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NEW OSTROWSKI TYPE INEQUALITIES FOR CO-ORDINATED S-CONVEX FUNCTIONS IN THE SECOND SENSE

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In this paper some new Ostrowski type inequalities for co-ordinated s-convex functions in the second sense are obtained.

1. Introduction

In 1938, A. Ostrowski proved the following interesting inequality [21]:

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then we have the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \left\|f'\right\|_\infty, \tag{1}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

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The inequality (1) can be rewritten in equivalent form as:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{(x-a)^2+(b-x)^2}{2(b-a)}\right]\left\|f'\right\|_\infty .$$

Since 1938 when A. Ostrowski proved his famous inequality, many mathematicians have been working about and around it, in many different directions and with a lot of applications in Numerical Analysis and Probability, etc. Several generalizations of the Ostrowski integral inequality for mappings of bounded variation, Lipschitzian, monotonic, absolutely continuous, convex mappings, s-convex mappings and n-times differentiable mappings with error estimates for some special means and for some numerical quadrature rules are considered by many authors. For recent results and generalizations concerning Ostrowski's inequality see [4, 5, 7, 8, 11, 12, 20, 23–26] and the references therein.

Let us consider now a bidimensional interval $\Delta =: [a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$, a mapping $f:\Delta\to\mathbb{R}$ is said to be convex on Δ if the inequality

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w) \le \lambda f(x,y)+(1-\lambda)f(z,w),$$

holds for all $(x,y),(z,w)\in\Delta$ and $\lambda\in[0,1]$. The mapping f is said to be concave on the co-ordinates on Δ if the above inequality holds in reversed direction, for all $(x,y),(z,w)\in\Delta$ and $\lambda\in[0,1]$.

A modification for convex (concave) functions on Δ, which is also known as co-ordinated convex (concave) functions, was introduced by S. S. Dragomir [9, 13] as follows:

A function $f:\Delta\to\mathbb{R}$ is said to be convex (concave) on the co-ordinates on Δ if the partial mappings $f_y:[a,b]\to\mathbb{R}, f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}, f_x(v)=f(x,v)$ are convex (concave) where defined for all $x\in[a,b], y\in[c,d]$.

A formal definition for co-ordinated convex (concave) functions may be stated in:

Definition 1.2. [18] A mapping $f:\Delta\to\mathbb{R}$ is said to be convex on the co-ordinates on Δ if the inequality

$$\begin{aligned}&f(tx+(1-t)y,ru+(1-r)w)\\&\le trf(x,u)+t(1-r)f(x,w)+r(1-t)f(y,u)+(1-t)(1-r)f(y,w),\end{aligned}\tag{2}$$

holds for all $t,r\in[0,1]$ and $(x,u),(y,w)\in\Delta$. The mapping f is concave on the co-ordinates on Δ if the inequality (2) holds in reversed direction for all $t,r\in[0,1]$ and $(x,y),(u,w)\in\Delta$.

Clearly, every convex (concave) mapping $f : \Delta \to \mathbb{R}$ is convex (concave) on the co-ordinates. Furthermore, there exists co-ordinated convex (concave) function which is not convex (concave), (see for instance [9, 13]).

The main result proved concerning the co-ordinated convex function from [9, 13] is given in:

Theorem 1.3. *[9] Suppose that $f : \Delta \to \mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequalities:*

$$\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
& \leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right] \\
& \leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\
& \leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right. \\
& \left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right] \\
& \leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \qquad (3)
\end{aligned}$$

The above inequalities are sharp. The inequalities in (3) hold in reverse direction if the mapping f is concave.

The concept of s-convex functions on the co-ordinates in the second sense was introduced by Alomari and Darus in [1] as a generalization of the co-ordinated convexity in:

Definition 1.4. [1] Consider the bidimensional interval $\Delta =: [a,b]\times[c,d]$ in $[0,\infty)^2$ with $a<b$ and $c<d$. The mapping $f : \Delta \to \mathbb{R}$ is s-convex in the second sense on Δ if

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\leq \lambda^s f(x,y)+(1-\lambda)^s f(z,w),$$

holds for all $(x,y),(z,w)\in\Delta$, $\lambda\in[0,1]$ with some fixed $s\in(0,1]$.

