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Hermite–Hadamard type inequalities for co-ordinated convex and qausi-convex functions and their applications

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Abstract

In the article, we present several Hermite–Hadamard type inequalities for the co-ordinated convex and quasi-convex functions and give an application to the product of the moment of two continuous and independent random variables. Our results are generalizations of some earlier results. Additionally, an illustrative example on the probability distribution is given to support our results.

MSC: 26D15; 26D20; 26D07

Keywords: Hermite–Hadamard inequality; Co-ordinated convex function; Co-ordinated quasi-convex function; Hölder integral inequality; Moment of random variable

1 Introduction

Let $\mathcal{I} \subseteq \mathbb{R}$ be a nonempty interval. Then a real-valued function $\Phi : \mathcal{I} \to \mathbb{R}$ is said to be convex (concave) if the inequality

$$\Phi\big[\varrho\mu + (1-\varrho)\nu\big] \le (\ge)\ \varrho\Phi(\mu) + (1-\varrho)\Phi(\nu)$$

holds for all $\mu, \nu \in \mathcal{I}$ and $\varrho \in [0,1]$.

It is a fact that the convex (concave) function is one of the most basic and important functions in the theory of geometric function, it has widely applications in pure and applied mathematics, physics, mechanics, statistics and economics, and meteorology [1–30]. Recently, the generalizations, extensions, variants and refinements for the convexity (concavity) have attracted the interest of several researchers [31–40]. In particular, many remarkable inequalities and properties in many branches of mathematics can be found in the literature [41–70] using the convexity (concavity) theory. The well-known Hermite–Hadamard inequality states that the double inequality

$$\Phi\left(\frac{\lambda_1+\lambda_2}{2}\right) \le (\ge)\ \frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2} \Phi(\mu)\,dx \le (\ge)\ \frac{\Phi(\lambda_1)+\Phi(\lambda_2)}{2} \tag{1.1}$$

holds for all $\lambda_1, \lambda_2 \in \mathcal{I}$ with $\lambda_1 \neq \lambda_2$ if $\Phi : \mathcal{I} \to \mathbb{R}$ is a convex (concave) function.

In the past hundred years, inequality (1.1) has inspired many researchers to estimate the bounds for

$$\left|\frac{\Phi(\lambda_1)+\Phi(\lambda_2)}{2}-\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\Phi(\mu)\,d\mu\right|$$

and

$$\left|\Phi\left(\frac{\lambda_1+\lambda_2}{2}\right)-\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\Phi(\mu)\,d\mu\right|,$$

and all the obtained results are called Hermite–Hadamard type inequalities.

It is well known that the multivariable functions also have the concept of convexity (concavity). An an example, we recall the definition of convexity (concavity) for the bivariate functions.

Let $\lambda_1,\lambda_2,\xi_1,\xi_2\in\mathbb{R}$ such that $\lambda_1<\lambda_2$ and $\xi_1<\xi_2$. Then a bivariate real-valued function $\Phi:[\lambda_1,\lambda_2]\times[\xi_1,\xi_2]\to\mathbb{R}$ is said to be convex (concave) if the inequality

$$\Phi\big(\varrho\mu+(1-\varrho)\rho,\varrho\nu+(1-\varrho)\omega\big)\le(\ge)\,\varrho\Phi(\mu,\nu)+(1-\varrho)\Phi(\rho,\omega)$$

holds for all $(\mu,\nu),(\rho,\omega)\in[\lambda_1,\lambda_2]\times[\xi_1,\xi_2]$ and $\varrho\in[0,1]$.

In order to establish the relation involving the convexity between the bivariate and univariate functions, Dragomir [71] introduced the definition of the bivariate co-ordinated convex function as follows.

Definition 1.1 (See [71]) Let $\lambda_1,\lambda_2,\xi_1,\xi_2\in\mathbb{R}$ such that $\lambda_1<\lambda_2$ and $\xi_1<\xi_2$. Then a bivariate real-valued function $\Phi:[\lambda_1,\lambda_2]\times[\xi_1,\xi_2]\to\mathbb{R}$ is said to be convex on the co-ordinates if both of the partial mappings $\Phi_\nu:[\lambda_1,\lambda_2]\to\mathbb{R}$ and $\Phi_\mu:[\xi_1,\xi_2]\to\mathbb{R}$ defined by

$$\Phi_\nu(\delta)=\Phi(\delta,\nu)$$

and

$$\Phi_\mu(\theta)=\Phi(\mu,\theta)$$

are convex for all $\mu\in[\lambda_1,\lambda_2]$ and $\nu\in[\xi_1,\xi_2]$.

Latif and Alomari [72] proved that the bivariate real-valued function $\Phi:[\lambda_1,\lambda_2]\times[\xi_1,\xi_2]\to\mathbb{R}$ is convex on the co-ordinates if and only if

$$\begin{aligned}&\Phi\big(\varrho\mu+(1-\varrho)\nu,\varsigma\rho+(1-\varsigma)\omega\big)\\&\quad\le\varrho\varsigma\Phi(\mu,\rho)+\varrho(1-\varsigma)\Phi(\mu,\omega)+\varsigma(1-\varrho)\Phi(\nu,\rho)+(1-\varrho)(1-\varsigma)\Phi(\nu,\omega)\end{aligned}$$

for all $\varrho,\varsigma\in[0,1]\times[0,1]$ and $(\mu,\rho),(\nu,\omega)\in[\lambda_1,\lambda_2]\times[\xi_1,\xi_2]$.

Dragomir [71] proved that every convex mapping $\Phi:[\lambda_1,\lambda_2]\times[\xi_1,\xi_2]\to\mathbb{R}$ is convex on the co-ordinates and the reverse is not true, and established the Hermite–Hadamard type inequality for the co-ordinated convex function on the rectangle of the plane $\mathbb{R}^2$.

Theorem 1.2 (See [71]) *Let $\lambda_1,\lambda_2,\xi_1,\xi_2 \in \mathbb{R}$ such that $\lambda_1 < \lambda_2$ and $\xi_1 < \xi_2$, and $\Phi : [\lambda_1,\lambda_2]\times[\xi_1,\xi_2] \to \mathbb{R}$ be a co-ordinated convex function. Then one has*

$$\begin{aligned}
&\Phi\left(\frac{\lambda_1+\lambda_2}{2},\frac{\xi_1+\xi_2}{2}\right) \\
&\le \frac{1}{2}\left[\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\Phi\left(\mu,\frac{\xi_1+\xi_2}{2}\right)d\mu + \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\Phi\left(\frac{\lambda_1+\lambda_2}{2},\nu\right)d\nu\right] \\
&\le \frac{1}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\mu,\nu)\,d\nu\,d\mu \\
&\le \frac{1}{4}\left[\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\Phi(\mu,\xi_1)\,d\mu + \frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\Phi(\mu,\xi_2)\,d\mu\right. \\
&\quad \left. + \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\Phi(\lambda_1,\nu)\,d\nu + \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\Phi(\lambda_2,\nu)\,d\nu\right] \\
&\le \frac{\Phi(\lambda_1,\xi_1)+\Phi(\lambda_1,\xi_2)+\Phi(\lambda_2,\xi_1)+\Phi(\lambda_2,\xi_2)}{4}
\end{aligned} \tag{1.2}$$

with the best possible constant $1/4$.

In [73], Latif et al. derived the variants of the Hermite–Hadamard type inequality (1.2), which are the weighted generalizations of (1.2).

Theorem 1.3 (See [73]) *Let $\triangle \subseteq \mathbb{R}^2$, $\triangle^\circ$ be the interior of $\triangle$, $\lambda_1 < \lambda_2$ and $\xi_1 < \xi_2$ such that $[\lambda_1,\lambda_2]\times[\xi_1,\xi_2] \in \triangle^\circ$, $\Phi : \Delta \to \mathbb{R}$ be a twice differentiable mapping on $\triangle^\circ$, $p : [\lambda_1,\lambda_2]\times[\xi_1,\xi_2] \to [0,\infty)$ be a continuous function symmetric with respect to $(\lambda_1+\lambda_2)/2$ and $(\xi_1+\xi_2)/2$, $\Phi_{\varrho\varsigma} \in L([\lambda_1,\lambda_2]\times[\xi_1,\xi_2])$ and $|\Phi_{\varrho\varsigma}|$ be a co-ordinated convex function on $[\lambda_1,\lambda_2]\times[\xi_1,\xi_2]$. Then*

$$\begin{aligned}
&\left|\Phi\left(\frac{\lambda_1+\lambda_2}{2},\frac{\xi_1+\xi_2}{2}\right)\int_{\xi_1}^{\xi_2}\int_{\lambda_1}^{\lambda_2}p(\mu,\nu)\,d\mu\,d\nu + \int_{\xi_1}^{\xi_2}\int_{\lambda_1}^{\lambda_2}\Phi(\mu,\nu)p(\mu,\nu)\,d\mu\,d\nu\right. \\
&\quad \left. - \int_{\xi_1}^{\xi_2}\int_{\lambda_1}^{\lambda_2}\Phi\left(\mu,\frac{\xi_1+\xi_2}{2}\right)p(\mu,\nu)\,d\mu\,d\nu - \int_{\xi_1}^{\xi_2}\int_{\lambda_1}^{\lambda_2}\Phi\left(\frac{\lambda_1+\lambda_2}{2},\nu\right)p(\mu,\nu)\,d\mu\,d\nu\right| \\
&\le \frac{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}{4}\left[\left|\Phi_{\varrho\varsigma}(\lambda_1,\xi_1)\right| + \left|\Phi_{\varrho\varsigma}(\lambda_1,\xi_2)\right| + \left|\Phi_{\varrho\varsigma}(\lambda_2,\xi_1)\right| + \left|\Phi_{\varrho\varsigma}(\lambda_1,\xi_2)\right|\right] \\
&\quad \times \int_0^1\int_0^1\left(\int_{\xi_1}^{L_2(\varsigma)}\int_{\lambda_1}^{L_1(\varrho)}p(\mu,\nu)\,d\mu\,d\nu\right)d\varrho\,d\varsigma,
\end{aligned} \tag{1.3}$$

where $L_1(\varrho) = \frac{1-\varrho}{2}\lambda_1 + \frac{1+\varrho}{2}\lambda_2$ and $L_2(\varsigma) = \frac{1-\varsigma}{2}\xi_1 + \frac{1+\varsigma}{2}\xi_2$.

Theorem 1.4 (See [73]) *Let $q > 1$, $\triangle \subseteq \mathbb{R}^2$, $\triangle^\circ$ be the interior of $\triangle$, $\lambda_1 < \lambda_2$ and $\xi_1 < \xi_2$ such that $[\lambda_1,\lambda_2]\times[\xi_1,\xi_2] \in \triangle^\circ$, $\Phi : \Delta \to \mathbb{R}$ be a twice differentiable mapping on $\triangle^\circ$, $p : [\lambda_1,\lambda_2]\times[\xi_1,\xi_2] \to [0,\infty)$ be a continuous function symmetric with respect to $(\lambda_1+\lambda_2)/2$ and $(\xi_1+\xi_2)/2$, $\Phi_{\varrho\varsigma} \in L([\lambda_1,\lambda_2]\times[\xi_1,\xi_2])$ and $|\Phi_{\varrho\varsigma}|^q$ be a co-ordinated convex function on*

$[\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$. *Then we have*

$$\begin{aligned}
&\left| \Phi\left(\frac{\lambda_1+\lambda_2}{2}, \frac{\xi_1+\xi_2}{2}\right) \int_{\xi_1}^{\xi_2} \int_{\lambda_1}^{\lambda_2} p(\mu,\nu)\,d\mu\,d\nu + \int_{\xi_1}^{\xi_2} \int_{\lambda_1}^{\lambda_2} \Phi(\mu,\nu) p(\mu,\nu)\,d\mu\,d\nu \right. \\
&\quad \left. - \int_{\xi_1}^{\xi_2} \int_{\lambda_1}^{\lambda_2} \Phi\left(\mu, \frac{\xi_1+\xi_2}{2}\right) p(\mu,\nu)\,d\mu\,d\nu - \int_{\xi_1}^{\xi_2} \int_{\lambda_1}^{\lambda_2} \Phi\left(\frac{\lambda_1+\lambda_2}{2}, \nu\right) p(\mu,\nu)\,d\mu\,d\nu \right| \\
&\quad \leq (\lambda_2-\lambda_1)(\xi_2-\xi_1) \left[\frac{|\Phi_{\varrho\varsigma}(\lambda_1,\xi_1)| + |\Phi_{\varrho\varsigma}(\lambda_1,\xi_2)| + |\Phi_{\varrho\varsigma}(\lambda_2,\xi_1)| + |\Phi_{\varrho\varsigma}(\lambda_2,\xi_2)|}{4} \right]^{\frac{1}{q}} \\
&\quad \times \int_0^1 \int_0^1 \left(\int_{\xi_1}^{L_2(\varsigma)} \int_{\lambda_1}^{L_1(\varrho)} p(\mu,\nu)\,d\mu\,d\nu \right) d\varrho\,d\varsigma,
\end{aligned} \tag{1.4}$$

where $L_1(\varrho)$ and $L_2(\varsigma)$ are defined as in Theorem 1.3.

Özdemir et al. [74] generalized the co-ordinated convex function to the co-ordinated quasi-convex function.

Definition 1.5 (See [74]) A real-valued function $\Phi : [\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \subset \mathbb{R}^2 \to \mathbb{R}$ is said to be quasi-convex if the inequality

$$\Phi\big(\varrho\mu + (1-\varrho)\rho, \varrho\nu + (1-\varrho)\omega\big) \leq \max\big\{\Phi(\mu,\nu), \Phi(\rho,\omega)\big\}$$

holds for all $(\mu,\nu), (\rho,\omega) \in [\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$ and $\varrho \in [0,1]$.

Definition 1.6 (See [74]) A real-valued function $\Phi : [\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \to \mathbb{R}$ is said to be quasi-convex on the co-ordinates if both of the partial mappings $\Phi_\nu : [\lambda_1, \lambda_2] \to \mathbb{R}$ and $\Phi_\mu : [\xi_1, \xi_2] \to \mathbb{R}$ defined by

$$\Phi_\nu(\delta) = \Phi(\delta, \nu), \qquad \Phi_\mu(\theta) = \Phi(\mu, \theta)$$

are quasi-convex for all $\mu \in [\lambda_1, \lambda_2]$ and $\nu \in [\xi_1, \xi_2]$.

In [75], Latif et al. provided an equivalent definition for the co-ordinated quasi-convex function as follows.

Definition 1.7 (See [75]) A real-valued function $\Phi : [\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be quasi-convex on the co-ordinates if the inequality

$$\Phi\big(\varrho\mu + (1-\varrho)\rho, \varsigma\nu + (1-\varsigma)\omega\big) \leq \max\big\{\Phi(\mu,\nu), \Phi(\mu,\omega), \Phi(\rho,\nu), \Phi(\rho,\omega)\big\}$$

holds for all $(\mu,\nu), (\rho,\omega) \in [\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$ and $(\varsigma,\varrho) \in [0,1] \times [0,1]$.

The class of co-ordinated quasi-convex functions on $[\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$ is denoted $QC([\lambda_1, \lambda_2] \times [\xi_1, \xi_2])$. Özdemir [74] proved that every quasi-convex function is also a co-ordinated quasi-convex function, but the converse does not hold true. The Hermite–Hadamard type inequality (1.2) was generalized to the co-ordinated quasi-convex function by Latif et al. [75].

