\Psi = -\ln$

Silvestru Sever Dragomir and Ian Gomm

Abstract. In this paper we establish some Hermite-Hadamard type inequalities for (m,M)-Ψ-convex functions when $\Psi = -\ln$. Applications for power functions and weighted arithmetic mean and geometric mean are also provided.

1. Introduction

The following integral inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}, \tag{1}$$

which holds for any convex function $f:[a,b]\to\mathbb{R}$, is well known in the literature as the Hermite-Hadamard inequality.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the Theory of Special Means and in Information Theory for divergence measures, for which we would like to refer the reader to [1]-[4], [15]-[30], the monograph [13] and the references therein.

Assume that the function $\Psi : I \subseteq \mathbb{R} \to \mathbb{R}$ (I is an interval) is convex on I and $m\in\mathbb{R}$. We shall say that the function $\Phi : I\to\mathbb{R}$ is m-Ψ-*lower convex* if $\Phi - m\Psi$ is a convex function on I. We may introduce (see [6]) the class of functions

$$\mathcal{L}(I,m,\Psi) := \{\Phi : I\to\mathbb{R} \mid \Phi - m\Psi \text{ is convex on } I\}. \tag{2}$$

Similarly, for $M\in\mathbb{R}$ and Ψ as above, we can introduce the class of M-Ψ-*upper convex* functions (see [6])

$$\mathcal{U}(I,M,\Psi) := \{\Phi : I\to\mathbb{R} \mid M\Psi - \Phi \text{ is convex on } I\}. \tag{3}$$

2010 *Mathematics Subject Classification.* Primary: 26D15; Secondary: 26D10.

Key words and phrases. Convex functions, special convexity, weighted arithmetic and geometric means, logarithmic function.

Full paper. Received 12 January 2018, revised 17 April 2018, accepted 11 May 2018, available online 25 June 2018.

©2018 Mathematica Moravica

The intersection of these two classes will be called the class of (m, M)-Ψ-*convex functions* and will be denoted by [6]

$$\mathcal{B}(I, m, M, \Psi) := \mathcal{L}(I, m, \Psi) \cap \mathcal{U}(I, M, \Psi). \tag{4}$$

Remark 1.1. If $\Psi \in \mathcal{B}(I, m, M, \Psi)$, then $\Phi - m\Psi$ and $M\Psi - \Phi$ are convex and then $(\Phi - m\Psi) + (M\Psi - \Phi)$ is also convex which shows that $(M - m)\Psi$ is convex, implying that $M \geq m$ (as Ψ is assumed not to be the trivial convex function $\Psi(t) = 0$, $t \in I$).

The above concepts may be introduced in the general case of a convex subset in a real linear space, but we do not consider this extension here.

In [12], S. S. Dragomir and N. M. Ionescu introduced the concept of *g-convex dominated functions,* for a function $f : I \to \mathbb{R}$. We recall this, by saying, for a given convex function $g : I \to \mathbb{R}$, the function $f : I \to \mathbb{R}$ is *g-convex dominated* iff $g + f$ and $g - f$ are convex functions on I. In [12], the authors pointed out a number of inequalities for convex dominated functions related to Jensen's, Fuch's, Pečarić's, Barlow-Proschan and Vasić-Mijalković results, etc.

We observe that the concept of g-convex dominated functions can be obtained as a particular case from (m, M)-Ψ-convex functions by choosing $m = -1$, $M = 1$ and $\Psi = g$.

The following lemma holds [6].

Lemma 1.1. *Let $\Psi, \Phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable functions on $\mathring{I}$, the interior of I and Ψ is a convex function on $\mathring{I}$.*

(i) *For $m \in \mathbb{R}$, the function $\Phi \in \mathcal{L}\left(\mathring{I}, m, \Psi\right)$ if and only if*

$$m\left[\Psi(t) - \Psi(s) - \Psi'(s)(t - s)\right] \leq \Phi(t) - \Phi(s) - \Phi'(s)(t - s), \tag{5}$$

for all $t, s \in \mathring{I}$.

(ii) *For $M \in \mathbb{R}$, the function $\Phi \in \mathcal{U}\left(\mathring{I}, M, \Psi\right)$ if and only if*

$$\Phi(t) - \Phi(s) - \Phi'(s)(t - s) \leq M\left[\Psi(t) - \Psi(s) - \Psi'(s)(t - s)\right], \tag{6}$$

for all $t, s \in \mathring{I}$.

(iii) *For $M, m \in \mathbb{R}$ with $M \geq m$, the function $\Phi \in \mathcal{B}\left(\mathring{I}, m, M, \Psi\right)$ if and only if both* (5) *and* (6) *hold.*

Another elementary fact for twice differentiable functions also holds [6].

