Developing a DSGE Consumption Function for a CGE Model

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by Peter B. Dixon and Maureen R. Rimmer

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Abstract

DSGE models incorporate attractive theoretical specifications of the behaviour of forward-looking households facing an uncertain future. Central to these specifications is the idea that households decide their consumption level in year t by applying a function (policy rule) whose arguments represent information available in year t. Using the insight that, under certain conditions, the policy rule (but not the resulting policy) is invariant through time, DSGE modellers have developed the perturbation and other methods for quantitatively specifying policy rules. They have applied these methods in small macro models. In this paper we adapt the perturbation method so that it can be used to specify a policy rule for household consumption in a full-scale CGE model. A novel feature of our method is the use of specially constructed CGE simulations to reveal key parameters used in deriving the policy rule. We apply our method in an illustrative simulation of the effects of a technology shock in a 70-sector version of the USAGE model of the U.S. economy.

JEL: E21; C61; C68; C63

Key words: Consumption function; Dynamic stochastic general equilibrium; Computable general equilibrium; Perturbation method

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Contents

1. Introduction

In the DSGE[1] theory of consumer behaviour, the household determines consumption in year t by applying a rule (the policy function) that takes account of all available information. While the household does not know the values of future variables with certainty, the household does know that it will be applying the same policy function in future years as in the current year. With a steady-growth baseline, this invariance of the policy function allows us to deduce its derivatives with respect to wealth and variables exogenous to the household such as technology and the terms of trade. From here, we can obtain a first-order approximation to the policy function which shows how aggregate consumption deviates from its baseline path in response to changes in wealth, and changes in variables exogenous to the household.

DSGE specifications of household behaviour are common in small macro models. We show how a DSGE specification can be formulated and applied in a full-scale CGE model. Our method is a variation of the DSGE perturbation approach (see Schmitt-Grohé and Uribe, 2004). However, we rely on CGE simulations to derive the elasticities of household wealth at the start of year t+1 with respect to household wealth at the start of year t, consumption in year t and exogenous variables in year t. These elasticities then become the core ingredients in formulas for the derivatives of the household's policy function. GEMPACK software (see Harrison *et al.* 2014, and Horridge *et al.* 2013) is ideal for our method which requires working with derivatives, elasticities and first-order approximations.

The rest of this paper is organized as follows.

In section 2, we study a 1-sector, 1-household neoclassical growth model. We refer to this as the standard model and use it to explain what we see as the central ideas in DSGE theory and the perturbation solution strategy. We hope this section will be useful to CGE modellers who may not be familiar with DSGE techniques.

We show that optimizing behaviour by households leads to equations that *relate* derivatives in year t of expected household welfare with respect to variations in endogenous predetermined variables *to* expected values of these derivatives in year t+1. By endogenous predetermined variables we mean those whose predetermined values in year t+1 are influenced by household decisions in year t or earlier. In the standard model, wealth is the only endogenous predetermined variable. We find that households plan consumption (their only decision variable) so that the welfare effect of having an extra unit of wealth at the start of year t, $V_K(t)$, is related in a particular way to the expected value of having an extra unit at the start of year t+1, $V_K(t+1)$. However, in deriving the consumption function for the standard model, we don't use directly the relationship between the current and expected future welfare effects of variations in wealth. Instead, we follow standard practice and eliminate $V_K(t)$ and $V_K(t+1)$, obtaining an equation that relates the current value of an extra unit of consumption to the expected future value. Correspondingly, we express the policy function directly as an equation describing aggregate household consumption.

We find that complete elimination of the rather abstract current and expected future marginal welfare variables (V derivatives) is possible only in the special case in which the number of predetermined endogenous variables matches the number of decision variables, e.g. one predetermined endogenous variable, wealth, and one decision variable, consumption. When this match does not occur, it is convenient to retain the V derivatives and derive policy functions for setting these derivatives. We explain these points by setting out a variation of the standard model in which there are two endogenous predetermined variables, wealth and the lagged wage rate, and one decision variable, consumption.

[1] Dynamic Stochastic General Equilibrium.

In section 3, we reformulate the theory from section 2 in a way that leads to a DSGE specification of consumer behaviour suitable for incorporation in a CGE model. As foreshadowed in section 2, we specify the household's policy as a function for determining V derivatives.

In section 4 we allow for multiple exogenous variables, and for correlations between different exogenous variables and between the realization of any given exogenous variable at different times.

Section 5 explains how we use a CGE model to derive first-order and second-order elasticities of consumer wealth at the start of year t+1 with respect to household wealth at the start of year t, consumption in year t and exogenous variables in year t. We then derive formulas that use these elasticities in specifying the derivatives of the household's policy function.

Section 6 shows how to generalize from a steady-state baseline to a steady-growth baseline.

Section 7 applies the theory from the previous sections to derive a DSGE consumption function for a 70-sector version of the USAGE CGE model of the U.S. economy (see for example Dixon *et al.* 2013). Illustrative simulations are provided.

Implementation of DSGE theory in a full-scale CGE model requires the adoption of restrictive assumptions, the most obvious being that the baseline exhibits steady growth. The paper concludes in section 8 with a discussion of these assumptions and how they might be relaxed in future research.

2. A standard DSGE model solved by using the perturbation method to find a policy rule for consumption

In this section, we study a 1-sector, 1-household neoclassical growth model. We refer to this as the standard model and use it to explain what we see as the central ideas in DSGE theory and the perturbation solution strategy.

2.1. The standard model

The standard model starts with an accumulation relationship:

$$K_{t+1} = K_t * (1-\delta) + A_t * K_t^{\alpha} - C_t \tag{2.1}$$

where

K_t is capital stock at the start of year t;
C_t is consumption during year t;
A_t is total factor productivity or technology in year t; and
δ and α are non-negative parameters with values less than or equal to 1. δ is the rate of depreciation and α can be thought of as the capital share in a Cobb-Douglas world in which the labour input is implicitly fixed on 1.

We treat technology as a stochastic variable determined by

$$A_{t+1} = A * \exp(\sigma\varepsilon_{t+1}) \tag{2.2}$$

where

ε_{t+1} is a draw made at the start of year t+1 from a normal distribution with mean zero and variance 1;
σ translates the ε draws for t+1 into a distribution with variance σ; and
A is a long-run or normal level of technology. In this section we will treat A as a fixed coefficient. In later sections we will allow A to be shocked.

In year t, the household knows the values of K_t, A_t, A, σ and the parameters δ and α. The household also knows the form of the accumulation relationship (2.1) and the nature of the stochastic determination of technology, (2.2). Given this information, the household chooses a strategy for determining consumption in each year to maximize the expected value of its lifetime welfare. Our aim in this section is to characterize that strategy.

In year t, expected lifetime welfare, V, is specified by:

$$V(t) = E_t\left[\sum_{\tau=0}^{\infty}\beta^{\tau} * \frac{C_{t+\tau}^{1-\gamma}}{1-\gamma}\right] \tag{2.3}$$

where

γ and β are parameters whose values we will assume are strictly between 0 and 1;[2] and
E_t indicates expectation held in year t.

γ introduces diminishing marginal utility to consumption in any given year and β introduces preference for current consumption relative to future consumption.

Under (2.3) we can write expected lifetime welfare in year t as:

$$V(K_t, A_t, \sigma) = \frac{C_t^{1-\gamma}}{1-\gamma} + \beta * E_t\left[V(K_{t+1}, A_{t+1}, \sigma)\right] \tag{2.4}$$

In (2.4), expected lifetime welfare in year t is a function of the information available in year t. This information consists of the capital stock, K_t, current technology, A_t, and the variance, σ, governing the stochastic process that determines technology in future years. The right hand side of (2.4) expresses lifetime welfare expected in year t as the sum of welfare generated in year t and the expectation held in year t for lifetime welfare onward from year t+1.

Why is σ an argument of V, but not A or the parameters α, β, γ and δ? This is simply a reflection of where we are going to take the analysis in the rest of this section. In year t, the household knows the values of σ, A, α, β, γ and δ, and assumes with certainty that they will never change. We will examine the effects of a change in σ that is unexpected by the household. Potentially the change in σ affects expected lifetime welfare. Thus, we make V an explicit function of σ. At least in this section, we won't be examining the effects of changes in A, α, β, γ and δ. Therefore there is no need for them to be included as explicit arguments of V.

We represent the household strategy for choosing consumption as:

$$C_t = H\left(K_t, A_t, \sigma\right) \tag{2.5}$$

$$C_{t+1} = H\left(K_{t+1}, A_{t+1}, \sigma\right) \tag{2.6}$$

(2.5) and (2.6) are often referred to as the policy rule. They say that the household uses the information available in year t to determine consumption in year t. Similarly, for year t+1. If the information available in year t+1 is the same as that for year t, then C_{t+1} will be the same as C_t. This is valid because the form of the V function describing expected lifetime welfare from year t+1 onwards is the same as that describing lifetime welfare from year t onwards.[3]

[2] It is possible to use values of $\gamma>1$, but conceptually simpler to assume that γ is <1.

[3] Specification (2.3) avoids the intertemporal inconsistency problem described by Strotz (1955). Intertemporal inconsistency arises when consumption planned in year t for year t+1 is not undertaken in t+1 even when there is no change in information in the transition for t to t+1. Specification (2.3) underlies (2.4) which legitimizes (2.5) and (2.6).

The first step in deriving properties of the policy rule, the H function, is to recognize that the household knows the trade-off between consumption in year t and capital in year t+1. Consequently, C_t must optimize the right hand side of (2.4). This requires

$$C_t^{-\gamma} + \beta * E_t\left[V_K(t+1)\frac{\partial K_{t+1}}{\partial C_t}\right] = 0 \tag{2.7}$$

where

$V_K(t+1)$ is the derivative of V with respect to its first argument evaluated at the t+1 values of its 3 arguments; and

$\partial K_{t+1}/\partial C_t$ is the partial derivative of K_{t+1} with respect to C_t evaluated from the right hand side of the accumulation relationship holding K_t constant. In the particular case of (2.1), this is -1. However, we will persist with the symbolic representation to point the way to more general cases.

Differentiating in (2.4) with respect to K_t gives

$$V_K(t) = C_t^{-\gamma} * \frac{\partial C_t}{\partial K_t} + \beta * E_t\left[V_K(t+1) * \left(\frac{\partial K_{t+1}}{\partial K_t} + \frac{\partial K_{t+1}}{\partial C_t} * \frac{\partial C_t}{\partial K_t}\right)\right] \tag{2.8}$$

where

$\partial K_{t+1}/\partial K_t$ is the partial derivative of K_{t+1} with respect to K_t evaluated from the right hand side of the accumulation relationship holding C_t constant; and

$\partial C_t/\partial K_t$ is the derivative of C_t with respect to K_t evaluated from the right hand side of (2.5) holding constant the other determinants of consumption, A_t and σ.

Simplify (2.8) using (2.7):

$$V_K(t) = \beta * E_t\left[V_K(t+1) * \frac{\partial K_{t+1}}{\partial K_t}\right] \tag{2.9}$$

From (2.9) we have

$$\beta * E_t\left(V_K(t+1)\right) = V_K(t) * \left(\frac{\partial K_{t+1}}{\partial K_t}\right)^{-1} \tag{2.10}$$

In deriving (2.10) from (2.9) we recognize that $\partial K_{t+1}/\partial K_t$ is non-stochastic. It is determined from values of variables known by the household in year t.

Substitute (2.10) into (2.7):

$$C_t^{-\gamma} + V_K(t) * \left(\frac{\partial K_{t+1}}{\partial K_t}\right)^{-1} \frac{\partial K_{t+1}}{\partial C_t} = 0 \tag{2.11}$$

Rearrange as:

$$V_K(t) = -C_t^{-\gamma} * \left(\frac{\partial K_{t+1}}{\partial C_t}\right)^{-1} * \left(\frac{\partial K_{t+1}}{\partial K_t}\right) \tag{2.12}$$

Hence

$$V_K(t+1) = -C_{t+1}^{-\gamma} * \left(\frac{\partial K_{t+2}}{\partial C_{t+1}}\right)^{-1} * \left(\frac{\partial K_{t+2}}{\partial K_{t+1}}\right) \tag{2.13}$$

Substituting into (2.7) gives

$$C_t^{-\gamma} + \beta * E_t\left[-C_{t+1}^{-\gamma} * \left(\frac{\partial K_{t+2}}{\partial C_{t+1}}\right)^{-1} * \left(\frac{\partial K_{t+2}}{\partial K_{t+1}}\right) * \frac{\partial K_{t+1}}{\partial C_t}\right] = 0 \tag{2.14}$$

Using (2.1) we can evaluate the derivatives in (2.14):

$$\frac{\partial K_{t+2}}{\partial C_{t+1}} = -1 \quad , \quad \frac{\partial K_{t+1}}{\partial C_t} = -1 \text{ and } \frac{\partial K_{t+2}}{\partial K_{t+1}} = (1-\delta) + A_{t+1} * \alpha * K_{t+1}^{\alpha-1} \tag{2.15}$$

Substituting into (2.14) gives the relationship

$$C_t^{-\gamma} = \beta * E_t\left[\; C_{t+1}^{-\gamma} * \left((1-\delta) + A_{t+1} * \alpha * K_{t+1}^{\alpha-1}\right)\right] \tag{2.16}$$

If we knew the form of H, then for any given values of σ, A, α, β, γ and δ, together with a starting values for capital, K_t, and technology, A_t, we could solve (2.1), (2.2), (2.5) and (2.6) to determine the paths of consumption and capital for any given path of the stochastic variable $\varepsilon_{t+\tau}$, τ= 1, 2, …, . To do this, we would start with (2.5). This would tell us C_t. Then we would go to (2.1) to find K_{t+1}. Next we would make a draw from the normal distribution to obtain ε_{t+1}, which via (2.2) would give us A_{t+1}. From there we would use (2.6) to compute C_{t+1}. At that stage we could move forward to year t+2. But what about (2.16)? The policy rule, H, must generate a path for consumption which is compatible with (2.16). This provides the key to determining the form of H.

There are various ways of finding the H function. We focus on the perturbation approach.[4] This method starts with linearized versions of equations (2.1), (2.2), (2.5), (2.6) and (2.16). The linearization is done around a known solution which we refer to as the baseline. Then the linear system is solved to give the effects on C_t and K_{t+1} of small perturbations in K_t, A_t and σ away from their baseline values. As we will see, analysing perturbation effects in the linearized system generates sufficient information to deduce derivatives of the H function, giving us a first-order approximation to the form of H in the vicinity of the baseline.

The perturbation approach is a natural choice for CGE modellers who use GEMPACK software.[5] GEMPACK solves CGE models using linear equations describing perturbations in variables away from a known solution.

2.2. Using the perturbation approach to find a first-order approximation to the policy rule, H, for the standard model

We use bars to denote values in the baseline solution. Then linearizing (2.1), (2.2), (2.5), (2.6) and (2.16) around this solution we obtain:

$$dK_{t+1} = dK_t * (1-\delta) + dA_t * \bar{K}_t^{\alpha} + \alpha\bar{A}_t * \bar{K}_t^{\alpha-1} * dK_t - dC_t \tag{2.17}$$

$$dA_{t+1} = A * \left[\bar{\varepsilon}_{t+1} * \exp(\bar{\sigma}\bar{\varepsilon}_{t+1}) * d\sigma + \bar{\sigma} * \exp(\bar{\sigma}\bar{\varepsilon}_{t+1}) * d\varepsilon_{t+1}\right] \tag{2.18}$$

$$dC_t = \bar{H}_K(t) * dK_t + \bar{H}_A(t) * dA_t + \bar{H}_{\sigma}(t) * d\sigma \tag{2.19}$$

$$dC_{t+1} = \bar{H}_K(t+1) * dK_{t+1} + \bar{H}_A(t+1) * dA_{t+1} + \bar{H}_{\sigma}(t+1) * d\sigma \tag{2.20}$$

[4] For learning about the perturbation approach we relied mainly on Schmitt-Grohé and Uribe (2004). For an overview of the perturbation approach and other methods for solving DSGE models, see Villaverde *et al.* (2016).

[5] See Harrison *et al.* (2014) and Horridge *et al.* (2013).

$$-\gamma * \bar{C}_t^{-\gamma-1} * dC_t = \beta * \left[\begin{array}{c} -\gamma * \bar{C}_{t+1}^{-\gamma-1} * E_t dC_{t+1} * \left((1-\delta) + \bar{A}_{t+1} * \alpha * \bar{K}_{t+1}^{\alpha-1} \right) \\ + \bar{C}_{t+1}^{-\gamma} * \left(E_t dA_{t+1} * \alpha * \bar{K}_{t+1}^{\alpha-1} + \bar{A}_{t+1} * \alpha * (\alpha-1) * \bar{K}_{t+1}^{\alpha-2} * dK_{t+1} \right) \end{array} \right] \quad (2.21)$$

In these equations, d denotes deviation from the baseline solution. Three aspects of the linearized system need clarification. First is the treatment of σ. In the baseline we have assumed that σ is fixed on $\bar{\sigma}$. In our linearized system we allow a change in σ, $d\sigma$, to occur in year t and to be viewed as permanent by the household. Second, we use the notation $\bar{H}_K(t)$, $\bar{H}_A(t), \bar{H}_\sigma(t)$ to denote derivatives of H with respect to K, A and σ evaluated at year-t baseline values of the variables. Third, the determination of dK_{t+1} follows in a non-stochastic way from the year t deviations in variables from their baseline values, see (2.17). Consequently, the expectation operator is not applied to dK_{t+1} in (2.21). On the other hand, it must be applied to dC_{t+1} and dA_{t+1}. By $E_t dC_{t+1}$ and $E_t dA_{t+1}$ in (2.21) we mean the expectation held by the household in year t for the deviations in C_{t+1} and of A_{t+1} for their baseline values. These are not the actual deviations which are given by (2.18) and (2.20).

Before we can use (2.17) to (2.21) we need to specify $E_t dC_{t+1}$ and $E_t dA_{t+1}$. Equation (2.2) implies that the household expects A_{t+1} to be A independently of changes in K_t, A_t and σ. Consequently, we assume that

$$E_t dA_{t+1} = A - \bar{A}_{t+1} \quad (2.22)$$

Then, in light of (2.20) we assume that

$$E_t dC_{t+1} = \bar{H}_K(t+1) * dK_{t+1} + \bar{H}_A(t+1) * \left(A - \bar{A}_{t+1} \right) + \bar{H}_\sigma(t+1) * d\sigma \quad (2.23)$$

Now we work with the expanded system (2.17) to (2.23), treating to dK_t, dA_t, $d\varepsilon_{t+1}$ and $d\sigma$ as exogenous variables. We can move these variables independently of each other and calculate the effects on other variables. Our aim is to use shocks to these variables to evaluate derivatives of the H function. We start by setting

$$dK_t = 1,\ dA_t = 0,\ d\varepsilon_{t+1} = 0 \text{ and } d\sigma = 0 \quad (2.24)$$

Under (2.24), (2.17) to (2.23) reduces to:

$$dK_{t+1} = (1-\delta) + \alpha \bar{A}_t * \bar{K}_t^{\alpha-1} - dC_t \quad (2.25)$$

$$dA_{t+1} = 0 \quad (2.26)$$

$$dC_t = \bar{H}_K(t) \quad (2.27)$$

$$dC_{t+1} = \bar{H}_K(t+1) * dK_{t+1} + \bar{H}_A(t+1) * dA_{t+1} \quad (2.28)$$

$$-\gamma * \bar{C}_t^{-\gamma-1} * dC_t = \beta * \left[\begin{array}{c} -\gamma * \bar{C}_{t+1}^{-\gamma-1} * E_t dC_{t+1} * \left((1-\delta) + \bar{A}_{t+1} * \alpha * \bar{K}_{t+1}^{\alpha-1} \right) \\ + \bar{C}_{t+1}^{-\gamma} * \left(E_t dA_{t+1} * \alpha * \bar{K}_{t+1}^{\alpha-1} + \bar{A}_{t+1} * \alpha * (\alpha-1) * \bar{K}_{t+1}^{\alpha-2} * dK_{t+1} \right) \end{array} \right] \quad (2.29)$$

$$E_t dA_{t+1} = A - \bar{A}_{t+1} \quad (2.30)$$

$$E_t dC_{t+1} = \bar{H}_K(t+1) * dK_{t+1} + \bar{H}_A(t+1) * \left(A - \bar{A}_{t+1} \right) \quad (2.31)$$

The unknowns in (2.24) to (2.31) are dK_{t+1}, dC_t, dC_{t+1}, dA_{t+1}, $\bar{H}_K(t)$, $\bar{H}_K(t+1)$, $\bar{H}_A(t+1)$, $E_t dA_{t+1}$ and $E_t dC_{t+1}$. Thus we have 9 unknowns in 7 equation. To proceed from here we need to add two pieces of information. We assume that in the baseline

$$\bar{A}_{t+1} = A \tag{2.32}$$

and that

$$\bar{H}_K(t) = \bar{H}_K(t+1) \equiv \bar{H}_K \tag{2.33}$$

Can we be sure that a solution satisfying (2.32) and (2.33) exists? Before we answer that question we note that if the baseline satisfies (2.32) and (2.33), then (2.25) to (2.31) becomes

$$dK_{t+1} = (1-\delta) + \alpha\bar{A}_t * \bar{K}_t^{\alpha-1} - dC_t \tag{2.34}$$

$$dA_{t+1} = 0 \tag{2.35}$$

$$dC_t = \bar{H}_K \tag{2.36}$$

$$dC_{t+1} = \bar{H}_K * dK_{t+1} \tag{2.37}$$

$$-\gamma * \bar{C}_t^{-\gamma-1} * dC_t = \beta * \begin{bmatrix} -\gamma * \bar{C}_{t+1}^{-\gamma-1} * E_t dC_{t+1} * \left((1-\delta) + \bar{A}_{t+1} * \alpha * \bar{K}_{t+1}^{\alpha-1}\right) \\ + \bar{C}_{t+1}^{-\gamma} * \left(E_t dA_{t+1} * \alpha * \bar{K}_{t+1}^{\alpha-1} + \bar{A}_{t+1} * \alpha * (\alpha-1) * \bar{K}_{t+1}^{\alpha-2} * dK_{t+1}\right) \end{bmatrix} \tag{2.38}$$

$$E_t dA_{t+1} = 0 \tag{2.39}$$

$$E_t dC_{t+1} = \bar{H}_K * dK_{t+1} \tag{2.40}$$

Now we have 7 equations which we anticipate can be solved for $\bar{H}_K$ together with the other 6 unknowns: dK_{t+1}, dC_t, dC_{t+1}, dA_{t+1}, $E_t dA_{t+1}$ and $E_t dC_{t+1}$.

