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Anthony Sofo
Victoria University
P. O. Box 14428, Melbourne City, Victoria 8001, Australia
E-Mail: anthony.sofo@vu.edu.au

Abstract

We provide an explicit analytical representation for Euler type sums of harmonic numbers with multiple arguments. We also explore the representation of integrals with logarithmic and hypergeometric integrand in terms of the polygamma function and other special functions. The integrals in question will be associated with harmonic numbers of positive terms. A few examples of integrals will be given an identity in terms of some special functions including the Riemann zeta function.

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1. Introduction

The study of the representation of infinite series in closed form goes back to antiquity, and it was the genius Euler who put the investigation of the analysis of series on a firm foundation. The celebrated Basel problem introduced the zeta function and Euler went on to represent binomial and harmonic number series of the type $\sum_{n\geq 1}\frac{H_n^{(m)}}{n^q}$ in closed form. The famous Euler recurrence expression states

$$2\sum_{n\geq 1}\frac{H_n}{n^m}=(m+2)\,\zeta\,(m+1)-\sum_{j=1}^{m-2}\zeta\,(j+1)\,\zeta\,(m-j)\,,$$

see [4]. Euler's harmonic number series have been extended and variations have been investigated by [2], [5], [10], [15], [19], [24] and many others. Binomial, inverse binomial and harmonic number series are of interest to physics and have been studied in order to perform calculations of higher order corrections to scattering processes in particle physics, see [9], [22], [28] and [45]. In [1], the authors explore the algorithmic and analytic properties of so-called generalized harmonic sums systematically, in order to compute the massive Feynman integrals which arise in quantum field theories and in certain combinatorial problems. Hence there is great motivation to study the representation of Euler series in closed form: a good account on closed form, what they are and why we care has been eloquently enunciated in [11]. There exists, in the literature, see [14], [23], [33] results on sums of the form $\sum_{n\geq 1} \frac{H_n^{(m)}}{n\binom{n+k}{k}}$, but very many fewer results incorporating series of harmonic numbers with multiple argument $H_{pn}^{(m)}$. A search of the literature yields no closed form representation of $T(k,m,p) = \sum_{n\geq 1} \frac{H_{pn}^{(m)}}{n\binom{n+k}{k}}$ for $p \geq 3$. In this paper we will develop analytical representations for Euler type series with inverse binomial coefficients and harmonic numbers of the type $T(k,m,p)$. The extra parameter p serves to unify a large number of previously published results, see [34], [46], [47] and references therein. Also, by association, we are able to demonstrate a representation for integrals of the type $\int_0^1 \frac{(1-x^p)}{x^p(1-x)} \ln^{m-1} x \ln(1-x^p)\ dx$.

2. Preliminaries

We define a harmonic number with multiple argument as H_{pn} for $p \in \mathbb{N}\backslash\{1\}$; $\mathbb{N} := \{1,2,3,\cdots\}$, the set of natural numbers. For $p = 1$, we write H_n as the n^{th} harmonic number with unitary argument. In this paper we will develop analytical representations for Euler type sums with inverse binomial coefficients of the type

$$T(k,m,p) = \sum_{n\geq 1} \frac{H_{pn}^{(m)}}{n\binom{n+k}{k}} \tag{2.1}$$

for $(m,k,p) \in \mathbb{N}$. Furthermore we discuss analytical representations of the integral

$$\int_0^1 \frac{x^p \ln^{m-1} x}{1-x}\ {}_2F_1\left[\left.\begin{matrix} 1,1 \\ 2+k \end{matrix}\right| x^p\right] dx \tag{2.2}$$

for (m,k,p) the set of positive integers and where ${}_2F_1\left[\left.\begin{matrix} \cdot,\cdot \\ \cdot \end{matrix}\right| z\right]$ is the classical Gauss hypergeometric function. Let $\mathbb{R}$ and $\mathbb{C}$ denote, respectively the sets of real and complex numbers and let $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$. The Riemann zeta function is defined, for $s \in \mathbb{C}$ with $\Re(s) > 1$ by $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$. For $q \in \mathbb{N}$ we define the generalized harmonic number of order m as

$$H_q^{(m)} = \zeta_q(m) = \sum_{j=1}^{q} \frac{1}{j^m}.$$

Let

$$H_n = \sum_{r=1}^{n} \frac{1}{r} = \gamma + \psi(n+1),\ H_0 := 0 \tag{2.3}$$

be the nth harmonic number, where γ denotes the Euler-Mascheroni constant and $\psi(z)$ is the Digamma (or Psi) function defined by

$$\psi(z) := \frac{d}{dz}\{\log\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \qquad \text{or} \qquad \log\Gamma(z) = \int_1^z \psi(t)\,dt.$$