Theorem 1.8 (See [75]) *Let $\triangle \subseteq \mathbb{R}^2$, $\triangle^\circ$ be the interior of $\triangle$, $\lambda_1 < \lambda_2$ and $\xi_1 < \xi_2$ such that $[\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \in \triangle^\circ$, $\Phi : \Delta \to \mathbb{R}$ be a differentiable mapping on $\triangle^\circ$, $\Phi_{\varrho\varsigma} \in L([\lambda_1, \lambda_2] \times [\xi_1, \xi_2])$ and $|\Phi_{\varrho\varsigma}|$ be a co-ordinated quasi-convex function on $[\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$. Then*

$$\begin{aligned}
&\Bigg| \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\mu, \nu)\, d\mu\, d\nu \\
&\quad + \frac{\Phi(\lambda_1, \xi_1) + \Phi(\lambda_1, \xi_2) + \Phi(\lambda_2, \xi_1) + \Phi(\lambda_2, \xi_2)}{4} \\
&\quad - \frac{1}{2}\bigg[\frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \big[\Phi(\mu, \xi_1) + \Phi(\mu, \xi_2)\big]\, d\mu \\
&\quad + \frac{1}{\xi_2 - \xi_1} \int_{\lambda_1}^{q_{\lambda_2}} \big[\Phi(\lambda_1, \nu) + \Phi(\lambda_2, \nu)\big]\, d\nu \bigg]\Bigg| \\
&\leq K\bigg[\sup\bigg\{ \big|\Phi_{\varrho\varsigma}(\lambda_1, \xi_1)\big|, \bigg|\Phi_{\varrho\varsigma}\bigg(\lambda_1, \frac{\xi_1 + \xi_2}{2}\bigg)\bigg|, \\
&\quad \bigg|\Phi_{\varrho\varsigma}\bigg(\frac{\lambda_1 + \lambda_2}{2}, \xi_1\bigg)\bigg|, \bigg|\Phi_{\varrho\varsigma}\bigg(\frac{\lambda_1 + \lambda_2}{2}, \frac{\xi_1 + \xi_2}{2}\bigg)\bigg|\bigg\} \\
&\quad + \sup\bigg\{ \big|\Phi_{\varrho\varsigma}(\lambda_1, \xi_2)\big|, \bigg|\Phi_{\varrho\varsigma}\bigg(\lambda_1, \frac{\xi_1 + \xi_2}{2}\bigg)\bigg|, \\
&\quad \bigg|\Phi_{\varrho\varsigma}\bigg(\frac{\lambda_1 + \lambda_2}{2}, \xi_2\bigg)\bigg|, \bigg|\Phi_{\varrho\varsigma}\bigg(\frac{\lambda_1 + \lambda_2}{2}, \frac{\xi_1 + \xi_2}{2}\bigg)\bigg|\bigg\} \\
&\quad + \sup\bigg\{ \big|\Phi_{\varrho\varsigma}(\lambda_2, \xi_1)\big|, \bigg|\Phi_{\varrho\varsigma}\bigg(\lambda_2, \frac{\xi_1 + \xi_2}{2}\bigg)\bigg|, \\
&\quad \bigg|\Phi_{\varrho\varsigma}\bigg(\frac{\lambda_1 + \lambda_2}{2}, \xi_1\bigg)\bigg|, \bigg|\Phi_{\varrho\varsigma}\bigg(\frac{\lambda_1 + \lambda_2}{2}, \frac{\xi_1 + \xi_2}{2}\bigg)\bigg|\bigg\} \\
&\quad + \sup\bigg\{ \big|\Phi_{\varrho\varsigma}(\lambda_2, \xi_2)\big|, \bigg|\Phi_{\varrho\varsigma}\bigg(\lambda_2, \frac{\xi_1 + \xi_2}{2}\bigg)\bigg|, \\
&\quad \bigg|\Phi_{\varrho\varsigma}\bigg(\frac{\lambda_1 + \lambda_2}{2}, \xi_2\bigg)\bigg|, \bigg|\Phi_{\varrho\varsigma}\bigg(\frac{\lambda_1 + \lambda_2}{2}, \frac{\xi_1 + \xi_2}{2}\bigg)\bigg|\bigg\}\bigg], \qquad (1.5)
\end{aligned}$$

where $K = (\lambda_2 - \lambda_1)(\xi_2 - \xi_1)/64$.

More recent results on Hermite–Hadamard type inequalities and their applications can be found in the literature [76–96].

Motivated by Theorems 1.2–1.4 and 1.8, it is natural to ask the question: what are the weighted versions of the Hermite–Hadamard type inequality for the co-ordinated convex and quasi-convex functions?

The main purpose of the article is to present several weighted versions of the Hermite–Hadamard type inequality for the co-ordinated convex and quasi-convex functions, and give an application to the moment of continuous random variables of bivariate distribution functions in the probability theory. Finally, we provide an example on the probability distribution to support our results.

2 Some auxiliary results

First of all, we introduce several symbols as follows.

Let $\omega(\mu,\nu) : [\lambda_1,\lambda_2] \times [\xi_1,\xi_2] \to [0,\infty)$ be a continuous real-valued function such that $\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2} \omega(\mu,\nu)\,d\nu\,d\mu = 1$. Then we denote the integral $\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2} \mu\omega(\mu,\nu)\,d\nu\,d\mu$ by Υ_1, the integral $\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2} \nu\omega(\mu,\nu)\,d\nu\,d\mu$ by Υ_2 and the integral $\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2} \mu\nu\omega(\mu,\nu)\,d\nu\,d\mu$ by Υ_3, that is,

$$\Upsilon_1 = \int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2} \mu\omega(\mu,\nu)\,d\nu\,d\mu, \qquad \Upsilon_2 = \int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2} \mu\nu\omega(\mu,\nu)\,d\nu\,d\mu,$$
$$\Upsilon_3 = \int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2} \nu\omega(\mu,\nu)\,d\mu.$$

We show an outcome in which the function $\omega(\mu,\nu)$ is symmetric on the co-ordinates with respect to $\frac{\lambda_1+\lambda_2}{2}$ and $\frac{\xi_1+\xi_2}{2}$.

Lemma 2.1 *If $\omega(\mu,\nu) : [\lambda_1,\lambda_2] \times [\xi_1,\xi_2] \to [0,\infty)$ is symmetric on the co-ordinates with respect to the midpoints $\frac{\lambda_1+\lambda_2}{2}$ and $\frac{\xi_1+\xi_2}{2}$. Then*

$$\Upsilon_1 = \frac{\lambda_1+\lambda_2}{2}, \qquad \Upsilon_3 = \frac{\xi_1+\xi_2}{2}, \qquad \Upsilon_2 = \left(\frac{\lambda_1+\lambda_2}{2}\right)\left(\frac{\xi_1+\xi_2}{2}\right).$$

Proof It follows from the hypothesis that

$$\begin{aligned}\Upsilon_1 &= \int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2} \mu\omega(\mu,\nu)\,d\nu\,d\mu = \int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2} \mu w(\lambda_1+\lambda_2-\mu,\nu)\,d\nu\,d\mu \\ &= \int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2} (\lambda_1+\lambda_2-\mu)\omega(\mu,\nu)\,d\nu\,d\mu,\end{aligned}$$

which gives the desired result due to

$$\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2} \omega(\mu,\nu)\,d\nu\,d\mu = 1.$$

Similarly, one can prove that

$$\Upsilon_3 = \frac{\xi_1+\xi_2}{2}$$

and

$$\Upsilon_3 = \int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2} \mu\nu\omega(\mu,\nu)\,d\nu\,d\mu = \left(\frac{\lambda_1+\lambda_2}{2}\right)\left(\frac{\xi_1+\xi_2}{2}\right). \qquad \square$$

Lemma 2.2 *Let $\Omega \subseteq \mathbb{R}^2$, Ω° be the interior of Ω, $\lambda_1 < \lambda_2$ and $\xi_1 < \xi_2$ such that $[\lambda_1,\lambda_2] \times [\xi_1,\xi_2] \subseteq \Omega^\circ$, $\omega : [\lambda_1,\lambda_2] \times [\xi_1,\xi_2] \to [0,\infty)$ be a continuous mapping, and $\Phi : \Omega \to \mathbb{R}$ be a twice partially differentiable mapping on Ω° such that $\Phi_{\varrho\varsigma} \in L([\lambda_1,\lambda_2] \times [\xi_1,\xi_2])$. Then*

$$\begin{aligned}&\frac{1}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\Big[\Phi(\lambda_1,\xi_1)\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2} (\lambda_2-\mu)(\xi_2-\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\ &\quad + \Phi(\lambda_1,\xi_2)\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2} (\lambda_2-\mu)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu\end{aligned}$$

$$
\begin{aligned}
&+ \Phi(\lambda_2,\xi_1)\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\mu-\lambda_1)(\xi_2-\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&+ \Phi(\lambda_2,\xi_2)\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu\Bigg] \\
&+ \int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\mu,\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&- \frac{1}{\xi_2-\xi_1}\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\lambda_2-\mu)\Phi(\lambda_1,\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&- \frac{1}{\xi_2-\xi_1}\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\mu-\lambda_1)\Phi(\lambda_2,\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&- \frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\xi_2-\nu)\Phi(\mu,\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu \\
&- \frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\nu-\xi_1)\Phi(\mu,\xi_2)\omega(\mu,\nu)\,d\nu\,d\mu \\
={}& \frac{(\Upsilon_1-\lambda_1)(\Upsilon_3-\xi_1)}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\int_0^1\int_0^1 H(\omega,\lambda_1,\xi_1,\Upsilon_1,\Upsilon_3;r,\rho) \\
&\times \Phi_{r\rho}\big((1-r)\lambda_1+\Upsilon_1 r,(1-\rho)\xi_1+\Upsilon_3\rho\big)\,d\rho\,dr \\
&+ \frac{(\lambda_2-\Upsilon_1)(\Upsilon_3-\xi_1)}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\int_0^1\int_0^1 H(\omega,\Upsilon_1,\xi_1,\lambda_2,\Upsilon_3;r,\rho) \\
&\times \Phi_{r\rho}\big((1-r)\Upsilon_1+\lambda_2 r,(1-\rho)\xi_1+\Upsilon_3\rho\big)\,d\rho\,dr \\
&+ \frac{(\Upsilon_1-\lambda_1)(\xi_2-\Upsilon_3)}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\int_0^1\int_0^1 H(\omega,\lambda_1,\Upsilon_3,\Upsilon_1,\xi_2;r,\rho) \\
&\times \Phi_{r\rho}\big((1-r)\lambda_1+\Upsilon_1 r,(1-\rho)\Upsilon_3+\xi_2\rho\big)\,d\rho\,dr \\
&+ \frac{(\lambda_2-\Upsilon_1)(\xi_2-\Upsilon_3)}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\int_0^1\int_0^1 H(\omega,\Upsilon_1,\Upsilon_3,\lambda_2,\xi_2;r,\rho) \\
&\times \Phi_{r\rho}\big((1-r)\Upsilon_1+\lambda_2 r,(1-\rho)\Upsilon_3+\xi_2\rho\big)\,d\rho\,dr,
\end{aligned}
\tag{2.1}
$$

where

$$
\begin{aligned}
&H(w,\Upsilon_1,\gamma,\beta,\epsilon;r,\rho) \\
&\quad= \int_{(1-r)\alpha+\beta r}^{\lambda_2}\int_{(1-\rho)\gamma+\epsilon\rho}^{\xi_2}(\lambda_2-\mu)(\xi_2-\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\qquad- \int_{(1-r)\alpha+\beta r}^{\lambda_2}\int_{\xi_1}^{(1-z)\gamma+\epsilon\rho}(\lambda_2-\mu)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\qquad- \int_{\lambda_1}^{(1-r)\alpha+\beta r}\int_{(1-\rho)\gamma+\epsilon\rho}^{\xi_2}(\mu-\lambda_1)(\xi_2-\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\qquad+ \int_{\lambda_1}^{(1-r)\alpha+\beta r}\int_{\xi_1}^{(1-\rho)\gamma+\epsilon\rho}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu
\end{aligned}
$$

and $(\alpha,\gamma),(\beta,\epsilon)\in[\lambda_1,\lambda_2]\times[\xi_1,\xi_2]$.

Proof Let

$$\sigma(\delta)=\begin{cases}0, & \delta<0,\\ 1, & \delta>0.\end{cases}$$

Then we clearly see that

$$\begin{aligned}
&\Phi(\mu,\nu)-\Phi(\lambda_1,\nu)-\Phi(\mu,\xi_1)+\Phi(\lambda_1,\xi_1)\\
&\quad=\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\sigma(\mu-\varrho)\sigma(\nu-\varsigma)\Phi_{\varrho\varsigma}(\varrho,\varsigma)\,d\varsigma\,d\varrho, \qquad (2.2)\\
&\Phi(\mu,\nu)-\Phi(\lambda_1,\nu)-\Phi(\mu,\xi_2)+\Phi(\lambda_1,\xi_2)\\
&\quad=-\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\sigma(\mu-\varrho)\sigma(\varsigma-\nu)\Phi_{\varrho\varsigma}(\varrho,\varsigma)\,d\varsigma\,d\varrho, \qquad (2.3)\\
&\Phi(\mu,\nu)-\Phi(\lambda_2,\nu)-\Phi(\mu,\xi_1)+\Phi(\lambda_2,\xi_1)\\
&\quad=-\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\sigma(\varrho-\mu)\sigma(\nu-\varsigma)\Phi_{\varrho\varsigma}(\varrho,\varsigma)\,d\varsigma\,d\varrho \qquad (2.4)
\end{aligned}$$

and

$$\begin{aligned}
&\Phi(\mu,\nu)-\Phi(\lambda_2,\nu)-\Phi(\mu,\xi_2)+\Phi(\lambda_2,\xi_2)\\
&\quad=\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\sigma(\varrho-\mu)\sigma(\varsigma-\nu)\Phi_{\varrho\varsigma}(\varrho,\varsigma)\,d\varrho\,d\varsigma. \qquad (2.5)
\end{aligned}$$

It follows from (2.2) that

$$\begin{aligned}
&\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\lambda_2-\mu)(\xi_2-\nu)\Phi(\mu,\nu)\omega(\mu,\nu)\,d\nu\,d\mu\\
&\qquad-\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\lambda_2-\mu)(\xi_2-\nu)\Phi(\lambda_1,\nu)\omega(\mu,\nu)\,d\nu\,d\mu\\
&\qquad-\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\lambda_2-\mu)(\xi_2-\nu)\Phi(\mu,\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu\\
&\qquad+\Phi(\lambda_1,\xi_1)\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\lambda_2-\mu)(\xi_2-\nu)\omega(\mu,\nu)\,d\nu\,d\mu\\
&\quad=\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\lambda_2-\mu)(\xi_2-\nu)\left(\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\sigma(\mu-\varrho)\sigma(\nu-\varsigma)\Phi_{\varrho\varsigma}(\varrho,\varsigma)\,d\varsigma\,d\varrho\right)\\
&\qquad\times\omega(\mu,\nu)\,d\nu\,d\mu\\
&\quad=\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\left(\int_{\varrho}^{\lambda_2}\int_{\varsigma}^{\xi_2}(\lambda_2-\mu)(\xi_2-\nu)\omega(\mu,\nu)\,d\nu\,d\mu\right)\Phi_{\varrho\varsigma}(\varrho,\varsigma)\,d\varsigma\,d\varrho. \qquad (2.6)
\end{aligned}$$

Similarly, from (2.3)–(2.5) we have

$$\begin{aligned}
&\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\lambda_2-\mu)(\nu-\xi_1)\Phi(\mu,\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad -\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\lambda_2-\mu)(\nu-\xi_1)\Phi(\lambda_1,\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad -\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\lambda_2-\mu)(\nu-\xi_1)\Phi(\mu,\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad +\Phi(\lambda_1,\xi_2)\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\lambda_2-\mu)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\ =-\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\left(\int_{\varrho}^{\lambda_2}\int_{\xi_1}^{\varsigma}(\lambda_2-\mu)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu\right)\Phi_{\varrho\varsigma}(\varrho,\varsigma)\,d\varsigma\,d\varrho,
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
&\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\mu-\lambda_1)(\xi_2-\nu)\Phi(\mu,\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad -\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\mu-\lambda_1)(\xi_2-\nu)\Phi(\lambda_2,\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad -\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\mu-\lambda_1)(\xi_2-\nu)\Phi(\mu,\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad +\Phi(\lambda_2,\xi_1)\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\mu-\lambda_1)(\xi_2-\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\ =-\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\left(\int_{\lambda_1}^{\varrho}\int_{\varsigma}^{\xi_2}(\mu-\lambda_1)(\xi_2-\nu)\omega(\mu,\nu)\,d\nu\,d\mu\right)\Phi_{\varrho\varsigma}(\varrho,\varsigma)\,d\varsigma\,d\varrho
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
&\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\mu-\lambda_1)(\nu-\xi_1)\Phi(\mu,\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad -\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\mu-\lambda_1)(\nu-\xi_1)\Phi(\lambda_2,\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad -\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\mu-\lambda_1)(\nu-\xi_1)\Phi(\mu,\xi_2)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad +\Phi(\lambda_2,\xi_2)\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\ =\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\left(\int_{\lambda_1}^{\varrho}\int_{\xi_1}^{\varsigma}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu\right)\Phi_{\varrho\varsigma}(\varrho,\varsigma)\,d\varsigma\,d\varrho.
\end{aligned} \tag{2.9}$$

