New generalized integral inequalities with applications

Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 4(3): 984–996.
DOI:10.3934/math.2019.3.984
Received: 06 March 2019
Accepted: 15 July 2019
Published: 31 July 2019

Research article

New generalized integral inequalities with applications

Artion Kashuri[1,*], **Rozana Liko**[1] **and Silvestru Sever Dragomir**[2]

[1] Department of Mathematics, Faculty of Technical Science, University Ismail Qemali, Vlora, Albania

[2] School of Engineering and Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia

* **Correspondence:** Email: artionkashuri@gmail.com.

Abstract: The authors have proved an identity for a generalized integral operator via differentiable function. By applying the established identity, the generalized trapezium type integral inequalities have been discovered. It is pointed out that the results of this research provide integral inequalities for almost all fractional integrals discovered in recent past decades. Various special cases have been identified. Some applications of presented results have been analyzed.

Keywords: Hermite-Hadamard inequality; general fractional integrals; Hölder's inequality; power mean inequality
Mathematics Subject Classification: Primary: 26A51; Secondary: 26A33, 26D07, 26D10, 26D15

1. Introduction

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be convex, $a_1, a_2 \in I$ and $a_1 < a_2$. Then*

$$f\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x)dx \leq \frac{f(a_1) + f(a_2)}{2}. \tag{1.1}$$

This inequality (1.1) is called Hermite-Hadamard inequality.

Authors of recent decades have studied (1.1) in the premises of newly invented definitions due to motivation of convex function. Interested readers see the references [1–20]. It is important to summarize the study of fractional integrals as follow:

Definition 1.2. The k-gamma function, where $k \in \mathbb{R}^+$ and $x \in \mathbb{C}$, is defined by

$$\Gamma_k(x) = \lim_{n \longrightarrow \infty} \frac{n! k^n (nk)^{\frac{x}{k} - 1}}{(x)_{n,k}}. \tag{1.2}$$

Its integral representation is given by

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt. \tag{1.3}$$

One can note that

$$\Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha). \tag{1.4}$$

Definition 1.3. [11] Let $f \in L[a_1, a_2]$. Then k-fractional integrals of order $\alpha, k > 0$, where $a_1 \geq 0$ is given as

$$I_{a_1^+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{a_1}^{x} (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a_1$$

and

$$I_{a_2^-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{x}^{a_2} (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \quad a_2 > x. \tag{1.5}$$

Let $\phi : [0, \infty) \longrightarrow [0, \infty)$ and

$$\int_0^1 \frac{\phi(t)}{t} dt < \infty, \tag{1.6}$$

$$\frac{1}{\mathbf{A}} \leq \frac{\phi(s)}{\phi(r)} \leq \mathbf{A} \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2 \tag{1.7}$$

$$\frac{\phi(r)}{r^2} \leq \mathbf{B}\frac{\phi(s)}{s^2} \text{ for } s \leq r \tag{1.8}$$

$$\left| \frac{\phi(r)}{r^2} - \frac{\phi(s)}{s^2} \right| \leq \mathbf{C}|r-s|\frac{\phi(r)}{r^2} \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2 \tag{1.9}$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C} > 0$ are independent of $r, s > 0$. If $\phi(r)r^\alpha$ is increasing for some $\alpha \geq 0$ and $\frac{\phi(r)}{r^\beta}$ is decreasing for some $\beta \geq 0$, then ϕ satisfies (1.6)–(1.9), see [15]. Therefore, the left-sided and right-sided generalized integral operators are defined as follows:

$$_{a_1^+}I_\phi f(x) = \int_{a_1}^{x} \frac{\phi(x-t)}{x-t} f(t)dt, \quad x > a_1, \tag{1.10}$$

$$_{a_2^-}I_\phi f(x) = \int_{x}^{a_2} \frac{\phi(t-x)}{t-x} f(t)dt, \quad x < a_2. \tag{1.11}$$

For other feature of generalized integrals, see [14].

The main objective of this paper is to discover in Section 2, an interesting identity in order to study some new bounds regarding trapezium type integral inequalities. By using the established identity as an auxiliary result, some new estimates for trapezoidal type integral inequalities via generalized integrals are obtained. In Section 3, some applications are given. At the end, a briefly conclusion is provided as well.

2. Main results

Let $a_1 < a_2$ and $m \in (0,1]$ be a fixed number. Throughout this study, for brevity, we define

$$\Lambda_m(t) := \int_0^t \frac{\phi\left((a_2 - ma_1)u\right)}{u} du < \infty, \quad \forall\, t \in [0,1], \quad \forall\, x \in P = [ma_1, a_2]. \tag{2.1}$$

The following lemma is crucial:

Lemma 2.1. *Let $f : P \longrightarrow \mathbb{R}$ be a differentiable mapping on (ma_1, a_2). If $f' \in L(P)$ and $\lambda \in (0,1]$, then*

$$\begin{aligned}
& f\left(ma_1 + \frac{\lambda}{2}(a_2 - ma_1)\right) - \frac{1}{2\Lambda_m\left(\frac{\lambda}{2}\right)} \\
& \times\Big[{}_{\left(ma_1 + \frac{\lambda}{2}(a_2 - ma_1)\right)^+}I_\phi f\left(ma_1(1-\lambda) + a_2\lambda\right) + {}_{\left(ma_1 + \frac{\lambda}{2}(a_2 - ma_1)\right)^-}I_\phi f\left(ma_1\right) \Big] \\
& = \frac{\lambda(a_2 - ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)} \times \left\{ \int_0^{\frac{1}{2}} \Lambda_m(\lambda t) f'\left(ma_1 + (\lambda t)(a_2 - ma_1)\right) dt \right. \\
& \left. - \int_{\frac{1}{2}}^1 \Lambda_m((1-t)\lambda) f'\left(ma_1 + (\lambda t)(a_2 - ma_1)\right) dt \right\}.
\end{aligned} \tag{2.2}$$