Justifying restrictions (2.32) and (2.33) and deriving the steady-state solution for years t and t+1

The usual justification for restrictions such as (2.32) and (2.33) is that the baseline is a non-stochastic steady state. Non-stochastic means that $\bar{\sigma} = 0$ which implies via (2.2) that $\bar{A}_{t+1} = A$. Steady state means that $\bar{K}_t = \bar{K}_{t+1}$ and $\bar{A}_t = \bar{A}_{t+1}$ which implies that $H_K(t+1) = H_K(t)$, and incidentally that $\bar{C}_t = \bar{C}_{t+1}$. With $\bar{\sigma} = 0$ we can remove the expectation operator from (2.16) and demonstrate that a steady state exists for years t and t+1 by solving the equations

$$\bar{C}^{-\gamma} = \beta * \left[\bar{C}^{-\gamma} * \left((1-\delta) + A * \alpha * \bar{K}^{\alpha-1}\right)\right] \tag{2.41}$$

$$\bar{K} = \bar{K}(1-\delta) + A * \bar{K}^{\alpha} - \bar{C} \tag{2.42}$$

giving

$$\left(\frac{1-\beta*(1-\delta)}{\beta*A*\alpha}\right)^{\frac{1}{(\alpha-1)}} = \bar{K} \tag{2.43}$$

and

$$\bar{C} = -\left(\frac{1-\beta*(1-\delta)}{\beta*A*\alpha}\right)^{\frac{1}{(\alpha-1)}} * \delta + A * \left(\frac{1-\beta*(1-\delta)}{\beta*A*\alpha}\right)^{\frac{\alpha}{(\alpha-1)}} \tag{2.44}$$

Evaluating $\bar{H}_K$ using (2.34) to (2.40) with the baseline solution for years t and t+1 being the non-stochastic steady state

Given our steady-state assumption, we can drop the t and t+1 subscripts from the barred coefficients in (2.34) to (2.40). Then we reduce this 7 equation system to 2 equations in 2 unknowns (dK_{t+1} and $\bar{H}_K$) by deleting (2.35) and (2.37) and substituting from (2.36) (2.39) and (2.40) into (2.38) and from (2.36) into (2.34):

$$dK_{t+1} = (1-\delta) + \alpha * A * \bar{K}^{\alpha-1} - \bar{H}_K \tag{2.45}$$

$$-\gamma * \bar{C}^{-1} * \bar{H}_K = \beta * \begin{bmatrix} -\gamma * \bar{C}^{-1} * \bar{H}_K * \left((1-\delta) + A * \alpha * \bar{K}^{\alpha-1}\right) \\ + A * \alpha * (\alpha-1) * \bar{K}^{\alpha-2} \end{bmatrix} * dK_{t+1} \tag{2.46}$$

Next, substitute from (2.45) into (2.46) to obtain:

$$-\gamma * \bar{H}_K = \beta * \begin{bmatrix} -\gamma * \bar{H}_K * \left((1-\delta) + A * \alpha * \bar{K}^{\alpha-1}\right) \\ + A * \bar{C} * \alpha * (\alpha-1) * \bar{K}^{\alpha-2} \end{bmatrix} * \left[(1-\delta) + \alpha * A * \bar{K}^{\alpha-1} - \bar{H}_K\right] \tag{2.47}$$

To relieve the algebraic load, it is useful to note from (2.41) that

$$1 = \beta\left((1-\delta) + A * \alpha * \bar{K}^{\alpha-1}\right) \tag{2.48}$$

Thus H_K satisfies the quadratic equation:

$$a2 * H_K^{\;2} + a1 * H_K + a0 = 0 \tag{2.49}$$

where

$$a2 = \gamma \tag{2.50}$$

$$a1 = \gamma - \frac{\gamma}{\beta} - \beta * \bar{C} * (\alpha-1) * A * \alpha * \bar{K}^{\alpha-2} \tag{2.51}$$

and

$$a0 = \bar{C} * (\alpha-1) * A * \alpha * \bar{K}^{\alpha-2} \tag{2.52}$$

A quadratic equation normally has two solutions. We must use information from outside the equation system to choose between these solutions. In experiments with the standard model, we have found that (2.49) to (2.52) gives two real solutions, one positive and one negative. When this happens, we accept the positive solution because we expect an increase in K_t to have a positive influence on C_t.

Evaluating $\bar{H}_A$ using (2.17) to (2.23) with the baseline solution for years t and t+1 being the non-stochastic steady state and with a known value for $\bar{H}_K$

Now we return to (2.17) to (2.23) and set:

$$dK_t = 0,\ dA_t = 1,\ d\varepsilon_{t+1} \text{ and } d\sigma = 0 \tag{2.53}$$

We continue to assume that the baseline is a non-stochastic ($\bar{\sigma} = 0$) steady state. In this case, we can assume that $\bar{H}_K(t)$ and $\bar{H}_K(t+1)$ have the value $\bar{H}_K$ computed in (2.49) to (2.52). We can also assume that

$$\bar{H}_A(t) = \bar{H}_A(t+1) \equiv \bar{H}_A \tag{2.54}$$

Then, (2.17) to (2.23) becomes

$$dK_{t+1} = \bar{K}^{\alpha} - dC_t \tag{2.55}$$

$$dA_{t+1} = 0 \tag{2.56}$$

$$dC_t = \bar{H}_A \tag{2.57}$$

$$dC_{t+1} = \bar{H}_K * dK_{t+1} + \bar{H}_A * dA_{t+1} \tag{2.58}$$

$$-\gamma * \bar{C}^{-\gamma-1} * dC_t = \beta * \left[\begin{array}{c} -\gamma * \bar{C}^{-\gamma-1} * E_t dC_{t+1} * \left((1-\delta) + A * \alpha * \bar{K}^{\alpha-1}\right) \\ + \bar{C}^{-\gamma} * \left(E_t dA_{t+1} * \alpha * \bar{K}^{\alpha-1} + A * \alpha * (\alpha-1) * \bar{K}^{\alpha-2} * dK_{t+1}\right) \end{array} \right] \tag{2.59}$$

$$E_t dA_{t+1} = 0 \tag{2.60}$$

$$E_t dC_{t+1} = \bar{H}_K * dK_{t+1} \tag{2.61}$$

We reduce this 7-equation system to one equation in 1 unknown ($\bar{H}_A$) by deleting (2.56) and (2.58) and substituting from (2.57) (2.55), (2.60) and (2.61) into (2.59):

$$-\gamma * \bar{H}_A = \beta * \left[\begin{array}{c} -\gamma * \bar{H}_K * \left((1-\delta) + A * \alpha * \bar{K}^{\alpha-1}\right) \\ + \bar{C} * \left(A * \alpha * (\alpha-1) * \bar{K}^{\alpha-2}\right) \end{array} \right] * \left(\bar{K}^{\alpha} - \bar{H}_A\right) \tag{2.62}$$

Simplifying using (2.48), we can solve (2.62) for $\bar{H}_A$:

$$\bar{H}_A = \frac{\left[-\gamma * \bar{H}_K + \beta * \bar{C} * A * \alpha * (\alpha-1) * \bar{K}^{\alpha-2}\right] * \left(\bar{K}^{\alpha}\right)}{\left[-\gamma - \gamma * \bar{H}_K + \beta * \bar{C} * A * \alpha * (\alpha-1) * \bar{K}^{\alpha-2}\right]} \tag{2.63}$$

Reassuringly we see that $\bar{H}_A$ is positive.

Evaluating $\bar{H}_{\sigma}$ using (2.17) to (2.23) with the baseline solution for years t and t+1 being the non-stochastic steady state and with known values for $\bar{H}_K$ and $\bar{H}_A$

We set:

$$dK_t = 0,\ dA_t = 0,\ d\varepsilon_{t+1} \text{ and } d\sigma = 1 \tag{2.64}$$

Again we assume that the baseline solution is a non-stochastic ($\bar{\sigma} = 0$) steady state. In this case we can assume that $\bar{H}_K(t)$ and $\bar{H}_K(t+1)$ have the value $\bar{H}_K$ computed in (2.49) to (2.52) and $\bar{H}_A(t)$ and $\bar{H}_A(t+1)$ have the value $\bar{H}_A$ computed in (2.63). We can also assume that

$$\bar{H}_{\sigma}(t) = \bar{H}_{\sigma}(t+1) \equiv \bar{H}_{\sigma} \tag{2.65}$$

Then, (2.17) to (2.23) becomes

$$dK_{t+1} = -dC_t \tag{2.66}$$

$$dA_{t+1} = A\varepsilon_{t+1} \tag{2.67}$$

$$dC_t = \bar{H}_{\sigma} \tag{2.68}$$

$$dC_{t+1} = \bar{H}_K * dK_{t+1} + \bar{H}_A * dA_{t+1} + \bar{H}_{\sigma} \tag{2.69}$$

$$-\gamma^* \bar{C}^{-\gamma-1} * dC_t = \beta^* \begin{bmatrix} -\gamma^* \bar{C}^{-\gamma-1} * E_t dC_{t+1} * \left((1-\delta) + A * \alpha * \bar{K}^{\alpha-1}\right) \\ + \bar{C}^{-\gamma} * \left(E_t dA_{t+1} * \alpha * \bar{K}^{\alpha-1} + A * \alpha * (\alpha-1) * \bar{K}^{\alpha-2} * dK_{t+1}\right) \end{bmatrix} \tag{2.70}$$

$$E_t dA_{t+1} = 0 \tag{2.71}$$

$$E_t dC_{t+1} = \bar{H}_K dK_{t+1} + \bar{H}_\sigma \tag{2.72}$$

We reduce this 7-equation system to one equation in 1 unknown ($\bar{H}_\sigma$) by deleting (2.67) and (2.69) and substituting from (2.66) (2.68), (2.71) and (2.72) into (2.70):

$$-\gamma^* \bar{H}_\sigma = \beta^* \begin{bmatrix} -\gamma^* (1-\bar{H}_K) * \bar{H}_\sigma * \left((1-\delta) + A * \alpha * \bar{K}^{\alpha-1}\right) \\ + \bar{C} * \left(-\bar{A} \;\; * \alpha * (\alpha-1) * \bar{K}^{\alpha-2} * \bar{H}_\sigma\right) \end{bmatrix} \tag{2.73}$$

Using (2.48) and making other simplifications we can rearrange (2.73) as

$$0 = \left[\; \gamma \bar{H}_K + \beta * \bar{C} * A * \alpha * (1-\alpha) * \bar{K}^{\alpha-2} \right] * \bar{H}_\sigma \tag{2.74}$$

The bracketed term on the RHS of (2.74) is positive. We can conclude that

$$\bar{H}_\sigma = 0 \tag{2.75}$$

This is a striking result. It means that a small increase in uncertainty from the zero level ($\bar{\sigma} = 0$) has no effect on consumption. This is disappointing. The "S" in DSGE holds out hope that models in this tradition will help us understand the role of uncertainty in determining macro-economic aggregates. However, this doesn't seem to be true when we start from a position of no uncertainty, and at this stage we don't know how to develop a known solution or baseline that incorporates a realistic level of uncertainty.

Numerical example: finding the first-order approximation to the consumption function in the standard model

The analysis so far has been heavily algebraic. A numerical example will provide a check on whether the numerous formulas produce reasonable results while at the same time assisting our general understanding.

We assume that β=0.9, δ=0.05, A =1, α=0.5 and γ=0.5.

Then from (2.43) and (2.44) we find that $\bar{K}$ = 9.63139 and $\bar{C}$ = 2.62188. $\bar{H}_K$ can now be evaluated from (2.49) which in this numerical example is

$$0.5 * \bar{H}_K^2 - 0.035819 * \bar{H}_K - 0.021929 \;\; = 0 \tag{2.76}$$

This gives two solutions: 0.248284 and -0.176645

We accept the positive solution: the quantity of consumption should increase if households are given an extra unit of capital. Hence

$$\bar{H}_K = 0.248284 \tag{2.77}$$

Using (2.63) we find that

$$\bar{H}_A = 0.683924 \tag{2.78}$$

Thus, the first-order approximation to the consumption function in the neighbourhood the non-stochastic steady-state solution is:

$$C_t = 2.62188 + 0.248284 * (K_t - 9.63139) + 0.693482 * (A_t - 1) \tag{2.79}$$

As expected, the coefficients on K_t and A_t are both positive. But why should a unit increase in A_t boost consumption by 2.79 times as much as a unit increase in K_t (2.79 = 0.693482/0.248284)?

One way to understand this result is to consider two experiments. In the first, we assume that the household receives a small increase in capital, say 0.01 units, and decides after allowing for depreciation, to devote all of this windfall to extra consumption in year t with no change in capital or consumption in future years. The year-t increase in consumption is

$$\left(dC_t\right)_{capital} = A*(\bar{K}+0.01)^{\alpha} - A*(\bar{K})^{\alpha} + (1-\delta)*0.01 = 0.011102 \tag{2.80}$$

In the second experiment, we assume that there is a temporary 0.01 increase in technology in year t, generating extra income. If the household decides to consume all this extra income in year t, then the increase in consumption is:

$$\left(dC_t\right)_{tech} = 0.01*\bar{K}^{\alpha} = 0.031034 \tag{2.81}$$

Equations (2.80) and (2.81) indicate that a 0.01unit transitory increase in technology potentially generates a consumption increase that is 2.79 (=0.031034/0.011102) times that generated by a 0.01 unit increase in capital. On this basis we would expect the ratio of the coefficient on A_t to be about 2.79 times the coefficient K_t.

2.3. A DSGE model with 2 predetermined variables solved with policy rules for welfare derivatives

Optimizing behaviour by households leads to equations that relate the effect on household welfare of variations in year-t levels of endogenous predetermined variables to the expected effects on welfare of variations in their year-t+1 levels. By endogenous predetermined variables we mean those whose predetermined values in year t+1 are influenced by household decisions in year t or earlier. In the standard model that we have been studying in the previous subsections, capital is the only endogenous predetermined variable. We found that households plan consumption (their only decision variable) so that the welfare effect of having an extra unit of capital in year t, $V_K(t)$, is related in a particular way to the expected value of having an extra unit in year t+1, $V_K(t+1)$, see (2.9). However, in deriving the consumption function for the standard model, we didn't use directly the relationship between the current and expected future welfare effects of variations in capital. Instead, we eliminated $V_K(t)$ and $V_K(t+1)$ and derived (2.16). This equation relates the current value of an extra unit of consumption to the expected future value. Correspondingly, we expressed the policy rule as a function for determining consumption, see (2.5) and (2.6).

In general, the number of policy rules must be the same as the number of endogenous predetermined variables. Complete elimination of the rather abstract current and expected future marginal welfare variables (V derivatives) is possible only in the special case in which the number of predetermined endogenous variables matches the number of decision variables, e.g. one predetermined endogenous variable, wealth, and one decision variable, consumption. When this match does not occur, it is convenient to retain the V derivatives and derive policy rules for setting these derivatives. In this section we explain these points by setting out a variation of the standard model in which there are two endogenous predetermined variables, capital and the lagged wage rate, and one decision variable, consumption.

In this model, labour input in year t (N_t) is explicit and output is a Cobb-Douglas constant-returns-to-scale function of capital and labour. The accumulation relationship is:

$$K_{t+1} = K_t*\left(1-\delta\right) + A_t*K_t^{\alpha}*N_t^{1-\alpha} - C_t \tag{2.82}$$

The wage rate (W_t) equals the marginal product of labour[6]:

$$W_t = (1-\alpha)*A_t*K_t^{\alpha}*N_t^{-\alpha} \tag{2.83}$$

We assume that the wage rate adjusts sluggishly through time:

$$W_t = WL_t * N_t^{\eta} \tag{2.84}$$

where
  η is a positive parameter; and
  WL_t is the lagged wage rate in year t, so that

$$WL_{t+1} = W_t \tag{2.85}$$

As in the standard model, see (2.2), stochastics is introduced via:

$$A_{t+1} = A*\exp(\sigma\varepsilon_{t+1}) \tag{2.86}$$

Equations (2.83) and (2.84) imply that labour input (employment) and the wage rate in year t are given by

$$N_t = \left(\frac{(1-\alpha)*A_t*K_t^{\alpha}}{WL_t}\right)^{\frac{1}{\eta+\alpha}} \tag{2.87}$$

$$W_t = WL_t^{\frac{\alpha}{\eta+\alpha}}\left((1-\alpha)*A_t*K_t^{\alpha}\right)^{\frac{\eta}{\eta+\alpha}} \tag{2.88}$$

Substituting (2.85) into (2.88) and (2.87) into (2.82) gives equations for the year t+1 values of the two predetermined variables (the lagged wage and capital) in terms of year t variables:

$$WL_{t+1} = WL_t^{\frac{\alpha}{\eta+\alpha}}*\left((1-\alpha)*A_t*K_t^{\alpha}\right)^{\frac{\eta}{\eta+\alpha}} \tag{2.89}$$

$$K_{t+1} = K_t*(1-\delta)+A_t*K_t^{\alpha}*\left(\frac{(1-\alpha)*A_t*K_t^{\alpha}}{WL_t}\right)^{\frac{1-\alpha}{\eta+\alpha}}-C_t \tag{2.90}$$

The household in this model understands (2.89) and (2.90). In deciding its consumption for year t, it takes account not only of K_t, as in the standard model, but also of WL_t. Both these variables have implications for the conditions that the household will face in year t+1. We continue to assume that (2.3) applies but this now leads to a version of (2.4) in which there is an extra argument in the V function:

$$V(K_t, WL_t, A_t, \sigma) = \frac{C_t^{1-\gamma}}{1-\gamma}+\beta*E_t\left[V(K_{t+1}, WL_{t+1}, A_{t+1}, \sigma)\right] \tag{2.91}$$

Optimizing with respect to C_t gives

$$C_t^{-\gamma}+\beta*E_t\left[V_K(t+1)\frac{\partial K_{t+1}}{\partial C_t}+V_{WL}(t+1)\frac{\partial WL_{t+1}}{\partial C_t}\right]=0 \tag{2.92}$$

where

[6] The product price is the numeraire, fixed on 1.

$V_K(t+1)$ and $V_{WL}(t+1)$ are the derivatives of V with respect to its first two argument evaluated at the t+1 values of its 4 arguments; and

$\partial K_{t+1}/\partial C_t$ and $\partial WL_{t+1}/\partial C_t$ are the partial derivatives of K_{t+1} and $WL_{t+1}$with respect to C_t evaluated from the right hand sides of (2.90) and (2.89) holding K_t, A_t, and WL_t constant.

Differentiating in (2.91) with respect to K_t gives

$$V_K(t) = C_t^{-\gamma} * \frac{\partial C_t}{\partial K_t} + \beta * E_t\left[V_K(t+1)*\left(\frac{\partial K_{t+1}}{\partial K_t} + \frac{\partial K_{t+1}}{\partial C_t}*\frac{\partial C_t}{\partial K_t}\right) + V_{WL}(t+1)*\left(\frac{\partial WL_{t+1}}{\partial K_t} + \frac{\partial WL_{t+1}}{\partial C_t}*\frac{\partial C_t}{\partial K_t}\right)\right] \tag{2.93}$$

where

$\partial K_{t+1}/\partial K_t$ is the partial derivative of K_{t+1} with respect to K_t evaluated from the right hand side of (2.90) holding C_t, A_t and WL_t constant;

$\partial WL_{t+1}/\partial K_t$ is the partial derivative of WL_{t+1} with respect to K_t evaluated from the right hand side of (2.89) holding A_t and WL_t constant;

$\partial C_t/\partial K_t$ is the partial derivative of year-t consumption with respect to K_t holding constant other determinants of current consumption which are A_t, WL_t and σ; and

$\partial K_{t+1}/\partial C_t$ and $\partial WL_{t+1}/\partial C_t$ are as defined earlier.

Differentiating in (2.91) with respect to WL_t gives

$$V_{WL}(t) = C_t^{-\gamma} * \frac{\partial C_t}{\partial WL_t} + \beta * E_t\left[V_K(t+1)*\left(\frac{\partial K_{t+1}}{\partial WL_t} + \frac{\partial K_{t+1}}{\partial C_t}*\frac{\partial C_t}{\partial WL_t}\right) + V_{WL}(t+1)*\left(\frac{\partial WL_{t+1}}{\partial WL_t} + \frac{\partial WL_{t+1}}{\partial C_t}*\frac{\partial C_t}{\partial WL_t}\right)\right] \tag{2.94}$$

where

$\partial K_{t+1}/\partial WL_t$ is the partial derivative of K_{t+1} with respect to WL_t evaluated from the right hand side of (2.90) holding C_t, A_t and K_t constant;

$\partial WL_{t+1}/\partial WL_t$ is the partial derivative of WL_{t+1} with respect to WL_t evaluated from the right hand side of (2.89) holding A_t and K_t constant; and

$\partial C_t/\partial WL_t$ is the partial derivative of year-t consumption with respect to WL_t holding constant other determinants of current consumption which are A_t, K_t and σ.