From (2.6)–(2.9), we get

$$\begin{aligned}
&\frac{1}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\Bigg[\Phi(\lambda_1,\xi_1)\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\lambda_2-\mu)(\xi_2-\nu)w(\mu,\nu)\,d\nu\,d\mu \\
&\quad +\Phi(\lambda_1,\xi_2)\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\lambda_2-\mu)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad +\Phi(\lambda_2,\xi_1)\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\mu-\lambda_1)(\xi_2-\nu)\omega(\mu,\nu)\,d\nu\,d\mu
\end{aligned}$$

$$
\begin{aligned}
&+ \Phi(\lambda_2,\xi_2)\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu\Big] \\
&- \frac{1}{\xi_2-\xi_1}\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\xi_2-\nu)\Phi(\mu,\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu \\
&- \frac{1}{\xi_2-\xi_1}\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\nu-\xi_1)\Phi(\mu,\xi_2)\omega(\mu,\nu)\,d\nu\,d\mu \\
&- \frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\lambda_2-\mu)\Phi(\lambda_1,\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&- \frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\mu-\lambda_1)\Phi(\lambda_1,\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&+ \int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\mu,\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
={}& \frac{1}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Big[\int_{\varrho}^{\lambda_2}\int_{\varsigma}^{\xi_2}(\lambda_2-\mu)(\xi_2-\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&- \int_{\varrho}^{\lambda_2}\int_{\xi_1}^{\varsigma}(\lambda_2-\mu)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu \\
&- \int_{\lambda_1}^{\varrho}\int_{\varsigma}^{\xi_2}(\mu-\lambda_1)(\xi_2-\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&+ \int_{\lambda_1}^{\varrho}\int_{\xi_1}^{\varsigma}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu\Big]\Phi_{\varrho\varsigma}(\varrho,\varsigma)\,d\varsigma\,d\varrho
\end{aligned}
\tag{2.10}
$$

and

$$
\begin{aligned}
&\frac{1}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Big[\int_{\varrho}^{\lambda_2}\int_{\varsigma}^{\xi_2}(\lambda_2-\mu)(\xi_2-\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad- \int_{\varrho}^{\lambda_2}\int_{\xi_1}^{\varsigma}(\lambda_2-\mu)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad- \int_{\lambda_1}^{\varrho}\int_{\varsigma}^{\xi_2}(\mu-\lambda_1)(\xi_2-\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad+ \int_{\lambda_1}^{\varrho}\int_{\xi_1}^{\varsigma}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu\Big]\Phi_{\varrho\varsigma}(\varrho,\varsigma)\,d\varsigma\,d\varrho \\
&= \frac{1}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\int_{\lambda_1}^{\Upsilon_1}\int_{\xi_1}^{\Upsilon_3}\Big[\int_{\varrho}^{\lambda_2}\int_{\varsigma}^{\xi_2}(\lambda_2-\mu)(\xi_2-\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad- \int_{\lambda_1}^{\varrho}\int_{\varsigma}^{\xi_2}(\mu-\lambda_1)(\xi_2-\nu)\omega(\mu,\nu)\,d\nu\,d\mu - \int_{\varrho}^{\lambda_2}\int_{\xi_1}^{\varsigma}(\lambda_2-\mu)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad+ \int_{\lambda_1}^{\varrho}\int_{\xi_1}^{\varsigma}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu\Big]\Phi_{\varrho\varsigma}(\varrho,\varsigma)\,d\varsigma\,d\varrho \\
&\quad+ \frac{1}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\int_{\Upsilon_1}^{\lambda_2}\int_{\xi_1}^{\Upsilon_3}\Big[\int_{\varrho}^{\lambda_2}\int_{\varsigma}^{\xi_2}(\lambda_2-\mu)(\xi_2-\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad- \int_{\lambda_1}^{\varrho}\int_{\varsigma}^{\xi_2}(\mu-\lambda_1)(\xi_2-\nu)\omega(\mu,\nu)\,d\nu\,d\mu - \int_{\varrho}^{\lambda_2}\int_{\xi_1}^{\varsigma}(\lambda_2-\mu)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\lambda_1}^{\varrho}\int_{\xi_1}^{\varsigma}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu\Bigg]\Phi_{\varrho\varsigma}(\varrho,\varsigma)\,d\varsigma\,d\varrho\\
&+\frac{1}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\int_{\lambda_1}^{\Upsilon_1}\int_{\Upsilon_3}^{\xi_2}\Bigg[\int_{\varrho}^{\lambda_2}\int_{\varsigma}^{\xi_2}(\lambda_2-\mu)(\xi_2-\nu)\omega(\mu,\nu)\,d\nu\,d\mu\\
&-\int_{\lambda_1}^{\varrho}\int_{\varsigma}^{\xi_2}(\mu-\lambda_1)(\xi_2-\nu)\omega(\mu,\nu)\,d\nu\,d\mu-\int_{\varrho}^{\lambda_2}\int_{\xi_1}^{\varsigma}(\lambda_2-\mu)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu\\
&+\int_{\lambda_1}^{\varrho}\int_{\xi_1}^{\varsigma}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu\Bigg]\Phi_{\varrho\varsigma}(\varrho,\varsigma)\,d\varsigma\,d\varrho\\
&+\frac{1}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\int_{\Upsilon_1}^{\lambda_2}\int_{\Upsilon_3}^{\xi_2}\Bigg[\int_{\varrho}^{\lambda_2}\int_{\varsigma}^{\xi_2}(\lambda_2-\mu)(\xi_2-\nu)\omega(\mu,\nu)\,d\nu\,d\mu\\
&-\int_{\lambda_1}^{\varrho}\int_{\varsigma}^{\xi_2}(\mu-\lambda_1)(\xi_2-\nu)\omega(\mu,\nu)\,d\nu\,d\mu-\int_{\varrho}^{\lambda_2}\int_{\xi_1}^{\varsigma}(\lambda_2-\mu)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu\\
&+\int_{\lambda_1}^{\varrho}\int_{\xi_1}^{\varsigma}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu\Bigg]\Phi_{\varrho\varsigma}(\varrho,\varsigma)\,d\varsigma\,d\varrho.
\end{aligned}
\tag{2.11}
$$

Therefore, inequality (2.1) follows from (2.10) and (2.11). □

Remark 2.3 Let $\omega(\mu,\nu)=\frac{1}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}$. Then (2.1) reduces to

$$
\begin{aligned}
&\frac{\Phi(\lambda_1,\xi_1)+\Phi(\lambda_1,\xi_2)+\Phi(\lambda_2,\xi_1)+\Phi(\lambda_2,\xi_2)}{4}\\
&\quad+\frac{1}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\mu,\nu)\,d\nu\,d\mu\\
&\quad-\frac{1}{2(\lambda_2-\lambda_1)}\int_{\lambda_1}^{\lambda_2}\big[\Phi(\mu,\xi_1)+\Phi(\mu,\xi_2)\big]\,d\mu\\
&\quad-\frac{1}{2(\xi_2-\xi_1)}\int_{\lambda_1}^{\lambda_2}\big[\Phi(\lambda_1,\nu)+\Phi(\lambda_2,\nu)\big]\,d\mu\\
&=\frac{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}{16}\Bigg[\int_0^1\int_0^1\varrho\varsigma\Phi_{\varrho\varsigma}\bigg(\frac{1-\varrho}{2}\lambda_1+\frac{1+\varrho}{2}\lambda_2,\frac{1-\varsigma}{2}\xi_1+\frac{1+\varsigma}{2}\xi_2\bigg)\,d\varsigma\,d\varrho\\
&\quad+\int_0^1\int_0^1(-\varrho)\varsigma\Phi_{\varsigma\varrho}\bigg(\frac{1+\varrho}{2}\lambda_1+\frac{1-\varrho}{2}\lambda_2,\frac{1-\varsigma}{2}\xi_1+\frac{1+\varsigma}{2}\xi_2\bigg)\,d\varsigma\,d\varrho\\
&\quad+\int_0^1\int_0^1\varrho(-\varsigma)\Phi_{\varrho\varsigma}\bigg(\frac{1-\varrho}{2}\lambda_1+\frac{1+\varrho}{2}\lambda_2,\frac{1+\varsigma}{2}\xi_1+\frac{1-\varsigma}{2}\xi_2\bigg)\,d\varsigma\,d\varrho\\
&\quad+\int_0^1\int_0^1(-\varsigma)(-\varrho)\Phi_{\varsigma\varrho}\bigg(\frac{1+\varrho}{2}\lambda_1+\frac{1-\varrho}{2}\lambda_2,\frac{1+\varsigma}{2}\xi_1+\frac{1-\varsigma}{2}\xi_2\bigg)\,d\varsigma\,d\varrho\Bigg].
\end{aligned}
\tag{2.12}
$$

The identity (2.12) was established in [75].

Corollary 2.4 *If the function $\omega(\mu,\nu)$ is symmetric with respect to $\frac{\lambda_1+\lambda_2}{2}$ and $\frac{\xi_1+\xi_2}{2}$ on the co-ordinates. Then Lemma 2.2 leads to*

$$
\begin{aligned}
&\frac{\Phi(\lambda_1,\xi_1)+\Phi(\lambda_1,\xi_2)+\Phi(\lambda_2,\xi_1)+\Phi(\lambda_2,\xi_2)}{4}+\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\mu,\nu)\omega(\mu,\nu)\,d\nu\,d\mu\\
&\quad-\frac{1}{\xi_2-\xi_1}\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\xi_2-\nu)\Phi(\mu,\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{\xi_2 - \xi_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\nu - \xi_1) \Phi(\mu, \xi_2) \omega(\mu, \nu) \, d\nu \, d\mu \\
&- \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\lambda_2 - \mu) \Phi(\lambda_2, \nu) \omega(\mu, \nu) \, d\nu \, d\mu \\
&- \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\mu - \lambda_1) \Phi(\lambda_2, \nu) \omega(\mu, \nu) \, d\nu \, d\mu \\
= &\int_0^1 \int_0^1 H(\omega, \lambda_1, \xi_1, \lambda_2, \xi_2; \varrho, \varsigma) \Phi_{\varrho\varsigma}\big(\lambda_1 \varrho + (1 - \varrho)\lambda_2, \xi_1 \varsigma + (1 - \varsigma)\xi_2\big) \, d\varsigma \, d\varrho. \qquad (2.13)
\end{aligned}
$$

Lemma 2.5 *Let $\mu \in [\lambda_1, \lambda_2]$, $\nu \in [\xi_1, \xi_2]$, $A : C([\lambda_1, \lambda_2] \times [\xi_1, \xi_2]) \to \mathbb{R}$ be a positive linear functional, $e_i(\mu) = \mu^i$ and $k_j(\nu) = \nu^j$ be the monomials ($i, j \in \mathbb{N}$), and g be a co-ordinated convex function on $[\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$. Then*

$$
\begin{aligned}
&A\big(g(e_1, k_1)\big) \\
&\leq \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \big[A\big((\lambda_2 - e_1)(\xi_2 - k_1)\big) g(\lambda_1, \xi_1) + A\big((\lambda_2 - e_1)(k_1 - \xi_1)\big) g(\lambda_1, \xi_2) \\
&\quad + A\big((e_1 - \lambda_1)(\xi_2 - k_1)\big) g(\lambda_2, \xi_1) + A\big((e_1 - \lambda_1)(k_1 - \xi_1)\big) g(\lambda_2, \xi_2)\big]. \qquad (2.14)
\end{aligned}
$$

Proof It follows from the convexity of g on the co-ordinates that

$$
\begin{aligned}
g(e_1, k_1) &\leq \frac{(\lambda_2 - e_1) g(\lambda_1, k_1) + (e_1 - \lambda_1) g(\lambda_2, k_1)}{\lambda_2 - \lambda_1} \\
&\leq \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \big[(\lambda_2 - e_1)(\xi_2 - k_1) g(\lambda_1, \xi_1) + (\lambda_2 - e_1)(k_1 - \xi_1) g(\lambda_1, \xi_2) \\
&\quad + (e_1 - \lambda_1)(\xi_2 - k_1) g(\lambda_2, \xi_1) + (e_1 - \lambda_1)(k_1 - \xi_1) g(\lambda_2, \xi_2)\big]. \qquad (2.15)
\end{aligned}
$$

Therefore, inequality (2.14) follows from (2.15) and the assumption of the functional A. □

3 Main results

In order to provide compact demonstration, we use the notations for our coming results as follows:

$$
\begin{aligned}
&B_1(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
&= \int_{\lambda_1}^{\Upsilon_1} \int_{\xi_1}^{\Upsilon_3} (\Upsilon_1 - \mu)^2 (\Upsilon_3 - \nu)^2 \omega(\mu, \nu) \, d\nu \, d\mu \\
&\quad - \int_{\lambda_1}^{\Upsilon_1} \int_{\Upsilon_3}^{\lambda_2} (\Upsilon_1 - \mu)^2 (\Upsilon_3 - \nu)^2 \omega(\mu, \nu) \, d\nu \, d\mu \\
&\quad - \int_{\Upsilon_1}^{\lambda_2} \int_{\xi_1}^{\Upsilon_3} (\Upsilon_1 - \mu)^2 (\Upsilon_3 - \nu)^2 \omega(\mu, \nu) \, d\nu \, d\mu \\
&\quad + \int_{\Upsilon_1}^{\lambda_2} \int_{\Upsilon_3}^{\xi_2} (\Upsilon_1 - \mu)^2 (\Upsilon_3 - \nu)^2 \omega(\mu, \nu) \, d\nu \, d\mu, \qquad (3.1)
\end{aligned}
$$

$B_2(\lambda_1, \xi_1, \lambda_2, \xi_2)$

$$
\begin{aligned}
&= \int_{\lambda_1}^{\Upsilon_1}\int_{\xi_1}^{\Upsilon_3}(\Upsilon_1-\mu)(\Upsilon_3-\nu)^2\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad - \int_{\lambda_1}^{\Upsilon_1}\int_{\Upsilon_3}^{\xi_2}(\Upsilon_1-\mu)(\Upsilon_3-\nu)^2\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad - \int_{\Upsilon_1}^{\lambda_2}\int_{\xi_1}^{\Upsilon_3}(\Upsilon_1-\mu)(\Upsilon_3-\nu)^2\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad + \int_{\Upsilon_1}^{\lambda_2}\int_{\Upsilon_3}^{\xi_2}(\Upsilon_1-\mu)(\Upsilon_3-\nu)^2\omega(\mu,\nu)\,d\nu\,d\mu, \qquad (3.2)
\end{aligned}
$$

$$
\begin{aligned}
&B_3(\lambda_1,\xi_1,\lambda_2,\xi_2) \\
&= \int_{\lambda_1}^{\Upsilon_1}\int_{\xi_1}^{\Upsilon_3}(\Upsilon_1-\mu)^2(\Upsilon_3-\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad - \int_{\lambda_1}^{\Upsilon_1}\int_{\Upsilon_3}^{\xi_2}(\Upsilon_1-\mu)^2(\Upsilon_3-\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad - \int_{\Upsilon_1}^{\lambda_2}\int_{\xi_1}^{\Upsilon_3}(\Upsilon_1-\mu)^2(\Upsilon_3-\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad + \int_{\Upsilon_1}^{\lambda_2}\int_{\Upsilon_3}^{\xi_2}(\Upsilon_1-\mu)^2(\Upsilon_3-\nu)\omega(\mu,\nu)\,d\nu\,d\mu \qquad (3.3)
\end{aligned}
$$

and

$$
\begin{aligned}
&B_4(\lambda_1,\xi_1,\lambda_2,\xi_2) \\
&= \int_{\lambda_1}^{\Upsilon_1}\int_{\xi_1}^{\Upsilon_3}(\Upsilon_1-\mu)(\Upsilon_3-\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad - \int_{\lambda_1}^{\Upsilon_1}\int_{\Upsilon_3}^{\xi_2}(\Upsilon_1-\mu)(\Upsilon_3-\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad - \int_{\Upsilon_1}^{\lambda_2}\int_{\xi_1}^{\Upsilon_3}(\Upsilon_1-\mu)(\Upsilon_3-\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad + \int_{\Upsilon_1}^{\lambda_2}\int_{\Upsilon_3}^{\xi_2}(\Upsilon_1-\mu)(\Upsilon_3-\nu)\omega(\mu,\nu)\,d\nu\,d\mu. \qquad (3.4)
\end{aligned}
$$