A function $f : \Delta =: [a,b]\times[c,d]\subseteq[0,\infty)^2\to\mathbb{R}$ is called s-convex in the second sense on the co-ordinates on Δ if the partial mappings $f_y : [a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ and $f_x : [c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$, are s-convex in the second sense for all $y\in[c,d]$, $x\in[a,b]$ and $s\in(0,1]$, i.e., the partial mappings f_y and f_x are s-convex in the second sense with some fixed $s\in(0,1]$.

A formal definition of co-ordinated s-convex function in second sense may be stated as follows:

Definition 1.5. A function $f:\Delta =: [a,b]\times[c,d]\subseteq[0,\infty]^2\to\mathbb{R}$ is called s-convex in the second sense on the co-ordinates on Δ if

$$\begin{aligned} & f(tx+(1-t)y, ru+(1-r)w) \\ & \leq t^s r^s f(x,u)+t^s(1-r)^s f(x,w)+r^s(1-t)^s f(y,u)+(1-t)^s(1-r)^s f(y,w), \end{aligned} \tag{4}$$

holds for all $t,r\in[0,1]$ and $(x,u),(y,w)\in\Delta$, for some fixed $s\in(0,1]$. The mapping f is s-concave on the co-ordinates on Δ if the inequality (4) holds in reversed direction for all $t,r\in[0,1]$ and $(x,y),(u,w)\in\Delta$ with some fixed $s\in(0,1]$.

In [5], Alomari et al. also proved a variant of inequalities given above by (3) for s-convex functions in the second sense on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$:

Theorem 1.6. [1] *Suppose $f:\Delta=[a,b]\times[c,d]\subseteq[0,\infty)^2\to[0,\infty)$ is s-convex function in the second sense on the co-ordinates on Δ. Then one has the inequalities:*

$$\begin{aligned} & 4^{s-1} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\ & \leq 2^{s-2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right] \\ & \leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\ & \leq \frac{1}{2(s+1)}\left[\frac{1}{b-a}\int_a^b [f(x,c)+f(x,d)]\,dx \right. \\ & \left. +\frac{1}{d-c}\int_a^b [f(a,y)+f(b,y)]\,dy\right] \\ & \leq \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{(s+1)^2}. \end{aligned} \tag{5}$$

In recent years, many authors have proved several inequalities for co-ordinated convex functions. These studies include, among others, the works in [1–3, 6, 9, 15, 17–20, 22, 27] (see also the references therein). Alomari et al. [1]-[3], proved several Hermite-Hadamard type inequalities for co-ordinated s-convex functions. Bakula et. al [6], proved Jensen's inequality for convex functions on the co-ordinates from the rectangle from the plan $\mathbb{R}^2$. Dragomir [9], proved the Hermite-Hadamard type inequalities for co-ordinated convex functions. Hwang et. al [15], also proved some Hermite-Hadamard type inequalities for co-ordinated convex function of two variables by considering some mappings directly associated with the Hermite-Hadamard type inequality for co-ordinated convex mappings of two variables. Latif et. al [17]-[20], proved some inequalities of Hermite-Hadamard type for differentiable co-ordinated convex functions, for product of two co-ordinated convex mappings, for co-ordinated h-convex mappings and also proved some Ostrowski type inequalities for co-ordinated convex mappings. Özdemir et. al [22], proved Hadamard's type inequalities for co-ordinated m-convex and (α,m)-convex functions. Sarikaya, et. al [27] proved Hermite-Hadamard type inequalities for differentiable co-ordinated convex function. For further inequalities on co-ordinated convex functions see also the references in the above cited papers.

In the present paper, we establish new Ostrowski type inequalities for co-ordinated s-convex functions in second sense similar to those from [20].

2. Main Results

To establish our main results we need the following identity:

Lemma 2.1. [20] *Let $f:\Delta\to\mathbb{R}$ be a twice partial differentiable mapping on Δ°. If $\frac{\partial^2 f}{\partial r\partial t}\in L(\Delta)$, then the following identity holds:*

$$\begin{aligned}
&f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\\
&=\frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1 rt\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)c)\,drdt\\
&-\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1 rt\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)d)\,drdt\\
&-\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1 rt\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)c)\,drdt\\
&+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1 rt\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)d)\,drdt, \qquad (6)
\end{aligned}$$

for all $(x,y)\in\Delta$, where

$$A=\frac{1}{d-c}\int_c^d f(x,v)\,dv+\frac{1}{b-a}\int_a^b f(u,y)\,du.$$

We begin with the following result:

Theorem 2.2. *Let $\Delta=[a,b]\times[c,d]\subseteq[0,\infty)^2\to\mathbb{R}$ be a twice partial differentiable mapping on Δ° such that $\frac{\partial^2 f}{\partial r\partial t}\in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial r\partial t}\right|$ is s-convex in the second* *sense on the co-ordinates on* Δ *with* $s\in(0,1]$ *and* $\left|\frac{\partial^2}{\partial r\partial t}f(x,y)\right|\le M$, $(x,y)\in\Delta$, *then the following inequality holds:*

$$\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\right| \le \frac{M}{(s+1)^2}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]\left[\frac{(y-c)^2+(d-y)^2}{d-c}\right], \tag{7}$$

for all $(x,y)\in\Delta$, *where* A *is defined in Lemma 2.1.*

Proof. By Lemma 2.1, we have that the following inequality holds:

$$\begin{aligned}
&\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\right|\\
&\le \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)c)\right|drdt\\
&+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)d)\right|drdt\\
&+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)c)\right|drdt\\
&+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)d)\right|drdt,
\end{aligned}\tag{8}$$

for all $(x,y)\in\Delta$.

Using the co-ordinated s-convexity of $\left|\frac{\partial^2 f}{\partial r\partial t}\right|$, we have that the following inequality holds:

$$\begin{aligned}
&\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)c)\right|drdt\\
&\le\left|\frac{\partial^2}{\partial r\partial t}f(x,y)\right|\int_0^1\int_0^1 t^{s+1}r^{s+1}drdt+\left|\frac{\partial^2}{\partial r\partial t}f(x,c)\right|\int_0^1\int_0^1 t^{s+1}r(1-r)^s\,drdt\\
&\quad+\left|\frac{\partial^2}{\partial r\partial t}f(a,y)\right|\int_0^1\int_0^1 r^{s+1}t(1-t)^s\,drdt\\
&\quad+\left|\frac{\partial^2}{\partial r\partial t}f(a,c)\right|\int_0^1\int_0^1 rt(1-t)^s(1-r)^s\,drdt.
\end{aligned}\tag{9}$$

Since

$$\int_0^1\int_0^1 t^{s+1}r^{s+1}drdt=\frac{1}{(s+2)^2},$$

$$\int_0^1\int_0^1 t^{s+1}r(1-r)^s\,drdt=\int_0^1\int_0^1 r^{s+1}t(1-t)^s\,drdt=\frac{1}{(s+1)(s+2)^2},$$

$$\int_0^1\int_0^1 rt(1-t)^s(1-r)^s\,drdt=\frac{1}{(s+1)^2(s+2)^2}$$

and

$$\left|\frac{\partial^2}{\partial r\partial t}f(x,y)\right|\leq M,(x,y)\in\Delta,$$

where we have used the Euler Beta function and its to evaluate the above integrals.
Hence from (9), we obtain

$$\begin{aligned}&\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)c)\right|drdt\\&\qquad\leq\frac{M}{(s+2)^2}+\frac{2M}{(s+1)(s+2)^2}+\frac{M}{(s+1)^2(s+2)^2}=\frac{M}{(s+1)^2}\end{aligned}\tag{10}$$

Analogously, we also have

$$\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)d)\right|drdt\leq\frac{M}{(s+1)^2},\tag{11}$$

$$\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)c)\right|drdt\leq\frac{M}{(s+1)^2}\tag{12}$$

and

$$\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)d)\right|drdt\leq\frac{M}{(s+1)^2}.\tag{13}$$

Now by making use of the inequalities (10)-(13) and the fact that

$$\begin{aligned}&(x-a)^2(y-c)^2+(x-a)^2(d-y)^2+(b-x)^2(y-c)^2+(b-x)^2(d-y)^2\\&=\left[(x-a)^2+(b-x)^2\right]\left[(y-c)^2+(d-y)^2\right],\end{aligned}$$

we get the inequality (7). This completes the proof. □

The corresponding version for powers of the absolute value of the partial derivative is incorporated in the following result:

Theorem 2.3. $\Delta=[a,b]\times[c,d]\subseteq[0,\infty)^2\to\mathbb{R}$ *be a twice partial differentiable mapping on* Δ° *such that* $\frac{\partial^2 f}{\partial r\partial t}\in L(\Delta)$. *If* $\left|\frac{\partial^2 f}{\partial r\partial t}\right|^q$ *is s-convex in the second sense* *on the co-ordinates on* Δ*,* $p,\ q>1$*,* $\frac{1}{p}+\frac{1}{q}=1$ *and* $\left|\frac{\partial^2}{\partial r\partial t}f(x,y)\right|\le M$*,* $(x,y)\in\Delta$*, then the following inequality holds:*

$$\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\right| \le \frac{M}{(1+p)^{\frac{2}{p}}}\left(\frac{2}{s+1}\right)^{\frac{2}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]\left[\frac{(y-c)^2+(d-y)^2}{d-c}\right], \tag{14}$$

for all $(x,y)\in\Delta$*, where A is defined in Lemma 2.1.*

Proof. By Lemma 2.1 and using the Hölder inequality for double integrals, we have that inequality holds:

$$\begin{aligned}
&\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\right|\le\left(\int_0^1\int_0^1 r^pt^p\,drdt\right)^{\frac{1}{p}}\\
&\times\left[\frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)c)\right|^q drdt\right)^{\frac{1}{q}}\right.\\
&+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)d)\right|^q drdt\right)^{\frac{1}{q}}\\
&+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,sy+(1-s)c)\right|^q drdt\right)^{\frac{1}{q}}\\
&\left.+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)d)\right|^q drdt\right)^{\frac{1}{q}}\right],
\end{aligned}\tag{15}$$

for all $(x,y)\in\Delta$.

Since $\left|\frac{\partial^2 f}{\partial r\partial t}\right|^q$ is s-convex in the second sense on the co-ordinates on Δ and $\left|\frac{\partial^2}{\partial r\partial t}f(x,y)\right|\le M$, $(x,y)\in\Delta$, we have

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,ry+(1-r)c)\right|^q drdt\\
&\le\left|\frac{\partial^2}{\partial r\partial t}f(x,y)\right|^q\int_0^1\int_0^1 t^sr^s\,drdt+\left|\frac{\partial^2}{\partial r\partial t}f(a,c)\right|^q\int_0^1\int_0^1(1-t)^s(1-r)^s\,drdt\\
&+\left|\frac{\partial^2}{\partial r\partial t}f(a,y)\right|^q\int_0^1\int_0^1 r^s(1-t)^s\,drdt+\left|\frac{\partial^2}{\partial r\partial t}f(x,c)\right|^q\int_0^1\int_0^1 t^s(1-r)^s\,drdt\\
&=\frac{4M^q}{(s+1)^2}.
\end{aligned}$$

Similarly, we also have the following inequalities:

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)d)\right|^q drdt\le\frac{4M^q}{(s+1)^2},$$

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)c)\right|^q drdt\le\frac{4M^q}{(s+1)^2}$$

and

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)d)\right|^q drdt\le\frac{4M^q}{(s+1)^2}.$$

Using the fact

$$\int_0^1\int_0^1 r^pt^p drdt=\frac{1}{(1+p)^2}$$

and the above inequalities in (15), we get (14). This completes the proof of the theorem. □

A different approach leads us to the following result:

Theorem 2.4. *Let* $\Delta=[a,b]\times[c,d]\subseteq[0,\infty)^2\to\mathbb{R}$ *be a twice partial differentiable mapping on* Δ° *such that* $\frac{\partial^2 f}{\partial r\partial t}\in L(\Delta)$. *If* $\left|\frac{\partial^2 f}{\partial r\partial t}\right|^q$ *is s-convex on the co-ordinates on* Δ, $q\ge1$ *and* $\left|\frac{\partial^2}{\partial r\partial t}f(x,y)\right|\le M$, $(x,y)\in\Delta$, *then the following inequality holds:*

$$\begin{aligned}&\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\right|\\&\le\frac{M}{4}\left(\frac{2}{s+1}\right)^{\frac{2}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]\left[\frac{(y-c)^2+(d-y)^2}{d-c}\right],\end{aligned}\tag{16}$$

for all $(x,y)\in\Delta$, *where* A *is defined in Lemma 2.1.*

Proof. Suppose $q\ge1$. From Lemma 2.1 and using the power mean inequality for double integrals, we have