In the case of non-integer values of n such as (for example) a value $\rho \in \mathbb{R}$, the generalized harmonic numbers $H_\rho^{(m+1)}$ may be defined, in terms of the polygamma functions

$$\psi^{(n)}(z) := \frac{d^n}{dz^n}\{\psi(z)\} = \frac{d^{n+1}}{dz^{n+1}}\{\log\Gamma(z)\} \qquad (n \in \mathbb{N}_0)$$

by

$$H_\rho^{(m+1)} = \zeta(m+1) + \frac{(-1)^m}{m!}\psi^{(m)}(\rho+1) \tag{2.4}$$

$$\rho \in \mathbb{R}\backslash\{-1,-2,-3,...\}; \ m \in \mathbb{N}$$

where $\zeta(z)$ is the Riemann zeta function. Whenever we encounter harmonic numbers of the form $H_\rho^{(m)}$ at admissible real values of ρ, they may be evaluated by means of this known relation (2.4). In the exceptional case of (2.4) when $m = 0$, we may define $H_\rho^{(1)}$ by

$$H_\rho^{(1)} = H_\rho = \gamma + \psi(\rho+1), \ \rho \in \mathbb{R}\backslash\{-1,-2,-3,...\}$$

We assume that $H_0^{(m)} = 0$, $m \in \mathbb{N}$. Some results, that are published, on Euler sums with multiple argument of the type (2.1) are, from [13]

$$\sum_{n\geq1}\frac{\binom{2n}{n}H_{2n}}{4^n(2n+1)} = G$$

where $G = 0.91596...$ is the Catalan constant, and

$$\sum_{n\geq1}\frac{\binom{2n}{n}H_{2n}}{8^n} = \frac{\sqrt{2}}{2}\ln\left(\frac{3+2\sqrt{2}}{2}\right).$$

The following identity is obtained in [36],

$$\sum_{n\geq1}\frac{H_{2n}^{(2)}}{(4n+1)^3(4n-1)^3} = \frac{35}{288}\zeta(3) + \frac{5\pi^3}{1152} - \frac{259}{432}\zeta(2) + \frac{7\pi}{27} - \frac{29}{54}G + \frac{5}{9}\ln 2,$$

and [12] obtained

$$\sum_{n\geq1}\left(\frac{\binom{2n}{n}}{4^n(2n-1)}\right)^2 H_{2n} = \frac{1}{\pi}(6 - 12\ln 2 + 4G).$$

In the following theorems we encounter harmonic numbers at possible rational values of the argument, of the form $H_{\frac{r}{q}}^{(\alpha)}$ they maybe evaluated by an available relation in terms of the polygamma function $\psi^{(\alpha)}(z)$ or, for rational arguments $z = \frac{r}{q}$, and we also define

$$H_{\frac{r}{q}}^{(1)} = \gamma + \psi\left(\frac{r}{q} + 1\right), \text{ and } H_0^{(\alpha)} = 0.$$

The evaluation of the polygamma function $\psi^{(\alpha)}\left(\frac{r}{a}\right)$ at rational values of the argument can be explicitly done via a formula as given by Kölbig [29], or Choi and Cvijovic [16] in terms of the polylogarithmic or other special functions. Polygamma functions at negative rational values of the argument can also be explicitly evaluated, some specific values are listed in the books [42], and [44]. Some results for sums of harmonic numbers may be seen in the works of [17], [35], [49] and references therein.

The following lemma is proved in [37].

Lemma 1. *Let k and m be positive integers. Then:*

$$B(k,m) = \sum_{n\geq 1} \frac{H_n^{(m)}}{n\binom{n+k}{k}}$$

$$= \frac{1}{k}\zeta(m) + \frac{(-1)^m}{(1+k)(m-1)!}\int_0^1 \frac{x\ln^{m-1}x}{1-x}\, {}_2F_1\left[\left.\begin{matrix}1,1\\2+k\end{matrix}\right| x\right] dx \tag{2.5}$$

$$= \zeta(m+1) + \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\left(\begin{matrix}\sum_{j=1}^{r-1}\frac{(-1)^{m+1}H_j}{j^m}\\ +\sum_{s=2}^{m}(-1)^{m-s}H_{r-1}^{(m+1-s)}\zeta(s)\end{matrix}\right). \tag{2.6}$$

We now prove the following Lemma which will be required in the proof of the main Theorem.