Theorem 3.1 *Let $\Omega \subseteq \mathbb{R}^2$, Ω° be the interior of Ω, $\lambda_1 < \lambda_2$ and $\xi_1 < \xi_2$ such that $[\lambda_1,\lambda_2] \times [\xi_1,\xi_2] \subseteq \Omega^\circ$, $\Phi : \Omega \to \mathbb{R}$ be a twice partially differentiable mapping on Ω° such that $\Phi_{\varrho\varsigma} \in L([\lambda_1,\lambda_2]\times[\xi_1,\xi_2])$ and $|\Phi_{\varrho\varsigma}|$ is co-ordinated convex on $[\lambda_1,\lambda_2]\times[\xi_1,\xi_2]$, and $\omega : [\lambda_1,\lambda_2]\times[\xi_1,\xi_2] \to [0,\infty)$ be a continuous mapping. Then one has*

$$
\begin{aligned}
&\biggl|\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\mu,\nu)\omega(\mu,\nu)\,d\nu\,d\mu - \int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\Upsilon_1,\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad - \int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\mu,\Upsilon_3)\omega(\mu,\nu)\,d\nu\,d\mu + \Phi(\Upsilon_1,\Upsilon_3)\biggr| \\
&\le \frac{1}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\bigl[\bigl|\Phi_{\varrho\varsigma}(\lambda_1,\xi_1)\bigr|A_1(\lambda_1,\xi_1,\lambda_2,\xi_2) + \bigl|\Phi_{\varrho\varsigma}(\lambda_1,\xi_2)\bigr|A_2(\lambda_1,\xi_1,\lambda_2,\xi_2) \\
&\quad + \bigl|\Phi_{\varrho\varsigma}(\lambda_2,\xi_1)\bigr|A_3(\lambda_1,\xi_1,\lambda_2,\xi_2) + \bigl|\Phi_{\varrho\varsigma}(\lambda_2,\xi_2)\bigr|A_4(\lambda_1,\xi_1,\lambda_2,\xi_2)\bigr], \qquad (3.5)
\end{aligned}
$$

where

$$\begin{aligned}
&A_1(\lambda_1,\xi_1,\lambda_2,\xi_2)\\
&\quad=\frac{1}{4}B_1(\lambda_1,\xi_1,\lambda_2,\xi_2)+\frac{(\lambda_2-\Upsilon_1)}{2}B_2(\lambda_1,\xi_1,\lambda_2,\xi_2)\\
&\qquad+\frac{(\xi_2-\Upsilon_3)}{2}B_3(\lambda_1,\xi_1,\lambda_2,\xi_2)+(\lambda_2-\Upsilon_1)(\xi_2-\Upsilon_3)B_4(\lambda_1,\xi_1,\lambda_2,\xi_2),\\
&A_2(\lambda_1,\xi_1,\lambda_2,\xi_2)\\
&\quad=\frac{1}{4}B_1(\lambda_1,\xi_1,\lambda_2,\xi_2)+\frac{(\lambda_2-\Upsilon_1)}{2}B_2(\lambda_1,\xi_1,\lambda_2,\xi_2)\\
&\qquad+\frac{(\Upsilon_3-\xi_1)}{2}B_3(\lambda_1,\xi_1,\lambda_2,\xi_2)+(\lambda_2-\Upsilon_1)(\Upsilon_3-\xi_1)B_4(\lambda_1,\xi_1,\lambda_2,\xi_2),\\
&A_3(\lambda_1,\xi_1,\lambda_2,\xi_2)\\
&\quad=\frac{1}{4}B_1(\lambda_1,\xi_1,\lambda_2,\xi_2)+\frac{(\Upsilon_1-\lambda_1)}{2}B_2(\lambda_1,\xi_1,\lambda_2,\xi_2)\\
&\qquad+\frac{(\xi_2-\Upsilon_3)}{2}B_3(\lambda_1,\xi_1,\lambda_2,\xi_2)+(\Upsilon_1-\lambda_1)(\xi_2-\Upsilon_3)B_4(\lambda_1,\xi_1,\lambda_2,\xi_2)
\end{aligned}$$

and

$$\begin{aligned}
&A_4(\lambda_1,\xi_1,\lambda_2,\xi_2)\\
&\quad=\frac{1}{4}B_1(\lambda_1,\xi_1,\lambda_2,\xi_2)+\frac{(\Upsilon_1-\lambda_1)}{2}B_2(\lambda_1,\xi_1,\lambda_2,\xi_2)\\
&\qquad+\frac{(\Upsilon_3-\xi_1)}{2}B_3(\lambda_1,\xi_1,\lambda_2,\xi_2)+(\Upsilon_1-\lambda_1)(\Upsilon_3-\xi_1)B_4(\lambda_1,\xi_1,\lambda_2,\xi_2).
\end{aligned}$$

Proof We clearly see that

$$\begin{aligned}
&\Phi(\mu,\nu)-\Phi(\Upsilon_1,\nu)-\Phi(\mu,\Upsilon_3)+\Phi(\Upsilon_1,\Upsilon_3)\\
&\quad=\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\bigl[\sigma(\mu-\varrho)\sigma(\nu-\varsigma)-\sigma(\Upsilon_1-\varrho)\sigma(\nu-\varsigma)\\
&\qquad-\sigma(\mu-\varrho)\sigma(\Upsilon_3-\varsigma)+\sigma(\Upsilon_1-\varrho)\sigma(\Upsilon_3-\varsigma)\bigr]\Phi_{\varrho\varsigma}(\varrho,\varsigma)\,d\varsigma\,d\varrho.
\end{aligned}\tag{3.6}$$

It follows from (3.6) that

$$\begin{aligned}
&\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\mu,\nu)\omega(\mu,\nu)\,d\nu\,d\mu-\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\Upsilon_1,\nu)\omega(\mu,\nu)\,d\nu\,d\mu\\
&\qquad-\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\mu,\Upsilon_3)\omega(\mu,\nu)\,d\nu\,d\mu+\Phi(\Upsilon_1,\Upsilon_3)\\
&\quad=\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\biggl(\int_{\varrho}^{\lambda_2}\int_{\varsigma}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu-\sigma(\Upsilon_1-\varrho)\int_{\lambda_1}^{\lambda_2}\int_{\varsigma}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu\\
&\qquad-\sigma(\Upsilon_3-\varsigma)\int_{\varrho}^{\lambda_2}\int_{\xi_1}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu+\sigma(\Upsilon_1-\varrho)\sigma(\Upsilon_3-\varsigma)\biggr)\\
&\qquad\times\Phi_{\varrho\varsigma}(\varrho,\varsigma)\,d\varsigma\,d\varrho.
\end{aligned}\tag{3.7}$$

From Lemma 2.5 and (3.7) we get

$$\begin{aligned}
&\left|\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\mu,\nu)\omega(\mu,\nu)\,d\nu\,d\mu-\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\Upsilon_1,\nu)\omega(\mu,\nu)\,d\nu\,d\mu\right.\\
&\quad\left.-\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\mu,\Upsilon_3)\omega(\mu,\nu)\,d\nu\,d\mu+\Phi(\Upsilon_1,\Upsilon_3)\right|\\
&\quad\leq\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\left|\int_{\varrho}^{\lambda_2}\int_{\varsigma}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu-\sigma(\Upsilon_1-\varrho)\int_{\lambda_1}^{\lambda_2}\int_{\varsigma}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu\right.\\
&\quad\left.-\sigma(\Upsilon_3-\varsigma)\int_{\varrho}^{\lambda_2}\int_{\xi_1}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu+\sigma(\Upsilon_1-\varrho)\sigma(\Upsilon_3-\varsigma)\right|\left|\Phi_{\varrho\varsigma}(\varrho,\varsigma)\right|ds\,dt\\
&\quad\leq\frac{1}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\left[\left|\Phi_{\varrho\varsigma}(\lambda_1,\lambda_2)\right|\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\left|\int_{\varrho}^{\lambda_2}\int_{\varsigma}^{d}\omega(\mu,\nu)\,d\nu\,d\mu\right.\right.\\
&\quad-\sigma(\Upsilon_1-\varrho)\int_{\lambda_1}^{\lambda_2}\int_{\varsigma}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu-\sigma(\Upsilon_3-\varsigma)\int_{\varrho}^{\lambda_2}\int_{\xi_1}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu\\
&\quad\left.+\sigma(\Upsilon_1-\varrho)\sigma(\Upsilon_3-\varsigma)\right|(\lambda_2-\varrho)(\xi_2-\varsigma)\,d\varsigma\,d\varrho\\
&\quad+\left|\Phi_{\varrho\varsigma}(\lambda_1,\xi_2)\right|\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\left|\int_{\varrho}^{\lambda_2}\int_{\varsigma}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu\right.\\
&\quad-\sigma(\Upsilon_1-\varrho)\int_{\lambda_1}^{\lambda_2}\int_{\varsigma}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu-\sigma(\Upsilon_3-\varsigma)\int_{\varrho}^{\lambda_2}\int_{\xi_1}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu\\
&\quad\left.+\sigma(\Upsilon_1-\varrho)\sigma(\Upsilon_3-\varsigma)\right|(\lambda_2-\varrho)(\varsigma-\xi_1)\,d\varsigma\,d\varrho\\
&\quad+\left|\Phi_{\varrho\varsigma}(\lambda_2,\xi_1)\right|\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\left|\int_{\varrho}^{\lambda_2}\int_{\varsigma}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu\right.\\
&\quad-\sigma(\Upsilon_1-\varrho)\int_{\lambda_1}^{\lambda_2}\int_{\varsigma}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu-\sigma(\Upsilon_3-\varsigma)\int_{\varrho}^{\lambda_2}\int_{\xi_1}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu\\
&\quad\left.+\sigma(\Upsilon_1-\varrho)\sigma(\Upsilon_3-\varsigma)\right|(\varrho-\lambda_1)(\xi_2-\varsigma)\,d\varsigma\,d\varrho\\
&\quad+\left|\Phi_{\varrho\varsigma}(\lambda_2,\xi_2)\right|\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\left|\int_{\varrho}^{\lambda_2}\int_{\varsigma}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu\right.\\
&\quad-\sigma(\Upsilon_1-\varrho)\int_{\lambda_1}^{\lambda_2}\int_{\varsigma}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu-\sigma(\Upsilon_3-\varsigma)\int_{\varrho}^{\lambda_2}\int_{\xi_1}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu\\
&\quad\left.\left.+\sigma(\Upsilon_1-\varrho)\sigma(\Upsilon_3-\varsigma)\right|(\varrho-\lambda_1)(\varsigma-\xi_1)\,d\varsigma\,d\varrho\right]
\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}
&\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\left|\int_{\varrho}^{\lambda_2}\int_{\varsigma}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu-\sigma(\Upsilon_1-\varrho)\int_{\lambda_1}^{\lambda_2}\int_{\varsigma}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu\right.\\
&\quad\left.-\sigma(\Upsilon_3-\varsigma)\int_{\varrho}^{\lambda_2}\int_{\xi_1}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu+\sigma(\Upsilon_1-\varrho)\sigma(\Upsilon_3-\varsigma)\right|(\lambda_2-\varrho)(\xi_2-\varsigma)\,d\varsigma\,d\varrho
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\lambda_1}^{\Upsilon_1}\int_{\xi_1}^{\Upsilon_3}\left(\int_{\lambda_1}^{\varrho}\int_{\xi_1}^{\varsigma}\omega(\mu,\nu)\,d\nu\,d\mu\right)(\lambda_2-\varrho)(\xi_2-\varsigma)\,d\varsigma\,d\varrho \\
&\quad + \int_{\Upsilon_1}^{\lambda_2}\int_{\xi_1}^{\Upsilon_3}\left(\int_{\varrho}^{\lambda_2}\int_{\xi_1}^{\varsigma}\omega(\mu,\nu)\,d\nu\,d\mu\right)(\lambda_2-\varrho)(\xi_2-\varsigma)\,d\varsigma\,d\varrho \\
&\quad + \int_{\lambda_1}^{\Upsilon_1}\int_{\Upsilon_3}^{\xi_2}\left(\int_{\lambda_1}^{\varrho}\int_{\varsigma}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu\right)(\lambda_2-\varrho)(\xi_2-\varsigma)\,d\varsigma\,d\varrho \\
&\quad + \int_{\Upsilon_1}^{\lambda_2}\int_{\Upsilon_3}^{\xi_2}\left(\int_{\varrho}^{\lambda_2}\int_{\varsigma}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu\right)(\lambda_2-\varrho)(\xi_2-\varsigma)\,d\varsigma\,d\varrho \\
&= \frac{(\xi_2-\Upsilon_3)^2(\lambda_2-\Upsilon_1)^2}{4}\int_{\lambda_1}^{\Upsilon_1}\int_{\xi_1}^{\Upsilon_3}\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad - \frac{(\xi_2-\Upsilon_3)^2(\lambda_2-\Upsilon_1)^2}{4}\int_{\Upsilon_1}^{\lambda_2}\int_{\xi_1}^{\Upsilon_3}\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad - \frac{(\xi_2-\Upsilon_3)^2(\lambda_2-\Upsilon_1)^2}{4}\int_{\lambda_1}^{\Upsilon_1}\int_{\Upsilon_3}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad + \frac{(\xi_2-\Upsilon_3)^2(\lambda_2-\Upsilon_1)^2}{4}\int_{\Upsilon_1}^{\lambda_2}\int_{\Upsilon_3}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad - \frac{(\xi_2-\Upsilon_3)^2}{2}\int_{\lambda_1}^{\Upsilon_1}\int_{\xi_1}^{\Upsilon_3}\frac{(\lambda_2-\mu)^2}{2}\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad + \frac{(\xi_2-\Upsilon_3)^2}{2}\int_{\Upsilon_1}^{\lambda_2}\int_{\xi_1}^{\Upsilon_3}\frac{(\lambda_2-\mu)^2}{2}\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad + \frac{(\xi_2-\Upsilon_3)^2}{2}\int_{\lambda_1}^{\Upsilon_1}\int_{\Upsilon_3}^{\xi_2}\frac{(\lambda_2-\mu)^2}{2}\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad - \frac{(\xi_2-\Upsilon_3)^2}{2}\int_{\Upsilon_1}^{\lambda_2}\int_{\Upsilon_3}^{\xi_2}\frac{(\lambda_2-\mu)^2}{2}\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad - \frac{(\lambda_2-\Upsilon_1)^2}{2}\int_{\lambda_1}^{\Upsilon_1}\int_{\xi_1}^{\Upsilon_3}\frac{(\xi_2-\nu)^2}{2}\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad + \frac{(\lambda_2-\Upsilon_1)^2}{2}\int_{\Upsilon_1}^{\lambda_2}\int_{\xi_1}^{\Upsilon_3}\frac{(\xi_2-\nu)^2}{2}\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad + \frac{(\lambda_2-\Upsilon_1)^2}{2}\int_{\lambda_1}^{\Upsilon_1}\int_{\Upsilon_3}^{\xi_2}\frac{(\xi_2-\nu)^2}{2}\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad - \frac{(\lambda_2-\Upsilon_1)^2}{2}\int_{\Upsilon_1}^{\lambda_2}\int_{\xi_1}^{\Upsilon_3}\frac{(\xi_2-\nu)^2}{2}\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad + \int_{\lambda_1}^{\Upsilon_1}\int_{\xi_1}^{\Upsilon_3}\frac{(\xi_2-\nu)^2(\lambda_2-\mu)^2}{4}\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad - \int_{\Upsilon_1}^{\lambda_2}\int_{\xi_1}^{\Upsilon_3}\frac{(\xi_2-\nu)^2(\lambda_2-\nu)^2}{4}\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad - \int_{\lambda_1}^{\Upsilon_1}\int_{\Upsilon_3}^{\xi_2}\frac{(\xi_2-\nu)^2(\lambda_2-\mu)^2}{4}\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad + \int_{\Upsilon_1}^{\lambda_2}\int_{\Upsilon_3}^{\xi_2}\frac{(\xi_2-\nu)^2(\lambda_2-\mu)^2}{4}\omega(\mu,\nu)\,d\nu\,d\mu.
\end{aligned}
\tag{3.9}
$$