Lemma 1.2. *Let $\Psi, \Phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable on $\mathring{I}$ and Ψ is convex on $\mathring{I}$.*

(i) *For $m \in \mathbb{R}$, the function $\Phi \in \mathcal{L}\left(\mathring{I}, m, \Psi\right)$ if and only if*

$$m\Psi''(t) \leq \Phi''(t) \quad \textit{for all } t \in \mathring{I}. \tag{7}$$

(ii) *For* $M \in \mathbb{R}$, *the function* $\Phi \in \mathcal{U}\left(\mathring{I}, M, \Psi\right)$ *if and only if*

$$\Phi''(t) \leq M\Psi''(t) \quad \text{for all } t \in \mathring{I}. \tag{8}$$

(iii) *For* $M, m \in \mathbb{R}$ *with* $M \geq m$, *the function* $\Phi \in \mathcal{B}\left(\mathring{I}, m, M, \Psi\right)$ *if and only if both* (7) *and* (8) *hold.*

For various inequalities concerning these classes of function, see the survey paper [11].

In what follows, we are considering the class of functions $\mathcal{B}(I, m, M, -\ln)$ for M, $m \in \mathbb{R}$ with $M \geq m$ that is obtained for $\Psi : I \subseteq (0, \infty) \to \mathbb{R}$, $\Psi(t) = -\ln t$.

If $\Phi : I \subseteq (0, \infty) \to \mathbb{R}$ is a differentiable function on $\mathring{I}$ then by Lemma 1.1 we have $\Phi \in \mathcal{B}(I, m, M, -\ln)$ if and only if

$$\begin{aligned} m\left[\ln s - \ln t - \frac{1}{s}(s-t)\right] &\leq \Phi(t) - \Phi(s) - \Phi'(s)(t-s) \\ &\leq M\left[\ln s - \ln t - \frac{1}{s}(s-t)\right], \end{aligned} \tag{9}$$

for any $s, t \in \mathring{I}$.

If $\Phi : I \subseteq (0, \infty) \to \mathbb{R}$ is a twice differentiable function on $\mathring{I}$ then by Lemma 1.2 we have $\Phi \in \mathcal{B}(I, m, M, -\ln)$ if and only if

$$m \leq t^2 \Phi''(t) \leq M, \tag{10}$$

which is a convenient condition to verify in applications.

In this paper we establish some Hermite-Hadamard type inequalities for (m, M)-Ψ-convex functions when $\Psi = -\ln$. Applications for power functions and weighted arithmetic mean and geometric mean are also provided.

2. Hermite-Hadamard Type Inequalities

In 2002, Barnett, Cerone and Dragomir [5] obtained the following refinement of the Hermite-Hadamard inequality for the convex function $f : [a, b] \to \mathbb{R}$:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \nu f\left(a + \nu\frac{b-a}{2}\right) + (1-\nu) f\left(\frac{a+b}{2} + \nu\frac{b-a}{2}\right) \\ &\leq \frac{1}{b-a}\int_a^b f(u)\, du \\ &\leq \frac{1}{2}\left[f((1-\nu)a + \nu b) + \nu f(a) + (1-\nu) f(b)\right] \leq \frac{f(a)+f(b)}{2}, \end{aligned} \tag{11}$$

for all $\nu \in [0, 1]$.

The inequality was also rediscovered in 2010 by A. E. Farissi in [14].

We give a simple proof by following [5].

Applying the Hermite-Hadamard inequality for the convex function f on each subinterval $[a,(1-\nu)a+\nu b]$, $[(1-\nu)a+\nu b,b]$, $\nu\in(0,1)$, we have,

$$\begin{aligned}&f\left(\frac{a+(1-\nu)a+\nu b}{2}\right)[(1-\nu)a+\nu b-a]\\&\leq\int_a^{(1-\nu)a+\nu b}f(u)\,du\\&\leq\frac{f((1-\nu)a+\nu b)+f(a)}{2}[(1-\nu)a+\nu b-a]\end{aligned}$$

and

$$\begin{aligned}&f\left(\frac{(1-\nu)a+\nu b+b}{2}\right)[b-(1-\nu)a-\nu b]\\&\leq\int_{(1-\nu)a+\nu b}^{b}f(u)\,du\\&\leq\frac{f(b)+f((1-\nu)a+\nu b)}{2}[b-(1-\nu)a-\nu b],\end{aligned}$$

which are clearly equivalent to

$$\begin{aligned}\nu f\left(a+\nu\frac{b-a}{2}\right)&\leq\frac{1}{b-a}\int_a^{(1-\nu)a+\nu b}f(u)\,du\\&\leq\frac{\nu f((1-\nu)a+\nu b)+\nu f(a)}{2}\end{aligned}\tag{12}$$

and

$$\begin{aligned}(1-\nu)f\left(\frac{a+b}{2}+\nu\frac{b-a}{2}\right)&\leq\frac{1}{b-a}\int_{(1-\nu)a+\nu b}^{b}f(u)\,du\\&\leq\frac{(1-\nu)f(b)+(1-\nu)f((1-\nu)a+\nu b)}{2}\end{aligned}\tag{13}$$

respectively.

Summing (12) and (13), we obtain the second and third inequality in (11).

By the convexity property of f, we obtain

$$\begin{aligned}&\nu f\left(a+\nu\frac{b-a}{2}\right)+(1-\nu)f\left(\frac{a+b}{2}+\nu\frac{b-a}{2}\right)\\&\geq f\left[\nu\left(a+\nu\frac{b-a}{2}\right)+(1-\nu)\left(\frac{a+b}{2}+\nu\frac{b-a}{2}\right)\right]=f\left(\frac{a+b}{2}\right)\end{aligned}$$

and the first inequality in (11) is proved.