Lemma 2. *Let $(k,m,p) \in \mathbb{N}$ and $j = 1,2,3,...,p-1$. Then:*

$$W(j,k,m,p) = \frac{(-1)^m}{(1+k)(m-1)!}\int_0^1 \frac{x^{1-\frac{j}{p}}\ln^{m-1}x}{1-x}\, {}_2F_1\left[\left.\begin{matrix}1,1\\2+k\end{matrix}\right| x\right] dx + \frac{1}{k}\zeta(m) \tag{2.7}$$

$$= \sum_{n\geq 1} \frac{H^{(m)}_{n-\frac{j}{p}}}{n \begin{pmatrix} n+k \\ k \end{pmatrix}}$$

$$= \frac{H^{(m)}_{-\frac{j}{p}}}{k} + (-1)^m \left(-H_{-\frac{j}{p}} H^{(m)}_{\frac{j}{p}-1} + \sum_{t=1}^{m-1} \frac{\psi^{(t)}\left(1-\frac{j}{p}\right)}{t! \left(\frac{j}{p}\right)^{m-t}} \right) \tag{2.8}$$

$$+ (-1)^m \sum_{r=1}^{k} (-1)^{r+1} \begin{pmatrix} k \\ r \end{pmatrix} \left(\begin{array}{c} H_{-\frac{j}{p}} H^{(m)}_{\frac{j}{p}+r-1} - \sum_{\mu=1}^{r-1} \frac{H_\mu}{\left(\mu+\frac{j}{p}\right)^m} \\ \\ + \sum_{\mu=1}^{r-1} \sum_{t=1}^{m-1} \frac{\psi^{(t)}\left(1-\frac{j}{p}\right)}{t!\left(\mu+\frac{j}{p}\right)^{m-t}} \end{array} \right).$$

In the case when $j = 0$, (2.7) reduces to (2.5).

Proof. Let $h^{(m)}_n = H^{(m)}_{n-a} - H^{(m)}_{-a}$ and put $a = \frac{j}{p}$, now consider the following expansion:

$$\sum_{n=1}^{\infty} \frac{h^{(m)}_n}{n \begin{pmatrix} n+k \\ k \end{pmatrix}} = \sum_{n=1}^{\infty} \frac{k!\ h^{(m)}_n}{n \prod_{r=1}^{k} (n+r)} = \sum_{n=1}^{\infty} \frac{k!\ h^{(m)}_n}{n\ (n+1)_{k+1}}.$$

Now

$$\sum_{n=1}^{\infty} \frac{h^{(m)}_n}{n \begin{pmatrix} n+k \\ k \end{pmatrix}} = \sum_{n=1}^{\infty} \frac{k!\ h^{(m)}_n}{n} \sum_{r=1}^{k} \left(\frac{A_r}{n+r} \right) \tag{2.9}$$

where

$$A_r = \lim_{n\to -r} \frac{n+r}{\prod_{r=1}^{k} n+r} = \frac{(-1)^{r+1}\ r}{k!} \begin{pmatrix} k \\ r \end{pmatrix}. \tag{2.10}$$

For an arbitrary positive sequence $X_{k,p}$ the following identity holds

$$\sum_{k=0}^{\infty} \sum_{p=0}^{n} X_{p,k} = \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} X_{p,k+p}$$

hence from (2.9)

$$\sum_{n=1}^{\infty}\frac{k!\ h_n^{(m)}}{n}\sum_{r=1}^{k}\left(\frac{A_r}{n+r}\right) = \sum_{r=1}^{k}(-1)^{r+1}\ r\ \binom{k}{r}\sum_{n=1}^{\infty}\frac{1}{n\,(n+r)} \times\sum_{\lambda=1}^{n}\frac{1}{(\lambda-a)^m}$$

where

$$h_n^{(m)} = H_{n-a}^{(m)}-H_{-a}^{(m)}=\sum_{\lambda=1}^{n}\frac{1}{(\lambda-a)^m}.$$

$$\sum_{n=1}^{\infty}\frac{h_n^{(m)}}{n\binom{n+k}{k}} = \sum_{r=1}^{k}(-1)^{r+1}\ r\ \binom{k}{r}\sum_{\lambda=1}^{\infty}\frac{1}{(\lambda-a)^m} \times\sum_{n=0}^{\infty}\frac{1}{(n+\lambda)\,(n+\lambda+r)}$$

$$= \sum_{r=1}^{k}(-1)^{r+1}\ r\ \binom{k}{r}\sum_{\lambda=1}^{\infty}\frac{1}{(\lambda-a)^m}\left[\frac{\psi\,(\lambda+r)-\psi\,(\lambda)}{r}\right].$$

Since we notice that

$$\psi\,(\lambda+r)-\psi\,(\lambda) = \sum_{\mu=0}^{r-1}\frac{1}{\mu+\lambda}$$

then

$$\sum_{n=1}^{\infty}\frac{h_n^{(m)}}{n\binom{n+k}{k}} = \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\sum_{\mu=0}^{r-1}\sum_{\lambda=1}^{\infty}\frac{1}{(\lambda-a)^m\,(\mu+\lambda)}$$

$$= \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\left(\begin{array}{c}\sum_{\lambda=1}^{\infty}\frac{1}{\lambda(\lambda-a)^m}\\ +\sum_{\mu=1}^{r-1}\sum_{\lambda=1}^{\infty}\frac{1}{(\lambda-a)^m(\mu+\lambda)}\end{array}\right).$$