Equation (3.9) can be rewritten as

$$\begin{aligned}
&\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\biggl|\int_{\varrho}^{\lambda_2}\int_{\varsigma}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu-\sigma(\Upsilon_1-\varrho)\int_{\lambda_1}^{\lambda_2}\int_{\varsigma}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu\\
&\quad-\sigma(\Upsilon_3-\varsigma)\int_{\varrho}^{\lambda_2}\int_{\xi_1}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu+\sigma(\Upsilon_1-\varrho)\sigma(\Upsilon_3-s)\biggr|\\
&\quad\times(\lambda_2-\varrho)(\xi_2-\varsigma)\,d\varsigma\,d\varrho\\
&=\frac{1}{4}B_1(\lambda_1,\xi_1,\lambda_2,\xi_2)+\frac{(\lambda_2-\Upsilon_1)}{2}B_2(\lambda_1,\xi_1,\lambda_2,\xi_2)+\frac{(\xi_2-\Upsilon_3)}{2}B_3(\lambda_1,\xi_1,\lambda_2,\xi_2)\\
&\quad+(\lambda_2-\Upsilon_1)(\xi_2-\Upsilon_3)B_4(\lambda_1,\xi_1,\lambda_2,\xi_2)=A_1(\lambda_1,\xi_1,\lambda_2,\xi_2).
\end{aligned}\tag{3.10}$$

Similarly, we have

$$\begin{aligned}
&\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\biggl|\int_{\varrho}^{\lambda_2}\int_{\varsigma}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu-\sigma(\Upsilon_1-\varrho)\int_{\lambda_1}^{\lambda_2}\int_{\varsigma}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu\\
&\quad-\sigma(\Upsilon_3-\varsigma)\int_{\varrho}^{\lambda_2}\int_{\xi_1}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu+\sigma(\Upsilon_1-\varrho)\sigma(\Upsilon_3-\varsigma)\biggr|\\
&\quad\times(\lambda_2-\varrho)(\varsigma-\xi_1)\,d\varsigma\,d\varrho\\
&=\frac{1}{4}B_1(\lambda_1,\xi_1,\lambda_2,\xi_2)+\frac{(\lambda_2-\Upsilon_1)}{2}B_2(\lambda_1,\xi_1,\lambda_2,\xi_2)+\frac{(\Upsilon_3-\xi_1)}{2}B_3(\lambda_1,\xi_1,\lambda_2,\xi_2)\\
&\quad+(\lambda_2-\Upsilon_1)(\Upsilon_3-\xi_1)B_4(\lambda_1,\xi_1,\lambda_2,\xi_2)=A_2(\lambda_1,\xi_1,\lambda_2,\xi_2),
\end{aligned}\tag{3.11}$$

$$\begin{aligned}
&\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\biggl|\int_{\varrho}^{\lambda_2}\int_{\varsigma}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu-\sigma(\Upsilon_1-\varrho)\int_{\lambda_1}^{\lambda_2}\int_{\varsigma}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu\\
&\quad-\sigma(\Upsilon_3-\varsigma)\int_{\varrho}^{\lambda_2}\int_{\xi_1}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu+\sigma(\Upsilon_1-\varrho)\sigma(\Upsilon_3-\varsigma)\biggr|\\
&\quad\times(\lambda_1-\varrho)(\xi_2-\varsigma)\,d\varsigma\,d\varrho\\
&=\frac{1}{4}B_1(\lambda_1,\xi_1,\lambda_2,\xi_2)+\frac{(\Upsilon_1-\lambda_1)}{2}B_2(\lambda_1,\xi_1,\lambda_2,\xi_2)+\frac{(\xi_2-\Upsilon_3)}{2}B_3(\lambda_1,\xi_1,\lambda_2,\xi_2)\\
&\quad+(\Upsilon_1-\lambda_1)(\xi_2-\Upsilon_3)B_4(\lambda_1,\xi_1,\lambda_2,\xi_2)=A_3(\lambda_1,\xi_1,\lambda_2,\xi_2)
\end{aligned}\tag{3.12}$$

and

$$\begin{aligned}
&\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\biggl|\int_{\varrho}^{\lambda_2}\int_{\varsigma}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu-\sigma(\Upsilon_1-\varrho)\int_{\lambda_1}^{\lambda_2}\int_{\varsigma}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu\\
&\quad-\sigma(\Upsilon_3-\varsigma)\int_{\varrho}^{\lambda_2}\int_{\xi_1}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu+\sigma(\Upsilon_1-\varrho)\sigma(\Upsilon_3-\varsigma)\biggr|\\
&\quad\times(\lambda_1-\varrho)(\varsigma-\xi_1)\,d\varsigma\,d\varrho\\
&=\frac{1}{4}B_1(\lambda_1,\xi_1,\lambda_2,\xi_2)+\frac{(\Upsilon_1-\lambda_1)}{2}B_2(\lambda_1,\xi_1,\lambda_2,\xi_2)+\frac{(\Upsilon_3-\xi_1)}{2}B_3(\lambda_1,\xi_1,\lambda_2,\xi_2)\\
&\quad+(\Upsilon_1-\lambda_1)(\Upsilon_3-\xi_1)B_4(\lambda_1,\xi_1,\lambda_2,\xi_2)=A_4(\lambda_1,\xi_1,\lambda_2,\xi_2).
\end{aligned}\tag{3.13}$$

Therefore, inequality (3.5) follows from (3.8) and (3.10)–(3.13). □

Corollary 3.2 *If all the conditions of Theorem 3.1 are satisfied and $\omega(\mu,\nu)$ is symmetric with respect to $\frac{\lambda_1+\lambda_2}{2}$ and $\frac{\xi_1+\xi_2}{2}$ on the co-ordinates. Then one has*

$$\begin{aligned}
&\biggl|\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\mu,\nu)\omega(\mu,\nu)\,d\nu\,d\mu-\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi\left(\frac{\lambda_1+\lambda_2}{2},\nu\right)\omega(\mu,\nu)\,d\nu\,d\mu\\
&\quad-\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi\left(\mu,\frac{\xi_1+\xi_2}{2}\right)\omega(\mu,\nu)\,d\nu\,d\mu+\Phi\left(\frac{\lambda_1+\lambda_2}{2},\frac{\xi_1+\xi_2}{2}\right)\biggr|\\
&\quad\le\left[\left|\Phi_{\varrho\varsigma}(\lambda_1,\xi_1)\right|+\left|\Phi_{\varrho\varsigma}(\lambda_1,\xi_2)\right|+\left|\Phi_{\varrho\varsigma}(\lambda_2,\xi_1)\right|+\left|\Phi_{\varrho\varsigma}(\lambda_2,\xi_2)\right|\right]\\
&\quad\quad\times\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}\int_{\frac{\xi_1+\xi_2}{2}}^{\xi_2}\left(\mu-\frac{\lambda_1+\lambda_2}{2}\right)\left(\nu-\frac{\xi_1+\xi_2}{2}\right)\omega(\mu,\nu)\,d\nu\,d\mu.
\end{aligned}\tag{3.14}$$

Proof Making use of the hypothesis of Theorem 3.1 and the symmetry of the function $\omega(\mu,\nu)$ with respect to $\frac{\lambda_1+\lambda_2}{2}$ and $\frac{\xi_1+\xi_2}{2}$ on the co-ordinates that the function $(\Upsilon_1-\mu)^2(\Upsilon_3-\nu)^2\omega(\mu,\nu)$ is symmetric with respect to $\frac{\lambda_1+\lambda_2}{2}$ and $\frac{\xi_1+\xi_2}{2}$ on the co-ordinates, the function $(\Upsilon_1-\mu)(\Upsilon_3-\nu)^2\omega(\mu,\nu)$ is symmetric with respect to $\frac{\xi_1+\xi_2}{2}$ on the co-ordinates, and the function $(\Upsilon_1-\mu)^2(\Upsilon_3-\nu)\omega(\mu,\nu)$ is symmetric with respect to $\frac{\lambda_1+\lambda_2}{2}$ on the co-ordinates. Therefore,

$$\begin{aligned}
&\int_{\lambda_1}^{\Upsilon_1}\int_{\xi_1}^{\Upsilon_3}(\Upsilon_1-\mu)^2(\Upsilon_3-\nu)^2\omega(\mu,\nu)\,d\nu\,d\mu\\
&\quad=\int_{\lambda_1}^{\Upsilon_1}\int_{\Upsilon_3}^{\xi_2}(\Upsilon_1-\mu)^2(\Upsilon_3-\nu)^2\omega(\mu,\nu)\,d\nu\,d\mu\\
&\quad=\int_{\Upsilon_1}^{\lambda_2}\int_{\xi_1}^{\Upsilon_3}(\Upsilon_1-\mu)^2(\Upsilon_3-\nu)^2\omega(\mu,\nu)\,d\nu\,d\mu\\
&\quad=\int_{\Upsilon_1}^{\lambda_2}\int_{\Upsilon_3}^{\xi_2}(\Upsilon_1-\mu)^2(\Upsilon_3-\nu)^2\omega(\mu,\nu)\,d\nu\,d\mu,\\
&\int_{\lambda_1}^{\Upsilon_1}\int_{\xi_1}^{\Upsilon_3}(\Upsilon_1-\mu)(\Upsilon_3-\nu)^2\omega(\mu,\nu)\,d\nu\,d\mu\\
&\quad=\int_{\lambda_1}^{\Upsilon_1}\int_{\Upsilon_3}^{\xi_2}(\Upsilon_1-\mu)(\Upsilon_3-\nu)^2\omega(\mu,\nu)\,d\nu\,d\mu,\\
&\int_{\Upsilon_1}^{\lambda_2}\int_{\xi_1}^{\Upsilon_3}(\Upsilon_1-\mu)(\Upsilon_3-\nu)^2\omega(\mu,\nu)\,d\nu\,d\mu\\
&\quad=\int_{\Upsilon_1}^{\lambda_2}\int_{\Upsilon_3}^{\xi_2}(\Upsilon_1-\mu)(\Upsilon_3-\nu)^2\omega(\mu,\nu)\,d\nu\,d\mu,\\
&\int_{\lambda_1}^{\Upsilon_1}\int_{\xi_1}^{\Upsilon_3}(\Upsilon_1-\mu)^2(\Upsilon_3-\nu)\omega(\mu,\nu)\,d\nu\,d\mu\\
&\quad=\int_{\lambda_1}^{\Upsilon_1}\int_{\Upsilon_3}^{\xi_2}(\Upsilon_1-\mu)^2(\Upsilon_3-\nu)\omega(\mu,\nu)\,d\nu\,d\mu
\end{aligned}$$

and

$$\int_{\Upsilon_1}^{\lambda_2}\int_{\xi_1}^{\Upsilon_3}(\Upsilon_1-\mu)^2(\Upsilon_3-\nu)\omega(\mu,\nu)\,d\nu\,d\mu=\int_{\Upsilon_1}^{\lambda_2}\int_{\Upsilon_3}^{\xi_2}(\Upsilon_1-\mu)^2(\Upsilon_3-\nu)\omega(\mu,\nu)\,d\nu\,d\mu.$$

From the above equations, we obtain

$$B_1(\lambda_1,\xi_1,\lambda_2,\xi_2) = B_2(\lambda_1,\xi_1,\lambda_2,\xi_2) = B_3(\lambda_1,\xi_1,\lambda_2,\xi_2) = 0$$

and

$$B_4(\lambda_1,\xi_1,\lambda_2,\xi_2) = 4\int_{\frac{\lambda_1+\lambda_2}{2}}^{b}\int_{\frac{\xi_1+\xi_2}{2}}^{d}\left(\mu-\frac{\lambda_1+\lambda_2}{2}\right)\left(\nu-\frac{\xi_1+\xi_2}{2}\right)\omega(\mu,\nu)\,d\nu\,d\mu.$$

Therefore, inequality (3.14) follows from (3.10) and the above quantities. □

Remark 3.3 Inequality (1.3) can be derived from (3.5) if

$$\omega(\mu,\nu) = \frac{g(\mu,\nu)}{\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2} g(\mu,\nu)\,d\nu\,d\mu}$$

and $g(\mu,\nu)$ is symmetric with respect to $\frac{\lambda_1+\lambda_2}{2}$ and $\frac{\xi_1+\xi_2}{2}$ on the co-ordinates.

Theorem 3.4 *Let $q>1$, $\Omega\subseteq\mathbb{R}^2$, Ω° be the interior of Ω, $\lambda_1<\lambda_2$ and $\xi_1<\xi_2$ such that $[\lambda_1,\lambda_2]\times[\xi_1,\xi_2]\subseteq\Omega^\circ$, $\Phi:\Omega\to\mathbb{R}$ be a twice partially differentiable mapping on Ω° such that $\Phi_{\varrho\varsigma}\in L([\lambda_1,\lambda_2]\times[\xi_1,\xi_2])$ and $|\Phi_{\varrho\varsigma}|^q$ is co-ordinated convex on $[\lambda_1,\lambda_2]\times[\xi_1,\xi_2]$, and $\omega:[\lambda_1,\lambda_2]\times[\xi_1,\xi_2]\to[0,\infty)$ be a continuous mapping. Then*

$$\begin{aligned}
&\left|\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\mu,\nu)\omega(\mu,\nu)\,d\nu\,d\mu-\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\Upsilon_1,\nu)\omega(\mu,\nu)\,d\nu\,d\mu\right.\\
&\quad\left.-\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\mu,\Upsilon_3)\omega(\mu,\nu)\,d\nu\,d\mu+\Phi(\Upsilon_1,\Upsilon_3)\right|\\
&\quad\le 4^{1-\frac{1}{q}}\left(\int_{\Upsilon_1}^{\lambda_2}\int_{\Upsilon_3}^{\lambda_2}(\mu-\Upsilon_1)(\nu-\Upsilon_3)\omega(\mu,\nu)\,d\nu\,d\mu\right)^{1-\frac{1}{q}}\left(\frac{1}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\right)^{\frac{1}{q}}\\
&\quad\times\Big[\big|\Phi_{\varrho\varsigma}(\lambda_1,\xi_1)\big|^qA_1(\lambda_1,\xi_1,\lambda_2,\xi_2)+\big|\Phi_{\varrho\varsigma}(\lambda_1,\xi_2)\big|^qA_2(\lambda_1,\xi_1,\lambda_2,\xi_2)\\
&\quad+\big|\Phi_{\varrho\varsigma}(\lambda_2,\xi_1)\big|^qA_3(\lambda_1,\xi_1,\lambda_2,\xi_2)+\big|\Phi_{\varrho\varsigma}(\lambda_2,\xi_2)\big|^qA_4(\lambda_1,\xi_1,\lambda_2,\xi_2)\Big]^{\frac{1}{q}},
\end{aligned}\tag{3.15}$$

where $A_1(\lambda_1,\xi_1,\lambda_2,\xi_2)$, $A_2(\lambda_1,\xi_1,\lambda_2,\xi_2)$, $A_3(\lambda_1,\xi_1,\lambda_2,\xi_2)$ and $A_4(\lambda_1,\xi_1,\lambda_2,\xi_2)$ are defined in Theorem 3.1.

