A new generalization of the trapezoid formula for N-time differentiable mappings and applications

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A NEW GENERALIZATION OF THE TRAPEZOID FORMULA FOR n-TIME DIFFERENTIABLE MAPPINGS AND APPLICATIONS

Abstract. A new generalization of the trapezoid formula for n-time differentiable mappings and applications in Numerical Analysis are given.

1. Introduction

In the recent paper [1], P. Cerone, S.S. Dragomir and J. Roumeliotis proved the following generalization of the trapezoid rule.

THEOREM 1. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Then we have the equality*

$$(1.1) \qquad \int_a^b f(t)\, dt$$

$$= \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} \left[\frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2} \right] + \int_a^b T_n(t) f^{(n)}(t)\, dt,$$

where

$$(1.2) \qquad T_n(t) := \frac{1}{n!} \left[\frac{(b-t)^n + (-1)^n (t-a)^n}{2} \right], \quad t \in [a,b].$$

In the same paper, the authors pointed out the following inequality which provides an approximation formula for the integral $\int_a^b f(t)\, dt$ whose error can be estimated in terms of the sup-norm of $f^{(n)}(t)$.

COROLLARY 1. *Under the above assumptions, we have the inequality*

$$(1.3) \qquad \left| \int_a^b f(t)\, dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} \left[\frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2} \right] \right|$$

$$\leq \frac{(b-a)^{n+1}}{(n+1)!} \| f^{(n)} \|_\infty \times \begin{cases} 1 & \text{if } n = 2r \\ \frac{2^n - 1}{2^n} & \text{if } n = 2r+1. \end{cases}$$

If, in the above corollary, we consider $n = 1$, then we get the known inequality [2]

$$(1.4) \qquad \left| \int_a^b f(t)\, dt - \frac{f(a) + f(b)}{2}(b-a) \right| \leq \frac{1}{4}(b-a)^2 \| f' \|_\infty.$$

For $n = 2$, we obtain

$$(1.5) \qquad \left| \int_a^b f(t)\, dt - \frac{f(a)+f(b)}{2}(b-a) - \frac{(b-a)^2}{2} \cdot \frac{f'(a) + f'(b)}{2} \right|$$

$$\leq \frac{(b-a)^3}{6} \| f'' \|_\infty.$$

For other recent results concerning the trapezoid formula, see the book [9] and the recent papers [1]-[8] and [10]-[11], where further references are given.

The main aim of this paper is to point out a generalization of the trapezoid rule and inequality in a different way. Applications in Numerical Analysis for quadrature formulae will also be provided. A perturbed trapezoidal type rule is presented in Section 4 in which a number of *premature* results are given that provide tighter bounds than the traditional Grüss, Chebychev and Lupaş inequalities.

2. Integral identities

We start with the following result:

THEOREM 2. *Let $f : [a, b] \to \mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ ($n \geq 1$) is absolutely continuous on $[a, b]$. Then*

$$(2.1) \qquad \int_a^b f(t)\, dt$$

$$= \sum_{k=0}^{n-1} \frac{1}{(k+1)!} [(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)]$$

$$+ \frac{1}{n!} \int_a^b (x-t)^n f^{(n)}(t)\, dt,$$

for all $x \in [a, b]$.

Proof. The proof is by mathematical induction.

For $n = 1$, we have to prove that

(2.2) $\qquad \int_a^b f(t)\,dt = (x - a)f(a) + (b - x)f(b) + \int_a^b (x - t)f^{(1)}(t)\,dt,$

which is straightforward as it may be seen by the integration by parts formula applied for the integral

(2.3) $\qquad\qquad \int_a^b (x - t)f^{(1)}(t)\,dt.$

Assume that (2.1) holds for "n" and let us prove it for "$n + 1$". That is, we wish to show that

(2.4) $\qquad \int_a^b f(t)\,dt$

$$= \sum_{k=0}^{n} \frac{1}{(k+1)!}[(x-a)^{k+1}f^{(k)}(a) + (-1)^k(b-x)^{k+1}f^{(k)}(b)]$$

$$+ \frac{1}{(n+1)!}\int_a^b (x-t)^{n+1}f^{(n+1)}(t)\,dt.$$

For this purpose, we apply formula (2.2) for the mapping $g(t) := (x - t)^n f^{(n)}(t)$, which is absolutely continuous on $[a, b]$, and then, we can write

(2.5) $\qquad \int_a^b (x - t)^n f^{(n)}(t)\,dt$

$$= (x - a)(x - a)^n f^{(n)}(a) + (b - x)(x - b)^n f^{(n)}(b)$$

$$+ \int_a^b (x - t)\frac{d}{dt}[(x - t)^n f^{(n)}(t)]\,dt$$

$$= \int_a^b (x - t)[-n(x - t)^{n-1}f^{(n)}(t) + (x - t)^n f^{(n+1)}(t)]\,dt$$

$$+ (x - a)^{n+1}f^{(n)}(a) + (-1)^n(b - x)^{n+1}f^{(n)}(b)$$

$$= -n\int_a^b (x - t)^n f^{(n)}(t)\,dt + \int_a^b (x - t)^{n+1}f^{(n+1)}(t)\,dt$$

$$+ (x - a)^{n+1}f^{(n)}(a) + (-1)^n(b - x)^{n+1}f^{(n+1)}(t)\,dt.$$

From (24) we can get

$$\int_a^b (x-t)^n f^{(n)}(t)\, dt$$

$$= \frac{1}{n+1} \int_a^b (x-t)^{n+1} f^{(n+1)}(t)\, dt$$

$$+ \frac{1}{n+1} [(x-a)^{n+1} f^{(n)}(a) + (-1)^n (b-x)^{n+1} f^{(n)}(b)].$$

Now, using the induction hypothesis, we have

$$\int_a^b f(t)\, dt = \sum_{k=0}^{n-1} \frac{1}{(k+1)!} [(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)]$$