Simplifying

$$\sum_{n=1}^{\infty}\frac{h_n^{(m)}}{n\binom{n+k}{k}} = (-1)^m\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\left[\frac{H_{-a}}{a^m}+\sum_{t=1}^{m-1}\frac{\psi^{(t)}\,(1-a)}{t!a^{m-t}}\right]$$

$$+(-1)^m \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} \left[\sum_{\mu=1}^{r-1} \left(\frac{H_{-a} - H_{-\mu}}{(\mu+a)^m} + \sum_{t=1}^{m-1} \frac{\psi^{(t)}(1-a)}{t!\,(\mu+a)^{m-t}} \right) \right]$$

$$= (-1)^m \left(\frac{H_{-a}}{a^m} + \sum_{t=1}^{m-1} \frac{\psi^{(t)}(1-a)}{t! a^{m-t}} \right)$$

$$+(-1)^m \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} \sum_{m=1}^{r-1} \begin{pmatrix} H_{-a}\left(H_{a+r-1}^{(m)} - H_a^{(m)}\right) \\ -\sum_{\mu=1}^{r-1} \frac{H_\mu}{(\mu+a)^m} \\ +\sum_{\mu=1}^{r-1} \sum_{t=1}^{m-1} \frac{\psi^{(t)}(1-a)}{t!(\mu+a)^{m-t}} \end{pmatrix}.$$

Now

$$\sum_{n=1}^{\infty} \frac{h_n^{(m)}}{n\binom{n+k}{k}} = (-1)^m \left(\frac{H_{-a}}{a^m} - H_{-a} H_a^{(m)} + \sum_{t=1}^{m-1} \frac{\psi^{(t)}(1-a)}{t! a^{m-t}} \right)$$

$$+(-1)^m \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} \sum_{m=1}^{r-1} \begin{pmatrix} H_{-a} H_{a+r-1}^{(m)} - \sum_{\mu=1}^{r-1} \frac{H_\mu}{(\mu+a)^m} \\ +\sum_{\mu=1}^{r-1} \sum_{t=1}^{m-1} \frac{\psi^{(t)}(1-a)}{t!(\mu+a)^{m-t}} \end{pmatrix}$$

and since, from

$$\sum_{n=1}^{\infty} \frac{h_n^{(m)}}{n\binom{n+k}{k}} = \sum_{n=1}^{\infty} \frac{H_{n-a}^{(m)} - H_{-a}^{(m)}}{n\binom{n+k}{k}} = \sum_{n=1}^{\infty} \frac{H_{n-a}^{(m)}}{n\binom{n+k}{k}} - \frac{H_{-a}^{(m)}}{k}$$

then

$$\sum_{n=1}^{\infty} \frac{H_{n-a}^{(m)}}{n\binom{n+k}{k}} = (-1)^m \left(\frac{H_{-a}}{a^m} - H_{-a} H_a^{(m)} + \sum_{t=1}^{m-1} \frac{\psi^{(t)}(1-a)}{t! a^{m-t}} \right) + \frac{H_{-a}^{(m)}}{k}$$

$$+(-1)^m \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} \sum_{m=1}^{r-1} \begin{pmatrix} H_{-a} H_{a+r-1}^{(m)} - \sum_{\mu=1}^{r-1} \frac{H_\mu}{(\mu+a)^m} \\ +\sum_{\mu=1}^{r-1} \sum_{t=1}^{m-1} \frac{\psi^{(t)}(1-a)}{t!(\mu+a)^{m-t}} \end{pmatrix},$$

and the identity (2.8) follows. □

The next few theorems relate the main results of this investigation, namely the closed form representation of the Euler sum (2.1) and integral (2.2).

3. The main theorem

The following main Theorem is proved

Theorem 1. *Let* $(k,m,p)\in\mathbb{N}$, *then*

$$T(k,m,p) = \sum_{n=1}^{\infty}\frac{H_{pn}^{(m)}}{n\binom{n+k}{k}}$$

$$= \frac{1}{k}\zeta(m) + \frac{(-1)^m}{(m-1)!\,(1+k)}\int_0^1 \frac{x^p\ln^{m-1}x}{1-x}\, {}_2F_1\left[\left.\begin{array}{c}1,1\\2+k\end{array}\right| x^p\right]dx \quad (3.1)$$

$$= \left(\frac{p^{m-1}-1}{kp^{m-1}}\right)\zeta(m) + \frac{1}{p^m}B(k,m) + \frac{1}{p^m}\sum_{j=1}^{p-1}W(j,k,m,p) \quad (3.2)$$

where $B(k,m)$ *is given by (2.6) and* $W(j,k,m,p)$ *is given by (2.8).*

Proof. For the integral representation (3.1), we recall that for $m\in\mathbb{N}$

$$H_n^{(m+1)} = \frac{(-1)^m}{m!}\int_0^1\frac{(1-x^{pn})\ln^m x}{1-x}dx.$$

We can now write

$$\sum_{n=1}^{\infty}\frac{H_{pn}^{(m)}}{n\binom{n+k}{k}} = \frac{(-1)^{m-1}}{(m-1)!}\int_0^1\frac{\ln^{m-1}x}{1-x}\sum_{n=1}^{\infty}\frac{(1-x^{pn})}{n\binom{n+k}{k}}dx$$