We denote

$$\begin{aligned}
T_{f,\Lambda_m}(\lambda; a_1, a_2) := & \frac{\lambda(a_2 - ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)} \times \left\{ \int_0^{\frac{1}{2}} \Lambda_m(\lambda t) f'\left(ma_1 + (\lambda t)(a_2 - ma_1)\right) dt \right. \\
& \left. - \int_{\frac{1}{2}}^1 \Lambda_m((1-t)\lambda) f'\left(ma_1 + (\lambda t)(a_2 - ma_1)\right) dt \right\}.
\end{aligned} \tag{2.3}$$

Proof. Integrating by parts Eq (2.3), we get

$$\begin{aligned}
T_{f,\Lambda_m}(\lambda; a_1, a_2) = & \frac{\lambda(a_2 - ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)} \times \left\{ \frac{\Lambda_m(\lambda t) f\left(ma_1 + (\lambda t)(a_2 - ma_1)\right)}{\lambda(a_2 - ma_1)} \right|_0^{\frac{1}{2}} \\
& - \frac{1}{(a_2 - ma_1)} \times \int_0^{\frac{1}{2}} \frac{\phi\left((a_2 - ma_1)(\lambda t)\right)}{\lambda t} f\left(ma_1 + (\lambda t)(a_2 - ma_1)\right) dt \\
& - \left. \frac{\Lambda_m((1-t)\lambda) f\left(ma_1 + (\lambda t)(a_2 - ma_1)\right)}{\lambda(a_2 - ma_1)} \right|_{\frac{1}{2}}^1 \\
& \left. - \frac{1}{(a_2 - ma_1)} \times \int_{\frac{1}{2}}^1 \frac{\phi\left((a_2 - ma_1)(1-t)\lambda\right)}{(1-t)\lambda} f\left(ma_1 + (\lambda t)(a_2 - ma_1)\right) dt \right\} \\
= & \frac{\lambda(a_2 - ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)}
\end{aligned}$$

$$\times\left\{\frac{\Lambda_m\left(\frac{\lambda}{2}\right)f\left(ma_1+\frac{\lambda}{2}(a_2-ma_1)\right)}{\lambda(a_2-ma_1)}-\frac{1}{(a_2-ma_1)}\times {}_{\left(ma_1+\frac{\lambda}{2}(a_2-ma_1)\right)^-}I_\phi f\,(ma_1)\right.$$

$$\left.+\frac{\Lambda_m\left(\frac{\lambda}{2}\right)f\left(ma_1+\frac{\lambda}{2}(a_2-ma_1)\right)}{\lambda(a_2-ma_1)}-\frac{1}{(a_2-ma_1)}\times {}_{\left(ma_1+\frac{\lambda}{2}(a_2-ma_1)\right)^+}I_\phi f\,(ma_1(1-\lambda)+a_2\lambda)\right\}$$

$$=f\left(ma_1+\frac{\lambda}{2}(a_2-ma_1)\right)-\frac{1}{2\Lambda_m\left(\frac{\lambda}{2}\right)}$$

$$\times\left[{}_{\left(ma_1+\frac{\lambda}{2}(a_2-ma_1)\right)^+}I_\phi f\,(ma_1(1-\lambda)+a_2\lambda)+{}_{\left(ma_1+\frac{\lambda}{2}(a_2-ma_1)\right)^-}I_\phi f\,(ma_1)\right].$$

The proof of Lemma 2.1 is completed. □

Remark 2.2. Taking $\lambda=1$ and $\phi(t)=t$ in Lemma 2.1, we have

$$T_f(a_1,a_2):=f\left(\frac{ma_1+a_2}{2}\right)-\frac{1}{(a_2-ma_1)}\int_{ma_1}^{a_2}f(t)dt$$

$$=(a_2-ma_1)\left\{\int_0^{\frac{1}{2}}tf'\,(ma_1+t(a_2-ma_1))\,dt-\int_{\frac{1}{2}}^1(1-t)f'\,(ma_1+t(a_2-ma_1))\,dt\right\}. \tag{2.4}$$

Theorem 2.3. *Let $f:P\longrightarrow\mathbb{R}$ be a differentiable mapping on (ma_1,a_2). If $|f'|^q$ is convex on P and $\lambda\in(0,1]$ for $q>1$ and $p^{-1}+q^{-1}=1$, then*

$$\left|T_{f,\Lambda_m}(\lambda;a_1,a_2)\right|\le\frac{\lambda(a_2-ma_1)}{2\sqrt[q]{8}\Lambda_m\left(\frac{\lambda}{2}\right)}\sqrt[p]{B_{\Lambda_m}(\lambda;p)} \tag{2.5}$$

$$\times\left\{\sqrt[q]{(4-\lambda)|f'(ma_1)|^q+\lambda|f'(a_2)|^q}+\sqrt[q]{(4-3\lambda)|f'(ma_1)|^q+3\lambda|f'(a_2)|^q}\right\},$$

where

$$B_{\Lambda_m}(\lambda;p):=\int_0^{\frac{1}{2}}\left[\Lambda_m(\lambda t)\right]^p dt. \tag{2.6}$$