Use (2.92) to simplify (2.93) and (2.94):

$$V_K(t) = \beta * E_t\left[V_K(t+1)*\left(\frac{\partial K_{t+1}}{\partial K_t}\right) + V_{WL}(t+1)*\left(\frac{\partial WL_{t+1}}{\partial K_t}\right)\right] \tag{2.95}$$

$$V_{WL}(t) = \beta * E_t\left[V_K(t+1)*\left(\frac{\partial K_{t+1}}{\partial WL_t}\right) + V_{WL}(t+1)*\left(\frac{\partial WL_{t+1}}{\partial WL_t}\right)\right] \tag{2.96}$$

That is:

$$\begin{bmatrix} V_K(t) \\ \\ V_{WL}(t) \end{bmatrix} = \beta * \begin{bmatrix} \dfrac{\partial K_{t+1}}{\partial K_t} & \dfrac{\partial WL_{t+1}}{\partial K_t} \\ \dfrac{\partial K_{t+1}}{\partial WL_t} & \dfrac{\partial WL_{t+1}}{\partial WL_t} \end{bmatrix} * \begin{bmatrix} E_t(V_K(t+1)) \\ \\ E_t(V_{WL}(t+1)) \end{bmatrix} \tag{2.97}$$

In compact notation we have:

$$VD(t) = \beta * Q(t) * E_t\left[VD(t+1)\right] \tag{2.98}$$

where

Q(t) is the matrix of derivatives on the right hand side of (2.97); and

VD(t) is the vector of partial derivatives of V evaluated at year-t values of variables.

Equation (2.98) corresponds to (2.9) in the standard model. In the standard model we manipulated (2.9) to eventually arrive at (2.16) which expresses year-t consumption in terms of the expected value of year t+1 consumption without the presence of V derivatives. We attempted to derive an equation similar to (2.16) for the present model. Starting from (2.98), and noting that Q(t) will generally be of full rank, we wrote

$$Q(t)^{-1} * VD(t) = \beta * E_t\left[VD(t+1)\right] \tag{2.99}$$

This corresponds to (2.10) in the standard model. Next we substituted from (2.99) into (2.92):

$$C_t^{-\gamma} + \left[\frac{\partial K_{t+1}}{\partial C_t}, \frac{\partial WL_{t+1}}{\partial C_t}\right] Q(t)^{-1} * VD(t) = 0 \tag{2.100}$$

This corresponds to (2.11) in the standard model. The next step with the standard model was to derive (2.12), which expresses $V_K(t)$ in terms of C_t. But the equivalent step is not possible here. VD(t) is a 2 by 1 vector of unknowns which cannot be solved using the single equation (2.100).

Rather than eliminating VD(t), the solution method we suggest for the current model depends on specifying a policy rule for VD, that is a vector that gives a policy rule for each of the partial derivatives of V. This rule can be written as:

$$VD(t) = M(K_t, WL_t, A_t, \sigma) \tag{2.101}$$

$$VD(t+1) = M(K_{t+1}, WL_{t+1}, A_{t+1}, \sigma) \tag{2.102}$$

Given this rule, we can simulate forward for any scenario for $A_{t+\tau}$ for τ= 1, 2, …, as follows.

Starting from K_t, WL_t, A_t and σ, evaluate VD(t) in (2.101). Then compute

$$C_t^{-\gamma} = -\left(\frac{\partial K_{t+1}}{\partial C_t} \quad \frac{\partial WL_{t+1}}{\partial C_t}\right) * Q(t)^{-1} * VD(t) \tag{2.103}$$

Substitute into (2.90) and (2.89) to obtain K_{t+1} and WL_{t+1}. Then proceed to year t+1.

But how do we determine the form of M?

A first-order approximation for M can be found by applying the perturbation method around a non-stochastic steady state. The required non-stochastic steady-state solution can be obtained by assuming $A_t = A$ and working with steady-state versions of (2.89), (2.90), (2.92) and (2.98):

$$\overline{WL} = \overline{WL}^{\frac{\alpha}{\eta+\alpha}} * \left((1-\alpha) * A * \overline{K}^{\alpha}\right)^{\frac{\eta}{\eta+\alpha}} \tag{2.104}$$

$$\overline{K} = \overline{K} * (1-\delta) + A * \overline{K}^{\alpha} * \left(\frac{(1-\alpha) * A * \overline{K}^{\alpha}}{\overline{WL}}\right)^{\frac{1-\alpha}{\eta+\alpha}} - \overline{C} \tag{2.105}$$

$$\overline{C}^{-\gamma} = \beta * \overline{V_K} \tag{2.106}$$

and

$$\overline{VD} = \quad \beta * \overline{Q} * \overline{VD} \tag{2.107}$$

In (2.106) we have simplified (2.92) by recognizing that $\partial K_{t+1}/\partial C_t = -1$ and $\partial WL_{t+1}/\partial C_t = 0$. In (2.107) we assume that $\overline{VD} \neq 0$. This means that

$$\det\left[I - \beta * \overline{Q}\right] = 0 \tag{2.108}$$

In (2.107) and (2.108), $\overline{Q}$ is a function of $\overline{K}$ and $\overline{WL}$ given by:

$$\overline{Q} = \begin{bmatrix} (1-\delta) + (A) * \left(\frac{(1-\alpha)A}{\overline{WL}}\right)^{\frac{(1-\alpha)}{(\alpha+\eta)}} * \frac{(1+\eta)\alpha}{(\alpha+\eta)} * \overline{K}^{\frac{\eta(\alpha-1)}{(\alpha+\eta)}} & \frac{\alpha\eta}{(\alpha+\eta)} \overline{WL}^{\frac{\alpha}{\eta+\alpha}} * ((1-\alpha) * A)^{\frac{\eta}{\eta+\alpha}} * \overline{K}^{\frac{\alpha\eta-\alpha-\eta}{(\alpha+\eta)}} \\ -\frac{1-\alpha}{\eta+\alpha} * A * \overline{K}^{\alpha} * \left((1-\alpha) * A * \overline{K}^{\alpha}\right)^{\frac{1-\alpha}{\eta+\alpha}} * \left(\frac{1}{\overline{WL}}\right)^{\frac{1+\eta}{(\alpha+\eta)}} & \frac{\alpha}{\eta+\alpha} * \overline{WL}^{\frac{\eta}{\eta+\alpha}} * \left((1-\alpha) * A * \overline{K}^{\alpha}\right)^{\frac{\eta}{\eta+\alpha}} \end{bmatrix} \tag{2.109}$$

With $\overline{Q}$ replaced by the right hand side of (2.109), the system (1.104) to (1.108) contains 6 equations and 5 unknowns: $\overline{K}$, $\overline{WL}$, $\overline{C}$, $\overline{V_K}$ and $\overline{V_{WL}}$. Under our assumption that $\overline{VD} \neq 0$, the 2 equations in (2.107) are linearly dependent. Consequently, one of them can be deleted, leaving us with 5 equations to determine the 5 unknowns.

Once the non-stochastic steady-state solution is in place, we can use it as the baseline and create a linearized version of the model in which the variables are deviations from this baseline. Rather than presenting this linearized version in its full algebraic complexity, we set it out in stylized form:

$$dWL_{t+1} = f_{2.89}(dWL_t, dA_t, dK_t) \tag{2.110}$$

$$dK_{t+1} = f_{2.90}(dK_t, dA_t, dWL_t, dC_t) \tag{2.111}$$

$$dC_t = f_{2.92}(E_t dVD(t+1), dK_t, dA_t, dWL_t) \tag{2.112}$$

$$E_t dVD(t+1) = f_{2.98}\left(dVD(t), dK_t, dWL_t, dA_t\right) \tag{2.113}$$

$$dVD(t) = \overline{M}_K * dK_t + \overline{M}_{WL} * dWL_t + \overline{M}_A * dA_t + \overline{M}_\sigma * d\sigma \tag{2.114}$$

$$E_t dVD(t+1) = \overline{M}_K * dK_{t+1} + \overline{M}_{WL} * dWL_{t+1} \tag{2.115}$$

Equations (2.110) and (2.111) are linearized versions of (2.89) and (2.90). Equation (2.112) is a linearized version of (2.92) with dC_t isolated on the left hand side. Equation (2.113) is a linearized version of (2.98) with $E_t dVD(t+1)$ isolated on the left hand side. In (2.113) we assume that the components of Q(t) do not depend on C_t. This assumption is valid in the current model, but its use here is convenient rather than essential. Equation (2.114) is a linearized version of the policy rule (2.101). Equation (2.115) shows in linearized form the household's expectation for dVD(t+1) on the assumption that the household expects to be following policy rule (2.101). We assume that the household's expectations for $d\sigma$ and dA_{t+1} are always zero and that dK_{t+1} and dWL_{t+1} are non-stochastic.

To determine the values of $\bar{M}_K$, $\bar{M}_{WL}$, $\bar{M}_A$ and $\bar{M}_\sigma$ to be used in a first order approximation to the policy rule, we conduct four experiments with the linearized equations (2.110) to (2.115). In these experiments, the four exogenous or predetermined variables, dK_t, dWL_t, dA_t and $d\sigma$, are shocked individually.

In the first experiment, $dK_t= 1$ and changes in the other exogenous variables are zero. This leads to:

$$dWL_{t+1} = f_{2.89}(0,0,1) \tag{2.116}$$

$$dK_{t+1} = f_{2.90}(1,0,0,dC_t) \tag{2.117}$$

$$dC_t = f_{2.92}(E_t dVD(t+1),1,0,0) \tag{2.118}$$

$$E_t dVD(t+1) = f_{2.98}(dVD(t), 1, 0,0) \tag{2.119}$$

$$dVD(t) = \bar{M}_K \tag{2.120}$$

$$E_t dVD(t+1) = \bar{M}_K * dK_{t+1} + \bar{M}_{WL} * dWL_{t+1} \tag{2.121}$$

After a series of substitutions, (2.116) to (2.121) reduces to:

$$f_{2.98}\left(\bar{M}_K,1,0,0\right) = \bar{M}_K * f_{2.90}(1,0,0,f_{2.92}(f_{2.98}\left(\bar{M}_K,1,0,0\right),1,0,0)) + \bar{M}_{WL} * f_{2.89}(0,0,1) \tag{2.122}$$

In the second experiment, $dWL_t= 1$ and changes in the other exogenous variables are zero. This leads to:

$$f_{2.98}\left(\bar{M}_{WL},0,1,0\right) = \bar{M}_K * f_{2.90}(0,0,1,f_{2.92}(f_{2.98}\left(\bar{M}_{WL},0,1,0\right),0,0,1)) + \bar{M}_{WL} * f_{2.89}(1,0,0) \tag{2.123}$$

Together, (2.122) and (2.123) give 4 equations that can be used to determine the 2 components of $\bar{M}_K$ and the 2 components of $\bar{M}_{WL}$.

In the third experiment $dA_t = 1$ and changes in the other exogenous variables are zero. This experiment reveals the value of $\bar{M}_A$ given that we already know the values of $\bar{M}_K$ and $\bar{M}_{WL}$.

In the fourth experiment $d\sigma = 1$ and changes in the other exogenous variables are zero. This experiment shows that $\bar{M}_\sigma = 0$.

3. Towards a DSGE consumption function for a full-scale CGE model

In this section, we reformulate the theory from section 2 in a way that we hope will lead to a DSGE specification of consumer behaviour suitable for incorporation in a CGE model. Towards this objective, we refer to wealth accumulation rather than capital accumulation and we work as much as possible with elasticities rather than derivatives. Following subsection 2.3, we specify the household's policy as a function for determining the derivatives of lifetime welfare with respect to pre-determined variables, and we use the perturbation method to derive the elasticities of this policy function with respect to variations in exogenous variables. Our derivation of these elasticities is long and potentially error prone. As a partial check, we apply our formulas to the standard neoclassical model and show that they lead to the same results as in subsection 2.2.

Model

CGE wealth accumulation equation:

$$X_{t+1} = J(X_t, Z_t, C_t) \tag{3.1}$$

In (3.1) we have in mind a CGE wealth accumulation relationship in which household wealth at the start of year t+1 (X_{t+1}) is a function of household wealth at the start of year t (X_t), consumption during year t (C_t) and other variables exogenous to households (Z_t). These other variables could include technologies, consumer preferences, the terms of trade, public expenditure, and the state of business confidence. We assume that the Z vector for year t is known to households in year t, but the Z vector for future years is known in year t only in a probabilistic form. As in previous sections, we indicate uncertainty by a variance/covariance variable denoted by σ. The exogenous Z variables through the CGE model can be thought of as determining components of household income such as wage rates, employment, profits and transfer payments.

Consumer lifetime welfare:

$$V(X_t, Z_t, \sigma) = U(C_t, X_t) + \beta * E_t\left[V(X_{t+1}, Z_{t+1}, \sigma)\right] \tag{3.2}$$

In (3.2) we generalize the earlier specification, e.g. (2.4), by making the utility contribution in year t a function not only of consumption (C_t) but also of wealth (X_t). We assume that wealth contributes to lifetime welfare not only by facilitating future consumption but also by providing a sense of security in each year. For concreteness, we give U the specific form

$$U(t) = \frac{\left(X_t^{1-\theta} * C_t^{\theta}\right)^{1-\gamma}}{(1-\gamma)} \tag{3.3}$$

where γ and θ are positive parameters, with $\gamma < 1$ and $\theta \leq 1$. We use the notation U(t) to denote the value of utility in year t. In view of the inability of the method being pursued in this paper to produce a consumption function that gives non-zero responses to variations in σ [see (2.75)], the inclusion of X_t in the utility function seems a potentially attractive path for capturing effects of uncertainty. For example, we might simulate growing uncertainty by decreasing θ.

Optimizing consumption between the current year and the future leads to

$$U_C(t) + \beta * \{E_t[V_X(t+1)]\} * \frac{\partial X_{t+1}}{\partial C_t} = 0 \tag{3.4}$$

where $U_C(t)$ and $V_X(t+1)$ denote partial derivatives of U and V with respect to C and X evaluated with the arguments in the U and V functions set at their year t and t+1 values.

Differentiating the V function in (3.2) with respect to X gives

$$V_X(t) = U_X(t) + U_C(t) * \frac{\partial C_t}{\partial X_t} + \beta * E_t\left[V_X(t+1) * \frac{\partial X_{t+1}}{\partial X_t} + V_X(t+1) * \frac{\partial X_{t+1}}{\partial C_t} * \frac{\partial C_t}{\partial X_t}\right] \quad (3.5)$$

By using (3.4) we can simplify (3.5) to:

$$V_X(t) = U_X(t) + \beta * \{E_t[V_X(t+1)]\} * \frac{\partial X_{t+1}}{\partial X_t} \quad (3.6)$$

As in subsection 2.3, we specify the policy rule for consumers as a function of the derivative of V, this time with respect to wealth rather than capital:

$$V_X(t) = M[X_t, Z_t, \sigma] \quad (3.7)$$

Finally, we assume that the household expects in year t to be implementing its policy rule in year t+1.

$$E_t V_X(t+1) = E_t M[X_{t+1}, Z_{t+1}, \sigma] \quad (3.8)$$

Linearizing (3.1), (3.4), (3.6), (3.7) and (3.8): an elasticity format

As in section 2, we assume that the DSGE model in this section has a non-stochastic steady state. Then we linearize the model's equations [(3.1), (3.4), (3.6), (3.7) and (3.8)] around this steady state. In CGE modelling, we are accustomed to specifying equations in terms of elasticities and percentage changes in variables rather than derivatives and changes in variables. Consequently, to ease the transfer of DSGE specifications into CGE modelling it is useful to express DSGE linearized equations mainly in elasticity percentage-change form.

We derive the linearized system in a general form and use the general form in two tasks:

(i) to determine the elasticity (M_X) of the policy rule with respect to wealth (X_t); and then

(ii) to specify the household's consumption function, that is the function relating C_t to X_t and Z_t.

As in section 2 we assume that σ is fixed on zero.

To derive the linearized system we start by writing the linearized version of (3.1) as:

$$x_{t+1} = J_X * x_t - J_C * c_t + J_Z * z_t \quad (3.9)$$

where

x_{t+1}, x_t, c_t and z_t are *percentage* deviations in X_{t+1}, X_t, C_t and Z_t from their steady-state values; and

J_X, J_C and J_Z (without t arguments) are elasticities of the J function in (3.1) evaluated at steady-state values of the variables. In general, the elasticities are defined by:

$$J_X(t) = \frac{\partial X_{t+1}}{\partial X_t} * \frac{X_t}{X_{t+1}} \quad (3.10)$$

$$J_C(t) = -\frac{\partial X_{t+1}}{\partial C_t} * \frac{C_t}{X_{t+1}} \quad (3.11)$$

and

$$J_Z(t) = \frac{\partial X_{t+1}}{\partial Z_t} * \frac{Z_t}{X_{t+1}} \quad (3.12)$$

In (3.10) to (3.12) we use the notation $J_X(t)$, $J_C(t)$ and $J_Z(t)$ to denote elasticities evaluated with variables set at their year t values. The absence of these t arguments in (3.9) means that the elasticities are evaluated at steady-state values. We assume that both $J_X(t)$ and $J_C(t)$ are non-negative. As becomes apparent shortly, we will need to compute percentage changes in $J_X(t)$ and $J_C(t)$. Percentage changes in negative quantities are not well defined. $J_X(t)$ presents no problem,. We expect the elasticity of future wealth with respect to current wealth to be positive. However, we expect an increase in current consumption to reduce future wealth. Consequently, to ensure that J_C is positive, we define it in (3.11) with a negative sign on the right hand side. The same problem does not arise with $J_Z(t)$. It can be either positive or negative. We don't need to compute percentage changes in it.

Next we use (3.3), (3.8) and (3.11) to obtain a non-stochastic version of (3.4):

$$\theta * C_t^{\theta-1} * X_t^{1-\theta}(X_t^{1-\theta} * C_t^{\theta})^{-\gamma} = \beta * \tilde{M}(t) * J_C(t) * \frac{X_{t+1}}{C_t} \tag{3.13}$$

where

$$\tilde{M}(t) = E_t\left[M(t+1)\right] \tag{3.14}$$

Linearizing (3.13) gives:

$$(\theta - 1 - \theta * \gamma) * c_t + (1-\theta)(1-\gamma) * x_t = \tilde{m}(t) + j_C(t) + x_{t+1} - c_t \tag{3.15}$$

where

$\tilde{m}(t)$ and $j_C(t)$ are percentage deviations in $\tilde{M}(t)$ and $J_C(t)$ from their steady-state values. Notice that by using the negative sign on the right hand side of (3.11) so that J_C is positive, we ensure the existence of the percentage change $j_C(t)$.

We calculate $j_C(t)$ via a linearized version of (3.11):

$$j_C(t) = J_{CX} * x_t + J_{CC} * c_t + J_{CZ} * z_t \tag{3.16}$$

J_{CX}, J_{CC} and J_{CZ} (without t arguments) are elasticities of the J_C function in (3.11) evaluated at steady-state values of the variables. In general, these elasticities are defined by:

$$J_{CX}(t) = \frac{\partial J_C(t)}{\partial X_t} * \frac{X_t}{J_C(t)} \tag{3.17}$$

$$J_{CC}(t) = \frac{\partial J_C(t)}{\partial C_t} * \frac{C_t}{J_C(t)} \tag{3.18}$$

and

$$J_{CZ}(t) = \frac{\partial J_C(t)}{\partial Z_t} * \frac{Z_t}{J_C(t)} \tag{3.19}$$

We use (3.3), (3.7), (3.8), (3.14) and (3.10) to obtain a non-stochastic version of (3.6):

$$M(t) = (1-\theta) * X_t^{-\theta} * C_t^{\theta} * (X_t^{1-\theta} * C_t^{\theta})^{-\gamma} + \beta * \{\tilde{M}(t)\} * J_X(t) * \frac{X_{t+1}}{X_t} \tag{3.20}$$

Linearizing (3.20) gives

$$M * m(t) = U_X * \left[(\theta*\gamma-\theta-\gamma)*x_t + (1-\gamma)*\theta*c_t \right] + \beta*M*J_X*\left[\tilde{m}(t) + j_X(t) + x_{t+1} - x_t \right] \tag{3.21}$$

where

m(t) and $j_X(t)$ are percentage deviations in M(t) and $J_X(t)$ from their steady-state values. In deriving (3.21) we use the facts that the steady-state values of M(t) and $E_t[M(t+1)]$ are the same and can be written as M, and that the steady state values of X_t and X_{t+1} are the same.