Proof It follows from (3.8) and the Hölder inequality that

$$\begin{aligned}
&\left|\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\mu,\nu)\omega(\mu,\nu)\,d\nu\,d\mu-\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\Upsilon_1,\nu)\omega(\mu,\nu)\,d\nu\,d\mu\right.\\
&\quad\left.-\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\mu,\Upsilon_3)\omega(\mu,\nu)\,d\nu\,d\mu+\Phi(\Upsilon_1,\Upsilon_3)\right|\\
&\quad\le\left(\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\left|\int_{\varrho}^{\lambda_2}\int_{\varsigma}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu-\sigma(\Upsilon_1-\varrho)\int_{\lambda_1}^{\lambda_2}\int_{\varsigma}^{\xi_2}\omega(\mu,\nu)\,d\nu\,d\mu\right.\right.\\
&\quad\left.\left.-\sigma(\Upsilon_1-\varsigma)\int_{\varrho}^{\lambda_2}\int_{\xi_1}^{\lambda_2}\omega(\mu,\nu)\,d\nu\,d\mu+\sigma(\Upsilon_1-\varrho)\sigma(\Upsilon_3-\varsigma)\right|d\varsigma\,d\varrho\right)^{1-\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
&\times \biggl(\int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \biggl| \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu,\nu)\,d\nu\,d\mu - \sigma(\Upsilon_1-\varrho) \int_{\lambda_1}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu,\nu)\,d\nu\,d\mu \\
&- \sigma(\Upsilon_3-\varsigma) \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\xi_2} \omega(\mu,\nu)\,d\nu\,d\mu + \sigma(\Upsilon_1-\varrho)\sigma(\Upsilon_3-\varsigma) \biggr| \\
&\times \bigl| \Phi_{\varrho\varsigma}(\varrho,\varsigma) \bigr|^q \,d\varsigma\,d\varrho \biggr)^{\frac{1}{q}}. \qquad (3.16)
\end{aligned}$$

Making use of Lemma 2.5, we have

$$\begin{aligned}
&\int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \biggl| \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu,\nu)\,d\nu\,d\mu - \sigma(\Upsilon_1-\varrho) \int_{\lambda_1}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad - \sigma(\Upsilon_3-\varsigma) \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\xi_2} \omega(\mu,\nu)\,d\nu\,d\mu + \sigma(\Upsilon_1-\varrho)\sigma(\Upsilon_3-\varsigma) \biggr| \bigl| \Phi_{\varrho\varsigma}(\varrho,\varsigma) \bigr|^q \,d\varsigma\,d\varrho \\
&\leq \frac{1}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)} \bigl[\bigl| \Phi_{\varrho\varsigma}(\lambda_1,\xi_1) \bigr|^q A_1(\lambda_1,\xi_1,\lambda_2,\xi_2) + \bigl| \Phi_{\varrho\varsigma}(\lambda_1,\xi_2) \bigr|^q A_2(\lambda_1,\xi_1,\lambda_2,\xi_2) \\
&\quad + \bigl| \phi_{\varrho\varsigma}(\lambda_2,\xi_1) \bigr|^q A_3(\lambda_1,\xi_1,\lambda_2,\xi_2) + \bigl| \Phi_{\varrho\varsigma}(\lambda_2,\xi_2) \bigr|^q A_4(\lambda_1,\xi_1,\lambda_2,\xi_2) \bigr]. \qquad (3.17)
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
&\int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \biggl| \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu,\nu)\,d\nu\,d\mu - \sigma(\Upsilon_1-\varrho) \int_{\lambda_1}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad - \sigma(\Upsilon_3-\varsigma) \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\xi_2} \omega(\mu,\nu)\,d\nu\,d\mu + \sigma(\Upsilon_1-\varrho)\sigma(\Upsilon_3-\varsigma) \biggr| \,d\varsigma\,d\varrho \\
&= \int_{\lambda_1}^{\Upsilon_1} \int_{\xi_1}^{\Upsilon_3} \biggl(\int_{\lambda_1}^{\varrho} \int_{\xi_1}^{\varsigma} \omega(\mu,\nu)\,d\nu\,d\mu \biggr) \,d\varsigma\,d\varrho \\
&\quad + \int_{\Upsilon_1}^{\lambda_2} \int_{\xi_1}^{\Upsilon_3} \biggl(\int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\varsigma} \omega(\mu,\nu)\,d\nu\,d\mu \biggr) \,d\varsigma\,d\varrho \\
&\quad + \int_{\lambda_1}^{\Upsilon_1} \int_{\Upsilon_3}^{\xi_2} \biggl(\int_{\lambda_1}^{\varrho} \int_{\varsigma}^{\xi_2} \omega(\mu,\nu)\,d\nu\,d\mu \biggr) \,d\varsigma\,d\varrho \\
&\quad + \int_{\Upsilon_1}^{\lambda_2} \int_{\Upsilon_3}^{\xi_2} \biggl(\int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu,\nu)\,d\nu\,d\mu \biggr) \,d\varsigma\,d\varrho \\
&= 4 \int_{\Upsilon_1}^{\lambda_2} \int_{\Upsilon_3}^{\xi_2} (\mu-\Upsilon_1)(\nu-\Upsilon_2)\omega(\mu,\nu)\,d\nu\,d\mu. \qquad (3.18)
\end{aligned}$$

Therefore, inequality (3.15) follows from (3.16)–(3.18). □

Remark 3.5 If $\omega(\mu,\nu)$ is symmetric with respect to $\frac{\lambda_1+\lambda_2}{2}$ and $\frac{\xi_1+\xi_2}{2}$ on the co-ordinates, then inequality (3.15) leads to

$$\begin{aligned}
&\biggl| \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\mu,\nu)\omega(\mu,\nu)\,d\nu\,d\mu - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi\biggl(\frac{\lambda_1+\lambda_2}{2},\nu\biggr)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\quad - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi\biggl(\mu,\frac{\xi_1+\xi_2}{2}\biggr)\omega(\mu,\nu)\,d\nu\,d\mu + \Phi\biggl(\frac{\lambda_1+\lambda_2}{2},\frac{\xi_1+\xi_2}{2}\biggr) \biggr|
\end{aligned}$$

$$\le 4\left[\frac{|\Phi_{\varrho\varsigma}(\lambda_1,\xi_1)|^q+|\Phi_{\varrho\varsigma}(\lambda_1,\xi_2)|^q+|\Phi_{\varrho\varsigma}(\lambda_2,\xi_1)|^q+|\Phi_{\varrho\varsigma}(\lambda_2,\xi_2)|^q}{4}\right]^{\frac{1}{q}}$$
$$\times\int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2}\int_{\frac{\xi_1+\xi_2}{2}}^{\xi_2}\left(\mu-\frac{\lambda_1+\lambda_2}{2}\right)\left(\nu-\frac{\xi_1+\xi_2}{2}\right)\omega(\mu,\nu)\,d\nu\,d\mu. \tag{3.19}$$

Remark 3.6 If $g(\mu,\nu)$ is symmetric with respect to $\frac{\lambda_1+\lambda_2}{2}$ and $\frac{\xi_1+\xi_2}{2}$ on the co-ordinates and

$$\omega(\mu,\nu)=\frac{g(\mu,\nu)}{\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}g(\mu,\nu)\,d\nu\,d\mu},$$

then inequality (3.15) gives the result proved in [75].

Next, we use the symbols for our upcoming results as follows:

$$\begin{aligned}
\Psi(\omega,\Phi)={}&\frac{1}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\left[\Phi(\lambda_1,\xi_1)\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\lambda_2-\mu)(\xi_2-\nu)\omega(\mu,\nu)\,d\nu\,d\mu\right.\\
&+\Phi(\lambda_1,\xi_2)\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\lambda_2-\mu)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu\\
&+\Phi(\lambda_2,\xi_1)\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\mu-\lambda_1)(\xi_2-\nu)\omega(\mu,\nu)\,d\nu\,d\mu\\
&\left.+\Phi(\lambda_2,\xi_2)\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu\right]\\
&+\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\mu,\nu)\omega(\mu,\nu)\,d\nu\,d\mu\\
&-\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\lambda_2-\mu)\Phi(\lambda_1,\nu)\omega(\mu,\nu)\,d\nu\,d\mu\\
&-\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\mu-\lambda_1)\Phi(\lambda_2,\nu)\omega(\mu,\nu)\,d\nu\,d\mu\\
&-\frac{1}{\xi_2-\xi_1}\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\xi_2-\nu)\Phi(\mu,\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu\\
&-\frac{1}{\xi_2-\xi_1}\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\nu-\xi_1)\Phi(\mu,\xi_2)\omega(\mu,\nu)\,d\nu\,d\mu.
\end{aligned}\tag{3.20}$$

From (3.20) we clearly see that

$$\begin{aligned}
&\Psi\left(\frac{1}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)},\Phi\right)\\
&\quad=\frac{\Phi(\lambda_1,\xi_1)+\Phi(\lambda_1,\xi_2)+\Phi(\lambda_2,\xi_1)+\Phi(\lambda_2,\xi_2)}{4}\\
&\qquad-\frac{1}{2(\lambda_2-\lambda_1)}\int_{\lambda_1}^{\lambda_2}\left[\Phi(\mu,\xi_2)+\Phi(\mu,\xi_1)\right]d\mu\\
&\qquad-\frac{1}{2(\xi_2-\xi_1)}\int_{\xi_1}^{\xi_2}\left[\Phi(\lambda_1,\nu)+\Phi(\lambda_2,\nu)\right]d\nu\\
&\qquad+\frac{1}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\mu,\nu)\,d\nu\,d\mu.
\end{aligned}\tag{3.21}$$

We also use the following notations:

$$\begin{aligned}
&\sup\bigl\{\Phi_{r\rho}(\lambda_1,\xi_1),\Phi_{r\rho}(\Upsilon_1,\xi_1),\Phi_{r\rho}(\lambda_1,\Upsilon_3),\Phi_{r\rho}(\Upsilon_1,\Upsilon_3)\bigr\}=\eta(\lambda_1,\xi_1,\lambda_2,\xi_2),\\
&\sup\bigl\{\Phi_{r\rho}(\Upsilon_1,\xi_1),\Phi_{r\rho}(\Upsilon_1,\Upsilon_3),\Phi_{r\rho}(\lambda_2,\xi_1),\Phi_{r\rho}(\lambda_2,\Upsilon_3)\bigr\}=\eta_2(\lambda_1,\xi_1,\lambda_2,\xi_2),\\
&\sup\bigl\{\Phi_{r\rho}(\lambda_1,\Upsilon_3),\Phi_{r\rho}(\lambda_1,\xi_2),\Phi_{r\rho}(\Upsilon_1,\Upsilon_3),\Phi_{r\rho}(\Upsilon_1,\xi_2)\bigr\}=\eta_3(\lambda_1,\xi_1,\lambda_2,\xi_2),\\
&\sup\bigl\{\Phi_{r\rho}(\Upsilon_1,\Upsilon_3),\Phi_{r\rho}(\Upsilon_1,\xi_2),\Phi_{r\rho}(\lambda_2,\Upsilon_3),\Phi_{r\rho}(\lambda_2,\xi_2)\bigr\}=\eta_4(\lambda_1,\xi_1,\lambda_2,\xi_2).
\end{aligned}$$

The following results present uppermost estimates for $|\Psi(\omega,\Phi)|$ if the function $\Phi(\mu,\nu)$ is quasi-convex on the co-ordinates.

Theorem 3.7 *Let $\Omega\subseteq\mathbb{R}^2$, Ω° be the interior of Ω, $\lambda_1<\lambda_2$ and $\xi_1<\xi_2$ such that $[\lambda_1,\lambda_2]\times[\xi_1,\xi_2]\subseteq\Omega^\circ$, $\Phi:\Omega\to\mathbb{R}$ be a twice partially differentiable mapping on Ω° such that $\Phi_{\varrho\varsigma}\in L([\lambda_1,\lambda_2]\times[\xi_1,\xi_2])$ and $|\Phi_{\varrho\varsigma}|$ is quasi-convex on the co-ordinates on $[\lambda_1,\lambda_2]\times[\xi_1,\xi_2]$, and $\omega:[\lambda_1,\lambda_2]\times[\xi_1,\xi_2]\to[0,\infty)$ be a continuous mapping. Then*

$$\begin{aligned}
\bigl|\Psi(\omega,\Phi)\bigr|\le{}&\frac{(\Upsilon_1-\lambda_1)(\Upsilon_3-\xi_1)}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\eta(\lambda_1,\xi_1,\lambda_2,\xi_2)\\
&\times\int_0^1\int_0^1\bigl|H(\omega,\lambda_1,\xi_1,\Upsilon_1,\Upsilon_3;r,\rho)\bigr|\,d\rho\,dr\\
&+\frac{(\lambda_2-\Upsilon_1)(\Upsilon_3-\xi_1)}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\eta_2(\lambda_1,\xi_1,\lambda_2,\xi_2)\\
&\times\int_0^1\int_0^1\bigl|H(\omega,\Upsilon_1,\xi_1,\lambda_2,\Upsilon_3;r,\rho)\bigr|\,d\rho\,dr\\
&+\frac{(\Upsilon_1-\lambda_1)(\xi_2-\Upsilon_3)}{(\lambda_2-\lambda_1)(\lambda_2-\xi_1)}\eta_3(\lambda_1,\xi_1,\lambda_2,\xi_2)\\
&\times\int_0^1\int_0^1\bigl|H(\omega,\lambda_1,\Upsilon_3,\Upsilon_1,\xi_2;r,\rho)\bigr|\,d\rho\,dr\\
&+\frac{(\lambda_2-\Upsilon_1)(\xi_2-\Upsilon_3)}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\eta_4(\lambda_1,\xi_1,\lambda_2,\xi_2)\\
&\times\int_0^1\int_0^1\bigl|H(\omega,\Upsilon_1,\Upsilon_3,\lambda_2,\xi_2;r,\rho)\bigr|\,d\rho\,dr.
\end{aligned}\tag{3.22}$$

Proof It follows from the quasi-convexity of the function $|\Phi_{\varrho\varsigma}|$ on the co-ordinates that

$$\begin{aligned}
&\Phi_{r\rho}\bigl((1-r)\lambda_1+\Upsilon_1 r,(1-\rho)\xi_1+\Upsilon_3\rho\bigr)\\
&\quad\le\sup\bigl\{\Phi_{r\rho}(\lambda_1,\xi_1),\Phi_{r\rho}(\Upsilon_1,\xi_1),\Phi_{r\rho}(\lambda_1,\Upsilon_3),\Phi_{r\rho}(\Upsilon_1,\Upsilon_3)\bigr\}=\eta(\lambda_1,\xi_1,\lambda_2,\xi_2),\\
&\Phi_{r\rho}\bigl((1-r)\Upsilon_1+br,(1-\rho)\xi_1+\Upsilon_3\rho\bigr)\\
&\quad\le\sup\bigl\{\Phi_{r\rho}(\Upsilon_1,\xi_1),\Phi_{r\rho}(\Upsilon_1,\Upsilon_3),\Phi_{r\rho}(\lambda_2,\xi_1),\Phi_{r\rho}(\lambda_2,\Upsilon_3)\bigr\}=\eta_2(\lambda_1,\xi_1,\lambda_2,\xi_2),\\
&\Phi_{r\rho}\bigl((1-r)\lambda_1+\Upsilon_1 r,(1-\rho)\Upsilon_3+\xi_2\rho\bigr)\\
&\quad\le\sup\bigl\{\Phi_{r\rho}(\lambda_1,\Upsilon_3),\Phi_{r\rho}(\lambda_1,\xi_2),\Phi_{r\rho}(\Upsilon_1,\Upsilon_3),\Phi_{r\rho}(\Upsilon_1,\xi_2)\bigr\}=\eta_3(\lambda_1,\xi_1,\lambda_2,\xi_2)
\end{aligned}$$

and

$$\begin{aligned}&\Phi_{r\rho}\big((1-r)\Upsilon_1+\lambda_2 r,(1-\rho)\Upsilon_3+\xi_2\rho\big)\\&\quad\le\sup\big\{\Phi_{r\rho}(\Upsilon_1,\Upsilon_3),\Phi_{r\rho}(\Upsilon_1,\xi_2),\Phi_{r\rho}(\lambda_2,\Upsilon_3),\Phi_{r\rho}(\lambda_2,\xi_2)\big\}=\eta_4(\lambda_1,\xi_1,\lambda_2,\xi_2)\end{aligned}$$

for all $(r,\rho)\in[0,1]\times[0,1]$.