Proof. Using Lemma 2.1, convexity of $|f'|^q$ and Hölder inequality, we get

$$\left|T_{f,\Lambda_m}(\lambda;a_1,a_2)\right|\le\frac{\lambda(a_2-ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)}\times\left\{\int_0^{\frac{1}{2}}\Lambda_m(\lambda t)\left|f'\,(ma_1+(\lambda t)(a_2-ma_1))\right|dt\right.$$

$$\left.+\int_{\frac{1}{2}}^1\Lambda_m((1-t)\lambda)\left|f'\,(ma_1+(\lambda t)(a_2-ma_1))\right|dt\right\}$$

$$\le\frac{\lambda(a_2-ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)}\times\left\{\left(\int_0^{\frac{1}{2}}\left[\Lambda_m(\lambda t)\right]^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'\,(ma_1+(\lambda t)(a_2-ma_1))\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^1\left[\Lambda_m((1-t)\lambda)\right]^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f'\,(ma_1+(\lambda t)(a_2-ma_1))\right|^q dt\right)^{\frac{1}{q}}\right\}$$

$$\leq \frac{\lambda(a_2 - ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)} \sqrt[p]{B_{\Lambda_m}(\lambda;p)} \times \left\{ \left(\int_0^{\frac{1}{2}} \left[(1-\lambda t)\left|f'(ma_1)\right|^q + (\lambda t)\left|f'(a_2)\right|^q \right] dt \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\int_{\frac{1}{2}}^1 \left[(1-\lambda t)\left|f'(ma_1)\right|^q + (\lambda t)\left|f'(a_2)\right|^q \right] dt \right)^{\frac{1}{q}} \right\}$$

$$= \frac{\lambda(a_2 - ma_1)}{2\sqrt[q]{8}\Lambda_m\left(\frac{\lambda}{2}\right)} \sqrt[p]{B_{\Lambda_m}(\lambda;p)}$$

$$\times \left\{ \sqrt[q]{(4-\lambda)|f'(ma_1)|^q + \lambda|f'(a_2)|^q} + \sqrt[q]{(4-3\lambda)|f'(ma_1)|^q + 3\lambda|f'(a_2)|^q} \right\}.$$

The proof of Theorem 2.3 is completed. □

Corollary 2.4. *For $p = q = 2$ in Theorem 2.3, we get*

$$\left|T_{f,\Lambda_m}(\lambda;a_1,a_2)\right| \leq \frac{\lambda(a_2 - ma_1)}{4\sqrt{2}\Lambda_m\left(\frac{\lambda}{2}\right)} \sqrt{B_{\Lambda_m}(\lambda;2)} \tag{2.7}$$

$$\times \left\{ \sqrt{(4-\lambda)|f'(ma_1)|^2 + \lambda|f'(a_2)|^2} + \sqrt{(4-3\lambda)|f'(ma_1)|^2 + 3\lambda|f'(a_2)|^2} \right\}.$$

Corollary 2.5. *For $\phi(t) = t$ in Theorem 2.3, we have*

$$\left|T_{f,\Lambda_m}(\lambda;a_1,a_2)\right| \leq \frac{\lambda(a_2 - ma_1)}{\sqrt[q]{8}\sqrt[p]{2^{p+1}(p+1)}} \tag{2.8}$$

$$\times \left\{ \sqrt[q]{(4-\lambda)|f'(ma_1)|^q + \lambda|f'(a_2)|^q} + \sqrt[q]{(4-3\lambda)|f'(ma_1)|^q + 3\lambda|f'(a_2)|^q} \right\}.$$

Remark 2.6. For $\lambda = 1$ in Corollary 2.5, we obtain

$$\left|T_f(a_1,a_2)\right| \leq \frac{(a_2 - ma_1)}{\sqrt[q]{8}\sqrt[p]{2^{p+1}(p+1)}} \tag{2.9}$$

$$\times \left\{ \sqrt[q]{3|f'(ma_1)|^q + |f'(a_2)|^q} + \sqrt[q]{|f'(ma_1)|^q + 3|f'(a_2)|^q} \right\}.$$

Corollary 2.7. *For $\phi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 2.3, we get*

$$\left|T_{f,\Lambda_m}(\lambda;a_1,a_2)\right| \leq \frac{2^{\alpha-1}\lambda(a_2 - ma_1)}{\sqrt[q]{8}\sqrt[p]{2^{p\alpha+1}(p\alpha+1)}} \tag{2.10}$$

$$\times \left\{ \sqrt[q]{(4-\lambda)|f'(ma_1)|^q + \lambda|f'(a_2)|^q} + \sqrt[q]{(4-3\lambda)|f'(ma_1)|^q + 3\lambda|f'(a_2)|^q} \right\}.$$

Corollary 2.8. *For $\phi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.3, we have*

$$\left|T_{f,\Lambda_m}(\lambda;a_1,a_2)\right| \leq \frac{2^{\frac{\alpha}{k}-1}\lambda(a_2 - ma_1)}{\sqrt[q]{8}\sqrt[p]{2^{\frac{p\alpha}{k}+1}(\frac{p\alpha}{k}+1)}} \tag{2.11}$$

$$\times \left\{ \sqrt[q]{(4-\lambda)|f'(ma_1)|^q + \lambda|f'(a_2)|^q} + \sqrt[q]{(4-3\lambda)|f'(ma_1)|^q + 3\lambda|f'(a_2)|^q} \right\}.$$