We calculate $j_X(t)$ via a linearized version of (3.10):

$$j_X(t) = J_{XX}*x_t + J_{XC}*c_t + J_{XZ}*z_t \tag{3.22}$$

J_{XX}, J_{XC} and J_{XZ} (without t arguments) are elasticities of the J_X function in (3.10) evaluated at steady-state values of the variables. In general, these elasticities are defined by:

$$J_{XX}(t) = \frac{\partial J_X(t)}{\partial X_t} * \frac{X_t}{J_X(t)} \tag{3.23}$$

$$J_{XC}(t) = \frac{\partial J_X(t)}{\partial C_t} * \frac{C_t}{J_X(t)} \tag{3.24}$$

and

$$J_{XZ}(t) = \frac{\partial J_X(t)}{\partial Z_t} * \frac{Z_t}{J_X(t)} \tag{3.25}$$

With σ fixed, we write the linearized form of (3.7) as

$$m(t) = M_X*x_t + M_Z*z_t \tag{3.26}$$

where

M_X and M_Z are the steady state values of the elasticity of the M function with respect to X and Z defined by:

$$M_X(t) = \frac{\partial M(t)}{\partial X_t} * \frac{X_t}{M(t)} \quad \text{and} \quad M_Z(t) = \frac{\partial M(t)}{\partial Z_t} * \frac{Z_t}{M(t)} \tag{3.27}$$

We assume in this section (but not in section 4) that year t contains no information about exogenous variables in year t+1. Thus, in this section we have

$$E_t[z_{t+1}]=0 \tag{3.28}$$

leading to a simple linearized form for (3.8):

$$\tilde{m}(t) = M_X*x_{t+1} \tag{3.29}$$

In deriving (3.29), we not only adopt (3.28) but we also note that x_{t+1} is non-stochastic: it is completely determined in (3.1) by year-t variables.

Evaluating M_X

We work with the seven equations (3.9), (3.15), (3.16), (3.21), (3.22), (3.26) and (3.29). We take the values of the parameters, γ, β and θ, as given and we assume that the coefficients J_X,

J_C, J_Z, J_{CC}, J_{CX}, J_{CZ}, J_{XC}, J_{XX}, J_{XZ}, M and U_X can be evaluated from the non-stochastic steady-state solution of our model. For the purposes of evaluating M_X, we treat x_t and z_t, as exogenous variables.

To obtain M_X, we set

$$x_t = 1 \text{ and } z_t = 0 \tag{3.30}$$

With the z variable set in this way, M_Z disappears from our 7 equations. This leaves 7 unknowns, x_{t+1}, c_t, $j_C(t)$, $j_X(t)$, m(t), $\tilde{m}(t)$ and M_X, in 7 equations. With x_t set on one and z_t set on zero, the valid solution for M_X (as we will see shortly there is more than one solution) reveals the steady-state elasticity of M(t) with respect to X_t: it is the percentage effect on M of a 1 per cent increase in X_t holding constant all other exogenous variables.

Under condition (3.30), we obtain a quadratic expression for M_X by eliminating the other 6 unknowns from the 7 equations. To do this, we start by using (3.26), (3.29), (3.16) and (3.22) to eliminate m(t), $\tilde{m}(t)$, $j_C(t)$ and $j_X(t)$ from (3.15) and (3.21):

$$(\theta - 1 - \theta * \gamma) * c_t + (1-\theta)(1-\gamma) = M_X * x_{t+1} + J_{CX} + J_{CC} * c_t + x_{t+1} - c_t \tag{3.31}$$

$$\begin{aligned} &M * M_X = \\ &U_X * \left[(\theta * \gamma - \theta - \gamma) + (1-\gamma) * \theta * c_t \right] + \beta * M * J_X * \left[M_X * x_{t+1} + J_{XX} + J_{XC} * c_t + x_{t+1} - 1 \right] \end{aligned} \tag{3.32}$$

Rearranging (3.9) we obtain

$$c_t = \left(\frac{x_{t+1} - J_X}{-J_C} \right) \tag{3.33}$$

Substituting from (3.33) into (3.31) and (3.32) gives

$$\frac{\left\{ -J_X(\theta - \theta * \gamma - J_{CC}) - J_C \left[(1-\theta)(1-\gamma) - J_{CX} \right] \right\}}{\left\{ -J_C \left[M_X + 1 \right] - (\theta - \theta * \gamma - J_{CC}) \right\}} = x_{t+1} \tag{3.34}$$

and

$$\begin{aligned} &\left\{ M * M_X - \beta * M * J_X * (J_{XX} - 1) - U_X * (\theta * \gamma - \theta - \gamma) \right\} = \\ &+ \left[U_X * (1-\gamma) * \theta + \beta * M * J_X * J_{XC} \right] * \left(\frac{x_{t+1} - J_X}{-J_C} \right) \\ &+ \left[\beta * M * J_X * (M_X + 1) \right] * x_{t+1} \end{aligned} \tag{3.35}$$

Rearrange (3.35) to obtain

$$\frac{\begin{bmatrix} J_C * M * M_X - J_C * \beta * M * J_X * (J_{XX} - 1) - J_C * U_X * (\theta * \gamma - \theta - \gamma) \\ -\left[J_X * U_X * (1-\gamma) * \theta + J_X * \beta * M * J_X * J_{XC} \right] \end{bmatrix}}{\left\{ \left[J_C * \beta * M * J_X * (M_X + 1) \right] - \left[U_X * (1-\gamma) * \theta + \beta * M * J_X * J_{XC} \right] \right\}} = x_{t+1} \tag{3.36}$$

Combine (3.34) and (3.36) to eliminate x_{t+1}:

$$\frac{\begin{bmatrix} J_C * M * M_X - J_C * \beta * M * J_X * (J_{XX} - 1) - J_C * U_X * (\theta * \gamma - \theta - \gamma) \\ -[J_X * U_X * (1-\gamma) * \theta + J_X * \beta * M * J_X * J_{XC}] \end{bmatrix}}{\{[J_C * \beta * M * J_X * (M_X + 1)] - [U_X * (1-\gamma) * \theta + \beta * M * J_X * J_{XC}]\}}$$

$$= \frac{\{-J_X(\theta - \theta * \gamma - J_{CC}) - J_C[(1-\theta)(1-\gamma) - J_{CX}]\}}{\{-J_C[M_X + 1] - (\theta - \theta * \gamma - J_{CC})\}} \tag{3.37}$$

Cross multiply in (3.37):

$$\begin{aligned} &\begin{bmatrix} J_C * M * M_X - J_C * \beta * M * J_X * (J_{XX} - 1) - J_C * [U_X * (\theta * \gamma - \theta - \gamma)] \\ -[J_X * U_X * (1-\gamma) * \theta + J_X * \beta * M * J_X * J_{XC}] \end{bmatrix} * \{-J_C M_X - J_C - (\theta - \theta * \gamma - J_{CC})\} \\ &= \{-J_X(\theta - \theta * \gamma - J_{CC}) - J_C[(1-\theta)(1-\gamma) - J_{CX}]\} \\ &\quad * \{J_C * \beta * M * J_X * M_X + J_C * \beta * M * J_X - [U_X * (1-\gamma) * \theta + \beta * M * J_X * J_{XC}]\} \end{aligned} \tag{3.38}$$

Working with (3.38) we find that M_X satisfies the quadratic equation:

$$e2 * M_X^2 + e1 * M_X + e0 = 0 \tag{3.39}$$

where

$$e2 = \{J_C^2 * M\} \tag{3.40}$$

$$\begin{aligned} e1 &= \{-J_X(\theta - \theta * \gamma - J_{CC}) - J_C[(1-\theta)(1-\gamma) - J_{CX}]\} * \{J_C * \beta * M * J_X\} \\ &+ \left\{ \begin{matrix} -J_C \beta * M * J_X * (J_{XX} - 1) - [U_X * (1-\gamma) * \theta + \beta * M * J_X * J_{XC}] * J_X \\ -J_C U_X * [(\theta * \gamma - \theta - \gamma)] \end{matrix} \right\} * \{J_C\} \\ &+ \{J_C M\} * \{J_C + (\theta - \theta * \gamma - J_{CC})\} \end{aligned} \tag{3.41}$$

and

$$\begin{aligned} e0 &= \{-J_X(\theta - \theta * \gamma - J_{CC}) - J_C[(1-\theta)(1-\gamma) - J_{CX}]\} * \{J_C \beta * M * J_X - [U_X * (1-\gamma) * \theta + \beta * M * J_X * J_{XC}]\} \\ &+ \left\{ \begin{matrix} -J_C \beta * M * J_X * (J_{XX} - 1) - [U_X * (1-\gamma) * \theta + \beta * M * J_X * J_{XC}] * J_X \\ -J_C U_X * [(\theta * \gamma - \theta - \gamma)] \end{matrix} \right\} * \{J_C + (\theta - \theta * \gamma - J_{CC})\} \end{aligned} \tag{3.42}$$

Equations (3.39) to (3.42) will normally give two real solutions for M_X. Which should we choose? In our admittedly limited experience, we have found that one of the solutions is negative and one positive. We choose the negative solution: an increase in X_t reduces the value of an extra unit of wealth.

The household consumption function

With the value of M_X now known, we can use our 7 equations to deduce how consumption (C_t) depends on wealth (X_t) and current values of exogenous variables (Z_t).

System (3.9), (3.15), (3.16), (3.21), (3.22), (3.26) and (3.29) gives 7 equations in 6 variables, x_{t+1}, c_t, $j_C(t)$, $j_X(t)$, $m(t)$ and $\tilde{m}(t)$ together with the unknown coefficient M_Z which appears only in (3.26). We delete (3.26): it could be used to back solve for M_Z if required. However, as we will see, M_Z is not required in the derivation of the consumption function. After deleting (3.26) and M_Z we are left with a system of 6 linear equations in 6 variables. We work towards the consumption function, starting by substituting from (3.16) and (3.29) into (3.15):

$$(\theta - 1 - \theta * \gamma) * c_t + (1-\theta)(1-\gamma) * x_t = M_X * x_{t+1} + J_{CX} * x_t + J_{CC} * c_t + J_{CZ} * z_t + x_{t+1} - c_t \quad (3.43)$$

Use (3.9) to eliminate x_{t+1} from (3.43)

$$\begin{aligned} &(\theta - \theta * \gamma) * c_t + (1-\theta)(1-\gamma) * x_t \\ &= (M_X + 1) * \left[J_X * x_t - J_C * c_t + J_Z * z_t\right] + J_{CX} * x_t + J_{CC} * c_t + J_{CZ} * z_t \end{aligned} \quad (3.44)$$

Rearrange (3.44):

$$\begin{aligned} &\left[(\theta - \theta * \gamma) + (M_X + 1) * J_C - J_{CC}\right] * c_t \\ &= \left[(M_X + 1)\right] * \left[J_X * x_t\right] + J_{CX} * x_t - (1-\theta)(1-\gamma) * x_t \\ &(M_X + 1) * \left[J_Z * z_t\right] + J_{CZ} * z_t \end{aligned} \quad (3.45)$$

From (3.45) we obtain the consumption function as:

$$c_t = \frac{\left[(M_X + 1)J_X + J_{CX} - (1-\theta)(1-\gamma)\right]}{\left[(\theta - \theta * \gamma) + (M_X + 1) * J_C - J_{CC}\right]} * x_t + \frac{\left[(M_X + 1)J_Z + J_{CZ}\right]}{\left[(\theta - \theta * \gamma) + (M_X + 1) * J_C - J_{CC}\right]} * z_t \quad (3.46)$$

Equation (3.46) confirms that M_Z does not play a role in the determination of consumption. How should we interpret this? We consider the standard neoclassical model in which Z_t is technology. An increase in Z_t causes the economy to produce more income in year t with the given capital stock X_t. How much of this transitory increase in income should be consumed and how much should be invested (the trade off between more C_t and more X_{t+1}) depends on: the productivity of capital in the future (reflected in the parameter α and the steady-state value of Z); the rate of diminishing marginal utility to consumption in the current year (reflected in the parameters γ and θ); and the rate at which future consumption is discounted relative to current consumption (reflected in the parameter β). None of these factors is affected by a transitory improvement in technology. These factors, together with the transitory increase in income generated by the increase in Z_t, determine the increase in C_t independently of the effect of the change in Z_t on the valuation of an extra unit of wealth in year t.

Checking formulas (3.39) – (3.42) for M_X and (3.46) for c_t by applying them in the standard neo-classical model

The model we are studying in this section, that is the model whose linearized form is given by (3.9), (3.15), (3.16), (3.21), (3.22), (3.26) and (3.29), is a generalization of the standard neoclassical model that we studied in sections 1 and 2. It becomes the standard model if we set

$$\theta = 1 \quad (3.47)$$

and adopt the specific form

$$X_{t+1} = X_t * (1-\delta) + Z_t * X_t^{\alpha} - C_t \quad (3.48)$$

for the wealth accumulation equation (3.1). We check that our more general model under conditions (3.47) and (3.48) produces results that are consistent with those in subsection 2.2.

We start by deriving formulas in the standard model for the elasticities of the accumulation relationship: J_C, J_X, J_Z, J_{XX}, J_{XC}, J_{XZ}, J_{CC}, J_{CX}, J_{CZ}. Under (3.48)

$$J_X(t)=\frac{\partial X_{t+1}}{\partial X_t}*\frac{X_t}{X_{t+1}}=\left[(1-\delta)+\alpha*Z_t*X_t^{\alpha-1}\right]*\frac{X_t}{X_{t+1}} \tag{3.49}$$

$$J_C(t)=-\frac{\partial X_{t+1}}{\partial C_t}*\frac{C_t}{X_{t+1}}=\frac{C_t}{X_{t+1}} \tag{3.50}$$

and

$$J_Z(t)=\frac{\partial X_{t+1}}{\partial Z_t}*\frac{Z_t}{X_{t+1}}=X_t^{\alpha}*\frac{Z_t}{X_{t+1}} \tag{3.51}$$

Now we derive the required elasticities of the elasticities

$$J_{XC}(t)=\frac{\partial\left(\frac{\partial X_{t+1}}{\partial X_t}*\frac{X_t}{X_{t+1}}\right)}{\partial C_t}*\frac{C_t}{J_X(t)}=\frac{C_t}{X_{t+1}}=J_C(t) \tag{3.52}$$

$$J_{XX}(t)=\frac{\partial\left(\frac{\partial X_{t+1}}{\partial X_t}*\frac{X_t}{X_{t+1}}\right)}{\partial X_t}*\frac{X_t}{J_X(t)}=\left\{\frac{\left[(1-\delta)+\alpha^2*Z_t*X_t^{\alpha-1}\right]}{\left[(1-\delta)+\alpha*Z_t*X_t^{\alpha-1}\right]}-\frac{\left(\left[(1-\delta)+\alpha*Z_t*X_t^{\alpha-1}\right]*X_t\right)}{X_{t+1}}\right\} \tag{3.53}$$

$$J_{CC}(t)=\frac{\partial\left(-\frac{\partial X_{t+1}}{\partial C_t}*\frac{C_t}{X_{t+1}}\right)}{\partial C_t}*\frac{C_t}{J_C(t)}=1+\frac{C_t}{X_{t+1}}=1+J_C(t) \tag{3.54}$$

$$J_{CX}(t)=\frac{\partial\left(-\frac{\partial X_{t+1}}{\partial C_t}*\frac{C_t}{X_{t+1}}\right)}{\partial X_t}*\frac{X_t}{J_C(t)}==-\left[(1-\delta)+\alpha*Z_t*X_t^{\alpha-1}\right]*\frac{X_t}{X_{t+1}} \tag{3.55}$$

$$J_{XZ}(t)=\frac{\partial\left(\frac{\partial X_{t+1}}{\partial X_t}*\frac{X_t}{X_{t+1}}\right)}{\partial Z_t}*\frac{Z_t}{J_X(t)}=\frac{\alpha*X_t^{\alpha-1}*Z_t}{\left[(1-\delta)+\alpha*Z_t*X_t^{\alpha-1}\right]}-\frac{X_t^{\alpha}*Z_t}{X_{t+1}} \tag{3.56}$$

$$J_{CZ}(t)=\frac{\partial\left(-\frac{\partial X_{t+1}}{\partial C_t}*\frac{C_t}{X_{t+1}}\right)}{\partial Z_t}*\frac{Z_t}{J_C(t)}=\frac{\partial\left(\frac{C_t}{X_{t+1}}\right)}{\partial Z_t}*\frac{Z_t*X_{t+1}}{C_t}=-J_Z(t) \tag{3.57}$$

We adopt the same parameter values as in subsection 2.2: β=0.9, δ=0.05, $\bar{Z}$=1, α=0.5 and γ=0.5. This gives the steady-state values $\bar{X}$ = 9.631385, $\bar{C}$ = 2.621878 and $\bar{V}_X = \bar{M}$ = 0.686201 [see (2.12)]. Also note that with θ = 1, U_X = 0. Now we have enough information to calculate the steady-state values of the elasticities in (3.49) to (3.57). From there we can evaluate e2, e1 and e0 in the (3.40) to (3.42). Substituting into (3.39) we obtained

$$0.050851*M_X{}^2+0.014164*M_X-0.006771=0 \tag{3.58}$$

The two solutions of (3.58) are -0.528531 and 0.251950. We accept the negative solution: we expect extra wealth to reduce the marginal value of wealth. Hence,

$$M_X=-0.528531 \tag{3.59}$$

Using this value in (3.46) we obtain the consumption function

$$c_t = 0.912063 * x_t + 0.264498 * z_t \tag{3.60}$$

In subsection 2.2 we found that a unit increase in capital (now wealth) causes an increase in consumption of 0.248284 units, see (2.77) and (2.79). With the steady-state values of wealth and consumption given by $\bar{X}$ = 9.631385, $\bar{C}$ = 2.621878, we can translate this result into an elasticity of consumption with respect to wealth:

$$\frac{\partial C_t}{\partial X_t} * \frac{X_t}{C_t} = 0.248284 * \frac{9.631385}{2.621878} = 0.912063 \tag{3.61}$$

This is consistent with (3.60).

We cannot check our result in (3.60) for elasticity of consumption with respect Z_t against results in subsection 2.2. There, we held Z_t constant. However, as a partial check we consider two experiments. In the first we assume that the household receives a small increase in wealth, say 0.01 per cent, and decides after allowing for depreciation, to devote all of this windfall to extra consumption in year t. Then the increase in consumption is

$$\begin{aligned}(dC_t)_{\text{wealth}} &= \sqrt{9.631385 * 1.0001} - \sqrt{9.631385} - 0.05 ** 0.0001 * 9.631385 + 0.0001 * 9.631385 \\ &= 0.001070\end{aligned} \tag{3.62}$$

In the second experiment we assume that there is a temporary 0.01 per cent increase in technology in year t, generating extra income. If the household decides to consume this extra income, then the increase to current consumption is.

$$(dC_t)_{\text{tech}} = 0.0001 * \sqrt{9.631385} = 0.000310 \tag{3.63}$$

Equations (3.62) and (3.63) indicate that a 1 per cent increase in wealth generates a welfare increase to the household of about 3.45 (=0.001070/0.000310) times that generated by a transitory 1 per cent improvement in technology. On this basis we would expect the ratio of the x_t coefficient to the z_t coefficient in (3.70) to be about 3.45. It is in fact, 3.45 (= 0.9121/0.2645).

4. Introducing contemporaneous and time-series correlations in exogenous variables

In this section we study a model that is the same as in section 3, but with two differences. We allow for more than one exogenous variable (a vector of Zs rather than a scalar) and we allow for correlations between different exogenous variables and between the realization of any given exogenous variable at different times. The model is as follows.

Accumulation equation

$$X_{t+1} = J(X_t, Z_{1t}, ..., Z_{nt}, C_t) \tag{4.1}$$

Exogenous variables with contemporaneous and time series correlation

$$Z_{i,t+1} = B_i(Z_{1t}, ..., Z_{nt}) * \exp(\sigma_i \varepsilon_{i,t+1}) \quad i = 1, \dots, n \tag{4.2}$$

In (4.2) we introduce a 1-year lag. Technology, terms of trade, etc, in year t+1 reflect values in year t and a multiplicative stochastic component.