$$+ \frac{1}{n!} \left[\frac{1}{n+1} \int_a^b (x-t)^{n+1} f^{(n+1)}(t)\, dt \right.$$

$$\left. + \frac{1}{n+1} [(x-a)^{n+1} f^{(n)}(a) + (b-x)^{n+1} f^{(n)}(b)] \right]$$

$$= \sum_{k=0}^{n-1} \frac{1}{(k+1)!} [(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)]$$

$$+ \frac{1}{(n+1)!} \int_a^b (x-t)^{n+1} f^{(n+1)}(t)\, dt$$

and the identity (2.4) is proved. This completes the proof. ■

The following corollary is useful in practice.

COROLLARY 2. *With the above assumptions for f and R, we have the particular identities (which can also be obtained by using Taylor's formula with the integral remainder)*

$$(2.6) \quad \int_a^b f(t)\, dt = \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)!} (b-a)^{k+1} f^{(k)}(b) + \frac{(-1)^n}{n!} \int_a^b (t-a)^n f^{(n)}(t)\, dt,$$

$$(2.7) \quad \int_a^b f(t)\, dt = \sum_{k=0}^{n-1} \frac{1}{(k+1)!} (b-a)^{k+1} f^{(k)}(a) + \frac{1}{n!} \int_a^b (b-t)^n f^{(n)}(t)\, dt,$$

and the identity (see also [11])

$$(2.8) \quad \int_a^b f(t)\, dt = \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{2} \right)^{k+1} [f^{(k)}(a) + (-1)^k f^{(k)}(b)]$$

$$+ \frac{(-1)^n}{n!} \int_a^b \left(t - \frac{a+b}{2} \right)^n f^{(n)}(t)\, dt.$$

REMARK 1. a) *For $n = 1$, we get the identity (2.2) which is a generalization of the trapezoid rule.*

i) *For $x = a$ in (2.2), we capture the "right rectangle rule"*

$$\int_a^b f(t)\,dt = (b-a)f(b) - \int_a^b (t-a)f'(t)\,dt.$$

ii) *For $x = b$ in (2.2), we obtain the "left rectangle rule"*

(2.9)
$$\int_a^b f(t)\,dt = (b-a)f(a) - \int_a^b (b-t)f'(t)\,dt.$$

iii) *Finally, for $x = \frac{a+b}{2}$, we get* [2]

(2.10)
$$\int_a^b f(t)\,dt = \frac{f(a)+f(b)}{2}(b-a) - \int_a^b \left(t - \frac{a+b}{2}\right)f'(t)\,dt$$

which is the "trapezoid rule".

b) *For $n = 2$, we get the identity:*

(2.11)
$$\int_a^b f(t)\,dt$$
$$= (x-a)f(a) + (b-x)f(b)$$
$$+ \frac{1}{2}[(x-a)^2 f'(a) - (b-x)^2 f'(b)] + \frac{1}{2}\int_a^b (x-t)^2 f''(t)\,dt.$$

i) *If in (2.11) we choose $x = b$, then we obtain the "perturbed left rectangle rule"*

(2.12)
$$\int_a^b f(t)\,dt = (b-a)f(a) + \frac{1}{2}(b-a)^2 f'(a) + \frac{1}{2}\int_a^b (t-a)^2 f''(t)\,dt,$$

which can also be obtained by using Taylor's formula with the integral remainder.

ii) *If in (2.11) we choose $x = a$, we can write the "perturbed right rectangle rule"*

(2.13)
$$\int_a^b f(t)\,dt = (b-a)f(b) - \frac{1}{2}(b-a)^2 f'(b) + \frac{1}{2}\int_a^b (t-b)^2 f''(t)\,dt.$$

iii) *Finally, for $x = \frac{a+b}{2}$, we capture the "perturbed trapezoid rule"* [11]

$$(2.14) \qquad \int_a^b f(t)\,dt = \frac{f(a) + f(b)}{2}(b - a) + \frac{(b - a)^2}{8}(f'(a) - f'(b))$$

$$+ \frac{1}{2} \int_a^b \left(t - \frac{a + b}{2}\right)^2 f''(t)\,dt.$$

3. Integral inequalities

Using the integral representation of Theorem 1, we can prove the following inequality

THEOREM 3. *Let* $f : [a, b] \to \mathbb{R}$ *be a mapping such that the derivative* $f^{(n-1)}$ *(*$n \geq 1$*) is absolutely continuous on* $[a, b]$*. Then*

$$(3.1) \quad \left| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!}[(x - a)^{k+1}f^{(k)}(a) + (-1)^k(b - x)^{k+1}f^{(k)}(b)] \right|$$

$$\leq \begin{cases} \dfrac{\|f^{(n)}\|_\infty}{(n+1)!}[(x - a)^{n+1} + (b - x)^{n+1}] & \text{if } f^{(n)} \in L_\infty[a, b]; \\[2ex] \dfrac{\|f^{(n)}\|_p}{n!}\left[\dfrac{(x - a)^{nq+1} + (b - x)^{nq+1}}{nq + 1}\right]^{\frac{1}{q}} & \text{if } p > 1, \ \dfrac{1}{p} + \dfrac{1}{q} = 1 \\[1ex] & \text{and } f^{(n)} \in L_p[a, b]; \\[2ex] \dfrac{\|f^{(n)}\|_1}{n!}\left[\dfrac{1}{2}(b - a) + \left|x - \dfrac{a + b}{2}\right|\right]^n \end{cases}$$

for all $x \in [a, b]$.