Corollary 2.9. *For $\phi(t) = t(a_2 - t)^{\alpha-1}$ for $\alpha \in (0,1)$ in Theorem 2.3, we obtain*

$$\left|T_{f,\Lambda_m}(\lambda; a_1, a_2)\right| \leq \frac{\lambda\alpha(a_2 - ma_1)}{2\sqrt[q]{8}\Big[a_2^\alpha - \Big((ma_1 - a_2)\frac{\lambda}{2} + a_2\Big)^\alpha\Big]} \sqrt[p]{B^\star_{\Lambda_m}(\lambda; p)} \tag{2.12}$$

$$\times\Big\{\sqrt[q]{(4-\lambda)|f'(ma_1)|^q + \lambda|f'(a_2)|^q} + \sqrt[q]{(4-3\lambda)|f'(ma_1)|^q + 3\lambda|f'(a_2)|^q}\Big\},$$

where

$$B^\star_{\Lambda_m}(\lambda; p) := \frac{1}{\lambda\alpha^p(a_2 - ma_1)} \int_{(ma_1-a_2)\frac{\lambda}{2}+a_2}^{a_2} (a_2^\alpha - t^\alpha)^p \, dt. \tag{2.13}$$

Corollary 2.10. *For $\phi(t) = \frac{t}{\alpha}\exp\Big[\Big(-\frac{1-\alpha}{\alpha}\Big)t\Big]$ for $\alpha \in (0,1)$ in Theorem 2.3, we get*

$$\left|T_{f,\Lambda_m}(\lambda; a_1, a_2)\right| \leq \frac{\lambda(\alpha-1)(a_2 - ma_1)}{2\sqrt[q]{8}\Big\{\exp\Big[\Big(-\frac{1-\alpha}{\alpha}\Big)(a_2 - ma_1)\frac{\lambda}{2}\Big] - 1\Big\}} \sqrt[p]{B^\diamond_{\Lambda_m}(\lambda; p)} \tag{2.14}$$

$$\times\Big\{\sqrt[q]{(4-\lambda)|f'(ma_1)|^q + \lambda|f'(a_2)|^q} + \sqrt[q]{(4-3\lambda)|f'(ma_1)|^q + 3\lambda|f'(a_2)|^q}\Big\},$$

where

$$B^\diamond_{\Lambda_m}(\lambda; p) := \frac{\alpha}{\lambda(\alpha-1)^{p+1}(a_2 - ma_1)} \int_0^{\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)(a_2-ma_1)\frac{\lambda}{2}\right]-1} \frac{t^p}{t+1} dt. \tag{2.15}$$

Theorem 2.11. *Let $f : P \longrightarrow \mathbb{R}$ be a differentiable mapping on (ma_1, a_2). If $|f'|^q$ is convex on P and $\lambda \in (0,1]$ for $q \geq 1$, then*

$$\left|T_{f,\Lambda_m}(\lambda; a_1, a_2)\right| \leq \frac{\lambda(a_2 - ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)} \left(B_{\Lambda_m}(\lambda; 1)\right)^{1-\frac{1}{q}} \tag{2.16}$$

$$\times\Bigg\{\sqrt[q]{[B_{\Lambda_m}(\lambda;1) - \lambda C_{\Lambda_m}(\lambda)]|f'(ma_1)|^q + \lambda C_{\Lambda_m}(\lambda)|f'(a_2)|^q}$$

$$+\sqrt[q]{[(1-\lambda)B_{\Lambda_m}(\lambda;1) - \lambda C_{\Lambda_m}(\lambda)]|f'(ma_1)|^q + \lambda[B_{\Lambda_m}(\lambda;1) - C_{\Lambda_m}(\lambda)]|f'(a_2)|^q}\Bigg\},$$

where

$$C_{\Lambda_m}(\lambda) := \int_0^{\frac{1}{2}} t\Lambda_m(\lambda t) dt \tag{2.17}$$

and $B_{\Lambda_m}(\lambda; 1)$ is defined as in Theorem 2.3.

Proof. Using Lemma 2.1, convexity of $|f'|^q$ and power mean inequality, we get

$$\left|T_{f,\Lambda_m}(\lambda; a_1, a_2)\right| \leq \frac{\lambda(a_2 - ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)} \times \Bigg\{\int_0^{\frac{1}{2}} \Lambda_m(\lambda t)\left|f'(ma_1 + (\lambda t)(a_2 - ma_1))\right| dt$$

$$+ \int_{\frac{1}{2}}^1 \Lambda_m((1-t)\lambda)\left|f'(ma_1 + (\lambda t)(a_2 - ma_1))\right| dt\Bigg\}$$

$$\leq \frac{\lambda(a_2-ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)} \times \left\{ \left(\int_0^{\frac{1}{2}} \Lambda_m(\lambda t)dt \right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{2}} \Lambda_m(\lambda t) \left| f'\left(ma_1 + (\lambda t)(a_2-ma_1)\right) \right|^q dt \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\int_{\frac{1}{2}}^1 \Lambda_m((1-t)\lambda)dt \right)^{1-\frac{1}{q}} \left(\int_{\frac{1}{2}}^1 \Lambda_m((1-t)\lambda) \left| f'\left(ma_1 + (\lambda t)(a_2-ma_1)\right) \right|^q dt \right)^{\frac{1}{q}} \right\}$$