Lifetime welfare

$$V(X_t, Z_{1t}, ..., Z_{nt}, \sigma) = U(X_t, C_t) + \beta * E_t\left[V(X_{t+1}, Z_{1,t+1}, ..., Z_{n,t+1}, \sigma)\right] \tag{4.3}$$

where

$$U(t)=\frac{\left(X_t^{1-\theta}*C_t^{\theta}\right)^{1-\gamma}}{(1-\gamma)} \tag{4.4}$$

Optimal allocation of consumption between years t and t+1 and the value, V_X, of an extra unit of wealth

We assume that variations in C_t do not affect Z_t. Then, under (4.2), variations in C_t do not affect expectations concerning Z_{t+1}. Hence, optimizing with respect to C_t:

$$U_C(t)+\beta*\left\{E_t\left[V_X(t+1)\right]\right\}*\frac{\partial X_{t+1}}{\partial C_t}=0 \tag{4.5}$$

Differentiating the V function in (4.3) with respect to X_t, and assuming that the Zs are independent of X_t, we obtain

$$V_X(t)=U_X(t)+U_C(t)*\frac{\partial C_t}{\partial X_t}+\beta*E_t\left[V_X(t+1)*\frac{\partial X_{t+1}}{\partial X_t}+V_X(t+1)*\frac{\partial X_{t+1}}{\partial C_t}*\frac{\partial C_t}{\partial X_t}\right] \tag{4.6}$$

Simplify (4.6) using (4.5):

$$V_X(t)=U_X(t)+\beta*\left\{E_t\left[V_X(t+1)\right]\right\}*\frac{\partial X_{t+1}}{\partial X_t} \tag{4.7}$$

Policy rule

We specify the policy rule for the household as:

$$V_X(t)=M\left(X_t,Z_{1t},...,Z_{nt},\sigma\right) \tag{4.8}$$

Finally, we assume that the household expects in year t to be implementing its policy rule in year t+1.

$$E_tV_X(t+1)=E_tM\left[X_{t+1},Z_{t+1},\sigma\right] \tag{4.9}$$

Linearizing around a non-stochastic steady state

We linearize (4.1), (4.2),(4.5), (4.7), (4.8) and (4.9) around a non-stochastic steady state, obtaining

$$x_{t+1}=J_X*x_t-J_C*c_t+\sum_{j=1}^{n}J_{Zj}*z_{j,t} \tag{4.10}$$

$$\tilde{z}_{i,t}=\sum_{j=1}^{n}B_{i,j}*z_{j,t} \qquad \text{for } i = 1, \ldots, n \tag{4.11}$$

$$(\theta-1-\theta*\gamma)*c_t+(1-\theta)(1-\gamma)*x_t=\tilde{m}(t)+j_C(t)+x_{t+1}-c_t \tag{4.12}$$

$$j_C(t)=J_{CX}*x_t+J_{CC}*c_t+\sum_{j=1}^{n}J_{CZj}*z_{j,t} \tag{4.13}$$

$$\begin{aligned}&M*m(t)=\\&U_X*\left[(\theta*\gamma-\theta-\gamma)*x_t+(1-\gamma)*\theta*c_t\right]+\beta*M*J_X*\left[\tilde{m}(t)+j_X(t)+x_{t+1}-x_t\right]\end{aligned} \tag{4.14}$$

$$j_X(t)=J_{XX}*x_t+J_{XC}*c_t+\sum_{j=1}^{n}J_{XZj}*z_{j,t} \tag{4.15}$$

$$m(t) = M_X * x_t + \sum_{k=1}^{n} M_{Zk} * z_{k,t} + M_\sigma * d\sigma \tag{4.16}$$

$$\tilde{m}(t) = M_X * x_{t+1} + \sum_{k=1}^{n} M_{Zk} * \sum_{j=1}^{n} B_{k,j} * z_{j,t} + M_\sigma * d\sigma \tag{4.17}$$

Most of the notation in these equations is familiar from corresponding equations in section 3, see (3.15), (3.16), (3.21), (3.22) , (3.26) and (3.29). New notation is as follows:

J_{Zj} is the elasticity of J with respect to Z_j;

J_{CZj} and J_{XZj} are elasticities of J_C and J_X with respect to Z_j;

$B_{i,j}$ is the elasticity of B_i with respect to Z_j;

M_{Zk} is the elasticity of M with respect to Z_k;

M_σ is the semi-elasticity of M with respect to σ, defined by $(100/M)*(\partial M/\partial\sigma)$; and

$\tilde{z}_{i,t}$ is the expectation held at time t for the percentage deviation in $Z_{i,t+1}$ from its steady-state value.

As previously, $\tilde{M}(t)$ is the expectation held in year t concerning the valuation, M(t+1), that will be given to an extra unit of wealth at the start of year t+1:

$$\tilde{M}(t) = E_t\left[M(X_{t+1}, Z_{1,t+1}, \ldots, Z_{n,t+1}, \sigma)\right] \tag{4.18}$$

This leads to (4.17). Notice that (4.17) contains percentage deviations in Zs. These are absent from the corresponding equation, (3.29), in section 3 where we assumed that realizations of year t exogenous variables contain no information about exogenous variables in year t+1.

Evaluating M_X

The only equation in our linearized system that contains $\tilde{z}_{i,t}$ is (4.11). We now delete this equation along with the variable and work with the 7-equation system, (4.10) & (4.12) to (4.17). We take the values of the parameters, γ, β and θ, as given and we assume that the coefficients J_X, J_C, J_{Zk}, J_{CC}, J_{CX}, J_{CZk}, J_{XC}, J_{XX}, J_{XZk}, $B_{i,k}$, M and U_X can be evaluated from the non-stochastic steady-state solution of our model. For the purposes of evaluating M_X, we treat x_t and $z_{i,t}$ for all i as exogenous variables.

To obtain M_X, we set

$$x_t = 1 \text{ and } z_{i,t} = 0 \quad \text{for } i = 1, \ldots, n \tag{4.19}$$

With the z variable set in this way, M_{Zk} disappears from our 7 equations. This leaves 7 unknowns, x_{t+1}, c_t, $j_C(t)$, $j_X(t)$, m(t), $\tilde{m}(t)$ and M_X, in 7 equations. These 7 equations and 7 unknowns form exactly the same system that we used in section 3 to evaluate M_X. Therefore (3.39) to (3.42) remains a valid system here for obtaining M_X..

Evaluating M_{Zk}, k = 1, …, n

Now we assume that M_X is known. We put

$$z_{q,t} = 1,\ z_{k,t} = 0 \text{ for all } k \neq q, \text{ and } x_t = 0 \tag{4.20}$$

Under (4.20), equations (4.10) & (4.12) to (4.17) reduce to

$$x_{t+1} = -J_C * c_t + J_{Zq} \tag{4.21}$$

$$(\theta-1-\theta*\gamma)*c_t=\tilde{m}(t)+j_C(t)+x_{t+1}-c_t \tag{4.22}$$

$$j_C(t)=J_{CC}*c_t+J_{CZq} \tag{4.23}$$

$$M*m(t)=U_X*(1-\gamma)*\theta*c_t+\beta*M*J_X*[\tilde{m}(t)+j_X(t)+x_{t+1}] \tag{4.24}$$

$$j_X(t)=J_{XC}*c_t+J_{XZq} \tag{4.25}$$

$$m(t)=M_{Zq} \tag{4.26}$$

$$\tilde{m}(t)=M_X*x_{t+1}+\sum_{k=1}^{n}M_{Zk}*B_{k,q} \tag{4.27}$$

Substitute out m(t), $\tilde{m}(t)$, jC(t) and jX(t):

$$x_{t+1}=-J_C*c_t+J_{Zq} \tag{4.28}$$

$$(\theta-1-\theta*\gamma)*c_t=M_X*x_{t+1}+\sum_{k=1}^{n}M_{Zk}*B_{k,q}+J_{CC}*c_t+J_{CZq}+x_{t+1}-c_t \tag{4.29}$$

$$\begin{aligned}&M*M_{Z_q}=\\&U_X*(1-\gamma)*\theta*c_t+\beta*M*J_X*\left[M_X*x_{t+1}+\sum_{k=1}^{n}M_{Zk}*B_{k,q}+J_{XC}*c_t+J_{XZq}+x_{t+1}\right]\end{aligned} \tag{4.30}$$

Rearrange (4.28):

$$c_t=\left(\frac{x_{t+1}-J_{Zq}}{-J_C}\right) \tag{4.31}$$

Use (4.31) to eliminate c_t from (4.29) and (4.30) and rearrange to obtain:

$$x_{t+1}=\frac{[(\theta-\theta*\gamma)-J_{CC}]*J_{Zq}-J_C*\sum_{k=1}^{n}M_{Zk}*B_{k,q}-J_C*J_{CZq}}{\mathbf{blue}} \tag{4.32}$$

and

$$\frac{\left\{\begin{aligned}&-J_C*M*M_{Zq}+[U_X*(1-\gamma)*\theta+\beta*M*J_X*J_{XC}]*J_{Zq}\\&+J_C*\beta*M*J_X*\sum_{k=1}^{n}M_{Zk}*B_{k,q}+J_C*\beta*M*J_X*J_{XZq}\end{aligned}\right\}}{\mathbf{red}}=x_{t+1} \tag{4.33}$$

where

$$\mathbf{red}=\left[U_X*(1-\gamma)*\theta+\beta*M*J_X*J_{XC}-J_C\beta*M*J_X*(M_X+1)\right] \tag{4.34}$$

$$\mathbf{blue}=\left[\theta-\theta*\gamma-J_{CC}+J_CM_X+J_C\right] \tag{4.35}$$

Eliminating x_{t+1} gives:

$$\begin{aligned}M_{Zq}=\frac{1}{-J_C*M*\mathbf{blue}}&\left[\begin{aligned}&\{-[U_X*(1-\gamma)*\theta+\beta*M*J_X*J_{XC}]*J_{Zq}-J_C*\beta*M*J_X*J_{XZq}\}*\mathbf{blue}\\&+\mathbf{red}*\{[(\theta-\theta*\gamma)-J_{CC}]*J_{Zq}-J_C*J_{CZq}\}\end{aligned}\right]\\&+\left\{\beta*J_X+\frac{\mathbf{red}}{M*\mathbf{blue}}\right\}*\sum_{k=1}^{n}M_{Zk}*B_{k,q}\end{aligned} \tag{4.36}$$

That is

$$M_{Zq} = T1(q) + T2 * \sum_{k=1}^{n} M_{Zk} * B_{k,q} \quad \text{for } q = 1, \ldots, n \tag{4.37}$$

where

$$T1(q) = \frac{1}{-J_C * M * \mathbf{blue}} \left[\begin{array}{l} \{-[U_X * (1-\gamma) * \theta + \beta * M * J_X * J_{XC}] * J_{Zq} - J_C * \beta * M * J_X * J_{XZq}\} * \mathbf{blue} \\ \quad + \mathbf{red} * \{[(\theta - \theta * \gamma) - J_{CC}] * J_{Zq} - J_C * J_{CZq}\} \end{array} \right]$$

(4.38)

and

$$T2 = \beta * J_X + \frac{\mathbf{red}}{M * \mathbf{blue}} \tag{4.39}$$

Values for the n elasticities, M_{Zq}, q = 1, …, n, can now be determined by solving the n equations in (4.37) with T1(q), T2, **red** and **blue** evaluated via (4.38), (4.39), (4.34) and (4.35).

The household consumption function

With the values of M_X and M_{Zq} for q = 1, …, n, now known, we can deduce how consumption (C_t) depends on wealth (X_t) and current values of exogenous variables (Z_t).

We substitute from (4.10), (4.13) and (4.17) into (4.12):

$$(\theta - \theta * \gamma) * c_t + (1-\theta)(1-\gamma) * x_t = \left([M_X + 1] * \left(J_X * x_t - J_C * c_t + \sum_{j=1}^{n} J_{Zj} * z_{j,t} \right) + \sum_{k=1}^{n} M_{Zk} * \sum_{j=1}^{n} B_{k,j} * z_{j,t} \right)$$

$$+ J_{CX} * x_t + J_{CC} * c_t + \sum_{j=1}^{n} J_{CZj} * z_{j,t}$$

(4.40)

Rearrange to obtain the consumption function:

$$c_t = \frac{\{[M_X + 1] * J_X - (1-\theta)(1-\gamma) + J_{CX}\}}{[(\theta - \theta * \gamma) + [M_X + 1] * J_C - J_{CC}]} * x_t + \sum_{j=1}^{n} \frac{\left[(M_X + 1) * J_{Zj} + J_{CZj} + \sum_{k=1}^{n} M_{Zk} * B_{k,j} \right]}{[(\theta - \theta * \gamma) + [M_X + 1] * J_C - J_{CC}]} * z_{j,t} \tag{4.41}$$

Numerical example: standard neoclassical model with multiple exogenous variables and lags

We test the formulas for M_X and M_{Zk} for k = 1,…,n, derived in this section by applying them to a version of the standard neoclassical model in which there are lags and two exogenous variables.

As in the earlier sections, we assume that

$$\theta = 1 \tag{4.42}$$

For (4.1) and (4.2) we adopt the specific forms

$$X_{t+1} = X_t * (1-\delta) + Z_{2,t} * X_t^{\alpha} - C_t \tag{4.43}$$

and

$$B_1(Z_{1,t}, Z_{2,t}) = Z_{1,t} \tag{4.44}$$

$$B_2(Z_{1,t}, Z_{2,t}) = Z_{1,t} * \left(\frac{Z_{2,t}}{Z_{1,t}}\right)^{\rho} \tag{4.45}$$

where $0 \le \rho < 1$.

In (4.44) and (4.45), $Z_{1,t}$ is the permanent level of technology and $Z_{2,t}$ is the actual level in year t. If the actual level in year t deviates from the permanent level, then in year t+1 there is a tendency for the actual level to move back towards the permanent level. While we think of $Z_{1,t}$ as the permanent level, we allow for changes in this level. With both $Z_{1,t}$ and $Z_{2,t}$ in the model, we can distinguish between the effects of permanent changes in technology and transitory changes. By setting ρ equal to zero and assuming that $Z_{1,t}$ never moves from its steady-state level ($z_{1,t} = 0$), we revert to case considered in earlier sections in which the technology deviation in year t+1 is determined independently of technology in year t.

The formulas for M_X, M_{Zk} and ultimately for the consumption function in (4.41) depend on elasticities of the accumulation relationship (J elasticities) and on elasticities of the B functions. Under (4.43) to (4.45), the required elasticities are as follows:

$$J_X(t) = \frac{\partial X_{t+1}}{\partial X_t} * \frac{X_t}{X_{t+1}} = \left[(1-\delta) + \alpha * Z_{2,t} * X_t^{\alpha-1}\right] * \frac{X_t}{X_{t+1}} \tag{4.46}$$

$$J_C(t) = -\frac{\partial X_{t+1}}{\partial C_t} * \frac{C_t}{X_{t+1}} = \frac{C_t}{X_{t+1}} \tag{4.47}$$

$$J_{Z1}(t) = \frac{\partial X_{t+1}}{\partial Z_{1,t}} * \frac{Z_{1,t}}{X_{t+1}} = 0 \tag{4.48}$$

$$J_{Z2}(t) = \frac{\partial X_{t+1}}{\partial Z_{2,t}} * \frac{Z_{2,t}}{X_{t+1}} = X_t^{\alpha} * \frac{Z_{2,t}}{X_{t+1}} \tag{4.49}$$

$$J_{XC}(t) = \frac{\partial\left(\frac{\partial X_{t+1}}{\partial X_t} * \frac{X_t}{X_{t+1}}\right)}{\partial C_t} * \frac{C_t}{J_X(t)} = \frac{C_t}{X_{t+1}} = J_C(t) \tag{4.50}$$

$$J_{XX}(t) = \frac{\partial\left(\frac{\partial X_{t+1}}{\partial X_t} * \frac{X_t}{X_{t+1}}\right)}{\partial X_t} * \frac{X_t}{J_X(t)} = \left\{\frac{\left[(1-\delta) + \alpha^2 * Z_{2,t} * X_t^{\alpha-1}\right]}{\left[(1-\delta) + \alpha * Z_{2,t} * X_t^{\alpha-1}\right]} - \frac{\left(\left[(1-\delta) + \alpha * Z_{2,t} * X_t^{\alpha-1}\right] * X_t\right)}{X_{t+1}}\right\} \tag{4.51}$$

$$J_{CC}(t) = \frac{\partial\left(-\frac{\partial X_{t+1}}{\partial C_t} * \frac{C_t}{X_{t+1}}\right)}{\partial C_t} * \frac{C_t}{J_C(t)} = 1 + \frac{C_t}{X_{t+1}} = 1 + J_C(t) \tag{4.52}$$

$$J_{CX}(t) = \frac{\partial\left(-\frac{\partial X_{t+1}}{\partial C_t} * \frac{C_t}{X_{t+1}}\right)}{\partial X_t} * \frac{X_t}{J_C(t)} == -\left[(1-\delta) + \alpha * Z_{2,t} * X_t^{\alpha-1}\right] * \frac{X_t}{X_{t+1}} \tag{4.53}$$

$$J_{XZ1}(t) = \frac{\partial\left(\frac{\partial X_{t+1}}{\partial X_t} * \frac{X_t}{X_{t+1}}\right)}{\partial Z_{1,t}} * \frac{Z_{1,t}}{J_X(t)} = \left(-\left[(1-\delta) + \alpha * Z_{2,t} * X_t^{\alpha-1}\right] * \frac{X_t}{X_{t+1}^{\ 2}} * 0\right) * \frac{Z_{1,t}}{J_X(t)} = 0 \tag{4.54}$$

$$J_{XZ2}(t)=\frac{\partial\left(\frac{\partial X_{t+1}}{\partial X_t}*\frac{X_t}{X_{t+1}}\right)}{\partial Z_{2,t}}*\frac{Z_{2,t}}{J_X(t)}=Z_{2,t}*X_t^{\alpha-1}*\left(\frac{\alpha*X_t}{J_X*X_{t+1}}-\frac{X_t}{X_{t+1}}\right) \tag{4.55}$$

$$J_{CZ1}(t)=\frac{\partial\left(-\frac{\partial X_{t+1}}{\partial C_t}*\frac{C_t}{X_{t+1}}\right)}{\partial Z_{1,t}}*\frac{Z_{1,t}}{J_C(t)}=\frac{\partial\left(\frac{C_t}{X_{t+1}}\right)}{\partial Z_{1,t}}*\frac{Z_{1t}*X_{t+1}}{C_t}=0 \tag{4.56}$$

$$J_{CZ2}(t)=\frac{\partial\left(-\frac{\partial X_{t+1}}{\partial C_t}*\frac{C_t}{X_{t+1}}\right)}{\partial Z_{2,t}}*\frac{Z_{2,t}}{J_C(t)}=\frac{\partial\left(\frac{C_t}{X_{t+1}}\right)}{\partial Z_{2,t}}*\frac{Z_{2,t}*X_{t+1}}{C_t}=\left(\frac{-Z_{2,t}}{X_{t+1}}*X_t^{\alpha}\right)=-J_{Z2}(t) \tag{4.57}$$

$$B_{1,1}(t)=\frac{\partial B_1(t)}{\partial Z_{1,t}}*\frac{Z_{1,t}}{B_1(t)}=1 \tag{4.58}$$

$$B_{1,2}(t)=\frac{\partial B_1(t)}{\partial Z_{2,t}}*\frac{Z_{2,t}}{B_1(t)}=0 \tag{4.59}$$

$$B_{2,1}(t)=\frac{\partial B_2(t)}{\partial Z_{1,t}}*\frac{Z_{1,t}}{B_2(t)}=1-\rho \tag{4.60}$$

$$B_{2,2}(t)=\frac{\partial B_2(t)}{\partial Z_{2,t}}*\frac{Z_{2,t}}{B_2(t)}=\rho \tag{4.61}$$

As in the numerical parts of sections 2 and 3 we assume that β=0.9, δ=0.05, α=0.5 and γ=0.5. Consistent with those earlier calculations in which $\bar{Z}=1$, here we assume that the steady-state values of Z_1 and Z_2 are one. With regard to ρ, we check that the earlier results are reproduced when $\rho = 0$. Then, we generate results with $\rho = 0.5$.

With these two values for ρ, our calculations produce the following consumption functions:

for $\rho = 0$, $c_t = 0.912063*x_t - 0.088623*z_{1t} + 0.264498*z_{2t}$ (4.62)

for $\rho = 0.5$ $c_t = 0.912063*x_t - 0.072437*z_{1t} + 0.248312*z_{2t}$ (4.63)

The numerical model in section 3 corresponds to the case in which $\rho = 0$: no effect on future technology of shocks to current technology (z_{2t}). Reassuringly, in (4.62), the coefficient on z_{2t} is the same as that in (3.60).

Four features of (4.62) and (4.63) stand out. First, the value of ρ makes no difference to how consumption responds to an increase in wealth (the coefficient on x_t is the same in both equations. This could be anticipated on the basis of (4.41): ρ plays no role in the determination of M_X or any of the J elasticities. More intuitively, ρ affects the persistence of technology shocks whereas the coefficient on x_t determines the effects of an extra unit of wealth holding technology constant (no technology shocks).

Second, the coefficient on z_{2t} in (4.63) is less than the corresponding coefficient in (4.62). This means that increased persistence of a technology shock (higher ρ) reduces the immediate consumption effect of a transitory technology improvement. Initially, we found this result surprising. With $\rho = 0.5$, a transitory 1 per cent improvement in technology (z_{2t} = 1) has favourable effects on technology into the future whereas when $\rho = 0$ the transitory improvement in technology has no effect on technology in the future. From the point of view of lifetime welfare, a transitory 1 per cent improvement in technology when $\rho = 0.5$ is more beneficial than when $\rho = 0$. On this basis we expected the immediate effect on consumption to be greater when $\rho = 0.5$ than when $\rho = 0$. However, with the technology improvement persisting into the future, capital stock in future years is more productive. Consequently, investment in year t becomes more attractive with technology persistence. This effect is strong enough to leave consumption slightly less stimulated in year t in the persistence case. As illustrated in Figure 4.1, beyond year t consumption and capital are higher when $\rho = 0.5$ than when $\rho = 0$.

Third, the coefficients on z_{1t} in (4.62) and (4.63) are negative, implying that a permanent improvement in technology (z_{1t} positive) reduces immediate consumption. The explanation is that the permanent technology improvement increases the attractiveness of building up capital (wealth) to take advantage of the technology improvement in future years. This requires extra investment in year t, with correspondingly lower consumption. As illustrated in Figures 4.2 and 4.3, beyond year t consumption is stimulated. The immediate reduction in consumption associated with a permanent improvement in technology is slightly more pronounced when $\rho = 0$ than when $\rho = 0.5$. The coefficient on z_{1t} in (4.62) is -0.088623 whereas in (4.63) it is -0.072437. As can be seen from the green lines in Figure 4.4, when $\rho = 0$, the permanent technology improvement is introduced in full immediately. By contrast, when $\rho = 0.5$, the technology improvement is phased in. The immediate technology improvement under $\rho = 0$ encourages investment in year t more strongly than is the case under the phase-in scenario, $\rho = 0.5$.

Fourth, the sum of the coefficient on z_{1t} and z_{2t} is the same in (4.62) and (4.63): 0.175875 in both cases. This indicates that the effect on immediate consumption of an increase of 1 per cent in both permanent and transitory technology ($z_{1t} = z_{2t} = 1$) does not depend on ρ. The response in year t of consumption to technology shocks depends on how these shocks affect the trade-off between immediate consumption and future consumption. When $z_{1t} = z_{2t} = 1$, both current and future technology is improved by 1 per cent, independently of ρ.