P r o o f. Using the representation (2.1) and the properties of the modulus, we have

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!}[(x - a)^{k+1}f^{(k)}(a) + (-1)^k(b - x)^{k+1}f^{(k)}(b)] \right|$$

$$\leq \frac{1}{n!}\int_a^b |x - t|^n |f^{(n)}(t)|\,dt =: R.$$

Observe that

$$R \leq \left[\frac{1}{n!}\int_a^b |x - t|^n\,dt\right]\|f^{(n)}\|_\infty$$

$$= \frac{\|f^{(n)}\|_\infty}{n!}\left[\int_a^b (x - t)^n\,dt + \int_a^b (t - x)^n\,dt\right]$$

$$= \frac{\|f^{(n)}\|_\infty}{n!}\left[\frac{(x - a)^{n+1} + (b - x)^{n+1}}{n + 1}\right]$$

and the first inequality in (3.1) is proved.

Using Hölder's integral inequality, we also have

$$R \leq \frac{1}{n!} \left(\int\limits_a^b |f^{(n)}(t)|^p \, dt \right)^{\frac{1}{p}} \left(\int\limits_a^b |x - t|^{nq} \, dt \right)^{\frac{1}{q}}$$

$$= \frac{1}{n!} \|f^{(n)}\|_p \left[\frac{(x-a)^{nq+1} + (b-x)^{nq+1}}{nq + 1} \right]^{\frac{1}{q}},$$

which proves the second inequality in (3.1).

Finally, let us observe that

$$R \leq \frac{1}{n!} \sup_{t \in [a,b]} |x - t|^n \int\limits_a^b |f^{(n)}(t)| \, dt$$

$$= \frac{1}{n!} [\sup_{t \in [a,b]} |x - t|]^n \|f^{(n)}\|_1$$

$$= \frac{1}{n!} [\max(x - a, b - x)]^n \|f^{(n)}\|_1$$

$$= \frac{1}{n!} \left[\frac{1}{2}(b - a) + \left| x - \frac{a+b}{2} \right| \right]^n \|f^{(n)}\|_1$$

and the theorem is completely proved. ∎

The following corollary is useful in practice.

COROLLARY 3. *With the above assumptions for f and n, we have the particular inequalities*

$$\left| \int\limits_a^b f(t) \, dt - \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)!} (b-a)^{k+1} f^{(k)}(b) \right|$$

$$\leq M := \begin{cases} \dfrac{\|f^{(n)}\|_\infty}{(n+1)!} (b-a)^{n+1} & \text{if } f^{(n)} \in L_\infty[a,b]; \\[2ex] \dfrac{\|f^{(n)}\|_p}{n!} \dfrac{(b-a)^{n+\frac{1}{q}}}{(nq+1)^{1/q}} & \text{if } p > 1, \dfrac{1}{p} + \dfrac{1}{q} = 1 \\ & \text{and } f^{(n)} \in L_p[a,b]; \\[2ex] \dfrac{\|f^{(n)}\|_1}{n!} (b-a)^n, \end{cases}$$

and

$$\left| \int\limits_a^b f(t) \, dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} (b-a)^{k+1} f^{(k)}(b) \right| \leq M$$

and (see also [11]*)*

$$(3.2) \quad \left| \int_a^b f(t)\, dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{2} \right)^{k+1} [f^{(k)}(a) + (-1)^k f^{(k)}(b)] \right|$$

$$\leq \begin{cases} \dfrac{\|f^{(n)}\|_\infty}{2^n (n+1)!} (b-a)^{n+1} & \text{if } f^{(n)} \in L_\infty[a,b]; \\[3mm] \dfrac{\|f^{(n)}\|_p}{2^n n! (nq+1)^{1/q}} (b-a)^{n+\frac{1}{q}} & \text{if } p > 1, \; \dfrac{1}{p} + \dfrac{1}{q} = 1 \\[2mm] & \text{and } f^{(n)} \in L_p[a,b]; \\[3mm] \dfrac{\|f^{(n)}\|_1}{2^n n!} (b-a)^n; \end{cases}$$

respectively.

REMARK 2. *If we put $n = 1$ in (3.1), we capture the inequality*

$$(3.3) \quad \left| \int_a^b f(t)\, dt - (x-a) f(a) - (b-x) f(b) \right|$$

$$\leq \begin{cases} \left[\dfrac{1}{4}(b-a)^2 + \left(x - \dfrac{a+b}{2} \right)^2 \right] \|f^{(1)}\|_\infty & \text{if } f' \in L_\infty[a,b]; \\[3mm] \|f'\|_p \left[\dfrac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}} & \text{if } p > 1, \; \dfrac{1}{p} + \dfrac{1}{q} = 1 \\[2mm] & \text{and } f' \in L_p[a,b]; \\[3mm] \left[\dfrac{1}{2}(b-a) + \left| x - \dfrac{a+b}{2} \right| \right] \|f'\|_1; \end{cases}$$

for all $x \in [a,b]$, and, in particular,

a) *the "left rectangle" inequality*

$$\left| \int_a^b f(t)\, dt - (b-a) f(a) \right| \leq \begin{cases} \dfrac{\|f'\|_\infty}{2} (b-a)^2 & \text{if } f' \in L_\infty[a,b]; \\[3mm] \dfrac{\|f'\|_p}{(q+1)^{1/q}} (b-a)^{1+\frac{1}{q}} & \text{if } f' \in L_p[a,b]; \\[3mm] \|f'\|_1 (b-a). \end{cases}$$