$$\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\right|\leq\left(\int_0^1\int_0^1 rtdrdt\right)^{1-\frac{1}{q}}$$
$$\times\left[\frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1 tr\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)c)\right|^q drdt\right)^{\frac{1}{q}}\right.$$
$$+\frac{(x-a)^2(y-d)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1 tr\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)d)\right|^q drdt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1 tr\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)c)\right|^q drdt\right)^{\frac{1}{q}}$$
$$\left.+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1 tr\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)d)\right|^q drdt\right)^{\frac{1}{q}}\right] \tag{17}$$

for all $(x,y)\in\Delta$.
By similar argument as in Theorem 2.3 that $\left|\frac{\partial^2 f}{\partial r\partial t}\right|^q$ is s-convex on the co-ordinates on Δ in the second sense and $\left|\frac{\partial^2}{\partial r\partial t}f(x,y)\right|\leq M$, $(x,y)\in\Delta$, we have

$$\int_0^1\int_0^1 tr\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)c)\right|^q drdt$$
$$\leq\left|\frac{\partial^2}{\partial r\partial t}f(x,y)\right|^q\int_0^1\int_0^1 t^{s+1}r^{s+1}drdt$$
$$+\left|\frac{\partial^2}{\partial r\partial t}f(x,c)\right|^q\int_0^1\int_0^1 t^{s+1}r(1-r)^s drdt$$
$$+\left|\frac{\partial^2}{\partial r\partial t}f(a,y)\right|^q\int_0^1\int_0^1 t(1-t)^s r^{s+1}drdt$$
$$+\left|\frac{\partial^2}{\partial r\partial t}f(a,c)\right|^q\int_0^1\int_0^1 t(1-t)^s r(1+r)^s drdt$$
$$=\frac{M^q}{(s+2)^2}+\frac{M^q}{(s+1)(s+2)^2}+\frac{M^q}{(s+1)(s+2)^2}+\frac{M^q}{(s+1)^2(s+2)^2}$$
$$=\frac{M^q}{(s+1)^2}.$$

In a similar way, we also have that the following inequalities:

$$\int_0^1\int_0^1 tr\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)d)\right|^q drdt\leq\frac{M^q}{(s+1)^2}$$

$$\int_0^1 \int_0^1 tr \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)c) \right|^q drdt \leq \frac{M^q}{(s+1)^2}$$

and

$$\int_0^1 \int_0^1 tr \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)d) \right|^q drdt \leq \frac{M^q}{(s+1)^2}.$$

Now using the above inequalities and

$$\int_0^1 \int_0^1 rtdrdt = \frac{1}{4}$$

in (17), we get the desired inequality (16). This completes the proof. □

Remark 2.5. Since $(1+p)^{\frac{1}{p}} < 2$, $p > 1$ and accordingly, we have

$$\frac{1}{2} < \frac{1}{(1+p)^{\frac{1}{p}}}, \; p > 1$$

which gives

$$\frac{1}{4} < \frac{1}{(1+p)^{\frac{2}{p}}}, \; p > 1.$$

This reveals that the the inequality (16) gives tighter estimate than that of the inequality (14).

Remark 2.6. From the inequalities proved above in Theorem 2.2-Theorem 2.4, one can get several midpoint type inequalities by setting $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$. However the details are left to the interested reader.

Now we drive some results with co-ordinated s-concavity property instead of co-ordinated s-convexity.

Theorem 2.7. $\Delta = [a,b] \times [c,d] \subseteq [0,\infty)^2 \to \mathbb{R}$ *be a twice partial differentiable mapping on* Δ° *such that* $\frac{\partial^2 f}{\partial r \partial t} \in L(\Delta)$. *If* $\left| \frac{\partial^2 f}{\partial r \partial t} \right|^q$ *is s-concave on the co-ordinates* *on* Δ *and* p, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, *then the inequality*

$$\begin{aligned}
&\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\right|\\
&\quad\le\frac{4^{\frac{s-1}{q}}}{(1+p)^{\frac{2}{p}}(b-a)(d-c)}\left[(x-a)^2(y-c)^2\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{x+a}{2},\frac{c+y}{2}\right)\right|\right.\\
&\qquad+(x-a)^2(d-y)^2\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{x+a}{2},\frac{d+y}{2}\right)\right|\\
&\qquad+(b-x)^2(y-c)^2\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{b+a}{2},\frac{y+c}{2}\right)\right|\\
&\qquad\left.+(b-x)^2(d-y)^2\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{b+x}{2},\frac{d+y}{2}\right)\right|\right],
\end{aligned}\tag{18}$$