$$= \frac{(-1)^{m-1}}{(m-1)!}\int_0^1\frac{\ln^{m-1}x}{1-x}\left(\frac{1}{k}-\frac{x^p}{1+k}\, {}_2F_1\left[\left.\begin{array}{c}1,1\\2+k\end{array}\right| x^p\right]\right)dx$$

$$= \frac{1}{k}\zeta(m) + \frac{(-1)^m}{(m-1)!\,(1+k)}\int_0^1\frac{x^p\ln^{m-1}x}{1-x}\, {}_2F_1\left[\left.\begin{array}{c}1,1\\2+k\end{array}\right| x^p\right]dx,$$

hence (3.1) follows. Now for $p\in\mathbb{N}$ and from the properties of the polygamma function with multiple argument

$$\psi^{(n)}(pz) = \delta_{n,0}\ln p + \frac{1}{p^{n+1}}\sum_{r=0}^{p-1}\psi^{(n)}(z+\frac{r}{p}),$$

where $\delta_{n,0}$ is the Kronecker delta, we are able to rewrite, in terms of harmonic numbers, and using the properties of the polygamma function, as

$$H_{pn}^{(m)} = \left(\frac{p^{m-1}-1}{p^{m-1}}\right)\zeta(m) + \frac{1}{p^m}H_n^{(m)} + \frac{1}{p^m}\sum_{j=1}^{p-1}H_{n-\frac{j}{p}}^{(m)}.$$

The harmonic numbers $H^{(m)}_{n-\frac{j}{p}}$ may be thought of as shifted harmonic numbers, other results on summing shifted harmonic numbers are published in [38], [39] and [41]. Now summing over the integers n

$$\sum_{n=1}^{\infty}\frac{H^{(m)}_{pn}}{n\binom{n+k}{k}} = \sum_{n=1}^{\infty}\frac{1}{n\binom{n+k}{k}}\left(\begin{array}{c}\left(\frac{p^{m-1}-1}{p^{m-1}}\right)\zeta(m)+\frac{1}{p^m}H^{(m)}_n \\ +\frac{1}{p^m}\sum_{j=1}^{p-1}H^{(m)}_{n-\frac{j}{p}}\end{array}\right)$$

$$= \left(\frac{p^{m-1}-1}{p^{m-1}}\right)\zeta(m)\sum_{n=1}^{\infty}\frac{1}{n\binom{n+k}{k}}+\frac{1}{p^m}\sum_{n=1}^{\infty}\frac{H^{(m)}_n}{n\binom{n+k}{k}}$$

$$+\frac{1}{p^m}\sum_{j=1}^{p-1}\sum_{n=1}^{\infty}\frac{H^{(m)}_{n-\frac{j}{p}}}{n\binom{n+k}{k}}$$

$$= \left(\frac{p^{m-1}-1}{kp^{m-1}}\right)\zeta(m)+\frac{1}{p^m}B(k,m)+\frac{1}{p^m}\sum_{j=1}^{p-1}W(j,k,m,p)$$

which is the result (3.2). □

We give an example to demonstrate the power of the above Theorem.

Example 1.

$$T(2,4,4)=\sum_{n=1}^{\infty}\frac{H^{(4)}_{4n}}{n\binom{n+2}{2}} = \frac{1}{256}\zeta(5)+\frac{787}{256}\zeta(4)-\frac{259667}{134400}\zeta(3)+\frac{599\pi^3}{11025}$$

$$+\frac{53526239}{32928000}\zeta(2)+\frac{59953072\pi}{121550625}-\frac{2217728}{1157625}G$$

$$-\frac{\psi^{(3)}\left(\frac{1}{4}\right)}{630}-\frac{383424682}{121550625}\ln 2+\frac{208307761}{31116960000},$$

$$T(2,7,2)=\sum_{n=1}^{\infty}\frac{H^{(7)}_{2n}}{n\binom{n+2}{2}} = \frac{1}{128}\zeta(8)+\frac{505}{384}\zeta(7)-\frac{223}{128}\zeta(6)+\frac{6421}{3456}\zeta(5)$$

$$-\frac{6373}{3456}\zeta(4)+\frac{53965}{31104}\zeta(3)-\frac{46349}{31104}\zeta(2)+\frac{4372}{2187}\ln 2-\frac{2315}{279936},$$

from which we obtain the integral identity

$$\int_0^1 \frac{x^2 \ln^6 x}{1-x} \, {}_2F_1 \left[\begin{array}{c} 1,1 \\ 4 \end{array} \middle| x^2 \right] dx = -\frac{45\zeta(8)}{4} - \frac{4695\zeta(7)}{4} + \frac{10035\zeta(6)}{4} - \frac{32105\zeta(5)}{12}$$

$$+\frac{31865\zeta(4)}{12} - \frac{269825\zeta(3)}{108} + \frac{231745\zeta(2)}{108} - \frac{699520\ln 2}{243} + \frac{11575}{972}.$$

The following proposition follows directly from Theorem 1 and is a comment on the evaluation of the integral in (3.1).