b) *the "right rectangle" inequality*

$$\left| \int_a^b f(t)\, dt - (b-a) f(b) \right| \leq \begin{cases} \dfrac{\|f'\|_\infty}{2} (b-a)^2 & \text{if } f' \in L_\infty[a,b]; \\[3mm] \dfrac{\|f'\|_p}{(q+1)^{1/q}} (b-a)^{1+\frac{1}{q}} & \text{if } f' \in L_p[a,b]; \\[3mm] \|f'\|_1 (b-a). \end{cases}$$

c) *the "trapezoid" inequality*

$$(3.4) \qquad \left| \int_a^b f(t)\, dt - \frac{f(a) + f(b)}{2}(b-a) \right|$$

$$\leq \begin{cases} \dfrac{\|f'\|_\infty}{4}(b-a)^2 & \text{if } f' \in L_\infty[a,b]; \\[2ex] \dfrac{\|f'\|_p}{2(q+1)^{1/q}}(b-a)^{1+\frac{1}{q}} & \text{if } f' \in L_p[a,b]; \\[2ex] \dfrac{\|f'\|_1}{2}(b-a). \end{cases}$$

REMARK 3. *If we put $n = 2$ in (3.1), we get the inequality*

$$(3.5) \qquad \left| \int_a^b f(t)\, dt - (x-a)f(a) - (b-x)f(b) \right.$$

$$\left. - \frac{1}{2}[(x-a)^2 f'(a) - (b-x)^2 f'(b)] \right|$$

$$\leq \begin{cases} \dfrac{\|f''\|_\infty}{6}[(b-a)^3 + (b-x)^3] & \text{if } f'' \in L_\infty[a,b]; \\[2ex] \dfrac{\|f''\|_p}{2}\left[\dfrac{(x-a)^{2q+1} + (b-x)^{2q+1}}{2q+1}\right]^{\frac{1}{q}} & \text{if } p > 1,\ \dfrac{1}{p} + \dfrac{1}{q} = 1; \\ & \text{and } f'' \in L_p[a,b]; \\[2ex] \dfrac{\|f''\|_1}{2}\left[\dfrac{1}{2}(b-a) + \left| x - \dfrac{a+b}{2} \right| \right]^2; \end{cases}$$

for all $x \in [a,b]$, and, in particular:

a) *the "perturbed left rectangle" inequality*

$$(3.6) \qquad \left| \int_a^b f(t)\, dt - (b-a)f(a) - \frac{1}{2}(b-a)^2 f'(a) \right|$$

$$\leq M_2 := \begin{cases} \dfrac{\|f''\|_\infty}{6}(b-a)^3 & \text{if } f'' \in L_\infty[a,b]; \\[2ex] \dfrac{\|f''\|_p}{2(2q+1)^{1/q}}(b-a)^{2+\frac{1}{q}} & \text{if } f'' \in L_p[a,b]; \\[2ex] \dfrac{\|f''\|_1}{2}(b-a)^2; \end{cases}$$

b) *the "perturbed right rectangle" inequality*

$$(3.7) \qquad \left| \int_a^b f(t)\, dt - (b-a)f(b) + \frac{1}{2}(b-a)^2 f'(b) \right| \leq M_2$$

c) *the "perturbed trapezoid" inequality*

(3.8) $\left| \int\limits_a^b f(t)\,dt - \dfrac{f(a)+f(b)}{2}(b-a) + \dfrac{(b-a)^2}{8}(f'(b)-f'(a)) \right|$

$$\leq \begin{cases} \dfrac{\|f''\|_\infty}{24}(b-a)^3 & \text{if } f'' \in L_\infty[a,b]; \\[2ex] \dfrac{\|f''\|_p}{8(2q+1)^{1/q}}(b-a)^{2+\frac{1}{q}} & \text{if } f'' \in L_p[a,b]; \\[2ex] \dfrac{\|f''\|_1}{8}(b-a)^2. \end{cases}$$

4. A perturbed version

A *premature* Grüss inequality is embodied in the following lemma (see papers [12] or [14] for a proof).

LEMMA 1. *Let f, g be integrable functions defined on $[a,b]$ and let $d \leq g(t) \leq D$. Then*

(4.1) $|T(f,g)| \leq \dfrac{D-d}{2}[T(f,f)]^{\frac{1}{2}},$

where

$$T(f,g) = \frac{1}{b-a}\int\limits_a^b f(t)g(t)\,dt - \frac{1}{b-a}\int\limits_a^b f(t)\,dt \cdot \frac{1}{b-a}\int\limits_a^b g(t)\,dt.$$

Using the above lemma, the following result may be stated.

THEOREM 4. *Let $f : [a,b] \to \mathbb{R}$ be such that the derivative $f^{(n-1)}$, $n \geq 1$ is absolutely continuous on $[a,b]$. Assume that there exist constants $\gamma, \Gamma \in \mathbb{R}$ such that $\gamma \leq f^{(n)}(t) \leq \Gamma$ a.e on $[a,b]$. Then, the following inequality holds*

(4.2) $|P_T(x)| := \left| \int\limits_a^b f(t)\,dt \right.$

$- \sum\limits_{k=0}^{n-1} \left(\dfrac{1}{(k+1)!} \times [(x-a)^{k+1}f^{(k)}(a) + (-1)^k(b-x)^{k+1}f^{(k)}(b)] \right)$

$\left. - \dfrac{(x-a)^{n+1}+(-1)^n(b-x)^{n+1}}{(n+1)!}\left[\dfrac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a} \right] \right|$