The latter inequality in (11) is obvious by the convexity property of f.

Let us recall the following means:

(i) The arithmetic mean
$$A=A(a,b):=\frac{a+b}{2},\qquad a,b\geq 0;$$
(ii) The geometric mean:
$$G=G(a,b):=\sqrt{ab},\qquad a,b\geq 0;$$
(iii) The harmonic mean:
$$H=H(a,b):=\frac{2}{\frac{1}{a}+\frac{1}{b}},\qquad a,b\geq 0;$$
(iv) The logarithmic mean:
$$L=L(a,b):=\begin{cases} a, & \text{if } a=b;\\ \dfrac{b-a}{\ln b-\ln a}, & \text{if } a\neq b;\end{cases}\qquad a,b>0;$$
(v) The identric mean:
$$I:=I(a,b)=\begin{cases} a, & \text{if } a=b;\\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & \text{if } a\neq b;\end{cases}\qquad a,b>0;$$
(vi) The p-logarithmic mean:
$$L_p=L_p(a,b):=\begin{cases} \left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & \text{if } a\neq b;\\ a, & \text{if } a=b;\end{cases}$$
where $p\in\mathbb{R}\backslash\{-1,0\}$ and $a,\ b>0$.

It is well known that L_p *is monotonic nondecreasing over* $p\in\mathbb{R}$ with $L_{-1}:=L$ and $L_0:=I$.

In particular, we have the inequalities $H\leq G\leq L\leq I\leq A$. We also notice that
$$\frac{1}{b-a}\int_a^b t^p dt=L_p^p(a,b),\ p\in\mathbb{R}\backslash\{-1,0\},\ \frac{1}{b-a}\int_a^b\frac{dt}{t}=L^{-1}(a,b)$$
and
$$\frac{1}{b-a}\int_a^b\ln t dt=\ln I(a,b).$$
We define the weighted arithmetic and geometric means
$$A_\nu(a,b):=(1-\nu)a+\nu b \text{ and } G_\nu(a,b):=a^{1-\nu}b^\nu$$
where $\nu\in[0,1]$ and $a,\ b>0$. If $\nu=\frac{1}{2}$, then we recapture $A(a,b)$ and $G(a,b)$.

Theorem 2.1. *Let M, $m \in \mathbb{R}$ with $M > m$ and $\Phi \in \mathcal{B}\left((0,\infty), m, M, -\ln\right)$. Then for any a, $b > 0$ and $\nu \in [0,1]$ we have*

(14)
$$\ln\left[\frac{\left(a+\nu\frac{b-a}{2}\right)^{\nu}\left(\frac{a+b}{2}+\nu\frac{b-a}{2}\right)^{1-\nu}}{I(a,b)}\right]^{m}$$
$$\leq \frac{1}{b-a}\int_a^b \Phi(u)\,du - \left[\nu\Phi\left(a+\nu\frac{b-a}{2}\right)+(1-\nu)\Phi\left(\frac{a+b}{2}+\nu\frac{b-a}{2}\right)\right]$$
$$\leq \ln\left[\frac{\left(a+\nu\frac{b-a}{2}\right)^{\nu}\left(\frac{a+b}{2}+\nu\frac{b-a}{2}\right)^{1-\nu}}{I(a,b)}\right]^{M}$$

and

$$\ln\left[\frac{I(a,b)}{\sqrt{A_\nu(a,b)\,G_{1-\nu}(a,b)}}\right]^{m} \tag{15}$$
$$\leq \frac{1}{2}\left[\Phi\left((1-\nu)a+\nu b\right)+\nu\Phi(a)+(1-\nu)\Phi(b)\right]-\frac{1}{b-a}\int_a^b \Phi(u)\,du$$
$$\leq \ln\left[\frac{I(a,b)}{\sqrt{A_\nu(a,b)\,G_{1-\nu}(a,b)}}\right]^{M}.$$

Proof. Since $\Phi \in \mathcal{B}\left((0,\infty), m, M, -\ln\right)$, then $\Phi_m := \Phi + m\ln$ is convex and by the second inequality in (11) we have

$$\begin{aligned}
&\nu\Phi\left(a+\nu\frac{b-a}{2}\right)+(1-\nu)\Phi\left(\frac{a+b}{2}+\nu\frac{b-a}{2}\right)\\
&+m\ln\left[\left(a+\nu\frac{b-a}{2}\right)^{\nu}\left(\frac{a+b}{2}+\nu\frac{b-a}{2}\right)^{1-\nu}\right]\\
&\leq \frac{1}{b-a}\int_a^b f(u)\,du + m\frac{1}{b-a}\int_a^b \ln u\,du\\
&= \frac{1}{b-a}\int_a^b f(u)\,du + m\ln I(a,b),
\end{aligned} \tag{16}$$

while from the third inequality in (11) we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(u)\,du + m\frac{1}{b-a}\int_a^b \ln u\,du\\
&\leq \frac{1}{2}\left[\Phi\left((1-\nu)a+\nu b\right)+\nu\Phi(a)+(1-\nu)\Phi(b)\right]\\
&+\frac{1}{2}m\ln\left[A_\nu(a,b)\,G_{1-\nu}(a,b)\right],
\end{aligned} \tag{17}$$

for any $a, b > 0$ and $\nu \in [0,1]$.