Therefore, inequality (3.22) follows from (2.1) and the above inequalities. □

Theorem 3.8 *Let $\Omega\subseteq\mathbb{R}^2$, Ω° be the interior of Ω, $\lambda_1<\lambda_2$ and $\xi_1<\xi_2$ such that $[\lambda_1,\lambda_2]\times[\xi_1,\xi_2]\subseteq\Omega^\circ$, $\Phi:\Omega\to\mathbb{R}$ be a twice partially differentiable mapping on Ω° such that $\Phi_{\varrho\varsigma}\in L([\lambda_1,\lambda_2]\times[\xi_1,\xi_2])$ and $|\Phi_{\varrho\varsigma}|$ is quasi-convex on the co-ordinates on $[\lambda_1,\lambda_2]\times[\xi_1,\xi_2]$, and $\omega:[\lambda_1,\lambda_2]\times[\xi_1,\xi_2]\to[0,\infty)$ be a continuous function symmetric with respect to $\frac{\lambda_1+\lambda_2}{2}$ and $\frac{\xi_1+\xi_2}{2}$ on the co-ordinates. Then*

$$\begin{aligned}&\bigg|\frac{\Phi(\lambda_1,\xi_1)+\Phi(\lambda_1,\xi_2)+\Phi(\lambda_2,\xi_1)+\Phi(\lambda_2,\xi_2)}{4}+\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\mu,\nu)\omega(\mu,\nu)\,d\nu\,d\mu\\&\quad-\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\lambda_2-\mu)\Phi(\lambda_1,\nu)\omega(\mu,\nu)\,d\nu\,d\mu\\&\quad-\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\mu-\lambda_1)\Phi(\lambda_2,\nu)\omega(\mu,\nu)\,d\nu\,d\mu\\&\quad-\frac{1}{\xi_2-\xi_1}\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\xi_2-\nu)\Phi(\mu,\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu\\&\quad-\frac{1}{\xi_2}(\nu-\xi_1)\Phi(\mu,\xi_2)\omega(\mu,\nu)\,d\nu\,d\mu\bigg|\\&\le\bigg(\int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}}\int_{\xi_1}^{\frac{\xi_1+\xi_2}{2}}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu\bigg)\\&\quad\times\bigg[\sup\bigg\{\Phi_{r\rho}(\lambda_1,\xi_1),\Phi_{r\rho}\bigg(\frac{\lambda_1+\lambda_2}{2},\xi_1\bigg),\\&\quad\Phi_{r\rho}\bigg(\lambda_1,\frac{\xi_1+\xi_2}{2}\bigg),\Phi_{r\rho}\bigg(\frac{\lambda_1+\lambda_2}{2},\frac{\xi_1+\xi_2}{2}\bigg)\bigg\}\\&\quad+\sup\bigg\{\Phi_{r\rho}\bigg(\frac{\lambda_1+\lambda_2}{2},\xi_1\bigg),\Phi_{r\rho}\bigg(\frac{\lambda_1+\lambda_2}{2},\frac{\xi_1+\xi_2}{2}\bigg),\Phi_{r\rho}(\lambda_2,\xi_1),\Phi_{r\rho}\bigg(\lambda_2,\frac{\xi_1+\xi_2}{2}\bigg)\bigg\}\\&\quad+\sup\bigg\{\Phi_{r\rho}\bigg(\lambda_1,\frac{\xi_1+\xi_2}{2}\bigg),\Phi_{r\rho}(\lambda_1,\xi_2),\Phi_{r\rho}\bigg(\frac{\lambda_1+\lambda_2}{2},\frac{\xi_1+\xi_2}{2}\bigg),\Phi_{r\rho}\bigg(\frac{\lambda_1+\lambda_2}{2},\xi_2\bigg)\bigg\}\\&\quad+\sup\bigg\{\Phi_{r\rho}\bigg(\frac{\lambda_1+\lambda_2}{2},\frac{\xi_1+\xi_2}{2}\bigg),\Phi_{r\rho}\bigg(\frac{\lambda_1+\lambda_2}{2},\xi_2\bigg),\\&\quad\Phi_{r\rho}\bigg(\lambda_2,\frac{\xi_1+\xi_2}{2}\bigg),\Phi_{r\rho}(\lambda_2,\xi_2)\bigg\}\bigg].\end{aligned}\tag{3.23}$$

Proof From the hypothesis of Theorem 3.8 we have

$$\begin{aligned}\Psi(\omega,\Phi)&=\frac{\Phi(\lambda_1,\xi_1)+\Phi(\lambda_1,\xi_2)+\Phi(\lambda_2,\xi_1)+\Phi(\lambda_2,\xi_2)}{4}\\&\quad+\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\mu,\nu)\omega(\mu,\nu)\,d\nu\,d\mu+\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\mu,\nu)\omega(\mu,\nu)\,d\nu\,d\mu\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\lambda_2-\mu)\Phi(\lambda_1,\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&- \frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\mu-\lambda_1)\Phi(\lambda_2,\nu)\omega(\mu,\nu)\,d\nu\,d\mu \\
&- \frac{1}{\xi_2-\xi_1}\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\xi_2-\nu)\Phi(\mu,\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu \\
&- \frac{1}{\xi_2-\xi_1}\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}(\nu-\xi_1)\Phi(\mu,\xi_2)\omega(\mu,\nu)\,d\nu\,d\mu.
\end{aligned}
\tag{3.24}
$$

We also observe that

$$
\begin{aligned}
&\frac{(\Upsilon_1-\lambda_1)(\Upsilon_3-\xi_1)}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\int_0^1\int_0^1\bigl|H(\omega,\lambda_1,\xi_1,\Upsilon_1,\Upsilon_3;r,\rho)\bigr|\,d\rho\,dr \\
&\quad= \frac{1}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\int_{\lambda_1}^{\Upsilon_1}\int_{\xi_1}^{\Upsilon_3}\biggl|\int_{\lambda_1}^{\varrho}\int_{\xi_1}^{\varsigma}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\qquad-\int_{\lambda_1}^{\varrho}\int_{\varsigma}^{\xi_2}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu-\int_{\varrho}^{\lambda_2}\int_{\xi_1}^{\varsigma}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\qquad+\int_{\varrho}^{\xi_2}\int_{\varsigma}^{\xi_1}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu\biggr|\,d\varsigma\,d\varrho, \\
&\frac{(\lambda_2-\Upsilon_1)(\Upsilon_3-\xi_1)}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\int_0^1\int_0^1\bigl|H(\omega,\Upsilon_1,\xi_1,\lambda_2,\Upsilon_3;r,\rho)\bigr|\,d\rho\,dr \\
&\quad= \frac{1}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\int_{\Upsilon_1}^{\lambda_2}\int_{\xi_1}^{\Upsilon_3}\biggl|\int_{\lambda_1}^{\varrho}\int_{\xi_1}^{\varsigma}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\qquad-\int_{\lambda_1}^{\varrho}\int_{\varsigma}^{\xi_2}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu-\int_{\varrho}^{\lambda_2}\int_{\xi_1}^{\varsigma}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\qquad+\int_{\varrho}^{\xi_2}\int_{\varsigma}^{\xi_1}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu\biggr|\,d\varsigma\,d\varrho, \\
&\frac{(\Upsilon_1-\lambda_1)(\xi_2-\Upsilon_3)}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\int_0^1\int_0^1\bigl|H(\omega,\Upsilon_1,\xi_1,\lambda_2,\Upsilon_3;r,\rho)\bigr|\,d\rho\,dr \\
&\quad= \frac{1}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\int_{\lambda_1}^{\Upsilon_1}\int_{\Upsilon_3}^{\xi_2}\biggl|\int_{\lambda_1}^{\varrho}\int_{\xi_1}^{\varsigma}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\qquad-\int_{\lambda_1}^{\varrho}\int_{\varsigma}^{\xi_2}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu-\int_{\varrho}^{\lambda_2}\int_{\xi_1}^{\varsigma}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu \\
&\qquad+\int_{\varrho}^{\xi_2}\int_{\varsigma}^{\xi_1}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu\biggr|\,d\varsigma\,d\varrho
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{(\lambda_2-\Upsilon_1)(\xi_2-\Upsilon_3)}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\int_0^1\int_0^1\bigl|H(\omega,\Upsilon_1,\Upsilon_3,\lambda_2,\xi_2;r,\rho)\bigr|\,d\rho\,dr \\
&\quad= \frac{1}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\int_{\Upsilon_1}^{\lambda_2}\int_{\Upsilon_3}^{\xi_2}\biggl|\int_{\lambda_1}^{\varrho}\int_{\xi_1}^{\varsigma}(\mu-\lambda_1)(\nu-\xi_1)\omega(\mu,\nu)\,d\nu\,d\mu
\end{aligned}
$$

$$- \int_{\lambda_1}^{\varrho} \int_{\varsigma}^{\xi_2} (\mu - \lambda_1)(\nu - \xi_1)\omega(\mu,\nu)\,d\nu\,d\mu - \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\varsigma} (\mu - \lambda_1)(\nu - \xi_1)\omega(\mu,\nu)\,d\nu\,d\mu$$
$$+ \int_{\varrho}^{\xi_2} \int_{\varsigma}^{\xi_1} (\mu - \lambda_1)(\nu - \xi_1)\omega(\mu,\nu)\,d\nu\,d\mu \Bigg|\,d\varsigma\,d\varrho.$$

Let $p : [\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \to \mathbb{R}$ defined by

$$\begin{aligned} p(\varrho,\varsigma) = {} & \int_{\lambda_1}^{\varrho} \int_{\xi_1}^{\varsigma} (\mu - \lambda_1)(\nu - \xi_1)\omega(\mu,\nu)\,d\nu\,d\mu \\ & - \int_{\lambda_1}^{\varrho} \int_{\varsigma}^{\xi_2} (\mu - \lambda_1)(\nu - \xi_1)\omega(\mu,\nu)\,d\nu\,d\mu \\ & - \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\varsigma} (\mu - \lambda_1)(\nu - \xi_1)\omega(\mu,\nu)\,d\nu\,d\mu \\ & + \int_{\varrho}^{\xi_2} \int_{\varsigma}^{\xi_1} (\mu - \lambda_1)(\nu - \xi_1)\omega(\mu,\nu)\,d\nu\,d\mu. \end{aligned}$$

Then we clearly see that

$$p_{\varrho\varsigma}(\varrho,\varsigma) = (\lambda_2 - \lambda_1)(\xi_2 - \xi_1)\omega(\varrho,\varsigma) > 0$$

for $(\varrho,\varsigma) \in [\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$, which implies that $p(\varrho,\varsigma)$ is an increasing function on $[\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$ and

$$p(\Upsilon_1, \Upsilon_3) = 0.$$

Now it is easy to see that

$$\begin{aligned} & \frac{(\Upsilon_1 - \lambda_1)(\Upsilon_3 - \xi_1)}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_0^1 \int_0^1 \big|H(\omega, \lambda_1, \xi_1, \Upsilon_1, \Upsilon_3; r, \rho)\big|\,d\rho\,dr \\ & \quad = \frac{(\lambda_2 - \Upsilon_1)(\Upsilon_3 - \xi_1)}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_0^1 \int_0^1 \big|H(\omega, \Upsilon_1, \xi_1, \lambda_2, \Upsilon_3; r, \rho)\big|\,d\rho\,dr \\ & \quad = \frac{(\Upsilon_1 - \lambda_1)(\xi_2 - \Upsilon_3)}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_0^1 \int_0^1 \big|H(\omega, \Upsilon_1, \xi_1, \lambda_2, \Upsilon_3; r, \rho)\big|\,d\rho\,dr \\ & \quad = \frac{(\lambda_2 - \Upsilon_1)(\xi_2 - \Upsilon_3)}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_0^1 \int_0^1 \big|H(\omega, \Upsilon_1, \Upsilon_3, \lambda_2, \xi_2; r, \rho)\big|\,d\rho\,dr \\ & \quad = \int_{\lambda_1}^{\frac{\lambda_1+\lambda_2}{2}} \int_{\xi_1}^{\frac{\xi_1+\xi_2}{2}} (\mu - \lambda_1)(\nu - \xi_1)\omega(\mu,\nu)\,d\nu\,d\mu. \end{aligned}$$

Thus, inequality (3.23) can be derived from (3.24). □

4 Applications to random variables

Let $\alpha, \beta \in \mathbb{R}$, $0 < \lambda_1 < \lambda_2$, $0 < \xi_1 < \xi_2$, $\mathcal{X}$ and $\mathcal{Y}$ be two independent continuous random variables with the common continuous probability density function $\omega : [\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \to [0, \infty)$. Then the α-moment of $\mathcal{X}$ and β-moment of $\mathcal{Y}$ about the origin are, respectively, defined by

$$\mathcal{E}_\alpha(\mathcal{X}) = \int_{\lambda_1}^{\lambda_2} \varrho^\alpha \omega_1(\varrho)\, d\varrho, \mathcal{E}_\beta(\mathcal{Y}) = \int_{\xi_1}^{\xi_2} \varsigma^\beta \omega_2(\varsigma)\, d\varsigma$$

if both $\mathcal{E}_\alpha(\mathcal{X})$ and $\mathcal{E}_\beta(\mathcal{Y})$ are finite, where $\omega_1 : [\lambda_1, \lambda_2] \to [0, \infty)$ and $\omega_2 : [\xi_1, \xi_2] \to [0, \infty)$ are, respectively, the marginal probability density functions of $\mathcal{X}$ and $\mathcal{Y}$. Since $\mathcal{X}$ and $\mathcal{Y}$ are two independent random variables, one has

$$\omega(\varrho, \varsigma) = \omega_1(\varrho)\omega_2(\varsigma)$$

for all $(\varrho, \varsigma) \in [\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$, and

$$\begin{aligned}
\mathcal{E}_{\alpha,\beta}(\mathcal{X}\mathcal{Y}) &= \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \varrho^\alpha \varsigma^\beta \omega(\varrho, \varsigma)\, d\varsigma\, d\varrho \\
&= \int_{\lambda_1}^{\lambda_2} \varrho^\alpha \omega_1(\varrho)\, d\varrho \int_{\xi_1}^{\xi_2} \varsigma^\beta \omega_2(\varsigma)\, d\varsigma = \mathcal{E}_\alpha(\mathcal{X})\mathcal{E}_\beta(\mathcal{Y}).
\end{aligned}$$