5. Estimation of the elasticities of the accumulation relationship

Implementation of consumption functions such as (3.46) and (4.41) requires steady-state values for the elasticities, J_X, J_C, J_Z, J_{CC}, J_{CX}, J_{CZ}, J_{XC}, J_{XX}, J_{XZ} associated with the accumulation relationship. This presents no difficulty when we are dealing with small-scale models such as those in sections 2 to 4 in which the accumulation relationship is simple and explicit, see (3.48) and the resulting elasticities in (3.49) to (3.57). But how do we evaluate the elasticities in a large-scale CGE model in which wealth accumulation is not represented by a simple explicit function, but instead is the outcome of a system of equations involving a large number of variables including wage rates, profits, taxes, interest rates, capital stocks and employment?

Before we can explain our plan, we need to fill in some background. Simulations with CGE models of the type we use consist of two runs, the baseline and policy runs.[7] Usually the baseline is intended as a business-as-usual, year-on-year picture of the paths for the myriad of variables in CGE models, such as employment and output by industry. CGE modellers often build into the baseline trends in technology, consumer preferences and commodity prices together with demographic projections. The policy run is usually undertaken with a different closure (choice of exogenous variables) from that in the baseline. For example, macro

[7] This includes models such as MONASH, USAGE and VU-NATIONAL. See for example, Dixon and Rimmer (2002) and Dixon *et al.* (2013).

Figure 4.1. A transitory improvement in technology (z_{2t} = 1, z_{1t} =0 for all t)

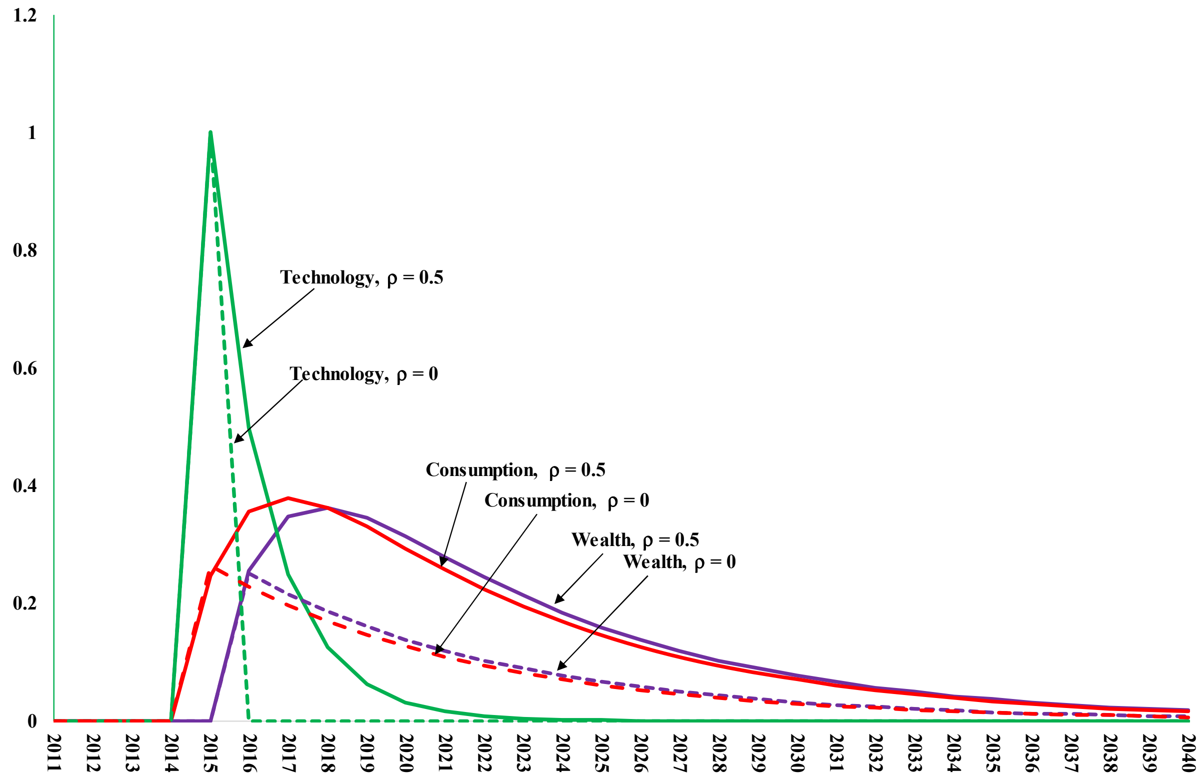


Figure 4.2. A permanent improvement in technology (z_{1t} = 1for all t) with ρ= 0.5

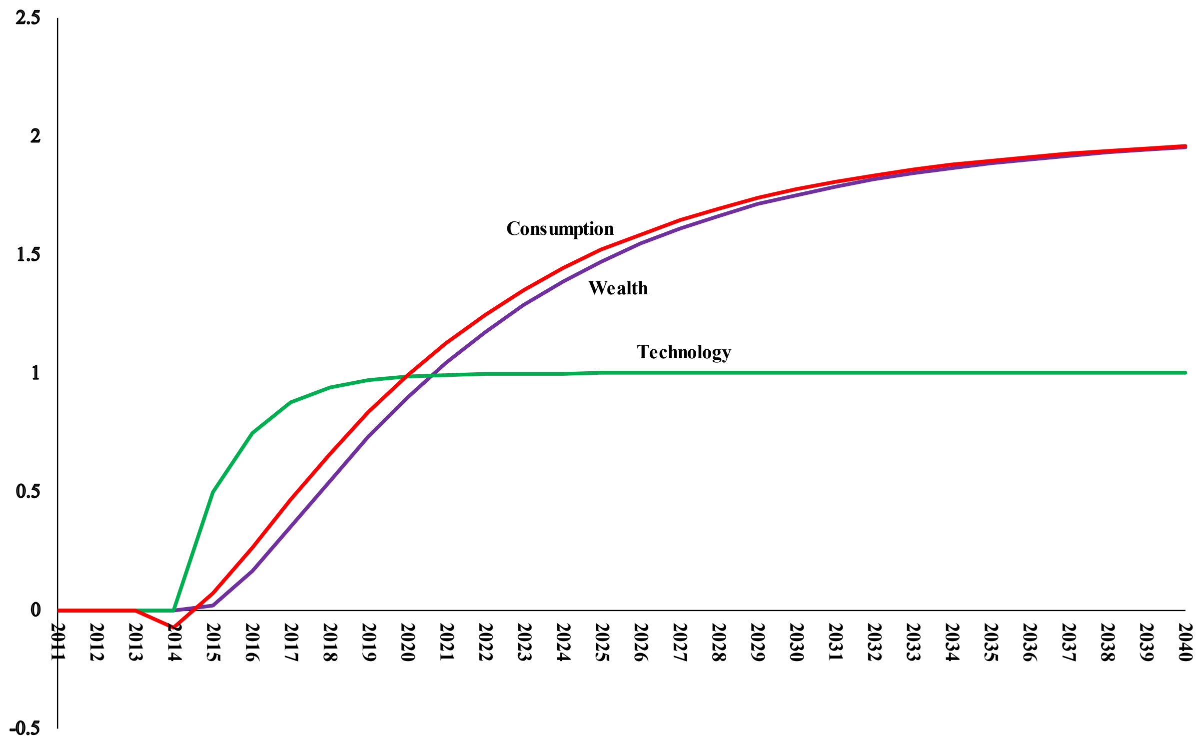


Figure 4.3. A permanent improvement in technology (z_{1t} = 1for all t) with ρ= 0

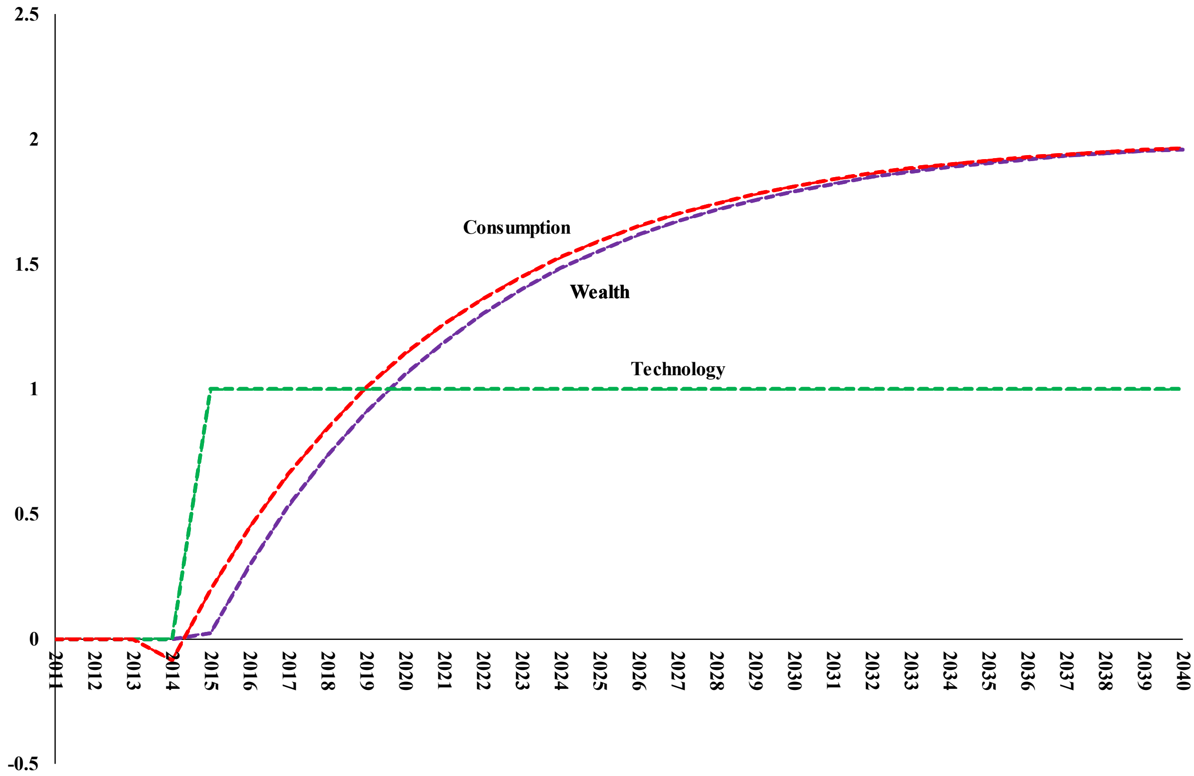
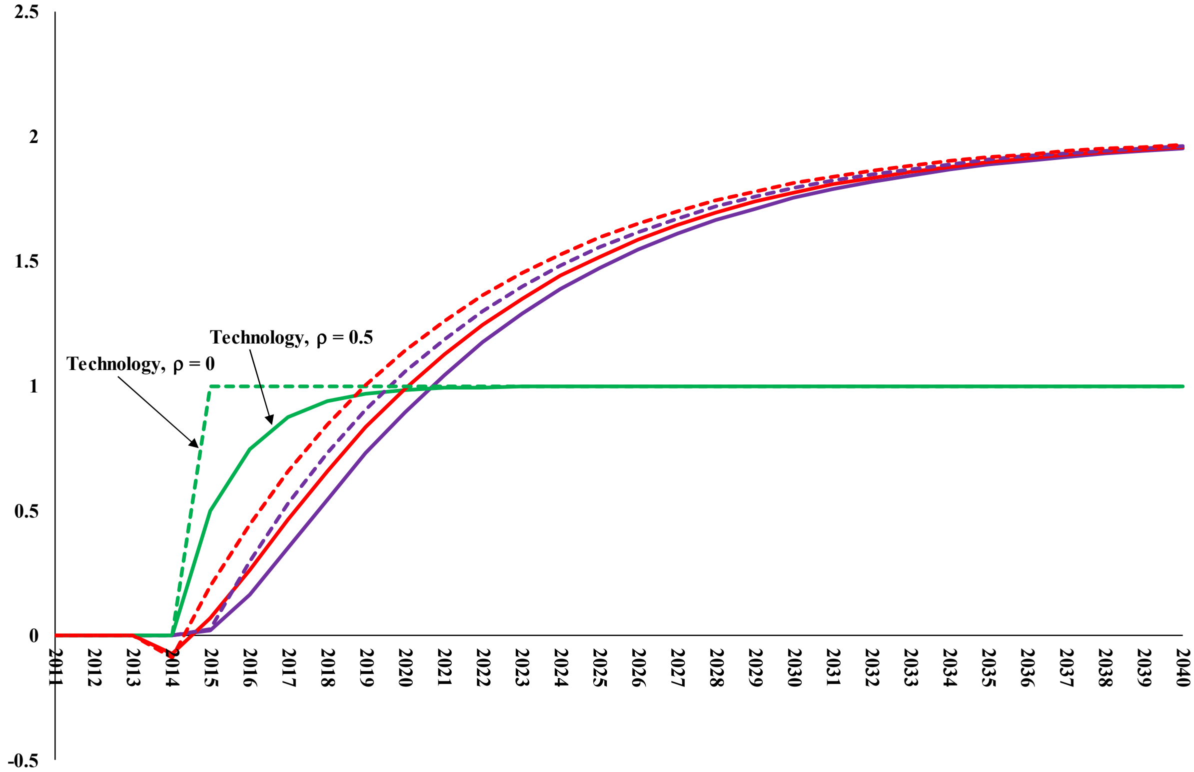


Figure 4.4. A permanent improvement in technology (z_{1t} = 1for all t): comparison of results for ρ= 0.5 and ρ= 0



variables in policy runs are normally endogenous, whereas they are often exogenous in the baseline so that the modeller can build into the baseline macro forecasts provided by specialist forecasting groups such as the IMF.

With key exceptions, all of the exogenous variables in the policy run follow the same paths that they had either endogenously or exogenously in the baseline. The key exceptions are usually policy variables. For example, if the purpose of the simulation is to determine the effects of proposed tariff changes, the relevant tariff variables are put on paths in the policy run different from their baseline paths. If none of the exogenous variables in the policy run is moved off its baseline path, then, despite a different closure, the policy run will give the same solution as the baseline run. Consequently, differences between policy and baseline results show the effects of deviations in policy variables (e.g. tariffs) from their baseline paths. Because we are normally interested in macro effects, macro variables must be endogenous in policy runs, although as mentioned earlier, they may be exogenous in the baseline. This is the reason that the policy closure is usually different from the baseline closure.

We return now to the problem of estimating J_X, J_C, J_Z, J_{CC}, J_{CX}, J_{CZ}, J_{XC}, J_{XX}, J_{XZ}.

The first step in our plan is to set up the CGE model with a steady growth baseline. For our U.S. model, we might set up a baseline in which each industry increases its output at 3 per cent a year. We can do this by assuming: 2 per cent annual labour-saving technical progress in each industry with no other changes in technology; 1 per cent annual growth in aggregate employment; 3 per cent annual outward movement in foreign demand curves for all U.S. exports; no changes in prices of imported products; 3 per cent annual growth in public expenditures; unitary consumer expenditure elasticities for all products; no changes in consumer preferences; and initial investment/capital ratios and depreciation rates implying 3 per cent capital growth. These assumptions reduce the realism of our baseline. Unfortunately, this seems to be an unavoidable cost of adopting DSGE theory which depends on steady-state assumptions. Also, as will be discussed in section 6, our formulas for estimating elasticities of consumer policy functions, e.g. (3.39) – (3.42), will need minor re-interpretation to accommodate a steady-growth baseline rather than a no-growth baseline.

The second step in our plan is to perform a series of policy runs to generate deviations away from the steady-growth baseline. In these policy runs, household consumption in year t (C_t) and household wealth at the start of year t (X_t) will be exogenous, together with the naturally exogenous variables (Z_t). By imposing a one per cent shock in X_t (i.e. moving X_t one per cent above its baseline value) while holding C_t and Z_t at their baseline values, we will be able to observe J_X. This will be done by looking at the percentage deviation result (x_{t+1}) for wealth at the start of year t+1. By imposing a one per cent shock in C_t (i.e. moving C_t one per cent above its baseline value) while holding X_t and Z_t at their baseline values, we will be able to observe J_C. Again, this will be done by looking at the percentage deviation result (x_{t+1}) for household wealth at the start of year t+1 but reversing its sign. Recall that J_C is the negative of $(\partial X_{t+1}/\partial C_t)*(C_t/X_{t+1})$. Finally, by imposing a one per cent shock in Z_t while holding C_t and X_t at their baseline values, we will be able to observe J_Z.

For evaluating the second-order elasticities, J_{XC} and J_{CC}, we will conduct two additional policy simulations with one per cent shocks to X_t and C_t imposed not on the baseline but on the situation reached in the simulation that revealed J_C. These two additional simulations will allow us to calculate J_{XC} and J_{CC} according to:

$$J_{XC} = 100 * \frac{J_X\left(\bar{C}*1.01, \bar{X}, \bar{Z}\right) - J_X\left(\bar{C}, \bar{X}, \bar{Z}\right)}{J_X\left(\bar{C}, \bar{X}, \bar{Z}\right)} \tag{5.1}$$

$$J_{CC} = 100 * \frac{J_C\left(\bar{C}*1.01, \bar{X}, \bar{Z}\right) - J_C\left(\bar{C}, \bar{X}, \bar{Z}\right)}{J_C\left(\bar{C}, \bar{X}, \bar{Z}\right)} \tag{5.2}$$

Similarly, we will conduct two additional simulations to reveal J_{XX} and J_{CX}, and two further simulations to reveal J_{XZ} and J_{CZ} according to:

$$J_{XX} = 100 * \frac{J_X\left(\bar{C}, \bar{X}*1.01, \bar{Z}\right) - J_X\left(\bar{C}, \bar{X}, \bar{Z}\right)}{J_X\left(\bar{C}, \bar{X}, \bar{Z}\right)} \tag{5.3}$$

$$J_{CX} = 100 * \frac{J_C\left(\bar{C}, \bar{X}*1.01, \bar{Z}\right) - J_C\left(\bar{C}, \bar{X}, \bar{Z}\right)}{J_C\left(\bar{C}, \bar{X}, \bar{Z}\right)} \tag{5.4}$$

$$J_{XZ} = 100 * \frac{J_X\left(\bar{C}, \bar{X}, \bar{Z}*1.01\right) - J_X\left(\bar{C}, \bar{X}, \bar{Z}\right)}{J_X\left(\bar{C}, \bar{X}, \bar{Z}\right)} \tag{5.5}$$

$$J_{CZ} = 100 * \frac{J_C\left(\bar{C}, \bar{X}, \bar{Z}*1.01\right) - J_C\left(\bar{C}, \bar{X}, \bar{Z}\right)}{J_C\left(\bar{C}, \bar{X}, \bar{Z}\right)} \tag{5.6}$$

The computations underlying (5.1) to (5.6) each require 2 comparisons. For example, to evaluate J_{XC}, we first compare the year-(t+1) wealth result from combined shocks to year-t wealth and consumption of one per cent imposed in the baseline situation with the result from a one per cent shock to consumption. This comparison reveals the elasticity [$J_X\left(\bar{C}*1.01, \bar{X}, \bar{Z}\right)$] of wealth in t+1 to year-t wealth when consumption is one per cent above its baseline value. Second, we compare this elasticity with J_X computed in the baseline situation [that is $J_X\left(\bar{C}, \bar{X}, \bar{Z}\right)$]. This second comparison reveals the sensitivity of J_X with respect to movements in consumption, giving us J_{XC}.

6. Steady-growth baseline versus no-growth baseline

The theory and computations that we have described so far are predicated on no-growth baselines. In these baselines, the value of every variable in year t+1 is the same as in year t. However, realism demands that we allow for economic growth. Assume, for example, that we are dealing with an economy such as Australia or the U.S. in which the investment share in GDP is about 20 per cent. This cannot be reproduced in a no-growth steady state. It is consistent with a situation typical of these countries in which the capital-to-output ratio is 2.5, the depreciation rate is 5 per cent and the growth rate is 3 per cent [20 = 2.5*(3 + 5)]. In a no-growth baseline the investment share of GDP is unrealistically low.

Although in the DSGE framework we can't go all the way to a realistic baseline, we can take a step in that direction by introducing steady growth. We do this by adopting a baseline in which the baseline value (denoted by b) of every variable Q can be described by

$$Q_b(t+1) = Q_b(t) * \xi(Q) \tag{6.1}$$

where

$\xi(Q)$ is the steady-state growth factor for variable Q.

As foreshadowed in section 5, we have set up the U.S. model with a steady growth baseline in which the growth factors are: 1.02 for labour-augmenting technology variables in each industry; 1.00 for all other technology variables; 1.01 for aggregate employment; 1.03 for the horizontal shifter on foreign demand curves for all U.S. exports; 1.00 for prices of imported products; 1.03 for public expenditures; and 1.00 for consumer preference variables.