Since $\Phi \in \mathcal{B}((0,\infty), m, M, -\ln)$, then also $f_M := -\Phi - M\ln$ is convex and by the second inequality in (11) we have

$$
\begin{aligned}
(18) \qquad & -\nu\Phi\left(a+\nu\frac{b-a}{2}\right)-(1-\nu)\Phi\left(\frac{a+b}{2}+\nu\frac{b-a}{2}\right) \\
& - M\ln\left[\left(a+\nu\frac{b-a}{2}\right)^{\nu}\left(\frac{a+b}{2}+\nu\frac{b-a}{2}\right)^{1-\nu}\right] \\
& \leq -\frac{1}{b-a}\int_a^b \Phi(u)\,du - M\ln I(a,b),
\end{aligned}
$$

while from the third inequality in (11) we have

$$
\begin{aligned}
(19) \qquad & -\frac{1}{b-a}\int_a^b \Phi(u)\,du - M\ln I(a,b) \\
& \leq -\frac{1}{2}\left[\Phi((1-\nu)a+\nu b)+\nu\Phi(a)+(1-\nu)\Phi(b)\right] \\
& -\frac{1}{2}M\ln\left[A_\nu(a,b)\,G_{1-\nu}(a,b)\right],
\end{aligned}
$$

for any a, $b>0$ and $\nu\in[0,1]$.

Making use of (16)-(19) we deduce the desired results (14) and (15). □

Remark 2.1. If we write the second inequality in (11) for the convex function $-\ln$ we have

$$
\ln I(a,b) \leq \ln\left[\left(a+\nu\frac{b-a}{2}\right)^{\nu}\left(\frac{a+b}{2}+\nu\frac{b-a}{2}\right)^{1-\nu}\right],
$$

which implies that

$$
I(a,b) \leq \left(a+\nu\frac{b-a}{2}\right)^{\nu}\left(\frac{a+b}{2}+\nu\frac{b-a}{2}\right)^{1-\nu}
$$

showing that

$$
\ln\left[\frac{\left(a+\nu\frac{b-a}{2}\right)^{\nu}\left(\frac{a+b}{2}+\nu\frac{b-a}{2}\right)^{1-\nu}}{I(a,b)}\right] \geq 0.
$$

If we use the third inequality in (11) for the convex function $-\ln$ we have

$$
\ln\sqrt{A_\nu(a,b)\,G_{1-\nu}(a,b)} \leq \ln I(a,b),
$$

which implies that

$$
\sqrt{A_\nu(a,b)\,G_{1-\nu}(a,b)} \leq I(a,b),
$$

showing that

$$
\ln\left[\frac{I(a,b)}{\sqrt{A_\nu(a,b)\,G_{1-\nu}(a,b)}}\right] \geq 0.
$$

Corollary 2.1. *With the assumptions of Theorem 2.1 we have*

$$\ln\left[\frac{A(a,b)}{I(a,b)}\right]^m \le \frac{1}{b-a}\int_a^b \Phi(u)\,du - \Phi\left(\frac{a+b}{2}\right) \le \ln\left[\frac{A(a,b)}{I(a,b)}\right]^M \tag{20}$$

and

$$\ln\left[\frac{I(a,b)}{G(a,b)}\right]^m \le \frac{\Phi(a)+\Phi(b)}{2} - \frac{1}{b-a}\int_a^b \Phi(u)\,du \le \ln\left[\frac{I(a,b)}{G(a,b)}\right]^M. \tag{21}$$

The inequality (20) was obtained in 2002 by Dragomir in [7], see also [11, p. 197].

Corollary 2.2. *With the assumptions of Theorem 2.1 we have*

$$\begin{aligned} &\ln\left[\frac{\sqrt{\left(\frac{3a+b}{4}\right)\left(\frac{a+3b}{4}\right)}}{I(a,b)}\right]^m \\ &\le \frac{1}{b-a}\int_a^b \Phi(u)\,du - \frac{1}{2}\left[\Phi\left(\frac{3a+b}{4}\right)+\Phi\left(\frac{a+3b}{4}\right)\right] \\ &\le \ln\left[\frac{\sqrt{\left(\frac{3a+b}{4}\right)\left(\frac{a+3b}{4}\right)}}{I(a,b)}\right]^M \end{aligned} \tag{22}$$

and

$$\begin{aligned} &\ln\left[\frac{I(a,b)}{\sqrt{A(a,b)\,G(a,b)}}\right]^m \\ &\le \frac{1}{2}\left[\Phi\left(\frac{a+b}{2}\right)+\frac{\Phi(a)+\Phi(b)}{2}\right] - \frac{1}{b-a}\int_a^b \Phi(u)\,du \\ &\le \ln\left[\frac{I(a,b)}{\sqrt{A(a,b)\,G(a,b)}}\right]^M. \end{aligned} \tag{23}$$

For related results see [8] and [11, p. 197].