Making use of the above notations, we clearly see that

$$\begin{aligned}
&B_1(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
&\quad = \int_{\lambda_1}^{\mathcal{E}(\mathcal{X})} \int_{\xi_1}^{\mathcal{E}(\mathcal{Y})} \big(\mathcal{E}(\mathcal{X}) - \mu\big)^2 \big(\mathcal{E}(\mathcal{Y}) - \nu\big)^2 \omega(\mu, \nu)\, d\nu\, d\mu \\
&\qquad - \int_{\lambda_1}^{\mathcal{E}(\mathcal{X})} \int_{\mathcal{E}(\mathcal{Y})}^{\xi_2} \big(\mathcal{E}(\mathcal{X}) - \mu\big)^2 \big(\mathcal{E}(\mathcal{Y}) - \nu\big)^2 \omega(\mu, \nu)\, d\nu\, d\mu \\
&\qquad - \int_{\mathcal{E}(\mathcal{X})}^{\lambda_2} \int_{\xi_1}^{\mathcal{E}(\mathcal{Y})} \big(\mathcal{E}(\mathcal{X}) - \mu\big)^2 \big(\mathcal{E}(\mathcal{Y}) - \nu\big)^2 \omega(\mu, \nu)\, d\nu\, d\mu \\
&\qquad + \int_{\mathcal{E}(\mathcal{X})}^{\lambda_2} \int_{\mathcal{E}(\mathcal{Y})}^{\xi_2} \big(\mathcal{E}(\mathcal{X}) - \mu\big)^2 \big(\mathcal{E}(\mathcal{Y}) - \nu\big)^2 \omega(\mu, \nu)\, d\nu\, d\mu, \\
&B_2(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
&\quad = \int_{\lambda_1}^{\mathcal{E}(\mathcal{X})} \int_{\xi_1}^{\mathcal{E}(\mathcal{Y})} \big(\mathcal{E}(\mathcal{X}) - \mu\big) \big(\mathcal{E}(\mathcal{Y}) - \nu\big)^2 \omega(\mu, \nu)\, d\nu\, d\mu \\
&\qquad - \int_{\lambda_1}^{\mathcal{E}(\mathcal{X})} \int_{\mathcal{E}(\mathcal{Y})}^{\xi_2} \big(\mathcal{E}(\mathcal{X}) - \mu\big) \big(\mathcal{E}(\mathcal{Y}) - \nu\big)^2 \omega(\mu, \nu)\, d\nu\, d\mu \\
&\qquad - \int_{\mathcal{E}(\mathcal{X})}^{\lambda_2} \int_{\xi_1}^{\mathcal{E}(\mathcal{Y})} \big(\mathcal{E}(\mathcal{X}) - \mu\big) \big(\mathcal{E}(\mathcal{Y}) - \nu\big)^2 \omega(\mu, \nu)\, d\nu\, d\mu \\
&\qquad + \int_{\mathcal{E}(\mathcal{X})}^{\lambda_2} \int_{\mathcal{E}(\mathcal{Y})}^{\xi_2} \big(\mathcal{E}(\mathcal{X}) - \mu\big) \big(\mathcal{E}(\mathcal{Y}) - \nu\big)^2 \omega(\mu, \nu)\, d\nu\, d\mu, \\
&B_3(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
&\quad = \int_{\lambda_1}^{\mathcal{E}(\mathcal{X})} \int_{\xi_1}^{\mathcal{E}(\mathcal{Y})} \big(\mathcal{E}(\mathcal{X}) - \mu\big)^2 \big(\mathcal{E}(\mathcal{Y}) - \nu\big) \omega(\mu, \nu)\, d\nu\, d\mu \\
&\qquad - \int_{\lambda_1}^{\mathcal{E}(\mathcal{X})} \int_{\mathcal{E}(\mathcal{Y})}^{\xi_2} \big(\mathcal{E}(\mathcal{X}) - \mu\big)^2 \big(\mathcal{E}(\mathcal{Y}) - \nu\big) \omega(\mu, \nu)\, d\nu\, d\mu
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\mathcal{E}(\mathcal{X})}^{\lambda_2} \int_{\xi_1}^{\mathcal{E}(\mathcal{Y})} \big(\mathcal{E}(\mathcal{X}) - \mu\big)^2 \big(\mathcal{E}(\mathcal{Y}) - \nu\big) \omega(\mu, \nu)\, d\nu\, d\mu \\
&+ \int_{\mathcal{E}(\mathcal{X})}^{\lambda_2} \int_{\mathcal{E}(\mathcal{Y})}^{\xi_2} \big(\mathcal{E}(\mathcal{X}) - \mu\big)^2 \big(\mathcal{E}(\mathcal{Y}) - \nu\big) \omega(\mu, \nu)\, d\nu\, d\mu
\end{aligned}
$$

and

$$
\begin{aligned}
&B_4(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
&\quad = \int_{\lambda_1}^{\mathcal{E}(\mathcal{X})} \int_{\xi_1}^{\mathcal{E}(\mathcal{Y})} \big(\mathcal{E}(\mathcal{X}) - \mu\big) \big(\mathcal{E}(\mathcal{Y}) - \nu\big) \omega(\mu, \nu)\, d\nu\, d\mu \\
&\qquad - \int_{\lambda_1}^{\mathcal{E}(\mathcal{X})} \int_{\mathcal{E}(\mathcal{Y})}^{\xi_2} \big(\mathcal{E}(\mathcal{X}) - \mu\big) \big(\mathcal{E}(\mathcal{Y}) - \nu\big) \omega(\mu, \nu)\, d\nu\, d\mu \\
&\qquad - \int_{\mathcal{E}(\mathcal{X})}^{\lambda_2} \int_{\xi_1}^{\mathcal{E}(\mathcal{Y})} \big(\mathcal{E}(\mathcal{X}) - \mu\big) (\Upsilon_3 - \nu) \omega(\mu, \nu)\, d\nu\, d\mu \\
&\qquad + \int_{\mathcal{E}(\mathcal{X})}^{\lambda_2} \int_{\mathcal{E}(\mathcal{Y})}^{\xi_2} \big(\mathcal{E}(\mathcal{X}) - \mu\big) \big(\mathcal{E}(\mathcal{Y}) - \nu\big) \omega(\mu, \nu)\, d\nu\, d\mu.
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&A_1(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
&\quad = \frac{1}{4} B_1(\lambda_1, \xi_1, \lambda_2, \xi_2) + \frac{(\lambda_2 - \mathcal{E}(\mathcal{X}))}{2} B_2(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
&\qquad + \frac{(\xi_2 - \mathcal{E}(\mathcal{Y}))}{2} B_3(\lambda_1, \xi_1, \lambda_2, \xi_2) + \big(\lambda_2 - \mathcal{E}(\mathcal{X})\big)\big(\xi_2 - \mathcal{E}(\mathcal{Y})\big) B_4(\lambda_1, \xi_1, \lambda_2, \xi_2), \\
&A_2(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
&\quad = \frac{1}{4} B_1(\lambda_1, \xi_1, \lambda_2, \xi_2) + \frac{(\lambda_2 - \mathcal{E}(\mathcal{X}))}{2} B_2(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
&\qquad + \frac{(\mathcal{E}(\mathcal{Y}) - \xi_1)}{2} B_3(\lambda_1, \xi_1, \lambda_2, \xi_2) + \big(\lambda_2 - \mathcal{E}(\mathcal{X})\big)\big(\mathcal{E}(\mathcal{Y}) - \xi_1\big) B_4(\lambda_1, \xi_1, \lambda_2, \xi_2), \\
&A_3(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
&\quad = \frac{1}{4} B_1(\lambda_1, \xi_1, \lambda_2, \xi_2) + \frac{(\mathcal{E}(\mathcal{X}) - \lambda_1)}{2} B_2(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
&\qquad + \frac{(\xi_2 - \mathcal{E}(\mathcal{Y}))}{2} B_3(\lambda_1, \xi_1, \lambda_2, \xi_2) + \big(\mathcal{E}(\mathcal{X}) - \lambda_1\big)\big(\xi_2 - \mathcal{E}(\mathcal{Y})\big) B_4(\lambda_1, \xi_1, \lambda_2, \xi_2)
\end{aligned}
$$

and

$$
\begin{aligned}
&A_4(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
&\quad = \frac{1}{4} B_1(\lambda_1, \xi_1, \lambda_2, \xi_2) + \frac{(\mathcal{E}(\mathcal{X}) - \lambda_1)}{2} B_2(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
&\qquad + \frac{(\mathcal{E}(\mathcal{Y}) - \xi_1)}{2} B_3(\lambda_1, \xi_1, \lambda_2, \xi_2) + \big(\mathcal{E}(\mathcal{X}) - \lambda_1\big)\big(\mathcal{E}(\mathcal{Y}) - \xi_1\big) B_4(\lambda_1, \xi_1, \lambda_2, \xi_2).
\end{aligned}
$$

Now, we give an application for our obtained results to the random variables.

Theorem 4.1 *The inequality*

$$\begin{aligned}&\left|\mathcal{E}_\alpha(\mathcal{X})\mathcal{E}_\beta(\mathcal{Y})-\left[\mathcal{E}(\mathcal{X})\right]^\alpha\mathcal{E}_\beta(\mathcal{Y})-\mathcal{E}_\alpha(\mathcal{X})\left[\mathcal{E}(\mathcal{Y})\right]^\beta+\left[\mathcal{E}(\mathcal{X})\right]^\alpha\left[\mathcal{E}(\mathcal{Y})\right]^\beta\right|\\&\quad\le\frac{\alpha\beta}{(\lambda_2-\lambda_1)(\xi_2-\xi_1)}\left[\lambda_1^{\alpha-1}\xi_1^{\beta-1}\Upsilon_1(\lambda_1,\xi_1,\lambda_2,\xi_2)+\lambda_1^{\alpha-1}\xi_2^{\beta-1}A_2(\lambda_1,\xi_1,\lambda_2,\xi_2)\right.\\&\qquad\left.+\lambda_2^{\alpha-1}\xi_1^{\beta-1}A_3(\lambda_1,\xi_1,\lambda_2,\xi_2)+\lambda_2^{\alpha-1}\xi_2^{\beta-1}A_4(\lambda_1,\xi_1,\lambda_2,\xi_2)\right]\end{aligned}\tag{4.1}$$

holds for $0<\lambda_1<\lambda_2$, $0<\xi_1<\xi_2$ *and* $\alpha,\beta\ge 2$, *where* $\Upsilon_1(\lambda_1,\xi_1,\lambda_2,\xi_2)$, $A_2(\lambda_1,\xi_1,\lambda_2,\xi_2)$, $A_3(\lambda_1,\xi_1,\lambda_2,\xi_2)$ *and* $A_4(\lambda_1,\xi_1,\lambda_2,\xi_2)$ *are defined by the previously shown equations.*

Proof Let $\alpha,\beta\ge 2$, $\Phi(\varrho,\varsigma)=\varrho^\alpha\varsigma^\beta$ be defined on $[\lambda_1,\lambda_2]\times[\xi_1,\xi_2]$. Then we clearly see that $|\Phi_{\varrho\varsigma}(\varrho,\varsigma)|=\alpha\beta\varrho^{\alpha-1}\varsigma^{\beta-1}$ is co-ordinated convex on $[\lambda_1,\lambda_2]\times[\xi_1,\xi_2]$ and

$$\begin{aligned}&\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\varrho,\varsigma)\omega(\varrho,\varsigma)\,d\varsigma\,d\varrho\\&\quad=\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\varrho^\alpha\varsigma^\beta\omega_1(\varrho)\omega_2(\varsigma)\,d\varsigma\,d\varrho\\&\quad=\int_{\lambda_1}^{\lambda_2}\varrho^\alpha\omega_1(\varrho)\,d\varrho\int_{\xi_1}^{\xi_2}\varsigma^\beta\omega_2(\varsigma)\,d\varsigma=\mathcal{E}_\alpha(\mathcal{X})\mathcal{E}_\beta(\mathcal{Y}),\\&\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\Upsilon_1,\varrho)\omega(\varrho,\varsigma)\,d\varsigma\,d\varrho=\left[\mathcal{E}(\mathcal{X})\right]^\alpha\mathcal{E}_\beta(\mathcal{Y}),\\&\int_{\lambda_1}^{\lambda_2}\int_{\xi_1}^{\xi_2}\Phi(\varrho,\Upsilon_3)\omega(\varrho,\varsigma)\,d\varsigma\,d\varrho=\mathcal{E}_\alpha(\mathcal{X})\left[\mathcal{E}(\mathcal{Y})\right]^\beta\end{aligned}$$

and

$$\Phi(\Upsilon_1,\Upsilon_3)=\left[\mathcal{E}(\mathcal{X})\right]^\alpha\left[\mathcal{E}(\mathcal{Y})\right]^\beta.$$

Therefore, inequality (4.1) follows from (3.5) and the above identities. □

Next, we provide an example to support our obtained results.

Example 4.2 Let

$$\Phi(\mu,\nu)=\begin{cases}\frac{4}{9}\mu\nu, & 1\le\mu,\nu\le 2,\\0, & \text{otherwise,}\end{cases}$$

be the joint probability density function of the random variables $\mathcal{X}$ and $\mathcal{Y}$. Then the marginal probability density functions of the random variables $\mathcal{X}$ and $\mathcal{Y}$ are given by

$$\mathcal{G}(\mu)=\begin{cases}\frac{2}{3}\mu, & 1\le\mu\le 2,\\0, & \text{otherwise,}\end{cases}$$

and

$$\mathcal{H}(\nu) = \begin{cases} \frac{2}{3}\nu, & 1 \le \nu \le 2, \\ 0, & \text{otherwise}, \end{cases}$$

respectively, and the random variables $\mathcal{X}$ and $\mathcal{Y}$ are independent due to $\Phi(\mu,\nu) = \mathcal{G}(\mu) \times \mathcal{H}(\nu)$. Elaborate computations give

$$\begin{aligned} &\mathcal{E}(\mathcal{X}) = \mathcal{E}(\mathcal{Y}) = \frac{14}{9}, \\ &B_1(\lambda_1,\xi_1,\lambda_2,\xi_2) = \frac{597{,}529}{13{,}947{,}137{,}604}, \\ &B_2(\lambda_1,\xi_1,\lambda_2,\xi_2) = \frac{618{,}400}{387{,}420{,}489}, \\ &B_3(\lambda_1,\xi_1,\lambda_2,\xi_2) = \frac{618{,}400}{387{,}420{,}489}, \end{aligned}$$

and

$$B_4(\lambda_1,\xi_1,\lambda_2,\xi_2) = \frac{2{,}560{,}000}{43{,}046{,}721}.$$

Hence,

$$\begin{aligned} &A_1(\lambda_1,\xi_1,\lambda_2,\xi_2) = \frac{77{,}281{,}681}{6{,}198{,}727{,}824}, \\ &A_2(\lambda_1,\xi_1,\lambda_2,\xi_2) = \frac{288{,}107{,}443}{18{,}596{,}183{,}472}, \\ &A_3(\lambda_1,\xi_1,\lambda_2,\xi_2) = \frac{288{,}107{,}443}{18{,}596{,}183{,}472}, \end{aligned}$$

and

$$A_4(\lambda_1,\xi_1,\lambda_2,\xi_2) = \frac{1{,}074{,}069{,}529}{55{,}788{,}550{,}416}.$$

Therefore, it follows from (4.1) that

$$\begin{aligned} &\left|\frac{4}{9}\left(\int_1^2 \mu^{\alpha+1}\,d\mu\right)\left(\int_1^2 \mu^{\beta+1}\,d\mu\right) - \frac{2}{3}\left(\frac{14}{9}\right)^{\alpha}\int_1^2 \mu^{\beta+1}\,d\mu \right. \\ &\quad \left. - \frac{2}{3}\left(\frac{14}{9}\right)^{\beta}\int_1^2 \mu^{\alpha+1}\,d\mu + \left(\frac{14}{9}\right)^{\alpha+\beta}\right| \\ &\quad \le \alpha\beta\left[\frac{77{,}281{,}681}{6{,}198{,}727{,}824} + \left(2^{\beta-1} + 2^{\alpha-1}\right) \times \frac{288{,}107{,}443}{18{,}596{,}183{,}472} \right. \\ &\quad \left. + 2^{\alpha+\beta-2} \times \frac{1{,}074{,}069{,}529}{55{,}788{,}550{,}416}\right] \end{aligned}$$

for $\alpha,\beta \geq 2$, that is,

$$\left|\frac{[2(9^{\alpha}-2^{2+\alpha}\times 9^{\alpha}+3\times 14^{\alpha})+3\times 14^{\alpha}\times\alpha][2(9^{\beta}-2^{2+\beta}\times 9^{\beta}+3\times 14^{\beta})+3\times 14^{\beta}\times\beta]}{9^{1+\alpha+\beta}(2+\alpha)(2+\beta)}\right|$$
$$\leq \frac{\alpha\beta(32{,}773\times 2^{\alpha}+52{,}746)(32{,}773\times 2^{\beta}+52{,}746)}{223{,}154{,}201{,}664}$$

for $\alpha,\beta \geq 2$.

5 Conclusion

In the article, we have established several weighted Hermite–Hadamard type inequalities for the co-ordinated convex and quasi-convex functions, provided an application to the moment of continuous random variables of bivariate distribution functions in the probability theory and presented an example on the probability distribution to support our results. Our results are generalizations of some previous results, and our approach may have further applications in the theory of convexity and Hermite–Hadamard inequality.

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