Proposition 1. *For* $(k,m,p) \in \mathbb{N}$,

$$I(k,m,p) = \frac{(-1)^m}{(m-1)!\,(1+k)} \int_0^1 \frac{x^p \ln^{m-1} x}{1-x} \, {}_2F_1 \left[\begin{array}{c} 1,1 \\ 2+k \end{array} \middle| x^p \right] dx$$

$$= \frac{1}{p^m} B(k,m) + \frac{1}{p^m} \sum_{j=1}^{p-1} W(j,k,m,p) - \frac{1}{kp^{m-1}} \zeta(m)$$

where $B(k,m)$ *is given by (2.6) and* $W(j,k,m,p)$ *is given by (2.8). An illustrative examples follows,*

$$I(2,4,3) = \frac{1}{18} \int_0^1 \frac{x^3 \ln^3 x}{1-x} \, {}_2F_1 \left[\begin{array}{c} 1,1 \\ 4 \end{array} \middle| x^3 \right] dx$$

$$= \frac{231361}{12960000} - \frac{2313}{8000}\psi'(\frac{1}{3}) - \frac{1}{360}\psi^{(3)}(\frac{1}{3}) - \frac{507357}{320000}\ln 3$$

$$+\frac{1}{81}\zeta(5) + \frac{277}{162}\zeta(4) - \frac{53303}{32400}\zeta(3) + \frac{97\sqrt{3}\pi^3}{5400} - \frac{1709}{648}\zeta(2) + \frac{49877\sqrt{3}\pi}{320000}.$$

The case $k=1$ is interesting in its own right and therefore we have the following result.

Corollary 1. *Under the assumptions of Theorem 1, with* $k=1$*, we have,*

$$T(1,m,p) = \sum_{n=1}^{\infty} \frac{H_{pn}^{(m)}}{n(n+1)}$$

$$= \frac{(-1)^m}{(m-1)!} \int_0^1 \frac{(1-x^p) \ln^{m-1} x \ln(1-x^p)}{x^p (1-x)} dx \qquad (3.3)$$

$$= \frac{1}{p^m}\zeta(m+1) + \frac{1}{p^m}\sum_{t=1}^{m-1}(-1)^{1+t} p^t H_{p-1}^{(t)} \zeta(m+1-t) \qquad (3.4)$$

$$+\frac{1}{p^m}\sum_{j=1}^{p-1}\sum_{t=1}^{m}(-1)^t \left(\frac{p}{j}\right)^t H_{-\frac{j}{p}}^{(m+1-t)}$$

$$= \frac{1}{p^m}\zeta(m+1) + 3pH_{p-1}^{(4)} - \frac{1}{p^m}\sum_{t=1}^{m-1}(-p)^t H_{p-1}^{(t)}\zeta(m+1-t) \quad (3.5)$$

$$+\frac{1}{p^m}\sum_{j=1}^{p-1}\sum_{t=1}^{m}\left(\frac{p}{j}\right)^t \left(\begin{array}{c} (-1)^m H_{\frac{j}{p}-1}^{(m+1-t)}\left(1+(-1)^{m+1-t}\right)\zeta(m+1-t) \\ + \frac{(-1)^t \pi}{(m-t)!}\frac{d^{m-t}}{dz^{m-t}}\left(\cot(\pi z)\right)\Big|_{z=\frac{j}{p}} \end{array} \right).$$

Proof. From (3.1)

$$\begin{aligned} T(1,m,p) &= \sum_{n=1}^{\infty}\frac{H_{pn}^{(m)}}{n(n+1)} \\ &= \zeta(m) + \frac{(-1)^m}{2(m-1)!}\int_0^1 \frac{x^p \ln^{m-1} x}{1-x}\ {}_2F_1\left[\begin{array}{c} 1,1 \\ 3 \end{array}\Big| x^p\right] dx \\ &= \zeta(m) + \frac{(-1)^m}{(m-1)!}\int_0^1 \frac{\ln^{m-1} x}{1-x}\left(1 + \frac{(1-x^p)}{x^p}\ln(1-x^p)\right) dx, \end{aligned}$$

and since $\int_0^1 \frac{\ln^{m-1} x}{1-x}\, dx = (-1)^{m-1}(m-1)!\zeta(m)$ then (3.3) follows. From (3.2)

$$T(1,m,p) = \left(\frac{p^{m-1}-1}{p^{m-1}}\right)\zeta(m) + \frac{1}{p^m}B(1,m) + \frac{1}{p^m}\sum_{j=1}^{p-1} W(j,1,m,p)$$