Theorem 2.2. *Let* $M, m \in \mathbb{R}$ *with* $M > m$ *and* $\Phi \in \mathcal{B}((0,\infty), m, M, -\ln)$. *Then for any* $a,\ b > 0$ *and* $\nu \in [0,1]$ *we have*

$$\begin{aligned} &m\left[\frac{(b-a)^2}{8ab} - \ln\left(\frac{I(a,b)}{G(a,b)}\right)\right] \\ &\le \frac{1}{8}\left[\Phi_-(b)-\Phi_+(a)\right](b-a) - \left[\frac{\Phi(a)+\Phi(b)}{2} - \frac{1}{b-a}\int_a^b \Phi(u)\,du\right] \\ &\le M\left[\frac{(b-a)^2}{8ab} - \ln\left(\frac{I(a,b)}{G(a,b)}\right)\right] \end{aligned} \tag{24}$$

and

$$m\left[\frac{(b-a)^2}{8ab}-\ln\left(\frac{A(a,b)}{I(a,b)}\right)\right] \tag{25}$$
$$\leq \frac{1}{8}\left[\Phi_-(b)-\Phi_+(a)\right](b-a)-\left[\frac{1}{b-a}\int_a^b \Phi(x)\,dx-\Phi\left(\frac{a+b}{2}\right)\right]$$
$$\leq M\left[\frac{(b-a)^2}{8ab}-\ln\left(\frac{A(a,b)}{I(a,b)}\right)\right].$$

Proof. The following reverses of the Hermite-Hadamard inequality hold [9] and [10]: Let $h:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then

$$(0\leq)\,\frac{h(a)+h(b)}{2}-\frac{1}{b-a}\int_a^b h(x)\,dx \tag{26}$$
$$\leq \frac{1}{8}\left[h_-(b)-h_+(a)\right](b-a)$$

and

$$(0\leq)\,\frac{1}{b-a}\int_a^b h(x)\,dx-h\left(\frac{a+b}{2}\right) \tag{27}$$
$$\leq \frac{1}{8}\left[h_-(b)-h_+(a)\right](b-a).$$

The constant $\frac{1}{8}$ is best possible in (26) and (27).

Since $\Phi\in\mathcal{B}((0,\infty),m,M,-\ln)$, then $\Phi_m:=\Phi+m\ln$ is convex and by (26) we have

$$\frac{\Phi(a)+\Phi(b)}{2}-\frac{1}{b-a}\int_a^b \Phi(x)\,dx+m\ln\left(\frac{G(a,b)}{I(a,b)}\right)$$
$$\leq \frac{1}{8}\left[\Phi_-(b)-\Phi_+(a)\right](b-a)-\frac{m}{8ab}(b-a)^2,$$

which proves the first inequality in (24).

Since $\Phi\in\mathcal{B}((0,\infty),m,M,-\ln)$, then also $f_M:=-\Phi-M\ln$ is convex and by (26) we have

$$(0\leq)\,\frac{-\Phi(a)-\Phi(b)}{2}+\frac{1}{b-a}\int_a^b \Phi(x)\,dx-M\ln\left(\frac{G(a,b)}{I(a,b)}\right)$$
$$\leq \frac{1}{8}\left[-\Phi_-(b)+\Phi_+(a)\right](b-a)+\frac{M}{8ab}(b-a)^2,$$

which proves the second inequality in (24).

Further on, since $\Phi\in\mathcal{B}((0,\infty),m,M,-\ln)$, then $\Phi_m:=\Phi+m\ln$ is convex and by (27) we have

$$(0\leq)\,\frac{1}{b-a}\int_a^b \Phi(x)\,dx-\Phi\left(\frac{a+b}{2}\right)-m\ln\left(\frac{A(a,b)}{I(a,b)}\right)$$

$$\leq \frac{1}{8}\left[\Phi_{-}(b)-\Phi_{+}(a)\right](b-a)-\frac{m}{8ab}(b-a)^{2},$$

which is equivalent to the first inequality in (25).

Since $\Phi \in \mathcal{B}((0,\infty), m, M, -\ln)$, then also $f_M := -\Phi - M\ln$ is convex and by (27) we have

$$(0\leq)-\frac{1}{b-a}\int_{a}^{b}\Phi(x)\,dx+\Phi\left(\frac{a+b}{2}\right)+M\ln\left(\frac{A(a,b)}{I(a,b)}\right)$$
$$\leq \frac{1}{8}\left[-\Phi_{-}(b)+\Phi_{+}(a)\right](b-a)+\frac{M}{8ab}(b-a)^{2},$$

which is equivalent to the second inequality in (25). □

Remark 2.2. If we write the inequality (26) for the convex function $-\ln$ we have

$$(0\leq)\ln I(a,b)-\ln G(a,b)\leq\frac{1}{8}\left(\frac{1}{a}-\frac{1}{b}\right)(b-a),$$