$$\leq \frac{\lambda(a_2-ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)} \left(B_{\Lambda_m}(\lambda;1)\right)^{1-\frac{1}{q}} \times \left\{ \left(\int_0^{\frac{1}{2}} \Lambda_m(\lambda t) \left[(1-\lambda t) \left| f'(ma_1) \right|^q + (\lambda t) \left| f'(a_2) \right|^q \right] dt \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\int_{\frac{1}{2}}^1 \Lambda_m((1-t)\lambda) \left[(1-\lambda t) \left| f'(ma_1) \right|^q + (\lambda t) \left| f'(a_2) \right|^q \right] dt \right)^{\frac{1}{q}} \right\}$$

$$= \frac{\lambda(a_2-ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)} \left(B_{\Lambda_m}(\lambda;1)\right)^{1-\frac{1}{q}}$$

$$\times \left\{ \sqrt[q]{\left[B_{\Lambda_m}(\lambda;1) - \lambda C_{\Lambda_m}(\lambda)\right] |f'(ma_1)|^q + \lambda C_{\Lambda_m}(\lambda) |f'(a_2)|^q} \right.$$

$$\left. + \sqrt[q]{\left[(1-\lambda) B_{\Lambda_m}(\lambda;1) - \lambda C_{\Lambda_m}(\lambda)\right] |f'(ma_1)|^q + \lambda\left[B_{\Lambda_m}(\lambda;1) - C_{\Lambda_m}(\lambda)\right] |f'(a_2)|^q} \right\}.$$

The proof of Theorem 2.11 is completed. □

Corollary 2.12. *For $q = 1$ in Theorem 2.11, we get*

$$\left| T_{f,\Lambda_m}(\lambda; a_1, a_2) \right| \leq \frac{\lambda(a_2-ma_1)}{2\Lambda_m\left(\frac{\lambda}{2}\right)} \tag{2.18}$$

$$\times \left\{ \left[(2-\lambda) B_{\Lambda_m}(\lambda;1) - 2\lambda C_{\Lambda_m}(\lambda)\right] |f'(ma_1)| + \lambda B_{\Lambda_m}(\lambda;1) |f'(a_2)| \right\}.$$

Corollary 2.13. *For $\phi(t) = t$ in Theorem 2.11, we have*

$$\left| T_{f,\Lambda_m}(\lambda; a_1, a_2) \right| \leq \frac{\lambda(a_2-ma_1)}{8\sqrt[q]{3}} \tag{2.19}$$

$$\times \left\{ \sqrt[q]{(3-\lambda)|f'(ma_1)|^q + \lambda|f'(a_2)|^q} + \sqrt[q]{(4\lambda-3)|f'(ma_1)|^q + 2\lambda|f'(a_2)|^q} \right\}.$$

Remark 2.14. For $\lambda = 1$ in Corollary 2.13, we obtain

$$\left| T_f(a_1, a_2) \right| \leq \frac{(a_2-ma_1)}{8\sqrt[q]{3}} \tag{2.20}$$

$$\times \left\{ \sqrt[q]{2|f'(ma_1)|^q + |f'(a_2)|^q} + \sqrt[q]{|f'(ma_1)|^q + 2|f'(a_2)|^q} \right\}.$$

Corollary 2.15. *For $\phi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$ in Theorem 2.11, we get*

$$\left|T_{f,\Lambda_m}(\lambda; a_1, a_2)\right| \le \frac{\lambda(a_2 - ma_1)}{4} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2)} \tag{2.21}$$

$$\times\left\{ \sqrt[q]{\left(1 - \frac{\lambda(\alpha+1)}{2(\alpha+2)}\right)|f'(ma_1)|^q + \frac{\lambda(\alpha+1)}{2(\alpha+2)}|f'(a_2)|^q}\right.$$

$$\left. + \sqrt[q]{\left((1-\lambda) - \frac{\lambda(\alpha+1)}{2(\alpha+2)}\right)|f'(ma_1)|^q + \frac{\lambda(\alpha+3)}{2(\alpha+2)}|f'(a_2)|^q}\right\}.$$

Corollary 2.16. *For $\phi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.11, we have*

$$\left|T_{f,\Lambda_m}(\lambda; a_1, a_2)\right| \le \frac{\lambda(a_2 - ma_1)}{4} \frac{\Gamma_k(\alpha+k)}{\Gamma_k(\alpha+k+1)} \tag{2.22}$$

$$\times\left\{ \sqrt[q]{\left(1 - \frac{\lambda(\frac{\alpha}{k}+1)}{2(\frac{\alpha}{k}+2)}\right)|f'(ma_1)|^q + \frac{\lambda(\frac{\alpha}{k}+1)}{2(\frac{\alpha}{k}+2)}|f'(a_2)|^q}\right.$$

$$\left. + \sqrt[q]{\left((1-\lambda) - \frac{\lambda(\frac{\alpha}{k}+1)}{2(\frac{\alpha}{k}+2)}\right)|f'(ma_1)|^q + \frac{\lambda(\frac{\alpha}{k}+3)}{2(\frac{\alpha}{k}+2)}|f'(a_2)|^q}\right\}.$$