$$= \left(\frac{p^{m-1}-1}{p^{m-1}}\right)\zeta(m) + \frac{1}{p^m}\zeta(m+1) + \frac{(-1)^m}{p^m}\sum_{j=1}^{p-1}\left(H_{-\frac{j}{p}}\left(H_{\frac{j}{p}}^{(m)} - H_{\frac{j}{p}-1}^{(m)}\right) + H_{-\frac{j}{p}}^{(m)}\right)$$

$$+\frac{(-1)^m}{p^m}\sum_{j=1}^{p-1}\sum_{t=1}^{m-1}(-1)^t\left(\frac{p}{j}\right)^{m-t}\left(H_{-\frac{j}{p}}^{(1+t)} - \zeta(1+t)\right).$$

since

$$\sum_{j=1}^{p-1} H_{-\frac{j}{p}}^{(m)} = -p\left(p^{m-1}-1\right)\zeta(m),$$

then

$$T(1,m,p) = \frac{1}{p^m}\zeta(m+1) + (-1)^m \sum_{j=1}^{p-1} \frac{1}{j^m} H_{-\frac{j}{p}}$$

$$+\frac{(-1)^m}{p^m} \sum_{t=1}^{m-1} (-1)^{1+t} p^{m-t} H_{p-1}^{(m-t)} \zeta(1+t)$$

$$+\frac{(-1)^m}{p^m} \sum_{j=1}^{p-1} \sum_{t=1}^{m-1} (-1)^t \left(\frac{p}{j}\right)^{m-t} H_{-\frac{j}{p}}^{(1+t)}.$$

Now by making a change in the summation index t we obtain

$$\begin{aligned} T(1,m,p) &= \frac{1}{p^m}\zeta(m+1) + (-1)^m \sum_{j=1}^{p-1} \frac{1}{j^m} H_{-\frac{j}{p}} \\ &+\frac{1}{p^m} \sum_{t=1}^{m-1} (-1)^{1+t} p^t H_{p-1}^{(t)} \zeta(m+1-t) \\ &+\frac{1}{p^m} \sum_{j=1}^{p-1} \sum_{t=1}^{m-1} (-1)^t \left(\frac{p}{j}\right)^t H_{-\frac{j}{p}}^{(m+1-t)} \end{aligned}$$

and hence the result (3.4) follows. From the reflection relation of the polygamma function, for $\upsilon \in \mathbb{N}$

$$\psi^{(\upsilon)}(1-z) + (-1)^{\upsilon+1} \psi^{(\upsilon)}(z) = (-1)^\upsilon \pi \frac{d^\upsilon}{dz^\upsilon}(\cot(\pi z))$$

we have, in terms of harmonic numbers

$$\begin{aligned} H_{-\frac{j}{p}}^{(m+1-t)} &= (-1)^{m-t} H_{\frac{j}{p}-1}^{(m+1-t)} + \left(1 + (-1)^{m+1-t}\right) \zeta(m+1-t) \\ &+\frac{\pi}{(m-t)!} \frac{d^{m-t}}{dz^{m-t}}(\cot(\pi z)) \Big|_{z=\frac{j}{p}} \end{aligned}$$

hence

$$\begin{aligned} T(1,m,p) &= \frac{1}{p^m}\zeta(m+1) + \frac{1}{p^m} \sum_{t=1}^{m-1} (-1)^{1+t} p^t H_{p-1}^{(t)} \zeta(m+1-t) \\ &+\frac{1}{p^m} \sum_{j=1}^{p-1} \sum_{t=1}^{m} \left(\frac{p}{j}\right)^t \left(\begin{array}{c} (-1)^m H_{\frac{j}{p}-1}^{(m+1-t)} \left(1 + (-1)^{m+1-t}\right) \zeta(m+1-t) \\ +\frac{(-1)^t \pi}{(m-t)!} \frac{d^{m-t}}{dz^{m-t}}(\cot(\pi z)) \Big|_{z=\frac{j}{p}} \end{array} \right), \end{aligned}$$

hence (3.5) follows. The identity (3.5) is noteworthy because it introduces finite cotangent and cosecant sums, which is a separate field of study in itself. Finite cotangent and cosecant sums of the form

$$\sum_{r=1}^{p-1} \cot^m \left(\frac{\pi r}{p}\right) \text{ and } \sum_{r=1}^{p-1} \csc^m \left(\frac{\pi r}{p}\right),$$

and their variations, have been investigated, see [3], [6], [20], [21], [25], [26] and [27]. Bettin and Conrey [7] prove a certain reciprocity formula for the cotangent sum

$$\sum_{r=1}^{p-1} \frac{r}{p} \cot \left(\frac{\pi r h}{p}\right). \tag{3.6}$$

The sum arises in connection with the Nyman-Beurling approach to the Riemann hypothesis. In another paper Bettin [8] give some simple arguments that confirm Maier and Rassias's [32] results on a distribution property and moments of the cotangent sum (3.6), a follow on paper [31] investigates the rate of growth of moments. The Rassias paper [30] also has some results on (3.6) related to the zeros of the Estermann zeta function. Finally, Chu and Marini [18] give many interesting examples of trigonometric sums. The author has not seen an investigation of

$$\sum_{r=1}^{p-1} r^q \cot^m \left(\frac{\pi r}{p}\right) \text{ and } \sum_{r=1}^{p-1} r^q \csc^m \left(\frac{\pi r}{p}\right),$$