hods for all $(x,y)\in\Delta$, *where* A *is defined in Lemma 2.1.*

Proof. From Lemma 2.1 and using the Hölder inequality for double integrals, we have that inequality holds:

$$\begin{aligned}
&\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\right|\\
&\le\left(\int_0^1\int_0^1 r^pt^p\,drdt\right)^{\frac{1}{p}}\\
&\times\left[\frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)c)\right|^q drdt\right)^{\frac{1}{q}}\right.\\
&+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)d)\right|^q drdt\right)^{\frac{1}{q}}\\
&+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)c)\right|^q drdt\right)^{\frac{1}{q}}\\
&\left.+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)d)\right|^q drdt\right)^{\frac{1}{q}}\right],
\end{aligned}\tag{19}$$

for all $(x,y)\in\Delta$.

Since $\left|\frac{\partial^2 f}{\partial r\partial t}\right|^q$ is s-concave on the co-ordinates on Δ, so an application of (5) with inequalities in reversed direction, gives us the following inequalities:

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(tx+(1-t)a,ry+(1-r)c\right)\right|^q drdt \\
&\qquad\leq 2^{s-2}\left[\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(tx+(1-t)a,\frac{y+c}{2}\right)\right|^q dt\right. \\
&\qquad\quad\left.+\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{x+a}{2},ry+(1-r)c\right)\right|^q dr\right] \\
&\qquad\qquad\leq 4^{s-1}\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{x+a}{2},\frac{y+c}{2}\right)\right|^q, \qquad (20)
\end{aligned}$$

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(tx+(1-t)a,ry+(1-r)d\right)\right|^q dsdt \\
&\qquad\leq 2^{s-2}\left[\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(tx+(1-t)a,\frac{d+y}{2}\right)\right|^q dt\right. \\
&\qquad\quad\left.+\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{x+a}{2},ry+(1-r)c\right)\right|^q dr\right] \\
&\qquad\qquad\leq 4^{s-1}\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{x+a}{2},\frac{d+y}{2}\right)\right|^q, \qquad (21)
\end{aligned}$$

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(tx+(1-t)b,ry+(1-r)c\right)\right|^q drdt \\
&\qquad\leq 2^{s-2}\left[\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(tx+(1-t)a,\frac{y+c}{2}\right)\right|^q dt\right. \\
&\qquad\quad\left.+\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{b+x}{2},sy+(1-s)c\right)\right|^q dr\right] \\
&\qquad\qquad\leq 4^{s-1}\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{b+a}{2},\frac{y+c}{2}\right)\right|^q \qquad (22)
\end{aligned}$$

and

$$\begin{aligned}\int_0^1\int_0^1 &\left|\frac{\partial^2}{\partial r\partial t}f\left(tx+(1-t)b, ry+(1-r)d\right)\right|^q drdt \\ &\le 2^{s-2}\left[\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(tx+(1-t)b,\frac{d+y}{2}\right)\right|^q dt\right. \\ &\quad\left.+\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{b+x}{2}, ry+(1-r)d\right)\right|^q dr\right] \\ &\le 4^{s-1}\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{b+x}{2},\frac{d+y}{2}\right)\right|^q. \qquad (23)\end{aligned}$$

By making use of (20)-(23) in (19), we obtain (18). Thus the proof of the theorem is complete. □

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REFERENCES

[1] M. Alomari - M. Darus, *The Hadamard's inequality for s-convex function of 2-variables on the co-ordinates*, Int. Journal of Math. Analysis 2 (13) (2008), 629–638.

[2] M. Alomari - M. Darus, *The Hadamard's inequality for s-convex functions*, Int. Journal of Math. Analysis 2 (13) (2008), 639–646.

[3] M. Alomari - M. Darus, *Hadamard-Type Inequalities for s-convex functions*, International Mathematical Forum 3 (2008), no. 40, 1965 - 1975.

[4] M. Alomari - M. Darus - S. S. Dragomir - P. Cerone, *Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense*, Applied Mathematics Letters 23, Issue 9, September 2010, 1071–1076.