which shows that

$$\frac{(b-a)^{2}}{8ab}-\ln\left(\frac{I(a,b)}{G(a,b)}\right)\geq 0.$$

Also, if we write the inequality (27) for the convex function $-\ln$ we have

$$(0\leq)\ln A(a,b)-\ln I(a,b)\leq\frac{1}{8}\left(\frac{1}{a}-\frac{1}{b}\right)(b-a),$$

which shows that

$$\frac{(b-a)^{2}}{8ab}-\ln\left(\frac{A(a,b)}{I(a,b)}\right)\geq 0.$$

3. Applications for Special Means

For m, M with $M>m>0$ we define

$$M_p := \begin{cases} M^p \text{ if } p>1 \\ \\ m^p \text{ if } p<0 \end{cases} \text{ and } m_p := \begin{cases} m^p \text{ if } p>1 \\ \\ M^p \text{ if } p<0 \end{cases}. \tag{28}$$

Consider the function $\Phi(t)=t^p$, $p\in(-\infty,0)\cup(1,\infty)$. This is a convex function and $\Phi''(t)=p(p-1)t^{p-2}$, $t>0$. Consider $\kappa(t):=t^2\Phi''(t)=p(p-1)t^p$. We observe that

$$\max_{t\in[m,M]}\kappa(t)=p(p-1)M_p \text{ and } \min_{t\in[m,M]}\kappa(t)=p(p-1)m_p.$$

By making use of the inequalities (14) and (15) for the function $\Phi(t)=t^p$, $p\in(-\infty,0)\cup(1,\infty)$, then for any a, $b\in[m,M]$ and $\nu\in[0,1]$ we have

$$\ln\left[\frac{\left(a+\nu\frac{b-a}{2}\right)^{\nu}\left(\frac{a+b}{2}+\nu\frac{b-a}{2}\right)^{1-\nu}}{I(a,b)}\right]^{p(p-1)m_p} \tag{29}$$

$$\leq L_p^p(a,b) - \left[\nu\left(a+\nu\frac{b-a}{2}\right)^p + (1-\nu)\left(\frac{a+b}{2}+\nu\frac{b-a}{2}\right)^p\right]$$
$$\leq \ln\left[\frac{\left(a+\nu\frac{b-a}{2}\right)^\nu\left(\frac{a+b}{2}+\nu\frac{b-a}{2}\right)^{1-\nu}}{I(a,b)}\right]^{p(p-1)M_p}$$

and

$$\ln\left[\frac{I(a,b)}{\sqrt{A_\nu(a,b)\,G_{1-\nu}(a,b)}}\right]^{p(p-1)m_p} \tag{30}$$
$$\leq \frac{1}{2}\left[((1-\nu)a+\nu b)^p + \nu a^p + (1-\nu)b^p\right] - L_p^p(a,b)$$
$$\leq \ln\left[\frac{I(a,b)}{\sqrt{A_\nu(a,b)\,G_{1-\nu}(a,b)}}\right]^{p(p-1)M_p},$$

where m_p and M_p are defined by (28).

If we take $p=2$ in (30), then we get

$$\ln\left[\frac{I(a,b)}{\sqrt{A_\nu(a,b)\,G_{1-\nu}(a,b)}}\right]^{2m^2} \tag{31}$$
$$\leq \frac{1}{2}\left[((1-\nu)a+\nu b)^2 + \nu a^2 + (1-\nu)b^2\right] - L_2^2(a,b)$$
$$\leq \ln\left[\frac{I(a,b)}{\sqrt{A_\nu(a,b)\,G_{1-\nu}(a,b)}}\right]^{2M^2}.$$

Since

$$\frac{1}{2}\left[((1-\nu)a+\nu b)^2 + \nu a^2 + (1-\nu)b^2\right] - L_2^2(a,b)$$
$$= \frac{1}{2}\left[(1-\nu)^2a^2 + 2\nu(1-\nu)ab + \nu^2b^2 + \nu a^2 + (1-\nu)b^2\right]$$
$$-\frac{1}{b-a}\int_a^b t^2dt$$
$$= \frac{1}{2}\left[(1-\nu)^2a^2 + 2\nu(1-\nu)ab + \nu^2b^2 + \nu a^2 + (1-\nu)b^2\right]$$
$$-\frac{1}{3}\left(a^2+ab+b^2\right) = \frac{1}{6}(b-a)^2\left(3\nu^2-3\nu+1\right),$$

then by (31) we get

$$\ln\left[\frac{I(a,b)}{\sqrt{A_\nu(a,b)\,G_{1-\nu}(a,b)}}\right]^{2m^2} \leq \frac{1}{6}(b-a)^2\left(3\nu^2-3\nu+1\right)$$

$$\leq \ln \left[\frac{I(a,b)}{\sqrt{A_\nu(a,b)\, G_{1-\nu}(a,b)}} \right]^{2M^2},$$

which is equivalent to

(32)
$$\exp\left[\left(\frac{1}{12} - \frac{1}{4}\nu(1-\nu)\right)\frac{(b-a)^2}{M^2}\right] \leq \frac{I(a,b)}{\sqrt{A_\nu(a,b)\, G_{1-\nu}(a,b)}} \leq \exp\left[\left(\frac{1}{12} - \frac{1}{4}\nu(1-\nu)\right)\frac{(b-a)^2}{m^2}\right],$$

for any $a,\ b \in [m, M]$ and $\nu \in [0,1]$.