Accommodating steady growth requires no change to the theory of the previous sections, just a reinterpretation. We assume now that this theory refers to relationships between growth discounted or gd variables. These are defined by

$$Q^{gd}(t) = \frac{Q(t)}{\xi(Q)^{t}} \tag{6.2}$$

With this definition, a steady-growth baseline becomes a no-growth baseline in gd variables:

$$Q_b^{gd}(t+1) = \frac{Q_b(t+1)}{\xi(Q)^{t+1}} = \frac{Q_b(t)}{\xi(Q)^{t}} = Q_b^{gd}(t) \tag{6.3}$$

In implementing DSGE theory in a full-scale CGE model, we plan to add definitions of gd variables at the end of the CGE code. The linearized DSGE equations that we will then add are relationship between these gd variables. Rather than (3.9), (3.15), (3.16), (3.21), (3.22), (3.26) and (3.29) we will include in the CGE model:

$$xend_t^{gd} = J_X * x_t^{gd} - J_C * c_t^{gd} + J_Z * z_t^{gd} \tag{6.4}$$

$$(\theta - 1 - \theta * \gamma) * c_t^{gd} + (1-\theta)(1-\gamma) * x_t^{gd} = \tilde{m}(t) + j_C(t) + xend_t^{gd} - c_t^{gd} \tag{6.5}$$

$$j_C(t) = J_{CX} * x_t^{gd} + J_{CC} * c_t^{gd} + J_{CZ} * z_t^{gd} \tag{6.6}$$

$$\begin{aligned} &M * m(t) = \\ &U_X * \left[(\theta * \gamma - \theta - \gamma) * x_t^{gd} + (1-\gamma) * \theta * c_t^{gd} \right] + \beta * M * J_X * \left[\tilde{m}(t) + j_X(t) + xend_t^{gd} - x_t^{gd} \right] \end{aligned} \tag{6.7}$$

$$j_X(t) = J_{XX} * x_t^{gd} + J_{XC} * c_t^{gd} + J_{XZ} * z_t^{gd} \tag{6.8}$$

$$m(t) = M_X * x_t^{gd} + M_Z * z_t^{gd} \tag{6.9}$$

$$\tilde{m}(t) = M_X * xend_t^{gd} \tag{6.10}$$

In these equations, we have added a gd superscript to all the variables that have counterparts in the CGE model. The gd superscript is not required for m(t), $\tilde{m}(t)$, $j_C(t)$ and $j_X(t)$. These variables do not have CGE counterparts, so they do not need to be distinguished from CGE variables by the gd superscript.

Apart from the use of gd superscripts, another change that we have made to our earlier representation of the DSGE linearized equations is the replacement of the variable (x_{t+1}) for start-of-year wealth in year t+1 with the variable ($xend_t$) for end-of-year wealth in year t. This brings the DSGE system (6.4) to (6.10) into line with the pervasive convention in our CGE models that all equations connect variables for the same year. Thus, wealth at the end of year t (a year t variable) is specified as wealth at the start of year t (another year t variable) plus the effects of other year t variables such as consumption in year t. This convention facilitates recursive dynamics in which the solution is generated by a sequence of single-year computations.

Under the convention, year t+1 is connected to year t by ensuring that in the year t+1 computation, start-of-year stock variables reflect end-of-year values for year t. Thus, in a recursive dynamic CGE computation we set start-of-year wealth in the year t+1 computation equal to end-of-year wealth from the year t computation:

$$X_{t+1} = Xend_t \tag{6.11}$$

But what does this mean in terms of gd variables? From (6.11), we have

$$\frac{X_{t+1}}{\xi(X)^{t+1}} = \frac{Xend_t}{\xi(X)^t} * \frac{1}{\xi(X)} \tag{6.12}$$

For it to be possible to have a steady-growth baseline, we must assume that

$$\xi(X) = \xi(Xend) \tag{6.13}$$

In view of (6.11) to (6.13), in our DSGE-CGE integration we will connect start-of-year t+1 gd stock variables with their end-of-year t versions via

$$X_{t+1}^{gd} = \frac{Xend_t^{gd}}{\xi(X)} \tag{6.14}$$

Returning to (6.4) to (6.10), we interpret all of the gd variables as percentage deviations from their steady-state (no growth) baseline values. What about m(t) and $\tilde{m}(t)$? What are the growth rates on the baseline paths from which they are percentage deviations?

In place of (3.2), we assume that

$$V\left(X_t^{gd}, Z_t^{gd}, \sigma\right) = U\left(C_t^{gd}, X_t^{gd}\right) + \beta * E_t\left[V\left(X_{t+1}^{gd}, Z_{t+1}^{gd}, \sigma\right)\right] \tag{6.15}$$

In (6.15), growth in consumption and wealth at the rates $\xi(C)$ and $\xi(X)$ maintains utility (U) at its year-t level. If $\xi(C)$ and $\xi(X)$ are greater than 1, then for maintenance of a given level of annual utility, consumption and wealth must grow. We can interpret this as meaning that maintenance of utility requires consumption and wealth to grow in line with population and community aspirations reflected in normal growth rates in per capita consumption and wealth. If X^{gd} and Z^{gd} are fixed on their no-growth steady-state values, then under (6.15) V is fixed on its steady-state value. With M in (6.7) now being interpreted as the derivative of V in (6.15) with respect to X^{gd}, we see that the baseline growth rate in M must be zero ($\xi(M) = 1$). If M is on a no-growth baseline, then $E_t[M(t+1)]$ must also be on a no-growth baseline. We conclude that m(t) and $\tilde{m}(t)$ are percentage deviations from no-growth steady-state values.

Finally, we consider the variables $j_C(t)$ and $j_X(t)$. In (6.4) to (6.10), the J coefficients are first and second-order elasticities of growth-discounted end-of-year wealth with respect to growth-discounted consumption, growth-discounted start-of-year wealth and growth-discounted exogenous variables. These coefficients are evaluated at the no-growth baseline values for the gd variables. Hence, in the baseline there is no growth in the J coefficients. Consequently, $j_C(t)$ and $j_X(t)$ are percentage deviations from no-growth steady-state values. What are these values? As is readily apparent, elasticities calculated for gd variables are the same as elasticities calculated for steady-growth variables, for example:

$$J_X^{gd}(t) = \frac{\partial\left(Xend_t / \xi(Xend)^t\right)}{\partial\left(X_t / \xi(X)^t\right)} * \frac{\left(X_t / \xi(X)^t\right)}{\left(Xend_t / \xi(Xend)^t\right)} = \frac{\partial Xend_t}{\partial X_t} * \frac{X_t}{Xend_t} = J_X(t) \tag{6.16}$$

Growth in the standard neo-classical model

In the standard model that we studied in the previous sections, the accumulation relationship is:

$$X_{t+1} = X_t * (1-\delta) + Z_t * X_t^{\alpha} - C_t \tag{6.17}$$

In terms of gd variables (6.17) can be written as:

$$Xend_t^{gd} * \xi(Xend)^t = X_t^{gd} * \xi(X)^t * (1-\delta) + Z_t^{gd} * \xi(Z)^t * \left[X_t^{gd} * \xi(X)^t\right]^{\alpha} - C_t^{gd} * \xi(C)^t \tag{6.18}$$

For steady-growth we require that

$$\xi(Z) = \xi(X)^{1-\alpha} \quad \text{and} \quad \xi(C) = \xi(X) = \xi(Xend) \tag{6.19}$$

Provided (6.19) is satisfied, (6.18) reduces to a relationship of the same form as (6.17), but between gd variables:

$$Xend_t^{gd} = X_t^{gd} * (1-\delta) + Z_t^{gd} * \left[X_t^{gd}\right]^{\alpha} - C_t^{gd} \tag{6.20}$$

As explained earlier, in a growth situation we interpret V as a function of gd variables, see (6.15). Then, adopting the same parameter values as those that led to (3.60), we find that

$$c_t^{gd} = 0.912063 * x_t^{gd} + 0.264498 * z_t^{gd} \tag{6.21}$$

Percentage deviations in gd variables from their no-growth baseline are the same as percentage deviations in growth variables from their steady-growth baseline. Thus, (6.21) gives us the consumption function in the growth situation as in (3.60).

7. Deriving a DSGE consumption function for the USAGE model of the U.S.

In this section we apply the theory from sections 3 to 6 to derive a DSGE consumption function for a 70-industry version of the USAGE model of the U.S. economy. Then we use this consumption function in an illustrative USAGE simulation.

USAGE is a dynamic CGE model that was initially created at the Centre of Policy Studies in 2002. Since then, it has been applied and further developed by, and on behalf of: the U.S. International Trade Commission; the U.S. Departments of Commerce, Agriculture, Transportation, Homeland Security and Energy; the Canadian government; the Mitre Corporation; and the Cato Institute. Application topics include trade policies, illegal immigration, road/rail/air infrastructure, energy policies, and terrorism.[8]

In standard applications of USAGE, household consumption in year t is proportional to household disposable income in year t (fixed average propensity to consume). Public consumption is usually linked in a linear way to private consumption. Investment in each industry in year t is specified as a function of the industry's expected rate of return on capital. So that the model can be solved recursively, expected rates of return are assumed to reflect current rates of return.[9] Imports of each commodity are modelled as imperfect substitutes for domestically produced products in the same industrial classification [the Armington assumption, Armington (1969)]. Exports of each commodity are modelled via constant-elasticity export demand functions.

As mentioned in section 5, we equipped USAGE with a 3 per cent steady-growth baseline. While leaving other features of USAGE unchanged, we replace the standard USAGE treatment of private and public consumption with a DSGE consumption function. The DSGE

[8] There are many published USAGE application papers. Recent examples are: Dixon *et al.* (2017a&b).

[9] Dixon *et al.* (2005) shows how models such as USAGE can be solved with forward-looking expectations for rates of return. The method involves a series of recursive dynamic simulations with adjustments in expectations between simulations.

consumption function determines shock-induced deviations in private and public consumption from the 3 per cent steady-growth baseline. In the initial application described here of DSGE theory to USAGE, shocks are allowed only to primary-factor-saving technology. This is the only Z variable. Thus, our DSGE consumption function takes the form:

$$c_t = ELAST(c,x) * x_t + ELAST(c,z) * z_t \tag{7.1}$$

In this equation,

c_t is the percentage deviation in real consumption in year t from its baseline value. This is private plus public consumption deflated a composite price index formed as value weighted average of the prices indexes for private consumption and public consumption.

x_t is the percentage deviation in real wealth at the start of year t from its baseline value. This is the deflated value of physical capital in the U.S. less net foreign liabilities. The deflator is the lagged value of the price index for private consumption, that is the deviation in wealth at the start of year t+1 is deflated by the deviation in the price of consumption in year t.

z_t is the percentage deviation in primary-factor-saving technology in year t from its baseline value. This variable applies uniformly across all industries. If, for example, z_t equals 2, then all industries can produce their baseline level of output for year t with 2 per cent less primary factor input than in the baseline and the baseline levels for all other inputs.

ELAST(c,x) and ELAST(c,z), treated as parameters, are the elasticities of consumption (private plus public) with respect to start-of-year wealth and primary-factor-saving technology.

7.1. Evaluation of the coefficients in the DSGE consumption function for USAGE

We evaluate the two elasticities in (7.1) by applying (4.41) and (3.39) - (3.42). This requires us to assign values to:

γ, the parameter introducing diminishing marginal utility to consumption in any year;

β, the parameter introducing preference for current consumption relative to future consumption;

B_{ZZ}, the elasticity of expected primary-factor-saving technology in year t+1 with respect to primary-factor-saving technology in year t;

J_X, J_C, J_Z, J_{XX}, J_{XC}, J_{XZ}, J_{CX}, J_{CC}, J_{CZ}, the first and second-order elasticities of wealth at the start of year t+1 with respect to wealth at the start of year t, consumption in year t and primary-factor-saving technology in year t;

θ, the parameter introduced to allow utility in each year to be a function of wealth as well as consumption;

M and U_X, the baseline values (constant through time) of the growth-discounted expected marginal values (derivatives) of lifetime welfare and current utility with respect to growth-discounted start-of-year wealth; and

M_X and M_Z, the elasticities of the household's policy function (the M function) with respect to growth-discounted start-of-year wealth and growth-discounted primary-factor-saving technology.

We set γ and β at 0.5 and 0.9. These values are representative of the values used in macro DSGE models.

We set B_{ZZ} at zero. Thus we assume in this preliminary application of DSGE theory to CGE modelling that the determination of primary-factor-saving technology is serially uncorrelated.

We determine the J elasticities by USAGE simulations. As described in section 5, in these simulations we treat real wealth at the start of 2018 as predetermined, and real consumption and primary-factor-saving technical change as exogenous. We apply shocks to these three variables to determine their effects on real wealth at the start of 2019. In any CGE simulation, closure (choice of exogenous variables) is important. This will be discussed more fully in section 8. For understanding the J values to be presented in this section, the most important aspects of the closure are that aggregate employment and aggregate capital are both exogenous. With this closure, the only avenues for movements in real GDP are changes in technology and changes in dead-weight losses associated with taxes and other distortions such as differences in rates of return on capital across industries. The J elasticities that we obtained for USAGE are in Tables 7.1 and 7.2. These tables are discussed in subsection 7.2.

To evaluate θ, we use (3.13) and (3.20) with X's, C's and $\tilde{M}$ replaced by their growth-discounted baseline values which are constant through time. Omitting time subscripts to indicate baseline growth-discounted values, and recognizing that on a non-stochastic steady-growth baseline $\tilde{M} = M$, we have:

$$\theta * C^{\theta-1} * X^{1-\theta}(X^{1-\theta} * C^{\theta})^{-\gamma} = \beta * M * J_C * \frac{X}{C} \tag{7.2}$$

and

$$M = (1-\theta) * X^{-\theta} * C^{\theta} * (X^{1-\theta} * C^{\theta})^{-\gamma} + \beta * \{M\} * J_X * \frac{X}{X} \tag{7.3}$$

leading to

$$\theta = \frac{\beta * J_C}{1-\beta*(J_X - J_C)} \tag{7.4}$$

With β assumed to be 0.9 and the values of J_C and J_X taken from Table 7.1, the value for θ obtained from (7.4) is 0.8335.

In the CGE database for the initial year (2018), the values for the X and C are \$US34.69 trillion and \$US16.58 trillion. M and U_X can now be evaluated via

$$M = \frac{(1-\theta) * X^{-\theta} * C^{\theta} * (X^{1-\theta} * C^{\theta})^{-\gamma}}{1-\beta * J_X} \tag{7.5}$$

and

$$U_X = (1-\theta) * X^{-\theta} * C^{\theta} * (X^{1-\theta} * C^{\theta})^{-\gamma} \quad , \tag{7.6}$$

giving M = 0.189 and U_X = 0.021.

We now have all of the coefficient and parameter values necessary for evaluating e2, e1 and e0 in (3.40) to (3.42), allowing us to solve the quadratic equation (3.39) for M_X. This gives two possible values:

$$M_X = -0.2128 \text{ and } M_X = -0.6775 \tag{7.7}$$

In our theory of consumer behaviour, M_X must be negative: diminishing marginal utility to wealth means that the increase in expected lifetime welfare from an additional unit of wealth must decline as wealth increases. In section 3, when we solved the quadratic equation for M_X in the standard neo-classical model [see (3.58) and (3.59)], the two solutions had different signs. Thus we were immediately able to choose the appropriate solution, the one with the negative sign. But here, with both solutions being negative that criterion doesn't help. Consequently we proceed to (4.41) and evaluate ELAST(c,x) under each possible M_X value. With $M_X = -0.2128$ we obtain -0.5995 and with $M_X = -0.6775$ we obtain 0.2669. We require ELAST(c,x) to be positive: an increase in wealth in year t should generate an increase in consumption in year t. On this basis we choose the second solution ($M_X = -0.6775$) in (7.7).

With M_X tied down we can now refer to (4.41) to evaluate ELAST(c,z). This leads to our DSGE consumption function for USAGE:

$$c_t = 0.2669 * x_t + 0.6724 * z_t \tag{7.8}$$

The value of M_Z could be determined via (4.34), (4.35) and (4.37) to (4.39). However, with B_{ZZ} set at zero, evaluating M_Z is unnecessary: it does not affect the values of ELAST(c,x) and ELAST(c,z).

7.2. Values for J elasticities

For understanding the values in Tables 7.1and 7.2 for first and second-order J elasticities calculated from USAGE simulations, it is useful to note that in USAGE:

Baseline consumption (private plus public) in 2018 is $US16.58 trillion;

Baseline real wealth at the start of 2018 is $US34.69 trillion; and

Baseline real wealth at the start of 2019 is $US35.73 trillion; and

Baseline GDP in 2018 is $US19.44 trillion.

In view of these baseline values, our first question is: why is J_C equal to 0.61, rather than about 0.46 (= 16.58/35.73)? What explains the discrepancy of 0.15 between the actual value of J_C and what we would expect simply on the basis of consuming an amount of wealth worth 1 per cent of 2018 consumption?

The answer involves two factors. The first is that a 1 per cent increase in consumption reallocates capital towards low-rate-of-return uses, especially housing. USAGE implies that this reduces GDP by 0.082 per cent, imparting a loss in next year's wealth of 0.04 per cent (= 0.082*19.44/35.73). The second factor is a reduction in the price of investment relative to the price of consumption. USAGE shows that a 1 per cent increase in consumption generates increases the price indexes for both investment and consumption. This is the mechanism by which USAGE introduces the real appreciation necessary to facilitate the transfer of resources towards consumption away from the trade balance. However, the increase in the price index for private consumption is greater than that for investment (0.373 per cent compared with 0.256 per cent), reflecting the greater import-intensity of private consumption relative to investment. With the investment price index being the major price in determining the value of the capital stock, and the consumer price index for year t being the chosen deflator for determining the real value of wealth at the start of year t+1, the movement in relative prices introduces a reduction in real wealth of about 0.12 per cent [= -100*(1.00256/1.00373-1)]. Together these two factors suggest that the loss of real wealth at the start of year t+1 should be about 0.16 per cent (= 0.04+0.12) greater than would be expected (0.46) on the basis of the relative sizes of consumption and wealth. This is close to the discrepancy of 0.15 per cent that we set out to explain.

Table 7.1. First-order real wealth elasticities

i	J_i
C	0.61228
X	0.98877
Z	0.71538

Table 7.2. Second-order real wealth elasticities

	$J_{i,s}$		
s / i	C	X	Z
C	1.34142	-0.42973	-0.71970
X	0.27038	0.01713	-0.26245

Why is J_X equal to 0.989? If the interest rate were zero then steady growth at 3 per cent suggests that J_X should be 0.97. In fact, the interest rate for the U.S. on foreign borrowing is about 2 per cent. This gives the result for J_X at 0.989 per cent.

We anticipated that one per cent primary-factor-saving technical change should increase GDP by about one per cent. So why is J_Z greater than 0.54 (= 19.44/35.73). In the USAGE simulation of the effect of a one per cent increase in primary-factor-saving technology, the price deflator for investment increases by 0.14 per cent relative to the price deflator for consumption. This reflects strong growth in wage rates induced by primary-factor-saving technical change combined with high labour intensity of investment and high import-intensity of consumption. The relative price movement explains an increase in real wealth of about 0.14 per cent, most of the discrepancy between 0.54 and 0.71.

Why is J_{CC} in Table 7.2 strongly positive (1.34)? J_C is the per cent damage to wealth in 2019 of a one per cent increase in consumption in 2018. J_{CC} is the per cent difference between J_C evaluated with consumption in year t above its baseline value by one per cent and J_C evaluated on the baseline. If consumption is elevated one per cent above baseline, then a one per cent increase in consumption uses up about one per cent more of next year's wealth than if the consumption increase were just one per cent of baseline. On this basis we would expect J_{CC} to be about one. However, there is an additional effect which takes J_{CC} above one. If 2018 consumption is elevated above baseline, then 2019 wealth will be below baseline. With 2019 wealth below baseline, any given destruction of wealth generated by consumption in 2018 produces a larger percentage effect on wealth than if wealth were on baseline.

J_{CX} is the per cent difference between J_C evaluated with real wealth at the start of year t above its baseline value by one per cent and J_C evaluated on the baseline, that is, as set out in (5.4),

$$J_{CX} = 100 * \frac{J_C\left(\overline{C}, \overline{X}*1.01, \overline{Z}\right) - J_C\left(\overline{C}, \overline{X}, \overline{Z}\right)}{J_C\left(\overline{C}, \overline{X}, \overline{Z}\right)} \tag{7.9}$$

We anticipated that the value of wealth used up at the start of year t+1 through a one per cent increase in consumption in year t over its baseline value would not depend on wealth. On this basis, we anticipated that J_{CX} would be about -1. So why is J_{CX} equal to -0.43?

We traced the answer to changes in the composition of wealth at the start of 2018. When we elevate real wealth at the start of 2018 so that we can calculate $J_C\left(\overline{C}, \overline{X}*1.01, \overline{Z}\right)$, we do it by reducing U.S. net foreign liabilities. Thus, our elevation of wealth slightly reduces the ratio of the value of physical assets to wealth. The effect on prices in 2018 of a one per cent increase in consumption is non-uniform. Although these price movements are independent of the level of wealth, nevertheless because of the change in the composition of wealth, the percentage effect on real wealth of consumption-induced price movements depends on whether real wealth is set at $\bar{X}*1.01$ or $\bar{X}$. In a spreadsheet not presented here we demonstrated that this price effect explains the discrepancy between our anticipated result of about -1 for J_{CX} and the actual result of -0.43.[10]

Continuing in this way we could explain all of the items in Table 7.2. However, we have done enough to be convinced that the computations underlying Table 7.2 are correct. The most important point about the explanations is that simple intuition is confounded by changes in relative prices and by seemingly innocuous but arbitrary assumptions concerning the composition of changes in wealth used in the calculations of elasticities of year t+1 real wealth with respect to year t real wealth (J_X, J_{XC}, J_{XX}, J_{XZ}).

7.3. Illustrative application: the effects of a one per cent shock to primary-factor-saving technology

Figure 7.1 shows USAGE results for the effects of a 1 per cent deviation occurring in 2018 in primary-factor saving technology from its baseline path. The shock is temporary: primary-factor saving technology returns to its baseline path from 2019 onwards.