$\leq \dfrac{\Gamma-\gamma}{2}\cdot\dfrac{1}{n!}I(x,n)$

$\leq \dfrac{\Gamma-\gamma}{2}\cdot\dfrac{n}{(n+1)!}\cdot\dfrac{(b-a)^{n+1}}{\sqrt{2n+1}},$

where

(4.3) $\quad I(x,n) = \dfrac{1}{(n+1)\sqrt{2n+1}}\{n^2(b-a)[(x-a)^{2n+1}+(b-x)^{2n+1}]$

$\qquad + (2n+1)(x-a)(b-x)[(x-a)^n - (x-b)^n]^2\}^{\frac{1}{2}}.$

P r o o f. Applying the premature Grüss result (4.1) on $(x-t)^n$ and $f^{(n)}(t)$, we have

$$\left|\frac{1}{b-a}\int_a^b (x-t)^n f^{(n)}(t)\,dt - \frac{1}{b-a}\int_a^b (x-t)^n\,dt \cdot \frac{1}{b-a}\int_a^b f^{(n)}(t)\,dt\right|$$

$$\leq \frac{\Gamma-\gamma}{2}\left\{\frac{1}{b-a}\int_a^b (x-t)^{2n}\,dt - \left[\frac{1}{b-a}\int_a^b (x-t)^n\,dt\right]^2\right\}^{\frac{1}{2}}.$$

Therefore,

$$\left|\frac{1}{b-a}\int_a^b (x-t)^n f^{(n)}(t)\,dt\right.$$

$$\left. -\frac{(x-a)^{n+1}+(-1)^n(b-x)^{n+1}}{(n+1)(b-a)}\cdot\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right|$$

$$\leq \frac{\Gamma-\gamma}{2}\left\{\frac{(x-a)^{2n+1}+(b-x)^{2n+1}}{(2n+1)(b-a)}-\left[\frac{(x-a)^{n+1}+(-1)^n(b-x)^{n+1}}{(b-a)(n+1)}\right]^2\right\}^{\frac{1}{2}}.$$

We get further simplification of the above result by multiplying throughout by $\frac{b-a}{n!}$. This gives

(4.4) $\qquad \left|\dfrac{1}{n!}\int_a^b (x-t)^n f^{(n)}(t)\,dt\right.$

$$-\frac{(x-a)^{n+1}+(-1)^n(b-x)^{n+1}}{(n+1)!}\cdot\left[\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right]\Bigg|$$

$$\leq \frac{\Gamma-\gamma}{2}\cdot\frac{1}{n!}J(x,n),$$

where

(4.5) $\quad J^2(x,n) = \dfrac{1}{(2n+1)(n+1)^2}\{(n+1)^2(A+B)(A^{2n+1}+B^{2n+1})$

$$-(2n+1)(A^{n+1}+(-1)^nB^{n+1})^2\}$$

with $A = x-a$, $B = b-x$.

Now, from (4.5),

$$(2n+1)(n+1)^2 J^2(x,n)$$

$$= n^2(A+B)(A^{2n+1}+B^{2n+1})$$

$$+(2n+1)[(A+B)(A^{2n+1}+B^{2n+1})-(A^{n+1}+(-1)^nB^{n+1})^2]$$

$$= n^2(A + B)(A^{2n+1} + B^{2n+1})$$
$$+(2n + 1)[AB(A^{2n} + B^{2n}) - 2A^{n+1} \cdot (-1)^n B^{n+1}]$$
$$= n^2(A + B)[A^{2n+1} + B^{2n+1}] + (2n + 1)AB[A^n - (-B)^n]^2$$

Now, substitution of $A = x - a$, $B = b - x$ and the fact that $A + B = b - a$ gives $I(x, n) = \frac{J(x,n)}{(n+1)\sqrt{2n+1}}$, as presented in (4.3). Substitution of identity (2.1) into (4.4) gives (4.2) and thus the first part of the theorem is proved.

The upper bound is obtained by taking either $I(a, n)$ or $I(b, n)$ since $I(x, n)$ is convex. Hence the theorem is completely proved. ∎

COROLLARY 4. *Let the conditions of Theorem 4 hold. Then the following result holds*

$$(4.6) \quad \left| \frac{1}{b-a} \int_a^b f(t)\, dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{2} \right)^{k+1} [f^{(k)}(a) + (-1)^k f^{(k)}(b)] \right.$$

$$\left. - \left(\frac{b-a}{2} \right)^n \frac{[1 + (-1)^n]}{(n+1)!} \left[\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \right] \right|$$

$$\leq \frac{\Gamma - \gamma}{2} \cdot \frac{1}{n!} \left(\frac{b-a}{2} \right)^{n+1} \cdot \frac{1}{\sqrt{2n+1}} \cdot \begin{cases} \frac{2n}{n+1}, & n \text{ even} \\ 2, & n \text{ odd.} \end{cases}$$

Proof. Taking $x = \frac{a+b}{2}$ in (4.2) gives (4.2), where

$$I\left(\frac{a+b}{2}, n \right) = \frac{1}{(n+1)\sqrt{2n+1}} \left(\frac{b-a}{2} \right)^{n+1} \{4n^2 + (2n+1)[1 + (-1)^n]^2\}^{\frac{1}{2}}.$$

Examining the above expression for n even or n odd readily gives the result (4.6). ∎

REMARK 4. *For n even, the third term in the modulus sign vanishes and thus there is no perturbation to the trapezoidal rule (4.6).*

THEOREM 5. *Let the conditions of Theorem 4 be satisfied. Further, suppose that $f^{(n)}$ is differentiable and be such that*

$$\|f^{(n+1)}\|_\infty := \sup_{t \in [a,b]} |f^{n+1}(t)| < \infty.$$

Then

$$(4.7) \qquad |P_T(x)| \leq \frac{b-a}{\sqrt{12}} \|f^{(n+1)}\|_\infty \cdot \frac{1}{n!} I(x, n),$$

where $P_T(x)$ is the perturbed trapezoidal type rule given by the left hand side of (4.2) and $I(x, n)$ is as given by (4.3).