Corollary 2.17. *For $\phi(t) = t(a_2 - t)^{\alpha-1}$ for $\alpha \in (0,1)$ in Theorem 2.11, we obtain*

$$\left|T_{f,\Lambda_m}(\lambda; a_1, a_2)\right| \le \frac{\lambda\alpha(a_2 - ma_1)}{2\left[a_2^{\alpha} - \left((ma_1 - a_2)\frac{\lambda}{2} + a_2\right)^{\alpha}\right]} \left(B^*_{\Lambda_m}(\lambda; 1)\right)^{1-\frac{1}{q}} \tag{2.23}$$

$$\times\left\{ \sqrt[q]{[B^*_{\Lambda_m}(\lambda;1) - \lambda C^*_{\Lambda_m}(\lambda)]|f'(ma_1)|^q + \lambda C^*_{\Lambda_m}(\lambda)|f'(a_2)|^q}\right.$$

$$\left. + \sqrt[q]{[(1-\lambda)B^*_{\Lambda_m}(\lambda;1) - \lambda C^*_{\Lambda_m}(\lambda)]|f'(ma_1)|^q + \lambda[B^*_{\Lambda_m}(\lambda;1) - C^*_{\Lambda_m}(\lambda)]|f'(a_2)|^q}\right\},$$

where

$$C^*_{\Lambda_m}(\lambda) := \frac{1}{\alpha}\int_0^{\frac{1}{2}} t[a_2^{\alpha} - ((ma_1 - a_2)\lambda t + a_2)^{\alpha}]dt \tag{2.24}$$

and $B^*_{\Lambda_m}(\lambda; 1)$ is defined by Eq (2.13) for $p = 1$.

Corollary 2.18. *For $\phi(t) = \frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$ for $\alpha \in (0,1)$ in Theorem 2.11, we get*

$$\left|T_{f,\Lambda_m}(\lambda; a_1, a_2)\right| \le \frac{\lambda(\alpha-1)(a_2 - ma_1)}{2\left\{\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)(a_2 - ma_1)\frac{\lambda}{2}\right] - 1\right\}} \sqrt[p]{B^{\diamond}_{\Lambda_m}(\lambda;1)} \tag{2.25}$$

$$\times\left\{ \sqrt[q]{[B^{\diamond}_{\Lambda_m}(\lambda;1) - \lambda C^{\diamond}_{\Lambda_m}(\lambda)]|f'(ma_1)|^q + \lambda C^{\diamond}_{\Lambda_m}(\lambda)|f'(a_2)|^q}\right.$$

$$+\sqrt[q]{[(1-\lambda)B^{\diamond}_{\Lambda_m}(\lambda;1)-\lambda C^{\diamond}_{\Lambda_m}(\lambda)]|f'(ma_1)|^q+\lambda[B^{\diamond}_{\Lambda_m}(\lambda;1)-C^{\diamond}_{\Lambda_m}(\lambda)]|f'(a_2)|^q}\Bigg\},$$

where

$$C^{\diamond}_{\Lambda_m}(\lambda) := \frac{1}{(\alpha-1)}\int_0^{\frac{1}{2}}\left\{\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)(a_2-ma_1)\lambda t\right]-1\right\}dt. \tag{2.26}$$

and $B^{\diamond}_{\Lambda_m}(\lambda;1)$ *is defined by eq (2.15) for* $p=1$.

3. Applications

1.
$$A := A(\wp_1,\wp_2) = \frac{\wp_1+\wp_2}{2},$$

2.
$$H := H(\wp_1,\wp_2) = \frac{2}{\frac{1}{\wp_1}+\frac{1}{\wp_2}},$$

3.
$$L := L(\wp_1,\wp_2) = \frac{\wp_2-\wp_1}{\ln|\wp_2|-\ln|\wp_1|},$$

4.
$$L_r := L_r(\wp_1,\wp_2) = \left[\frac{\wp_2^{r+1}-\wp_1^{r+1}}{(r+1)(\wp_2-\wp_1)}\right]^{\frac{1}{r}};\quad r\in\mathbb{Z}\setminus\{-1,0\}.$$

From Section 2, we obtain:

Proposition 3.1. *Let* $m\in(0,1]$ *and* $a_1,a_2\in\mathbb{R}\setminus\{0\}$, *where* $a_1<a_2$. *Then for* $r\in\mathbb{N}$ *and* $r\geq 2$, *where* $q>1$ *and* $p^{-1}+q^{-1}=1$,

$$\left|A^r(ma_1,a_2)-L_r^r(ma_1,a_2)\right| \leq \frac{r(a_2-ma_1)}{\sqrt[q]{4}\sqrt[p]{2^{p+1}(p+1)}} \tag{3.1}$$

$$\times\left\{\sqrt[q]{A\left(3|ma_1|^{q(r-1)},|a_2|^{q(r-1)}\right)}+\sqrt[q]{A\left(|ma_1|^{q(r-1)},3|a_2|^{q(r-1)}\right)}\right\}.$$

Proof. Taking $\lambda=1,\ f(t)=t^r$ and $\phi(t)=t$, in Theorem 2.3. □

Proposition 3.2. *Let* $m\in(0,1]$ *and* $a_1,a_2\in\mathbb{R}\setminus\{0\}$, *where* $a_1<a_2$. *Then for* $q>1$ *and* $p^{-1}+q^{-1}=1$,

$$\left|\frac{1}{A(ma_1,a_2)}-\frac{1}{L(ma_1,a_2)}\right| \leq \sqrt[q]{\frac{3}{4}}\frac{(a_2-ma_1)}{\sqrt[p]{2^{p+1}(p+1)}} \tag{3.2}$$

$$\times\left\{\frac{1}{\sqrt[q]{H\left(3|ma_1|^{2q},|a_2|^{2q}\right)}}+\frac{1}{\sqrt[q]{H\left(|ma_1|^{2q},3|a_2|^{2q}\right)}}\right\}.$$