[5] M. Alomari - M. Darus, *Some Ostrowski type inequalities for convex functions with applications*, RGMIA 13 (1) (2010), Article 3.
http://ajmaa.org/RGMIA/v13n1.php

[6] M. K. Bakula - J. Pečarić, *On the Jensen's inequality for convex functions on the coordinates in a rectangle from the plane*, Taiwanese Journal of Math. 5 (2006), 1271–1292.

[7] N. S. Barnett - S. S. Dragomir, *An Ostrowski type inequality for double integrals and applications for cubature formulae*, Soochow J. Math. 27 (1) (2001), 109–114.

[8] P. Cerone - S. S. Dragomir, *Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions*, Demonstratio Math. 37 (2) (2004), 299–308.

[9] S.S. Dragomir, *On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane*, Taiwanese Journal of Mathematics 5 (2001), 775–788.

[10] S. S. Dragomir - S. Fitzpatrick, *The Hadamard's inequality for s-convex functions in the second sense*, Demonstratio Math. 32 (4) (1999), 687–696.

[11] S.S. Dragomir - A. Sofo, *Ostrowski type inequalities for functions whose derivatives are convex*, Proceedings of the 4th International Conference on Modelling and Simulation, November 11-13, 2002. Victoria University, Melbourne, Australia. RGMIA Res. Rep. Coll. 5 (2002), Supplement, Article 30. http://rgmia.vu.edu.au/v5(E).html

[12] S.S. Dragomir - N.S. Barnett - P. Cerone, *An n-dimensional version of Ostrowski's inequality for mappings of Hölder type*, RGMIA Res. Pep. Coll.2 (2) (1999), 169–180.

[13] S. S. Dragomir - C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000. Online: http://www.staff.vu.edu.au/RGMIA/monographs/hermite_hadamard.html

[14] H. Hudzik - L. Maligranda, *Some remarks on s-convex functions*, Aequationes Math. 48 (1994), 100–111.

[15] D. Y. Hwang - K.L. Tseng - G.S. Yang, *Some Hadamard's inequalities for co-ordinated convex functions in a rectangle from the plane*, Taiwanese Journal of Mathematics 11 (2007), 63–73.

[16] U. S. Kirmaci - M.K. Bakula - M.E. Ozdemir - J. Pečarić, *Hadamard-type inequalities for s-convex functions*, Appl. Math. and Compt. 93 (2007), 26–35.

[17] M. A. Latif - M. Alomari, *On Hadamard-type inequalities for h-convex functions on the co-ordinates*, International Journal of Math. Analysis 3 (33) (2009), 1645–1656.

[18] M. A. Latif - M. Alomari, *Hadamard-type inequalities for product two convex functions on the co-ordinates*, International Mathematical Forum 4 (47) (2009), 2327–2338.

[19] M. A. Latif - S. S. Dragomir, *On some new inequalities for differentiable co-ordinated convex functions*, RGMIA Research. Report. Collection 14 (2011), Article 45.[online:

[20] M. A. Latif - S. Hussain - S. S. Dragomir, *New Ostrowski type inequalities for co-ordinated convex functions*, RGMIA Research Report Collection 14 (2011), Article 49. http://www.ajmaa.org/RGMIA/v14.php

[21] A. M. Ostrowski, *Über die absolutabweichung einer differentiebaren funktion von*

ihrem integralmitelwert, Comment. Math. Helv. 10 (1938), 226–227.

[22] M. E. Özdemir - E. Set - M. Z. Sarıkaya, *Some new Hadamard's type inequalities for co-ordinated m-convex and* (α, m)*-convex functions*, Accepted.

[23] B. G. Pachpatte, *On an inequality of Ostrowski type in three independent variables*, J. Math. Anal. Appl. 249 (2000), 583–591.

[24] B. G. Pachpatte, *On a new Ostrowski type inequality in two independent variables*, Tamkang J. Math. 32 (1) (2001), 45–49

[25] B. G. Pachpatte, *A new Ostrowski type inequality for double integrals*, Soochow J. Math. 32 (2) (2006), 317–322.

[26] M. Z. Sarıkaya, *On the Ostrowski type integral inequality*, Acta Math. Univ. Comenianae LXXIX [1] (2010), 129–134.

[27] M. Z. Sarıkaya - E. Set - M. E. Özdemir - S. S. Dragomir, *New some Hadamard's type inequalities for co-ordinated convex functions*, Accepted.

[28] N. Ujević, *Some double integral inequalities and applications*, Appl. Math. E-Notes 7 (2007), 93–101.

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