Proof. Let $f, g : [a, b] \to \mathbb{R}$ be absolutely continuous and f', g' be bounded. Then Chebychev's inequality holds (see [13, p. 207])

$$|T(f, g)| \leq \frac{(b - a)^2}{12} \sup_{t \in [a,b]} |f'(t)| \cdot \sup_{t \in [a,b]} |g'(t)|.$$

In [14] Matić, Pečarić and Ujević, using a *premature* Grüss type argument, proved that

$$(4.8) \qquad |T(f, g)| \leq \frac{(b - a)}{\sqrt{12}} \sup_{t \in [a,b]} |g'(t)| \sqrt{T(f, f)}.$$

Thus, associating $f^{(n)}(\cdot)$ with $g(\cdot)$ and $(x - t)^n$ with f in (4.8) readily produces (4.7) where $I(x, n)$ is as given by (4.3). ∎

THEOREM 6. *Let the conditions of Theorem 4 be satisfied. Further, suppose that $f^{(n)}$ is locally absolutely continuous on (a, b) and let $f^{(n+1)} \in L_2(a, b)$. Then*

$$(4.9) \qquad |P_T(x)| \leq \frac{b - a}{\pi} \|f^{(n+1)}\|_2 \cdot \frac{1}{n!} I(x, n),$$

where $P_T(x)$ is the perturbed trapezoidal type rule given by the left hand side of (4.2) and $I(x, n)$ is as given in (4.3).

Proof. The following result was obtained by Lupaş (see [13, p. 210]). For $f, g : (a, b) \to \mathbb{R}$ being locally absolutely continuous on (a, b) and $f', g' \in L_2(a, b)$, then

$$|T(f, g)| \leq \frac{(b - a)^2}{\pi^2} \|f'\|_2 \|g'\|_2,$$

where

$$\|h\|_2 := \left(\frac{1}{b - a} \int_a^b |h(t)|^2 \right)^{\frac{1}{2}} \quad \text{for} \quad h \in L_2(a, b).$$

In [14] Matić, Pečarić and Ujević further show that

$$(4.10) \qquad |T(f, g)| \leq \frac{(b - a)}{\pi} \|g'\|_2 \sqrt{T(f, f)}.$$

Now, associating $f^{(n)}(\cdot)$ with $g(\cdot)$ and $(x - t)^n$ with f in (4.10) gives (4.9), where $I(x, n)$ is found in (4.3). ∎

REMARK 5. *Results (4.7) and (4.9) are not readily comparable to that obtained in Theorem 4 since the bound now involves the behaviour of $f^{(n+1)}(\cdot)$ rather than $f^{(n)}(\cdot)$.*

5. Application in numerical integration

Consider the partition $I_m : a = x_0 < x_1 < ... < x_{m-1} < x_m = b$ of the interval $[a, b]$ and the intermediate points $\xi = (\xi_0, ..., \xi_{m-1})$, where $\xi_j \in [x_j, x_{j+1}]$ $(j = 0, ..., m - 1)$. Put $h_j := x_{j+1} - x_j$ and $\vartheta(h) = \max\{h_j | j = 0, ..., m - 1\}$.

In [1], the authors considered the following generalization of the trapezoid formula

$$(5.1) \quad T_{m,n}(f, I_m) := \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{h_j^{k+1}}{(k+1)!} \left[\frac{f^{(k)}(x_j) + (-1)^k f^{(k)}(x_{j+1})}{2} \right]$$

and proved the following theorem:

THEOREM 7. *Let $f : [a, b] \to \mathbb{R}$ be such that it's derivative $f^{(n-1)}$ is absolutely continuous on $[a, b]$. Then we have*

$$(5.2) \qquad \int_a^b f(t)\, dt = T_{m,n}(f, I_m) + R_{m,n}(f, I_m),$$

where the reminder $R_{m,n}(f, I_m)$ satisfies the estimate

$$(5.3) \qquad |R_{m,n}(f, I_m)| \le \frac{C_n}{(n+1)!} \|f^{(n)}\|_\infty \sum_{j=0}^{m-1} h_j^{n+1},$$

and

$$C_n := \begin{cases} 1 & \text{if } n = 2r \\ \dfrac{2^{2r+1} - 1}{2^{2r+1}} & \text{if } n = 2r + 1. \end{cases}$$

Now, let us define the even more generalized quadrature formula

$$\widetilde{T}_{m,n}(f, \xi, I_m) := \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{1}{(k+1)!} [(\xi_j - x_j)^{k+1} f^{(k)}(x_j)$$
$$+ (-1)^k (x_{j+1} - \xi_j)^{k+1} f^{(k)}(x_{j+1})],$$

where x_j, ξ_j $(j = 0, ..., m - 1)$ are as above.
The following theorem holds.