If we take in (32) $\nu = 0$, then we get

$$\exp\left[\frac{1}{12}\frac{(b-a)^2}{M^2}\right] \leq \frac{I(a,b)}{G(a,b)} \leq \exp\left[\frac{1}{12}\frac{(b-a)^2}{m^2}\right], \tag{33}$$

for any $a,\ b \in [m, M]$.

If we take in (32) $\nu = \frac{1}{2}$, then we get

$$\exp\left[\frac{1}{48}\frac{(b-a)^2}{M^2}\right] \leq \frac{I(a,b)}{\sqrt{A(a,b)\, G(a,b)}} \leq \exp\left[\frac{1}{48}\frac{(b-a)^2}{m^2}\right], \tag{34}$$

for any $a,\ b \in [m, M]$.

If $a,\ b > 0$ then my taking $M = \max\{a,b\}$ and $m = \min\{a,b\}$ in (33) and (34) and since

$$\frac{(b-a)^2}{\max^2\{a,b\}} = \left(\frac{\min\{a,b\}}{\max\{a,b\}} - 1\right)^2 \text{ and } \frac{(b-a)^2}{\min^2\{a,b\}} = \left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2,$$

for any $a,\ b > 0$, then we have

$$\exp\left[\frac{1}{12}\left(\frac{\min\{a,b\}}{\max\{a,b\}} - 1\right)^2\right] \leq \frac{I(a,b)}{G(a,b)} \leq \exp\left[\frac{1}{12}\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2\right] \tag{35}$$

and

$$\exp\left[\frac{1}{48}\left(\frac{\min\{a,b\}}{\max\{a,b\}} - 1\right)^2\right] \leq \frac{I(a,b)}{\sqrt{A(a,b)\, G(a,b)}} \leq \exp\left[\frac{1}{48}\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2\right], \tag{36}$$

for any $a,\ b > 0$.

By making use of the inequalities (24) and (25) for the function $\Phi(t) = t^p$, $p \in (-\infty, 0) \cup (1, \infty)$, then for any a, $b \in [m, M]$ and $\nu \in [0, 1]$ we have

$$\begin{aligned} &p(p-1) m_p \left[\frac{(b-a)^2}{8ab} - \ln \left(\frac{I(a,b)}{G(a,b)} \right) \right] \\ &\leq \frac{1}{8} p \left(b^{p-1} - a^{p-1} \right)(b-a) - \left[\frac{a^p + b^p}{2} - L_p^p(a,b) \right] \\ &\leq p(p-1) M_p \left[\frac{(b-a)^2}{8ab} - \ln \left(\frac{I(a,b)}{G(a,b)} \right) \right] \end{aligned} \tag{37}$$

and

$$\begin{aligned} &p(p-1) m_p \left[\frac{(b-a)^2}{8ab} - \ln \left(\frac{A(a,b)}{I(a,b)} \right) \right] \\ &\leq \frac{1}{8} p \left(b^{p-1} - a^{p-1} \right)(b-a) - \left[L_p^p(a,b) - A^p(a,b) \right] \\ &\leq p(p-1) M_p \left[\frac{(b-a)^2}{8ab} - \ln \left(\frac{A(a,b)}{I(a,b)} \right) \right], \end{aligned} \tag{38}$$

where m_p and M_p are defined by (28).

The case $p = 2$ provides some simpler inequalities, however the details are left to the interested reader.

References

[1] A. Akkurt, H. Yldirim, *On Hermite-Hadamard-Fejér inequality type for convex functions via fractional integrals*, Mathematica Moravica, 21 (1) (2017), 105–123.

[2] M. Alomari, M. Darus, *The Hadamard's inequality for s-convex function*, Int. J. Math. Anal. (Ruse), 2 (13-16) (2008), 639–646.

[3] M. Alomari, M. Darus, *Hadamard-type inequalities for s-convex functions*, Int. Math. Forum, 3 (37-40) (2008), 1965–1975.

[4] G. A. Anastassiou, *Univariate Ostrowski inequalities, revisited*, Monatsh. Math., 135 (3) (2002), 175–189.

[5] N. S. Barnett, P. Cerone and S. S. Dragomir, *Some new inequalities for Hermite-Hadamard divergence in information theory,* Preprint RGMIA Res. Rep. Coll. **5** (2002), No. 4, Art. 8, Published in Y. J. Cho, J. K. Kim and Y. K. Choi (Eds.), *Stochastic Analysis and Applications*, Voulme 3, Nova Science Publishers, 2003, pp. 7-19.

[6] S. S. Dragomir, *On a reverse of Jessen's inequality for isotonic linear functionals*, J. Ineq. Pure & Appl. Math., 2 (3) (2001), Article ID: 36.

[7] S. S. Dragomir, *On the Jessen's inequality for isotonic linear functionals*, Nonlinear Anal. Forum, 7 (2) (2002), 139–151.