Proof. Taking $\lambda=1,\ f(t)=\frac{1}{t}$ and $\phi(t)=t$, in Theorem 2.3. □

Proposition 3.3. *Let $m \in (0,1]$ and $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, where $a_1 < a_2$. Then for $r \in \mathbb{N}$ and $r \geq 2$, where $q \geq 1$,*

$$\left|A^r(ma_1, a_2) - L_r^r(ma_1, a_2)\right| \leq \sqrt[q]{\frac{2}{3}}\frac{r(a_2 - ma_1)}{8} \tag{3.3}$$

$$\times\left\{\sqrt[q]{A\left(2|ma_1|^{q(r-1)}, |a_2|^{q(r-1)}\right)} + \sqrt[q]{A\left(|ma_1|^{q(r-1)}, 2|a_2|^{q(r-1)}\right)}\right\}.$$

Proof. Taking $\lambda = 1$, $f(t) = t^r$ and $\phi(t) = t$, in Theorem 2.11. □

Proposition 3.4. *Let $m \in (0,1]$ and $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, where $a_1 < a_2$. Then for $q \geq 1$, the following inequality hold:*

$$\left|\frac{1}{A(ma_1, a_2)} - \frac{1}{L(ma_1, a_2)}\right| \leq \sqrt[q]{\frac{4}{3}}\frac{(a_2 - ma_1)}{8} \tag{3.4}$$

$$\times\left\{\frac{1}{\sqrt[q]{H\left(2|ma_1|^{2q}, |a_2|^{2q}\right)}} + \frac{1}{\sqrt[q]{H\left(|ma_1|^{2q}, 2|a_2|^{2q}\right)}}\right\}.$$

Proof. Taking $\lambda = 1$, $f(t) = \frac{1}{t}$ and $\phi(t) = t$, in Theorem 2.11. □

Next, we provide some new error estimates for the midpoint formula. Let Q be the partition of $a_1 = \ell_0 < \ell_1 < \ldots < \ell_k = a_2$ of $[a_1, a_2]$. Let consider the following quadrature formula:

$$\int_{a_1}^{a_2} f(x)dx = M(f, Q) + E(f, Q),$$

where

$$M(f, Q) = \sum_{i=0}^{k-1} f\left(\frac{\ell_i + \ell_{i+1}}{2}\right)(\ell_{i+1} - \ell_i)$$

is the midpoint version and $E(f, Q)$ is denote their associated approximation error.

Proposition 3.5. *Let $f : [a_1, a_2] \longrightarrow \mathbb{R}$ be a differentiable function on (a_1, a_2), where $a_1 < a_2$. If $|f'|^q$ is convex on $[a_1, a_2]$ for $q > 1$ and $p^{-1} + q^{-1} = 1$, then*

$$\left|E(f, Q)\right| \leq \frac{1}{\sqrt[q]{8}\sqrt[p]{2^{p+1}(p+1)}} \times \sum_{i=0}^{k-1}(\ell_{i+1} - \ell_i)^2 \tag{3.5}$$

$$\times\left\{\sqrt[q]{3|f'(\ell_i)|^q + |f'(\ell_{i+1})|^q} + \sqrt[q]{|f'(\ell_i)|^q + 3|f'(\ell_{i+1})|^q}\right\}.$$

Proof. Applying Theorem 2.3 for $\lambda = m = 1$ and $\phi(t) = t$ on $[\ell_i, \ell_{i+1}]$ $(i = 0, \ldots, k-1)$ of Q, we have

$$\left|f\left(\frac{\ell_i + \ell_{i+1}}{2}\right) - \frac{1}{(\ell_{i+1} - \ell_i)}\int_{\ell_i}^{\ell_{i+1}} f(x)dx\right| \leq \frac{(\ell_{i+1} - \ell_i)}{\sqrt[q]{8}\sqrt[p]{2^{p+1}(p+1)}} \tag{3.6}$$

$$\times\left\{\sqrt[q]{3|f'(\ell_i)|^q + |f'(\ell_{i+1})|^q} + \sqrt[q]{|f'(\ell_i)|^q + 3|f'(\ell_{i+1})|^q}\right\}.$$

Hence from (3.6), we get

$$\left|E(f,Q)\right| = \left|\int_{a_1}^{a_2} f(x)dx - M(f,Q)\right|$$

$$\leq \left|\sum_{i=0}^{k-1}\left\{\int_{\ell_i}^{\ell_{i+1}} f(x)dx - f\left(\frac{\ell_i+\ell_{i+1}}{2}\right)(\ell_{i+1}-\ell_i)\right\}\right|$$

$$\leq \sum_{i=0}^{k-1}\left|\left\{\int_{\ell_i}^{\ell_{i+1}} f(x)dx - f\left(\frac{\ell_i+\ell_{i+1}}{2}\right)(\ell_{i+1}-\ell_i)\right\}\right|$$

$$\leq \frac{1}{\sqrt[q]{8}\sqrt[p]{2^{p+1}(p+1)}} \times \sum_{i=0}^{k-1}(\ell_{i+1}-\ell_i)^2$$

$$\times\left\{\sqrt[q]{3|f'(\ell_i)|^q + |f'(\ell_{i+1})|^q} + \sqrt[q]{|f'(\ell_i)|^q + 3|f'(\ell_{i+1})|^q}\right\}.$$