THEOREM 8. *Let f be as in Theorem 7. Then we have the formula*

$$(5.4) \qquad \int_a^b f(t)\, dt = \widetilde{T}_{m,n}(f, \xi, I_m) + \widetilde{R}_{m,n}(f, \xi, I_m),$$

where the reminder satisfies the estimate

$(5.5) \; |\widetilde{R}_{m,n}(f, \xi, I_m)|$

$$:= \begin{cases} \dfrac{1}{(n+1)!} \|f^{(n)}\|_\infty \displaystyle\sum_{j=0}^{m-1} [(\xi_j - x_j)^{n+1} + (x_{j+1} - \xi_j)^{n+1}], \\[3mm] \dfrac{1}{n!(nq+1)^{1/q}} \|f^{(n)}\|_p \left[\displaystyle\sum_{j=0}^{m-1} (\xi_j - x_j)^{nq+1} + \sum_{j=0}^{m-1} (x_{j+1} - \xi_j)^{nq+1} \right]^{\frac{1}{q}}, \\[3mm] \dfrac{1}{n!} \|f^{(n)}\|_1 \left[\dfrac{1}{2} \vartheta(h) + \displaystyle\max_{j=0,...,m-1} \left| \xi_j - \dfrac{x_j + x_{j+1}}{2} \right| \right]^n. \end{cases}$$

Proof. Apply the inequality (3.1) on the subinterval $[x_j, x_{j+1}]$ to get

$$\left| \int_{x_j}^{x_{j+1}} f(t)\, dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \right.$$

$$\left. \times [(\xi_j - x_j)^{k+1} f^{(k)}(x_j) + (-1)^k (x_{j+1} - \xi_j)^{k+1} f^{(k)}(x_{j+1})] \right|$$

$$\leq \begin{cases} \dfrac{1}{(n+1)!} \sup\limits_{t \in [x_j, x_{j+1}]} |f^{(n)}(t)| [(\xi_j - x_j)^{n+1} + (x_{j+1} - \xi_j)^{n+1}], \\[2ex] \dfrac{1}{n!} \left(\int_{x_j}^{x_{j+1}} |f^{(n)}(s)|^p ds \right)^{\frac{1}{p}} \left[\dfrac{(\xi_j - x_j)^{nq+1} + (x_{j+1} - \xi_j)^{nq+1}}{nq+1} \right]^{\frac{1}{q}}, \\[2ex] \dfrac{1}{n!} \left(\int_{x_j}^{x_{j+1}} |f^{(n)}(s)| ds \right) \left[\dfrac{1}{2} h_j + \left| \xi_j - \dfrac{x_j + x_{j+1}}{2} \right| \right]^n. \end{cases}$$

Summing over j from 0 to $m-1$ and using the generalized triangle inequality, we have

$$|\widetilde{R}_{m,n}(f, \xi, I_m)|$$

$$\leq \left| \sum_{j=0}^{m-1} \int_{x_j}^{x_{j+1}} f(t)\, dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \right.$$

$$\left. \times [(\xi_j - x_j)^{k+1} f^{(k)}(x_j) + (-1)^k (x_{j+1} - \xi_j)^{k+1} f^{(k)}(x_{j+1})] \right|$$

$$:= \begin{cases} \dfrac{1}{(n+1)!} \sum\limits_{j=0}^{m-1} \sup\limits_{t \in [x_j, x_{j+1}]} |f^{(n)}(t)| [(\xi_j - x_j)^{n+1} + (x_{j+1} - \xi_j)^{n+1}], \\[2ex] \dfrac{1}{n!} \sum\limits_{j=0}^{m-1} \left(\int_{x_j}^{x_{j+1}} |f^{(n)}(s)|^p ds \right)^{\frac{1}{p}} \left[\dfrac{(\xi_j - x_j)^{nq+1} + (x_{j+1} - \xi_j)^{nq+1}}{nq+1} \right]^{\frac{1}{q}}, \\[2ex] \dfrac{1}{n!} \sum\limits_{j=0}^{m-1} \left(\int_{x_j}^{x_{j+1}} |f^{(n)}(s)| ds \right) \left[\dfrac{1}{2} h_j + \left| \xi_j - \dfrac{x_j - x_{j+1}}{2} \right| \right]^n. \end{cases}$$

Since $\sup\limits_{t \in [x_j, x_{j+1}]} |f^{(n)}(t)| \leq \|f^{(n)}\|_\infty$, the first inequality is obvious.

Using the discrete Hölder inequality, we have

$$\frac{1}{(nq+1)^{1/q}} \sum_{j=0}^{m-1} \left(\int_{x_j}^{x_{j+1}} |f^{(n)}(s)|^p ds \right)^{\frac{1}{p}} [(\xi_j - x_j)^{nq+1} + (x_{j+1} - \xi_j)^{nq+1}]^{\frac{1}{q}}$$

$$\leq \frac{1}{(nq+1)^{1/q}} \left[\sum_{j=0}^{m-1} \left[\left(\int_{x_j}^{x_{j+1}} |f^{(n)}(s)|^p ds \right)^{\frac{1}{p}} \right]^p \right]^{\frac{1}{p}}$$

$$\times \Big[\sum_{j=0}^{m-1} [[(\xi_j - x_j)^{nq+1} + (x_{j+1} - \xi_j)^{nq+1}]^{\frac{1}{q}}]^q \Big]^{\frac{1}{q}}$$

$$= \frac{1}{(nq+1)^{1/q}} \|f^{(n)}\|_p \Big[\sum_{j=0}^{m-1} (\xi_j - x_j)^{nq+1} + \sum_{j=0}^{m-1} (x_{j+1} - \xi_j)^{nq+1} \Big]^{\frac{1}{q}}$$

and the second inequality in (5.5) is proved.