[8] S. S. Dragomir, *On the Lupaş–Beesack–Pečarić inequality for isotonic linear functionals*, Nonlinear Funct. Anal. Appl., 7 (2) (2002), 285–298.

[9] S. S. Dragomir, *An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products*, J. Inequal. Pure and Appl. Math., 3 (2) (2002), Article ID: 31.

[10] S. S. Dragomir, *An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products*, J. Inequal. Pure and Appl. Math., 3 (3) (2002), Article ID: 35.

[11] S. S. Dragomir, *A survey on Jessen's type inequalities for positive functionals*, in P. M. Pardalos et al. (eds.), *Nonlinear Analysis*, Springer Optimization and Its Applications 68, In Honor of Themistocles M. Rassias on the Occasion of his 60th Birthday, DOI 10.1007/978-1-4614-3498-6_12, © Springer Science+Business Media, LLC 2012.

[12] S. S. Dragomir, N. M. Ionescu, *On some inequalities for convex-dominated functions*, L'Anal. Num. Théor. L'Approx., 19 (1) (1990), 21–27.

[13] S. S. Dragomir, C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, 2000.

[14] A. El Farissi, *Simple proof and refinement of Hermite-Hadamard inequality*, J. Math. Inequal., 4 (3) (2010), 365–369.

[15] E. K. Godunova, V. I. Levin, I*nequalities for functions of a broad class that contains convex, monotone and some other forms of functions*, Numerical mathematics and mathematical physics, Moskov. Gos. Ped. Inst., Moscow, 166 (1985), 138–142 (in Russian).

[16] H. Hudzik, L. Maligranda, *Some remarks on s-convex functions*, Aequationes Math., 48 (1) (1994), 100–111.

[17] E. Kikianty, S. S. Dragomir, *Hermite-Hadamard's inequality and the p-HH-norm on the Cartesian product of two copies of a normed space*, Math. Inequal. Appl., 13 (1) (2010), 1–32.

[18] U. S. Kirmaci, M. Klaričić Bakula, M. E. Özdemir, J. Pečarić, *Hadamard-type inequalities for s-convex functions*, Appl. Math. Comput., 193 (1) (2007), 26–35.

[19] M. A. Latif, *On some inequalities for h-convex functions*, Int. J. Math. Anal. (Ruse), 4 (29-32) (2010), 1473–1482.

[20] D. S. Mitrinović, I. B. Lacković, *Hermite and convexity*, Aequationes Math., 28 (1985), 229–232.

[21] D. S. Mitrinović, J. E. Pečarić, *Note on a class of functions of Godunova and Levin*, C. R. Math. Rep. Acad. Sci. Canada, 12 (1) (1990), 33–36.

[22] C. E. M. Pearce, A. M. Rubinov, *P-functions, quasi-convex functions, and Hadamard-type inequalities*, J. Math. Anal. Appl., 240 (1) (1999), 92–104.

[23] J. E. Pečarić, S. S. Dragomir, *On an inequality of Godunova-Levin and some refinements of Jensen integral inequality*, Itinerant Seminar on Functional Equations, Approximation and Convexity, Univ. "Babeş-Bolyai", Cluj-Napoca, 89 (6) (1989), 263–268.

[24] J. Pečarić, S. S. Dragomir, *A generalization of Hadamard's inequality for isotonic linear functionals*, Radovi Mat. (Sarajevo), 7 (1991), 103–107.

[25] M. Radulescu, S. Radulescu, P. Alexandrescu, *On the Godunova-Levin-Schur class of functions*, Math. Inequal. Appl., 12 (4) (2009), 853–862.

[26] M. Z. Sarikaya, A. Saglam, H. Yildirim, *On some Hadamard-type inequalities for h-convex functions*, J. Math. Inequal., 2 (3) (2008), 335–341.

[27] E. Set, M. E. Özdemir, M. Z. Sarıkaya, *New inequalities of Ostrowski's type for s-convex functions in the second sense with applications*, Facta Univ. Ser. Math. Inform., 27 (1) (2012), 67–82.

[28] M. Z. Sarikaya, E. Set, M. E. Özdemir, *On some new inequalities of Hadamard type involving h-convex functions*, Acta Math. Univ. Comenian. (N.S.), 79 (2) (2010), 265–272.

[29] M. Tunç, *Ostrowski-type inequalities via h-convex functions with applications to special means*, J. Inequal. Appl., 2013, 2013:326.

[30] S. Varošanec, *On h-convexity*, J. Math. Anal. Appl., 326 (1) (2007), 303–311.

Silvestru Sever Dragomir
Mathematics
College of Engineering & Science
Victoria University
PO Box 14428 Melbourne City, MC 8001
Australia
Secondary address:
School of Computer Science & Applied Mathematics
University of the Witwatersrand
Private Bag 3, Johannesburg 2050
South Africa
E-mail address: sever.dragomir@vu.edu.au

Ian Gomm
Mathematics
College of Engineering & Science
Victoria University
PO Box 14428 Melbourne City, MC 8001
Australia
E-mail address: ian.gomm@vu.edu.au