With aggregate employment set exogenously (unaffected by the shock) and capital predetermined (and therefore also unaffected by the shock in the first year), a one per cent improvement in primary-factor-saving technology must cause a deviation in GDP in 2018 from its baseline path of approximately one per cent. In fact, the USAGE result was an increase of 1.03 per cent.[11]

An increase in GDP in 2018 of 1.03 per cent is worth \$0.20 trillion (= 19.44*0.0103). The increase in consumption, dictated by (7.8) is 0.6781 per cent.[12] This uses up \$0.11 trillion of the GDP increase (= 16.58*0.006781), leaving \$0.09 trillion as a contribution to an increase in wealth at the start of 2019. This contribution is a percentage increase in real wealth of 0.25 per cent (=100*0.09/35.73). The actual increase projected by USAGE and shown in Figure 7.1 is 0.30 per cent. The extra 0.05 per cent (= 0.30 – 0.25) comes from price changes.

As explained in subsection 7.2 in our discussion of J_Z, a primary-factor saving improvement in technology generates an increase in the price of capital goods relative to consumption goods. With our chosen price deflator for real wealth at the start of 2019 being the price deflator for consumption in 2018 and the price of wealth being predominately the price of capital goods, the relative price movement in 2018 imparts an increase in real wealth at the start of 2019. In the USAGE simulation, the increase in the price of capital goods relative to the price of consumption goods in 2018 is 0.06 per cent[13], closely explaining the bonus real wealth increase (0.05 per cent) beyond that generated by extra saving in 2018.

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[10] The relevant spreadsheet is C:\dixon\consult\DSGE\Present101219\DSGEwork150120.xlsx

[11] The discrepancy of 0.03 percentage points is not important from the point of view of illustrating the workings of our DSGE consumption function. Nevertheless, we traced the source to a reallocation of the capital stock in 2018 towards industries that happen to have relatively high rates of return on capital.

[12] At first glance, (7.8) appears to dictate an increase of 0.6724 per cent. However, in GEMPACK, (7.8) is interpreted as the non-linear equation : $\frac{C}{Cbase} = \left(\frac{RWEALTH}{RWEALTHbase}\right)^{0.2669} * \left(\frac{TECH}{TECHbase}\right)^{-0.6724}$. In 2018 the first term in brackets on the RHS is one and the second term in brackets is 0.99. This produces a percentage consumption deviation of 0.6781 [=100*($0.99^{-0.6724}$ -1)].

[13] The increase in the price of capital goods relative to consumption goods in the simulation that revealed J_Z was 0.14 per cent. In the simulation being discussed here it is only 0.06 per cent. In generating J_Z we held consumption constant. In the current simulation, consumption moves. This damps the increase in the ratio of the price of capital goods to the price of consumption goods.

Figure 7.1. Effects of a 1% temporary improvement in primary-factor technology with a DSGE consumption function (percentage deviations from baseline)

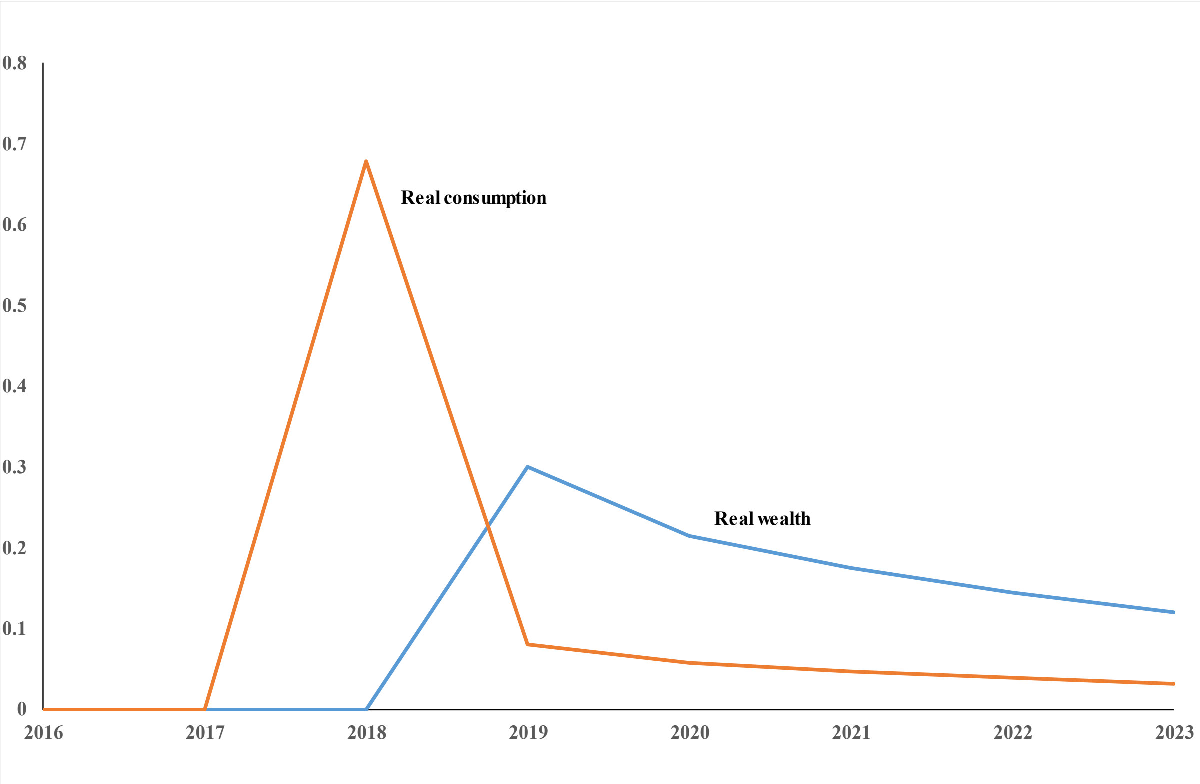
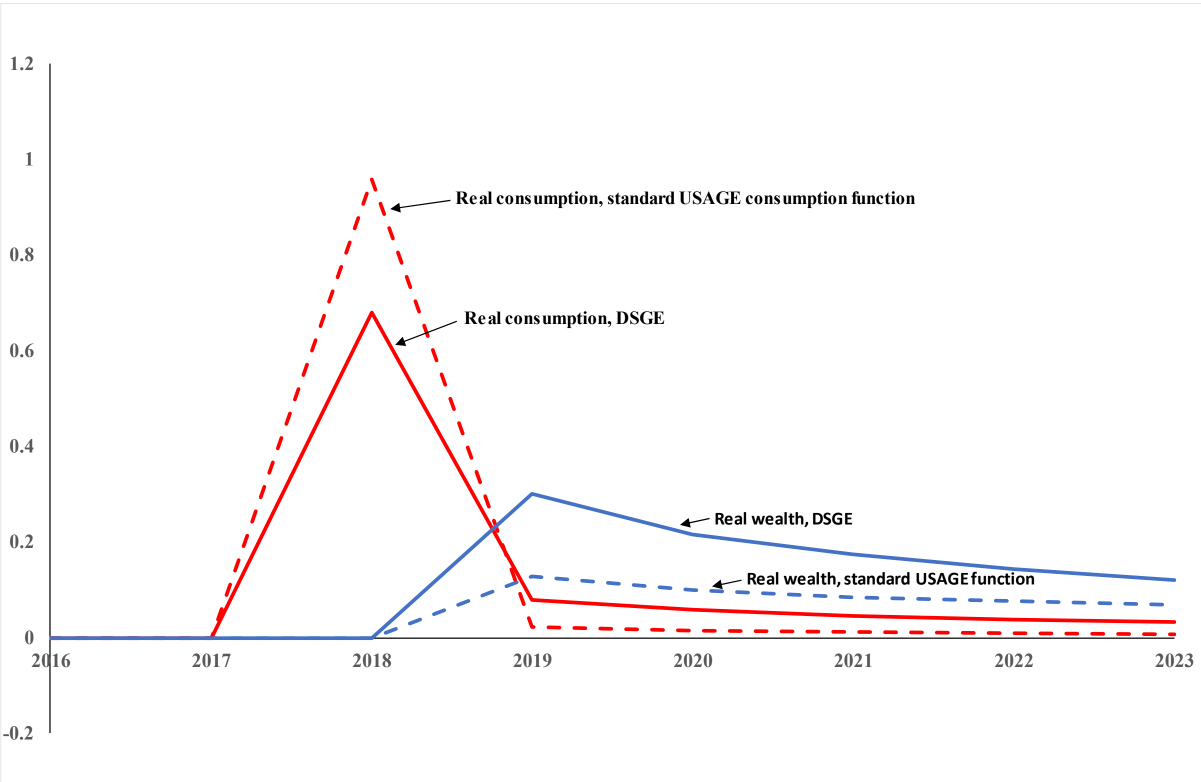


Figure 7.2. Effects of a 1% temporary improvement in primary-factor technology with DSGE and a standard USAGE consumption functions (percentage deviations from baseline)



As mentioned earlier, in standard applications of USAGE private and public consumption move proportionately with income. Under this treatment, the benefits of a good-news temporary shock are absorbed almost entirely as an immediate increase in consumption. In our DSGE specification, the year-t contribution to lifetime welfare is a diminishing-marginal-utility function ($1-\gamma = 0.5$) of wealth at the start of year t and consumption in year t, see (3.3). With diminishing marginal utility, we anticipated that the replacement in USAGE of the standard consumption function by a DSGE consumption function would spread the consumption response to temporary good-news across years. With time-preference discounting [β in (3.2) is 0.9] we anticipated that the DSGE deviation path for consumption would be declining after the first year, but in a relatively smooth manner.

Figure 7.2 compares the DSGE results from Figure 7.1 with results under a standard USAGE consumption function. With the standard USAGE treatment, the consumption deviation is 0.9589 per cent in 2018 falling to 0.0210 per cent in 2019. With the DSGE treatment, the consumption deviation is 0.6781 per cent in 2018 falling to 0.0799 per cent in 2019. Thus, as anticipated, the introduction of the DSGE consumption function has a smoothing effect on the consumption deviations and, as was also anticipated, the consumption deviations decline over time.

Although the introduction of the DSGE consumption function smooths out the consumption response to the temporary shock to primary-factor technology, we were surprised that the smoothing was not more pronounced. Even with the DSGE consumption specification, the 2018 consumption deviation is 8.48 times the 2019 deviation (0.6781/0.0799).

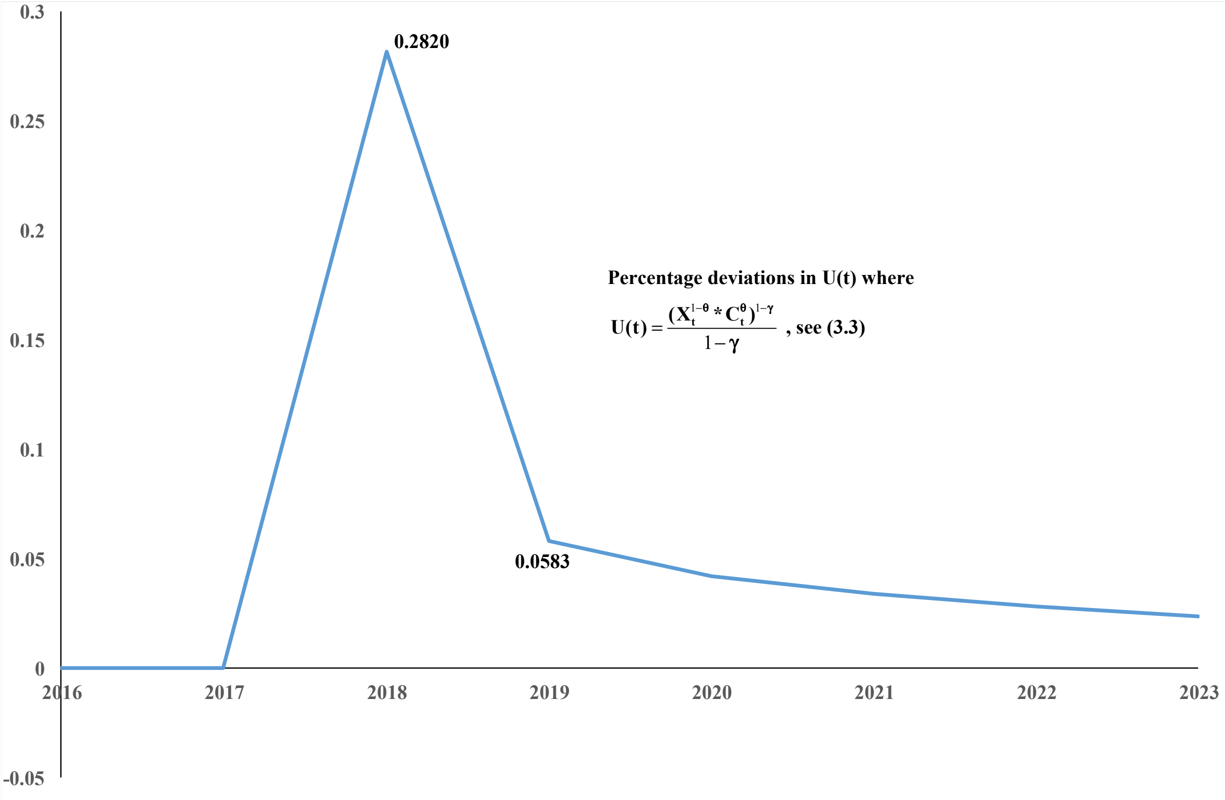
Why, even with the DSGE specification, does consumption increase so sharply in 2018 relative to 2019? There are three reasons.

The first is that all of the benefit to be enjoyed in 2018 from the temporary shock must be generated by a consumption increase. Wealth in 2018 is predetermined. From 2019 onwards, some of the benefit can be taken in the form of extra wealth. Thus, to smooth out utility contributions through time, the DSGE household must make a relatively large consumption increase in 2018 when this is the only avenue for generating utility. However, this is not the whole story. As shown in Figure 7.3, the deviation path for the annual utility contribution is far from smooth. The utility deviation in 2018 is 4.840 times that in 2019 (=0.2820/0.0583).

The second reason for the large consumption deviation in 2018 relative to that in 2019 relates to relative prices in 2018 compared with 2019 and later years. As explained already, the primary-factor technology shock in 2018 generates a wage increase with a resulting increase in the price of capital goods relative to consumption goods. This effect on relative prices is temporary, creating an incentive for increased consumption in 2018 when consumption goods are relatively cheap.

The third reason relates to the increase in real wealth at the start of 2019 generated by the relative price change in 2018. This is similar to a gift at the start of 2019 that is withdrawn in subsequent years. There is strongly diminishing marginal utility to extra wealth in any given year [$(1-\theta)$ in (3.3) is 0.1665]. The gift of wealth at the start of 2019 makes it difficult to transfer utility from 2018 to 2019 through extra saving in 2018. Thus the household takes a disproportionate share of the good news from the temporary technology improvement in 2018 as a utility increase in 2018, rather than in subsequent years.

Figure 7.3. Effects on annual utility of a 1% temporary improvement in primary-factor technology with a DSGE consumption function (percentage deviations from baseline)



8. Concluding remarks

In DSGE modelling, agents make decisions in year t applying rules (policy functions) that take account of: year t values of predetermined stock variables; year t values of exogenous variables; accumulation relationships determining future values of stock variables; and probability distributions for future values of exogenous variables. The key idea in DSGE modelling is that under rational expectations agents know that the rules that they apply in year t will also be applicable in future years.

DSGE models are generally small with little or no sectoral, trade, technology and tax detail. We find DSGE ideas attractive. This has lead to the research reported in this paper in which we incorporate a DSGE consumption function in a CGE model that contains a high level of sectoral disaggregation and considerable detail on trade, technology and taxes.

Making DSGE ideas operational requires numerical determination of the policy functions that describe agent behaviour in year t. For CGE modelling, especially with GEMPACK software, the perturbation method seems the most natural way to determine these policy functions. As an introduction to DSGE modelling and the perturbation method, we reviewed the DSGE version of the standard neoclassical growth model. We showed how the policy rule for the household in that model can be found by the perturbation method. Then we generalized the standard model in two directions and again using the perturbation method we derived the policy rules. The first generalization was the addition of a sticky wage equation under which the real wage rate (and consequently employment) in year t depends on the real wage rate in year t-1. The second generalization concerned the role of wealth. In the standard model, wealth is simply a vehicle through which consumption can be transferred between years. This is done by foregoing consumption in one year, thereby generating capital which can be used to produce income to support consumption in future years. In our generalized treatment, wealth retains its original role but also joins consumption in directly creating utility in each year. We have in mind that for households, wealth gives a sense of security.

The perturbation method for finding the policy rule requires evaluation of either derivatives or elasticities of the wealth accumulation relationship. In the standard neoclassical growth model and our generalizations we can evaluate the required elasticities simply via formulas expressed in terms of known parameters. However, this option isn't available for a full-scale CGE model. For these models, it is not possible to express wealth at the start of year t+1 as an explicit algebraic function of variables in year t. Consequently, it is not possible to obtain explicit formulas for the required elasticities. To overcome this problem we showed how the elasticities can be evaluated by suitable CGE simulations. For example, to obtain the elasticity of start-of-year wealth for year t+1 with respect to primary-factor-saving technology, we conducted a simulation in which primary-factor-saving technology in year t was shocked by one per cent and other exogenous variables and consumption were held fixed. The deviation result for start-of-year wealth in year t+1 revealed the required elasticity.

With elasticities of year t+1wealth evaluated by CGE simulations, we were able to compute coefficients for a DSGE consumption function. We used this function in an illustrative CGE simulation of the effects of a temporary improvement in primary-factor-saving technology. The technology shock produced a once-off windfall increase in income in year t. Under the usual CGE specification in which consumption moves in line with income, a windfall income gain in year t is almost entirely consumed in year t. By contrast, with the DSGE consumption function, some of the windfall is devoted to wealth accumulation, allowing consumption benefits to be spread across time.

While the spread effect was clearly visible in our USAGE simulation, it was weak. The introduction of the DSGE consumption function did not prevent a high proportion of the windfall income gain in year t from being consumed in year t. The main explaining factor was a temporary increase in the price of capital goods relative to consumption goods. This is a CGE effect that would not be captured by a small-scale DSGE model.

Our illustrative USAGE simulation with a DSGE consumption function demonstrates the feasibility of transferring key DSGE ideas into a full-scale CGE model. We think that the integration of these two types of models has the potential to produce insights of value to researchers in both modelling streams. Our illustrative simulation raises the possibility for CGE modellers of adopting consumption functions that imply intertemporal spreading of consumption effects tailored to specific shocks. For DSGE modellers, it shows the potential importance of relative price effects.

However, the transfer of DSGE specifications into a CGE model comes at a cost.
To make the transfer feasible, we introduced realism-reducing simplifications to the CGE model. Most obviously, we used a steady-growth baseline. This sacrifices important achievements in CGE modelling concerning the identification of different trends in technology and consumer preferences across sectors. Although we retained considerable sectoral detail (70 industries) in our illustrative simulation with the DSGE-enhanced USAGE model, we simplified the model by effectively eliminating lagged responses. We assumed that the technology shock in 2018 caused an immediate adjustment in wage rates to maintain employment exogenously on its baseline path and that the shock had no effect on aggregate investment. These assumptions can be compared with our standard and more realistic CGE specifications in which wage rates respond with a lag, aggregate employment moves

endogenously from its baseline path and investment reacts to changes in rates of return causing a lagged response in industry capital stocks. While it seems that a steady-growth baseline is fundamental to DSGE theory and the perturbation solution method[14], it may be possible eventually to handle lags in a CGE model containing DSGE-features. But it is clear to us that this will require a further major research effort.

Another area in which major research effort could be made is in the measurement of real wealth. In this paper we specified real wealth as nominal wealth deflated by the price of consumer goods. Nominal wealth was defined as the value of the nation's physical capital less net foreign liabilities. We didn't offer a theoretical justification for this measure of real wealth, and we didn't distinguish between private and public wealth.

DSGE ideas seem just as applicable to the specification of investment as consumption. But again, progress towards a DSGE specification of investment in a CGE model will require a major research effort.

In the meantime, what can be done with the research reported in this paper? So far, we have included only one Z term (a temporary primary-factor-saving technology shock) in our USAGE-based DSGE consumption function. We are planning to build an inventory of Z terms in the DSGE consumption function by applying the method set out in this paper. These new Z terms could include a permanent rather than temporary technology shock, a technology shock that is sector-specific rather than economy-wide, a terms-of-trade shock, and a shock to aggregate employment. Then we propose to broaden the comparison started in section 7.2 between CGE results with and without the DSGE consumption function. This might help us characterize the sort of shocks for which the DSGE-approach is important. For example, it might tell us that the DSGE-approach produces results that are significantly different from the standard CGE approach only if we are dealing with temporary shocks.

Another direction that could be pursued immediately is an investigation of the effect of using realistic lags and a non-steady-growth baseline in a CGE model incorporating a DSGE consumption function derived without lags and with a steady-growth baseline. For example, if we simply imposed (7.8) in a USAGE simulation with lags and a realistic baseline, would we obtain significantly different deviation results for the effects of a primary-factor-saving technical change from those in Figure 7.1? If the answer is no, then we might be encouraged to use realistic lag and baseline specifications in combination with a DSGE consumption function even though the function is strictly legitimate only in a model without lags and with a steady-growth baseline. However, we would still be left with unanswered questions about how the consumption function itself should be adjusted in the light of lags and unbalanced growth.

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[14] In section 4 we saw that implementation of the DSGE consumption function was built around a 6 equation system with 7 unknowns. The steady-growth assumption introduced the seventh piece of information necessary to make this system solvable. Relaxation of the steady-growth assumption will require a breakthrough that produces an alternative seventh piece of information.

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