Finally, let us observe that

$$\frac{1}{n!} \sum_{j=0}^{m-1} \Big(\int_{x_j}^{x_{j+1}} |f^{(n)}(s)| ds \Big) \Big[\frac{1}{2} h_j + \Big| \xi_j - \frac{x_j + x_{j+1}}{2} \Big| \Big]^n \le$$

$$\le \max_{j=0,\dots,m-1} \Big[\frac{1}{2} h_j + \Big| \xi_j - \frac{x_j + x_{j+1}}{2} \Big| \Big]^n \sum_{j=0}^{m-1} \Big(\int_{x_j}^{x_{j+1}} |f^{(n)}(s)| ds \Big)$$

$$\le \Big[\frac{1}{2} h_j + \max_{j=0,\dots,m-1} \Big| \xi_j - \frac{x_j + x_{j+1}}{2} \Big| \Big]^n \|f^{(n)}\|_1$$

and the last part of (5.5) is proved. ∎

REMARK 6. *Since $(x - a)^\alpha + (b - x)^\alpha \le (b - a)^\alpha$ for $\alpha \ge 1$, $x \in [a, b]$, then we can remark that the first branch of (5.5) can be bounded by*

$$(5.6) \qquad \frac{1}{(n+1)!} \|f^{(n)}\|_\infty \sum_{j=0}^{m-1} h_j^{n+1}.$$

The second branch can be upper bounded by

$$(5.7) \qquad \frac{1}{n!(nq+1)^{1/q}} \|f^{(n)}\|_p \Big[\sum_{j=0}^{m-1} h_j^{nq+1} \Big]^{\frac{1}{q}}$$

and finally, the last branch in (5.5) can be upper bounded by

$$(5.8) \qquad \frac{1}{n!} [\vartheta(h)]^n \|f^{(n)}\|_1.$$

Note that all the bounds provided by (5.6)–(5.8) are uniform bounds for $\tilde{R}_{m,n}(f, \xi, I_m)$ in terms of the intermediate points ξ.

The last inequality we can get from (5.5) is that one for which we have $\xi_j = \frac{x_j + x_{j+1}}{2}$. Consequently, we can state the following corollary (see also [11]):

COROLLARY 5. *Let f be as in Theorem 8. Then we have the formula*

$$(5.9) \qquad \int_a^b f(t)\, dt = \tilde{T}_{m,n}(f, I_m) + \tilde{R}_{m,n}(f, I_m),$$

where

$$(5.10) \quad \widetilde{T}_{m,n}(f, I_m) = \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{1}{2^{k+1}(k+1)!} [f^{(k)}(x_j) + (-1)^k f^{(k)}(x_{j+1})] h_j^{n+1}$$

and the remainder $\widetilde{R}$ satisfies the estimate

$$|\widetilde{R}_{m,n}(f, I_m)| \leq \begin{cases} \dfrac{1}{2^n(n+1)!} \|f^{(n)}\|_\infty \sum_{j=0}^{m-1} h_j^{n+1}, \\[2ex] \dfrac{1}{2^n n! (nq+1)^{1/q}} \|f^{(n)}\|_p \Big[\sum_{j=0}^{m-1} h_j^{n+1} \Big]^{\frac{1}{q}}, \\[2ex] \dfrac{1}{2^n n!} [\vartheta(h)]^n \|f^{(n)}\|_1. \end{cases}$$

REMARK 7. *Similar results can be stated by using the "perturbed" versions embodied in Theorems 4, 5 and 6, but we omit the details.*

References

[1] P. Cerone, S. S. Dragomir and J. Roumeliotis, *Some Ostrowski type inequalities for n-time differentiable mappings and applications*, Demonstratio Math., 32(4) (1999), 697–712.

[2] S. S. Dragomir, *On the trapezoid formula for Lipschitzian mappings and applications*, Tamkang J. Math., 30(2) (1999), 133–138.

[3] S. S. Dragomir, P. Cerone and A. Sofo, *Some remarks on the trapezoid rule in numerical integration*, Indian J. Pure Appl. Math., 31(5) (2000), 415–494.

[4] S. S. Dragomir and T. C. Peachey, *New estimation of the remainder in the trapezoidal formula with applications*, accepted Studia Math. Babes-Bolyai Univ.

[5] P. Cerone, S. S. Dragomir and C. E. M. Pearce, *A generalized trapezoid inequality for functions of bounded variation*, Turkish. J. Math. 24 (2000), 1–17.

[6] N. S. Barnett, S. S. Dragomir and C. E. M. Pearce, *A quasi-trapezoid inequality for double integrals*, submitted J. Austral. Math. Sec. (B).

[7] S. S. Dragomir and A. McAndrew, *On trapezoid inequality via a Grüss type result and applications*, accepted in Tamkang J. Math.

[8] S. S. Dragomir, J. E. Pečarić and S. Wang, *The unified treatment of trapezoid, Simpson and Ostrowski type inequality for monotonic mappings and applications*, Math. Comput. Modelling, 31 (2000), 61–70.

[9] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Inequalities for Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, 1994.

[10] V. Čuljak, C.E.M. Pearce and J. P. Pečarić, *The unified treatment of some inequalities of Ostrowski and Simpson's type*, submitted.

[11] S. S. Dragomir, *A Taylor like formula and application in numerical integration*, submitted.

[12] P. Cerone and S. S. Dragomir, *Three point quadrature rules involving, at most, a first derivative*, Preprint. RGMIA Res. Rep. Coll., 2(4) (1999), Article 8.

[13] J. E. Pečarić, F. Proschan and Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Presss, 1992.

[14] M. Matić, J. E. Pečarić and N. Ujević, *On New estimation of the remainder in Generalised Taylor's Formula*, M.I.A., Vol. 2 No. 3 (1999), 343–361.

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