$q \in \mathbb{Z} \setminus \{0\}, m \in \mathbb{N}$, in the published literature. The general integrals (3.1) and (3.3) cannot be evaluated with mathematical packages such as *Mathematica.* □

An example follows.

Example 2. *From (3.3) we have*

$$\begin{aligned} T(1,5,4) &= \sum_{n=1}^{\infty} \frac{H_{4n}^{(5)}}{n(n+1)} = \frac{1}{1024}\zeta(6) + \frac{4061}{1536}\zeta(5) - \frac{975}{256}\zeta(4) \\ &-\frac{2705}{1728}\zeta(2) + \frac{6461}{3456}\zeta(3) - \frac{5}{1152}\pi^5 - \frac{13}{216}\pi^3 - \frac{121}{243}\pi \\ &+\frac{3985}{1296}\ln 2 + \frac{1}{432}\psi^{(3)}\left(\frac{1}{4}\right) + \frac{160}{81}G, \end{aligned}$$

and

$$T(1,5,4) = -\frac{1}{24}\int_0^1 \frac{(1-x^4)\ln^4 x \ln(1-x^4)}{x^4(1-x)}\, dx.$$

The closed form (3.2) of the integral (3.1) is an exact identity which is expressed in finite sums of harmonic numbers and special functions. The following Theorem gives a bound on the integral (3.1).

Theorem 2. *Let* $k,\ p \in \mathbb{N}$ *and* m *integer* ≥ 2, *then,*

$$\frac{H_p^{(m)}}{1+k} < T(k,m,p) \leq \frac{1}{k}\zeta(m) + \frac{\beta}{p(m-1)!}\psi'(p+1) \tag{3.7}$$

where

$$\begin{aligned} T(k,m,p) &= \frac{1}{k}\zeta(m) + \frac{(-1)^m}{(m-1)!(1+k)}\int_0^1 \frac{x^p \ln^{m-1} x}{1-x}\ {}_2F_1\left[\left.\begin{matrix}1,1\\2+k\end{matrix}\right| x^p\right] dx \\ &= \sum_{n=1}^{\infty} \frac{H_{pn}^{(m)}}{n\binom{n+k}{k}}. \end{aligned}$$

Here $\beta = \left|\frac{\alpha \ln^{m-1}\alpha}{1-\alpha}\right|$ *and* $\alpha \in (0,1)$ *is the unique zero of the non algebraic equation* $\ln x + (1-m)x - (1-m) = 0$.

Proof. The infinite sum $T(k,m,p)$ is one of positive terms, monotonic increasing and therefore

$$T(k,m,p) > \frac{H_p^{(m)}}{1+k}.$$

Consider the integral inequality

$$\int_{x_0}^{x_1} |f(x)\, g(x)|\ dx \leq \sup_{x\in[x_0,x_1]} |f(x)| \int_{x_0}^{x_1} |g(x)|\ dx$$

for integrable functions $f(x)$ and $g(x)$ and $0 \leq x_0 < x_1 \in \mathbb{R}$. Now

$$\sup_{x\in[x_0,x_1]} |f(x)| = \sup_{x\in[0,1]} \left|\frac{\alpha \ln^{m-1}\alpha}{1-\alpha}\right| = \beta,$$

$\alpha \in (0,1)$ is the unique zero of the non algebraic equation

$$\ln x + (1-m)x - (1-m) = 0.$$

Also

$$\int_{x_0}^{x_1} |g(x)|\ dx = \int_0^1 \left| x^{p-1}\ {}_2F_1\left[\left.\begin{matrix}1,1\\2+k\end{matrix}\right| x^p\right]\right| dx = \frac{(1+k)\psi'(1+k)}{p},$$

therefore

$$\frac{H_p^{(m)}}{1+k} < T(k,m,p) \leq \frac{1}{k}\zeta(m) + \frac{(k+1)\beta}{p(k+1)(m-1)!}.\psi'(1+k)$$

and (3.7) follows. $\square$

Remark 1. *We have expressed Euler series of harmonic numbers with multiple argument and inverse binomial coefficients in closed form. Introducing the parameter p in the harmonic number $H_{pn}^{(m)}$ has allowed for the unification of a number of published results. It may be possible to further consider quadratic harmonic numbers of the form $\left(H_{pn}^{(m)}\right)^2$, and obtain some closed form representations, thereby generalizing the results of* [43] *and* [48].

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