□

Proposition 3.6. *Let* $f : [a_1, a_2] \longrightarrow \mathbb{R}$ *be a differentiable function on* (a_1, a_2), *where* $a_1 < a_2$. *If* $|f'|^q$ *is convex on* $[a_1, a_2]$ *for* $q \geq 1$, *then*

$$\left|E(f,Q)\right| \leq \frac{1}{8\sqrt[q]{3}} \times \sum_{i=0}^{k-1}(\ell_{i+1}-\ell_i)^2 \tag{3.7}$$

$$\times\left\{\sqrt[q]{2|f'(\ell_i)|^q + |f'(\ell_{i+1})|^q} + \sqrt[q]{|f'(\ell_i)|^q + 2|f'(\ell_{i+1})|^q}\right\}.$$

Proof. The proof is analogous as to that of Proposition 3.5 but use Theorem 2.11. □

Remark 3.7. Applying our Theorems 2.3 and 2.11, where $m = 1$, for appropriate choices of function $\phi(t)$, we can deduce some new bounds for midpoint formula using above ideas and techniques. The details are left to the interested reader.

4. Conclusion

The authors have proved an identity for a generalized integral operator via differentiable function. By applying the established identity, the generalized trapezium type integral inequalities have been discovered. Some applications of presented results have been analyzed. Interested reader can establish new inequalities by using different integral operators and they can be applied in convex analysis, optimization and different area of pure and applied mathematics.

Acknowledgments

The authors would like to thank the referees for valuable comments and suggestions for improved our manuscript.

Conflict of interest

The authors declare that they have no competing interests.

References

1. S. M. Aslani, M. R. Delavar and S. M. Vaezpour, *Inequalities of Fejér type related to generalized convex functions with applications*, Int. J. Anal. Appl., **16** (2018), 38–49.
2. F. X. Chen and S. H. Wu, *Several complementary inequalities to inequalities of Hermite-Hadamard type for s-convex functions*, J. Nonlinear Sci. Appl., **9** (2016), 705–716.
3. Y. M. Chu, M. A. Khan, T. U. Khan, et al. *Generalizations of Hermite-Hadamard type inequalities for MT-convex functions*, J. Nonlinear Sci. Appl., **9** (2016), 4305–4316.
4. M. R. Delavar and M. De La Sen, *Some generalizations of Hermite-Hadamard type inequalities*, SpringerPlus, **5** (2016), 1–9.
5. S. S. Dragomir and R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and trapezoidal formula*, Appl. Math. Lett., **11** (1998), 91–95.
6. G. Farid and A. U. Rehman, *Generalizations of some integral inequalities for fractional integrals*, Ann. Math. silesianae, **31** (2017), 14.
7. M. A. Khan, Y. M. Chu, A. Kashuri, et al. *New Hermite-Hadamard inequalities for conformable fractional integrals*, J. Funct. Space., **2018** (2018), Article ID 6928130.
8. W. Liu, W. Wen and J. Park, *Hermite-Hadamard type inequalities for MT-convex functions via classical integrals and fractional integrals*, J. Nonlinear Sci. Appl., **9** (2016), 766–777.
9. C. Luo, T. S. Du, M. A. Khan, et al. *Some k-fractional integrals inequalities through generalized $\lambda_{\phi m}$-MT-preinvexity*, J. Comput. Anal. Appl., **27** (2019), 690–705.
10. M. V. Mihai, *Some Hermite-Hadamard type inequalities via Riemann-Liouville fractional calculus*, Tamkang J. Math, **44** (2013), 411–416.
11. S. Mubeen and G. M. Habibullah, *k-Fractional integrals and applications*, Int. J. Contemp. Math. Sci., **7** (2012), 89–94.
12. O. Omotoyinbo and A. Mogbodemu, *Some new Hermite-Hadamard integral inequalities for convex functions*, Int. J. Sci. Innovation Tech., **1** (2014), 1–12.
13. M. E. Özdemir, S. S. Dragomir and C. Yildiz, *The Hadamard's inequality for convex function via fractional integrals*, Acta Math. Sci., **33** (2013), 153–164.
14. M. Z. Sarikaya and F. Ertuğral, *On the generalized Hermite-Hadamard inequalities*, 2017. Available from: `https://www.researchgate.net/publication/321760443.`
15. M. Z. Sarikaya and H. Yildirim, *On generalization of the Riesz potential*, Indian J. Math. Mathematical Sci., **3** (2007), 231–235.
16. E. Set, M. A. Noor, M. U. Awan, et al. *Generalized Hermite-Hadamard type inequalities involving fractional integral operators*, J. Inequal. Appl., **2017** (2017), 1–10.
17. H. Wang, T. S. Du and Y. Zhang, *k-fractional integral trapezium-like inequalities through (h, m)-convex and (α, m)-convex mappings*, J. Inequal. Appl., **2017** (2017), 1–20.

18. B. Y. Xi and F. Qi, *Some integral inequalities of Hermite-Hadamard type for convex functions with applications to means*, J. Funct. Spaces Appl., **2012** (2012), Article ID 980438.

19. X. M. Zhang, Y. M. Chu and X. H. Zhang, *The Hermite-Hadamard type inequality of GA-convex functions and its applications*, J. Inequal. Appl., **2010** (2010), Article ID 507560.

20. Y. Zhang, T. S. Du, H. Wang, et al. *Extensions of different type parameterized inequalities for generalized* (m, h)*-preinvex mappings via k-fractional integrals*, J. Inequal. Appl., **2018** (2018), 1–30.